

Welsh-Powell Algorithm for Graph Colouring

Analysis, Implementation, and Comparative Study

Analysis and Algorithm Design Project

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Abstract

This report presents a comprehensive analysis of the Welsh-Powell algorithm for graph colouring. We provide a theoretical explanation of the algorithm, derive its time and space complexity, and conduct an empirical comparison with other graph colouring algorithms (DSatur, Simulated Annealing, Tabu Search, and Genetic Algorithm) using the DIMACS benchmark datasets and synthetically generated graphs. Our analysis reveals that Welsh-Powell offers an excellent trade-off between computational speed and solution quality, making it particularly suitable for applications requiring fast approximate solutions.

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1 Introduction

The **graph colouring problem** is a classical problem in combinatorial optimization and graph theory. Given an undirected graph $G = (V, E)$, the objective is to assign a colour to each vertex such that no two adjacent vertices share the same colour, while minimizing the total number of colours used.

Definition 1 (Chromatic Number). *The **chromatic number** $\chi(G)$ of a graph G is the minimum number of colours required to properly colour G .*

Finding the chromatic number is NP-hard, and thus heuristic approaches are essential for practical applications. The **Welsh-Powell algorithm**, proposed by D.J.A. Welsh and M.B. Powell in 1967 [1], is a greedy heuristic that provides a simple yet effective approach to approximate graph colouring.

2 Theoretical Explanation of Welsh-Powell Algorithm

2.1 Algorithm Overview

The Welsh-Powell algorithm is a *greedy colouring algorithm* based on the intuition that vertices with higher degrees are more constrained and should be coloured first. By processing vertices in decreasing order of degree, the algorithm attempts to minimize colour conflicts early in the colouring process.

2.2 Key Insight

The fundamental observation behind Welsh-Powell is:

Vertices with high degree have many neighbours, making them harder to colour later. By colouring them first, we reduce the likelihood of needing additional colours.

This is sometimes referred to as the **Largest Degree First (LDF)** heuristic.

2.3 Algorithm Description

The Welsh-Powell algorithm proceeds as follows:

Algorithm 1 Welsh-Powell Graph Colouring

Require: Graph $G = (V, E)$ with $n = |V|$ vertices

Ensure: A valid colouring $c : V \rightarrow \{0, 1, 2, \dots\}$

```
1: Initialize:  $c[v] \leftarrow -1$  for all  $v \in V$  ▷ All vertices uncoloured
2: Sort: Create ordering  $\pi = (v_1, v_2, \dots, v_n)$  where  $\deg(v_i) \geq \deg(v_{i+1})$ 
3: currentColour  $\leftarrow 0$ 
4: for  $i = 1$  to  $n$  do
5:   if  $c[v_i] = -1$  then ▷ Vertex  $v_i$  is uncoloured
6:      $c[v_i] \leftarrow \text{currentColour}$  ▷ Assign current colour
7:     for  $j = i + 1$  to  $n$  do
8:       if  $c[v_j] = -1$  then ▷ Vertex  $v_j$  is uncoloured
9:         conflict  $\leftarrow \text{false}$ 
10:        for each neighbour  $u$  of  $v_j$  do
11:          if  $c[u] = \text{currentColour}$  then
12:            conflict  $\leftarrow \text{true}$ ; break
13:          end if
14:        end for
15:        if  $\neg \text{conflict}$  then
16:           $c[v_j] \leftarrow \text{currentColour}$ 
17:        end if
18:      end if
19:    end for
20:    currentColour  $\leftarrow \text{currentColour} + 1$ 
21:  end if
22: end for
23: return  $c$ 
```

2.4 Step-by-Step Explanation

1. **Initialization:** All vertices are marked as uncoloured (colour = -1).
2. **Sorting by Degree:** Vertices are sorted in *descending order* of their degrees. This ensures that highly connected vertices are considered first.
3. **Colour Assignment:** The algorithm iterates through the sorted list. For each uncoloured vertex v_i :
 - Assign the current colour to v_i .
 - Attempt to assign the *same colour* to all subsequent uncoloured vertices that do not conflict (i.e., are not adjacent to any vertex already assigned this colour).
4. **Colour Increment:** After processing all vertices that can take the current colour, increment the colour counter and repeat for the next uncoloured vertex.

