Problem Set 2 — State space, trends, and policy regime change

Important

We **do not collect** this problem set, and you **do not need to do all exercises**. We will cover the solutions in class and provide written solutions afterwards for your study. The goal is to give you more material to prepare for the exam.

Problem 1 (Long-run risk: state-space and likelihood)**.** *Consider a version of Bansal and Yaron's (2004) long-run risk model,*

$$x_t = \rho x_{t-1} + \sigma_\varepsilon \varepsilon_t,$$
$$g_t = x_t + \sigma_\eta \eta_t,$$

where $(\varepsilon_t, \eta_t)'$ is an i.i.d. standard normal random vector. The free parameters are

$$\theta = [\rho, \sigma_\varepsilon, \sigma_\eta]'.$$

(a) Derive a VAR representation for $(x_t, g_t)'$. What condition on θ is necessary for covariance stationarity?

(b) Assuming that the stationarity condition is satisfied and that x_t and g_t are both observable, explain how you would estimate θ by maximum likelihood.

(c) Now assume that g_t is observable but x_t is not. Assuming stationarity, explain how you would estimate θ by maximum likelihood.

Problem 2 (Beveridge–Nelson stochastic trend via a VAR(2))**.** *Beveridge and Nelson (1981) define a stochastic trend τ_t as the level to which a series x_t is expected to converge in the long run,*

$$\tau_t \; = \; x_t + E_t \sum_{j=1}^{\infty} (\Delta x_{t+j} - \mu_x),$$

where μ_x is the unconditional mean of Δx_t. Suppose that $y_t = [\Delta x_t, w_t']'$, where w_t is a vector of variables that might be helpful for predicting Δx_t. Assume that y_t is covariance stationary and for simplicity that its unconditional mean is zero. Furthermore, suppose that y_t is well approximated by a VAR(2) representation with homoskedastic gaussian innovations.

(a) Explain how you choose variables to include in w_t.

(b) How would you estimate the VAR parameters by MLE?

(c) Derive an expression for the trend τ_t in terms of the VAR parameters and current and lagged values of y_t.

(d) How would you approximate the asymptotic distribution of the coefficients mapping current and lagged values of y_t into τ_t?

(e) Prove that τ_t is a random walk (with drift if $\mu_{\Delta x} \neq 0$).

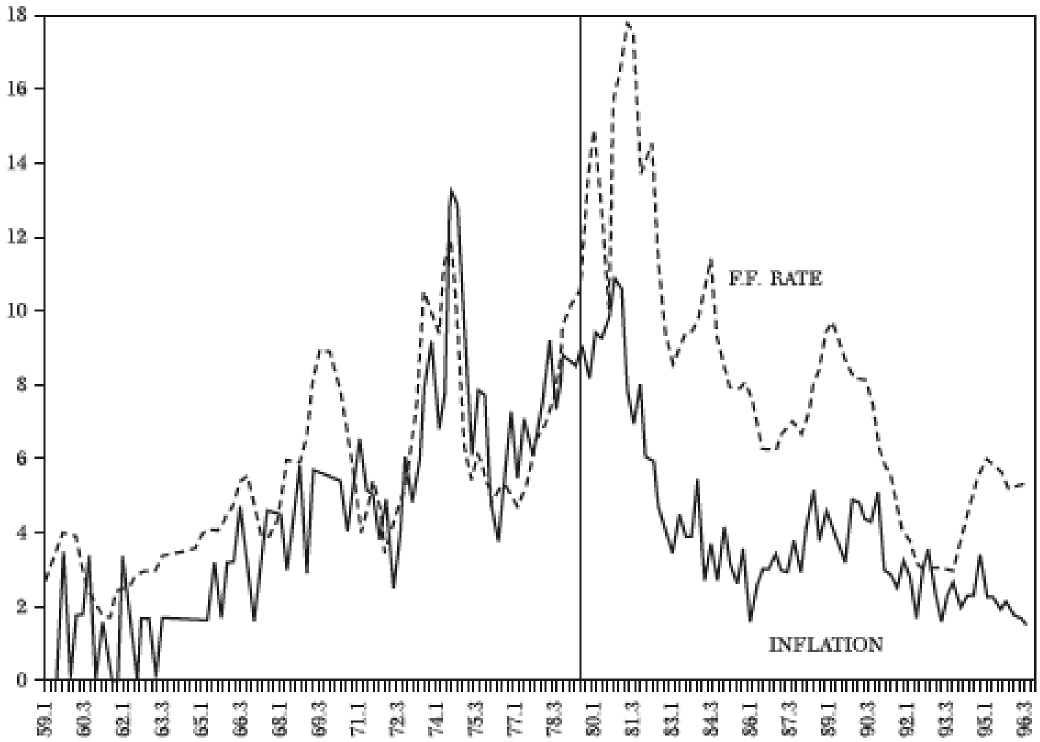

Figure 1: Federal funds rate and inflation (see Clarida, Galí, and Gertler, 1999).

