Problem Set 1 — Practice for the Exam (Not for Submission)

Important

We **do not collect** this problem set, and you **do not need to do all exercises**.
We will cover the solutions in class and provide written solutions afterwards for your study.
The goal is to give you more material to prepare for the exam.

Problem 1 (Nonstationary bivariate system & non-invertibility). *Consider the non-stationary processes*

$$y_{1t} = y_{1,t-1} + \varepsilon_{1t} + \theta\varepsilon_{1,t-1}, \qquad y_{2t} = \phi y_{1t} + \varepsilon_{2t},$$

where $|\theta| < 1$ *and* $\phi \neq 0$.

(a) Show that the vector of first differences $[\Delta y_{1t}, \Delta y_{2t}]'$ *is a non-invertible MA(1) process.*

(b) Are there any observationally equivalent, invertible representations? Why or why not?

Problem 2 (Measurement error in MA(1)). *Let* y_t, $t = 1, \ldots, T$, *be a time series described by an MA(1) process:*

$$y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}, \quad \varepsilon_t \sim IID(0, \sigma_\varepsilon^2).$$

Suppose that y_t *is observed with* ***additive measurement noise****, i.e.,* $x_t = y_t + u_t$, *where* $u_t \sim IID(0, \sigma_u^2)$ *and* $\mathbb{E}(\varepsilon_t u_s) = 0$ *for all* t, s.

(a) Compute the mean and variance of x_t.

(b) Compute the autocovariances of x_t.

(c) Which $ARMA(p, q)$ *process best describes* x_t*?*

Problem 3 (Finite MA(q): stationarity and ACF truncation). *A finite-order MA(q) process can be expressed as*

$$y_t = \sum_{j=0}^{q} \theta_j \varepsilon_{t-j},$$

where ε_t *is white noise and* $\theta_0 = 1$. *Show that all finite-order moving-average processes:*

(a) are covariance stationary;

(b) have autocovariance functions that truncate at lag q, i.e.,

$$\mathbb{E}(y_t y_{t-k}) = \begin{cases} \Gamma_k \neq 0 & \text{for } k \leq q, \\ 0 & \text{for } k > q; \end{cases}$$

(c) have absolutely summable autocovariances (equivalently, ACFs),

$$\sum_{j=-\infty}^{\infty} |\Gamma_j| = \sum_{j=-q}^{q} |\Gamma_j| < \infty.$$

Problem 4 (Sum of two AR(1) processes)**.** *Let $\{y_t\}$ and $\{z_t\}$, $t = 1, \dots, T$, be two stochastic processes:*

$$y_t = \phi_1 y_{t-1} + \varepsilon_t, \quad z_t = \phi_2 z_{t-1} + u_t,$$

where $\varepsilon_t \sim IID(0, \sigma_\varepsilon^2)$, $u_t \sim IID(0, \sigma_u^2)$, and $\mathbb{E}(\varepsilon_t u_s) = 0$ for all t, s.

(a) Show that if $\phi_1 = \phi_2$, then $x_t = y_t + z_t$ is an AR(1) process.

(b) Show that if $\phi_1 \neq \phi_2$, then $x_t = y_t + z_t$ is an $ARMA(2, q)$ process with $q \leq 1$.

Problem 5 (MA(2): invertibility and observational equivalence)**.** *Consider a second-order moving average process*

$$x_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2},$$

where the innovation ε_t is iid $N(0, \sigma^2)$.

(a) State conditions on θ_1, θ_2 for the process to be invertible.

(b) Characterize the family of observationally equivalent, non-invertible representations.

Problem 6 (Sum of independent AR(1) processes)**.** *Consider two independent AR(1) processes x_t and y_t defined by:*

$$(1 - \rho_1 L)x_t = u_t, \ |\rho_1| < 1 \quad \text{and} \quad (1 - \rho_2 L)y_t = v_t, \ |\rho_2| < 1$$

where $\mathbb{E}(u_t) = \mathbb{E}(v_t) = 0$, $\mathbb{E}(u_t v_\tau) = 0$ for all t, τ, $\mathbb{E}(u_t u_\tau) = \sigma_u^2$ if $t = \tau$ (0 otherwise), and $\mathbb{E}(v_t v_\tau) = \sigma_v^2$ if $t = \tau$. Let $z_t = x_t + y_t$.

(a) Compute $\mathbb{E}(x_{t+1}|\mathcal{F}_t)$, $\mathbb{E}(y_{t+1}|\mathcal{F}_t)$, $\mathbb{E}(z_{t+1}|\mathcal{F}_t)$, and the forecast error variance V_D.

(b) Show that z_t is ARMA(2,1) and find θ.

(c) Give conditions for z_t to be AR(2). Show that if ρ_1, ρ_2 have the same sign, then it cannot be AR(2).

(d) Show that $\theta \neq \pm 1$.

Problem 7 (Partial adjustment vs. adaptive expectations)**.** *The desired level y_t^* is related to x_t by $y_t^* = \beta x_t$. Assume x_t is covariance-stationary. A fraction γ of the disequilibrium is removed through partial adjustment:*

$$y_t - y_{t-1} = \gamma(y_t^* - y_{t-1}) + e_t, \quad e_t \sim N(0, \sigma_e^2),\ 0 < \gamma < 1.$$

Compare with the adaptive expectations model:

$$y_t = \beta x_{t+1}^e + v_t, \quad v_t \sim N(0, \sigma_v^2),$$

$$x_{t+1}^e - x_t^e = \lambda(x_t - x_t^e), \quad 0 < \lambda < 1.$$

(a) Compare the two models (their reduced forms and error terms).

(b) Show that OLS of y_t on y_{t-1} and x_t is consistent for the partial adjustment model, but inconsistent for the adaptive expectations model.

Problem 8 (Vector model: inflation, output gap, and policy)**.** *Consider the macroeconomic model*

$$\begin{aligned}
(1):&\quad \pi_t = \lambda y_t + \pi_t^e + u_{1t}, \quad 0 < \lambda < 1,\\
(2):&\quad y_t = \gamma(i_{t-1} - \pi_t^e) + u_{2t}, \quad -1 < \gamma < 0,\\
(3):&\quad \pi_t^e = \pi_{t-1},\\
(4):&\quad i_t = i^* + \rho(\pi_t - \pi^*), \quad \rho \geq 0,
\end{aligned}$$

where π_t is inflation, y_t is the output gap, i_t is the nominal interest rate, and $(u_{1t}, u_{2t})' \sim N(0, \Sigma)$.

(a) Show that π_t follows an $ARMA(p, q)$. Relate the coefficients to the structural parameters.

(b) Consider $\rho = 0, 1, 2$. What can be said about inflation and monetary policy?

(c) Compute $\hat{\pi}_{t+j|t} = \mathbb{E}[\pi_{t+j}|\mathcal{F}_t]$ and $\mathrm{Var}(\pi_{t+j} - \hat{\pi}_{t+j|t})$.

(d) Let $\mathbf{x}_t = (\pi_t, y_t)'$. Find $\mathbb{D}(\mathbf{x}_t|\mathbf{x}_{t-1})$ and $\mathbb{E}(\pi_t|y_t, \mathbf{x}_{t-1})$.

(e) What would change if $\pi_t^e = \alpha\pi_{t-1} + (1-\alpha)\pi_{t-1}^e$?