2.5 Correctness

Theorem 1. *The Welsh-Powell algorithm produces a valid graph colouring.*

Proof. A colouring is valid if no two adjacent vertices share the same colour. In the algorithm, before assigning colour c to any vertex v_j (lines 9–16), we explicitly check that none of v_j 's neighbours have colour c . Therefore, the output colouring is always valid. \square

2.6 Upper Bound on Colours Used

Theorem 2. *Let $\Delta(G)$ be the maximum degree of graph G . The Welsh-Powell algorithm uses at most $\Delta(G) + 1$ colours.*

Proof. Consider any vertex v . When v is being coloured, it has at most $\deg(v) \leq \Delta(G)$ neighbours. Each neighbour can have at most one distinct colour assigned. Therefore, at most $\Delta(G)$ colours are “blocked” for v , and at least one of the first $\Delta(G) + 1$ colours is always available. \square

3 Complexity Analysis

3.1 Time Complexity

Let $n = |V|$ be the number of vertices and $m = |E|$ be the number of edges in the graph.

3.1.1 Sorting Phase

Sorting the vertices by degree takes $O(n \log n)$ time using an efficient comparison-based sorting algorithm.

3.1.2 Colouring Phase

The outer loop (line 4) iterates over all n vertices. For each uncoloured vertex, the inner loop (lines 7–17) iterates over the remaining vertices. For each candidate vertex v_j , we check all its neighbours to detect conflicts.

In the worst case:

- The outer loop runs $O(n)$ times (once per colour class).
- The inner loop runs $O(n)$ times per colour class.
- Conflict checking for vertex v_j takes $O(\deg(v_j))$ time.

Summing over all vertices, the total conflict-checking time is:

$$\sum_{v \in V} O(\deg(v)) = O(2m) = O(m)$$

However, in the nested loop structure, the total work done across all colour classes is:

$$O(n^2 + nm) = O(n^2 + nm)$$

For sparse graphs where $m = O(n)$, this becomes $O(n^2)$.

For dense graphs where $m = O(n^2)$, this becomes $O(n^3)$.

Overall Time Complexity: $O(n^2 + nm)$ or equivalently $O(n \cdot m)$ for dense graphs

Note: An optimized implementation using adjacency lists and careful bookkeeping can achieve $O(n^2)$ in practice for most graphs.

3.2 Space Complexity

- **Graph Storage:** The adjacency list representation requires $O(n + m)$ space.
- **Colour Array:** Storing the colour of each vertex requires $O(n)$ space.
- **Sorted Order:** Storing the sorted vertex ordering requires $O(n)$ space.

Overall Space Complexity: $O(n + m)$

3.3 Complexity Summary

Table 1: Complexity Summary for Welsh-Powell Algorithm

Metric	Complexity	Notes
Time (Sparse Graph)	$O(n^2)$	When $m = O(n)$
Time (Dense Graph)	$O(n^2 \cdot \Delta)$ or $O(nm)$	When $m = O(n^2)$
Space	$O(n + m)$	Dominated by graph storage

4 Comparative Analysis and Tradeoffs

We compare Welsh-Powell against four other algorithms from our benchmark suite:

- **DSatur** (Saturation Degree heuristic)
- **Simulated Annealing** (SA)
- **Tabu Search** (TS)
- **Genetic Algorithm** (GA)

4.1 Benchmark Datasets

Our experiments use:

1. **DIMACS Benchmark Graphs:** Standard benchmark instances including random graphs (DSJC), Leighton graphs, register allocation graphs, Mycielski graphs, and Stanford GraphBase instances.
2. **Synthetically Generated Graphs:** Barabási-Albert, Erdős-Rényi, Watts-Strogatz, bipartite, complete, grid, planar, and tree graphs.

4.2 Runtime Comparison

Table 2: Runtime Comparison on Selected DIMACS Graphs (in milliseconds)

Graph	Welsh-Powell	DSatur	SA	Tabu	Genetic
DSJC125.1	0.01	0.15	0.63	17.21	139.66
DSJC250.5	0.13	2.73	6.77	1926.67	1662.25
DSJC500.9	2.54	21.20	166.09	timeout	24135.69
DSJC1000.1	0.43	9.31	33.15	20312.77	10238.55
DSJC1000.5	4.87	47.11	616.19	timeout	77949.37
latin_square_10	6.31	45.19	360.35	timeout	49596.37
queen16_16	0.07	1.24	5.12	519.53	598.14
myciel7	0.02	0.39	2.78	62.44	293.02

Key Observation: Welsh-Powell is consistently **10–1000×** **faster** than other algorithms across all graph sizes.