Problem 3 (Clarida–Galí–Gertler (1999))**.** *A common interpretation of U.S. monetary history is that policy became more aggressive against inflation from the Burns/Miller era to the Volcker era. In a structural New Keynesian (NK) model, changes in the Taylor-rule parameter can alter the reduced-form persistence of inflation, which is closely related to the Lucas critique. Consider the 3-equation NK model:*

$$\text{(IS)} \qquad x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\Big(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n\Big), \tag{1}$$

$$\text{(NKPC)} \qquad \pi_t = \beta\, \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t, \tag{2}$$

$$\text{(Taylor)} \qquad i_t = r_t^n + \phi_\pi \pi_t, \qquad \phi_\pi > 0, \tag{3}$$

where x_t is the output gap, π_t inflation, i_t the nominal interest rate, and u_t a cost-push shock. The discount factor satisfies

$$\beta = \frac{1}{1+\rho}, \qquad \rho > 0.$$

Assume throughout:

(A1) $r^n_t = r^n$ *is constant;*

(A2) $\mathbb{E}_t x_{t+1} = 0$*;*

(A3) $\mathbb{E}_t \pi_{t+1} = \pi_{t-1}$*;*

(A4) u_t *is i.i.d. with* $\mathbb{E}[u_t] = 0$ *and* $\mathrm{Var}(u_t) = \sigma_u^2$*.*

(a) ***Derive the reduced-form AR(1) for inflation and relate it to policy.*** *Eliminate* x_t *and* i_t *in (1)–(3) and show that inflation can be written as an AR(1):*

$$\pi_t = a(\phi_\pi, \kappa, \rho)\, \pi_{t-1} + \varepsilon_t,$$

for some innovation ε_t*. Give explicit expressions for* $a(\phi_\pi, \kappa, \rho)$ *and* ε_t*. Then determine the sign of* $\partial a / \partial \phi_\pi$ *and interpret it (one sentence).*

(b) ***Burns vs. Volcker: structural hypothesis stated as a reduced-form restriction.*** *Let* T_b *be a break date (e.g. 1979:Q3). Assume* $(\beta, \kappa, \sigma, \sigma_u^2, r^n)$ *are constant over time, but policy may change:*

$$\phi_\pi = \begin{cases} \phi_{\pi,1}, & t \le T_b, \\ \phi_{\pi,2}, & t > T_b. \end{cases}$$

Using your mapping from part (a), translate "policy changed from Burns to Volcker" into a null and alternative hypothesis in terms of the reduced-form persistence parameters a_1 *and* a_2*.*

(c) ***Reduced-form likelihood test with dependent data.*** *Assume the reduced form holds with a break in the AR(1) coefficient:*

$$\pi_t = c + a_1 \pi_{t-1} + \varepsilon_t, \quad t \le T_b, \qquad \pi_t = c + a_2 \pi_{t-1} + \varepsilon_t, \quad t > T_b,$$

with

$$\varepsilon_t \mid \mathcal{F}_{t-1} \sim \mathcal{N}(0, \sigma^2), \qquad \mathcal{F}_{t-1} = \sigma(\pi_{t-1}, \pi_{t-2}, \ldots).$$

(i) *Write the conditional log-likelihood* $\ell_n(\theta)$ *conditioning on* π_0*, where* $\theta = (c, a_1, a_2, \sigma^2)$*.*

(ii) *Describe an LR test of* $H_0 : a_1 = a_2$ *and state its asymptotic null distribution.*

(d) ***Wald and LM versions of the same restriction.*** *Let* $r(\theta) = a_1 - a_2$*.*

(i) *Write the Wald statistic for testing* $r(\theta) = 0$ *using the unrestricted MLE.*

(ii) *Write the LM statistic in the standard score–information form evaluated at the restricted MLE (no need to simplify fully).*

(e) ***Interpretation.*** *If you estimate* $\hat{a}_2 < \hat{a}_1$ *and reject* $H_0 : a_1 = a_2$*, interpret the result in terms of* $\phi_{\pi,2}$ *vs.* $\phi_{\pi,1}$ *and monetary policy aggressiveness.*

(f) ***Lucas critique (short).*** *Explain why an AR(1) estimated under the Burns regime may fail to forecast inflation under the Volcker regime. In your answer, identify which objects are "structural" and which are "reduced-form" in this problem.*

Problem 4 (Kalman filter with correlated state and measurement innovations)**.** *Consider a state-space model with correlated state and measurement innovations,*

$$S_t = AS_{t-1} + B\varepsilon_{1t}, \qquad X_t = CS_t + D\varepsilon_{2t},$$

where X_t is an $m \times 1$ vector of observables and S_t is a $n \times 1$ vector that may include latent variables. The innovations ε_{1t} and ε_{2t} have dimensions $p \times 1$ and $q \times 1$, respectively, and are i.i.d. normal with mean zero and covariance matrix

$$E\left(\begin{bmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{bmatrix}[\varepsilon_{1t}', \varepsilon_{2t}']\right) = \begin{bmatrix} Q & F \\ F' & R\end{bmatrix}.$$

Derive the Kalman filter for this model.

Problem 5 (ARMA(2,2) $\rightarrow$ state space)**.** *Suppose y_t has an ARMA(2,2) representation*

$$(1 - \rho_1 L - \rho_2 L^2)y_t = (1 + \theta_1 L + \theta_2 L^2)\varepsilon_t,$$

where ε_t is i.i.d. $N(0, \sigma^2)$. Put this into state-space form and verify your solution.