# Nonlinear Poisson Equation

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## 1 Problem

Reference Paper: `https://bertoldi.seas.harvard.edu/files/bertoldi/files/jmps_bas.pdf`
PDE formualtion:

$$\nabla \cdot \left[\left(1 + 0.1 \times u^2\right) \nabla u\right] = f \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

The domain $\Omega$ is defined in polar coordinates using geometric parameters $r_\Omega = (c_1, c_2)$:

$$\theta \in [0, 2\pi]$$
$$r \in [0, r_0]\,,\; r_0 = 1 + c_1 \cos(4\theta) + c_2 \sin(8\theta)$$

The source term $f$ is determined given source parameters $r_S \in \mathbb{R}^{(2,3)}$:

$$r_S = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{bmatrix}$$

$$f(x, y) = r_{13} \cdot \exp\left[-(x - r_{11})^2 - (y - r_{12})^2\right] + r_{23} \cdot \exp\left[-(x - r_{21})^2 - (y - r_{22})^2\right]$$

The Dirichlet boundary condition (in polar coordinates) $g$ takes 4 parameters $r_B = (r_1, r_2, r_3, r_4)$:

$$g(\theta) = r_0 + \frac{1}{4}\left(r_1 \cos\theta + r_2 \sin\theta + r_3 \cos 2\theta + r_4 \cos 2\theta\right)$$

## 2 Variational Formulation

We multiply the PDE by a test function $v \in \hat{V}$ ($\hat{V} = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$), integrate over the domain, and integrate by parts:

$$\int_\Omega v\, \nabla \cdot \left[\left(1 + 0.1 \times u^2\right) \nabla u\right]\, dx = \int_\Omega v f\, dx$$
$$\int_\Omega \nabla \cdot \left[v \left(1 + 0.1 \times u^2\right) \nabla u\right]\, dx - \int_\Omega \left[\left(1 + 0.1 \times u^2\right) \nabla u\right] \cdot \nabla v\, dx = \int_\Omega v f\, dx$$

Apply divergence theorem on the first term of the LHS:

$$\int_{\Omega} \nabla \cdot \left[v \left(1 + 0.1 \times u^2\right) \nabla u\right] \, dx = \oint_{\partial\Omega} v \left(1 + 0.1 \times u^2\right) \nabla u \cdot \, d\mathbf{S}$$

Since we force $v$ to be 0 on the domain boundary, the term evaluates to 0. We then obtain the variational formulation for nonlinear poisson equation.

$$\int_{\Omega} \left(1 + 0.1 \times u^2\right) \nabla u \cdot \nabla v \, dx = \int_{\Omega} v f \, dx$$

Therefore, we need to find $F(u; v) = 0 \; \forall v \in \hat{V}$ where:

$$F(u; v) = \int_{\Omega} \left[\left(1 + 0.1 \times u^2\right) \nabla u \cdot \nabla v - v f\right] \, dx$$

# 3 NN Loss Function

**Loss in the domain**
For estimated potential function $\hat{\phi}(x, y)$, the loss in the domain is calculated as:

$$\mathcal{L}_{\Omega} = \frac{1}{n} \sum_{(x,y) \in S_{\Omega}} \{\nabla \cdot \left[\left(1 + 0.1 \times \hat{\phi}^2(x, y)\right) \nabla \hat{\phi}(x, y)\right] - f(x, y)\}^2$$

where $S_{\Omega}$ is the set of the randomly sampled points in the domain ($|S_{\Omega}| = n$).
**Loss on the boundary**
For estimated potential function $\hat{\phi}(x, y)$, the loss on the boundary is calculated as:

$$\mathcal{L}_{\partial\Omega} = \frac{1}{n} \sum_{(x,y) \in S_{\partial\Omega}} \left[g(x, y) - \hat{\phi}(x, y)\right]^2$$

where $S_{\partial\Omega}$ in the set of the randomly sampled points in the domain ($|S_{\partial\Omega}| = |S_{\Omega}| = n$).
**Total Loss Function**
Total loss function used in training is the sum of $\mathcal{L}_{\Omega}$ and $\mathcal{L}_{\partial\Omega}$ with $\mathcal{L}_{\partial\Omega}$ scaled (optionally) by hyper-parameter `bc_weight`:

$$\mathcal{L} = \mathcal{L}_{\Omega} + \texttt{bc\_weight} \times \mathcal{L}_{\partial\Omega}$$