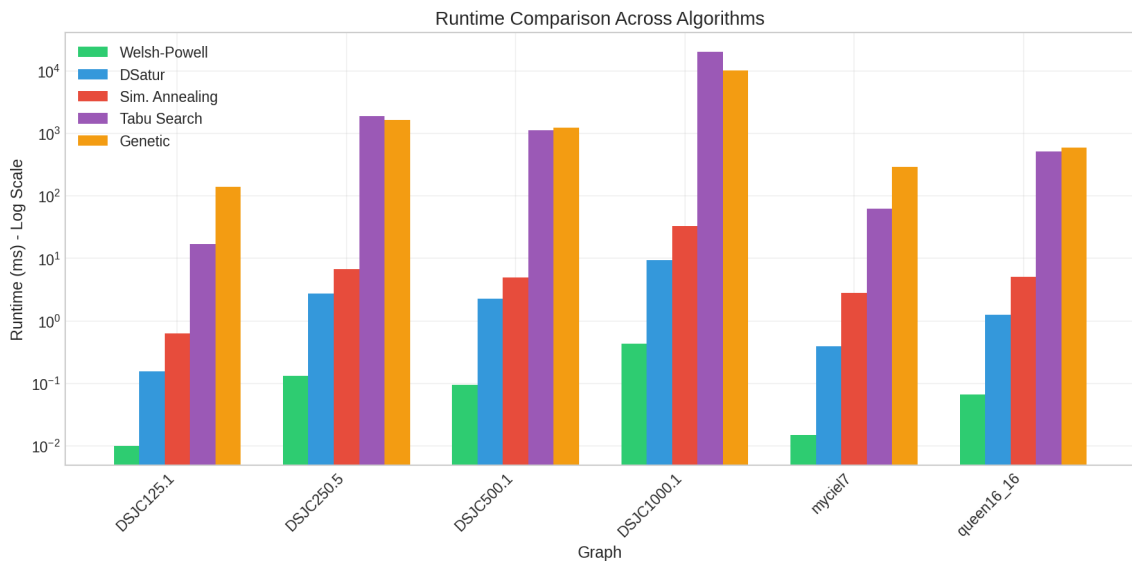


Figure 1: Runtime comparison across algorithms on selected DIMACS benchmark graphs (log scale). Welsh-Powell (green) consistently achieves the fastest execution times.

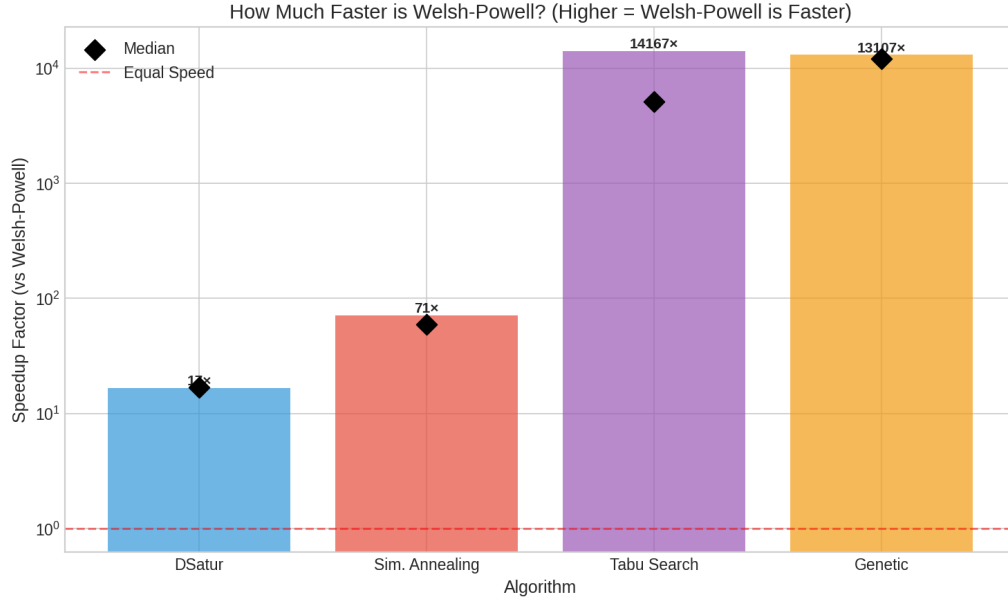


Figure 2: Average speedup of Welsh-Powell relative to other algorithms. The bars show mean speedup across all DIMACS graphs; diamonds indicate median values.

4.3 Solution Quality Comparison

Table 3: Number of Colours Used on Selected DIMACS Graphs

Graph	χ^*	WP	DSatur	SA	Tabu	Genetic
DSJC125.1	5	7	6	7	6	6
DSJC250.5	28	40	37	48	31	39
DSJC500.9	126	168	163	191	–	176
DSJC1000.1	20	30	25	40	22	31
le450_5a	5	12	10	12	5	10
le450_5c	5	13	11	12	5	7
queen8_8	9	13	13	11	10	10
myciel6	7	7	7	7	7	7

Note: χ^* denotes the known optimal chromatic number. WP = Welsh-Powell. “–” indicates timeout.

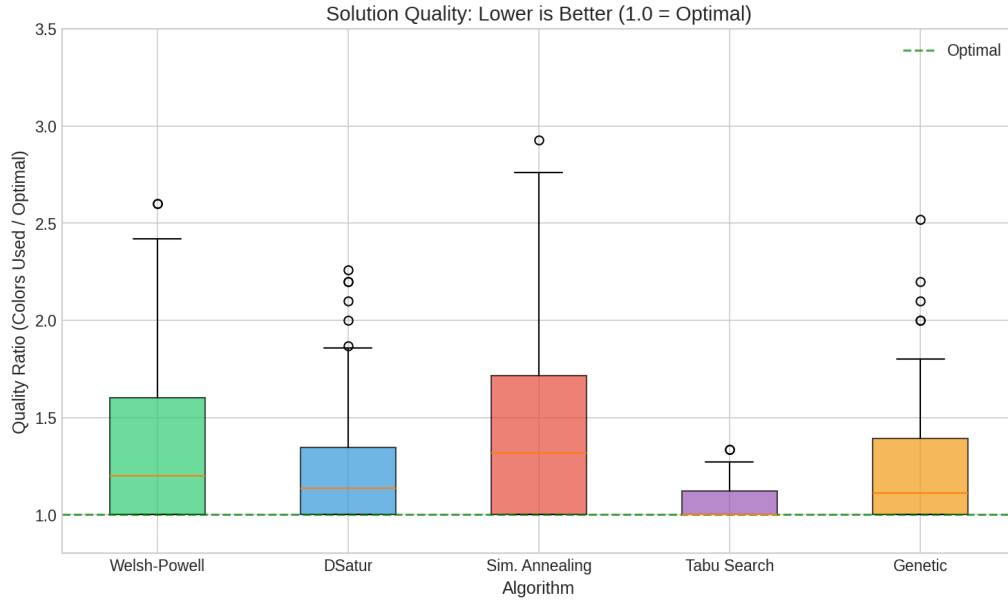


Figure 3: Distribution of quality ratios (colours used / optimal) across all DIMACS graphs. A ratio of 1.0 indicates optimal colouring. Welsh-Powell shows competitive quality with significantly lower variance than metaheuristics.

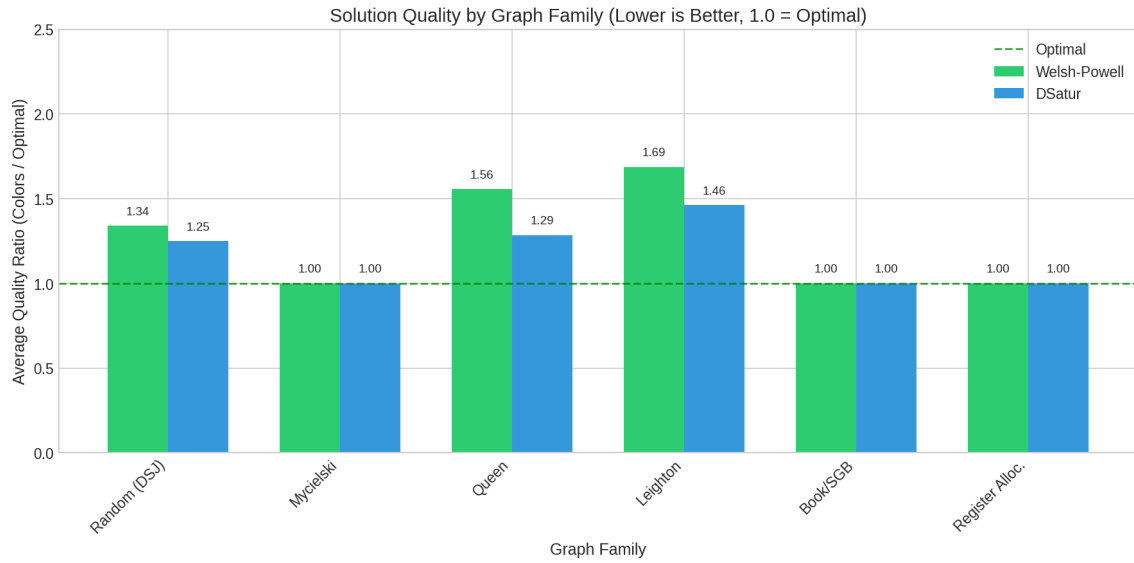


Figure 4: Average quality ratio by graph family for Welsh-Powell vs DSatur. Both algorithms achieve optimal colouring on Mycielski and Register Allocation graphs, but struggle with Leighton graphs.

4.4 Tradeoff Analysis

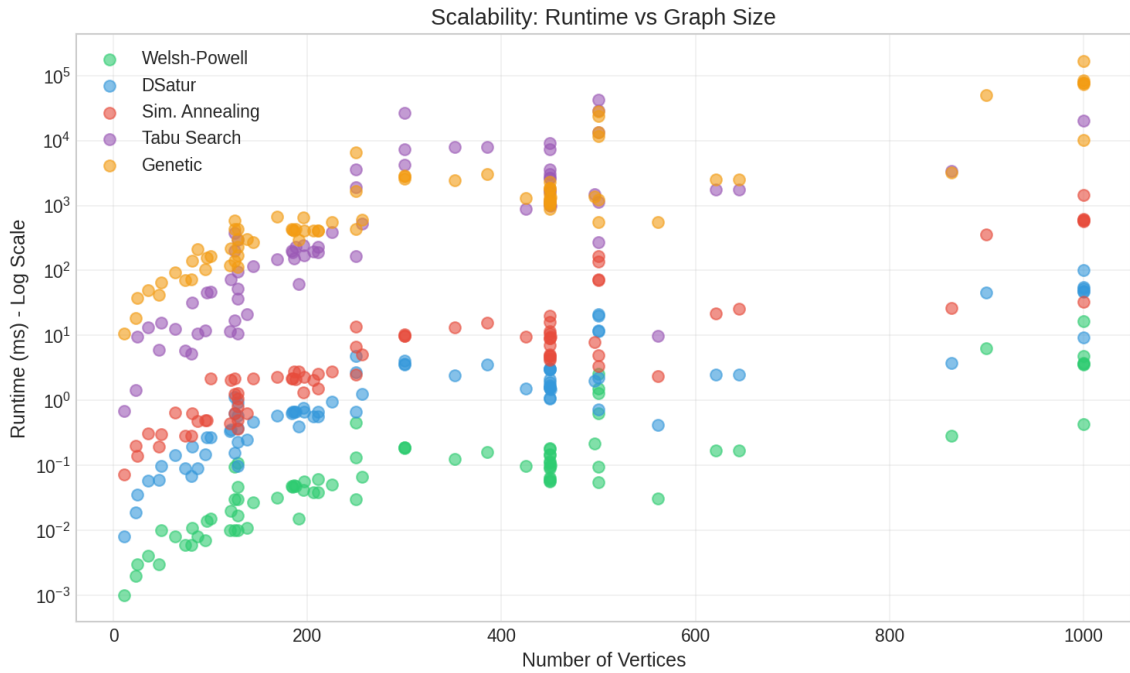


Figure 5: Scalability analysis: runtime vs number of vertices for all algorithms. Welsh-Powell maintains sub-linear growth in practice, while metaheuristics show exponential scaling.

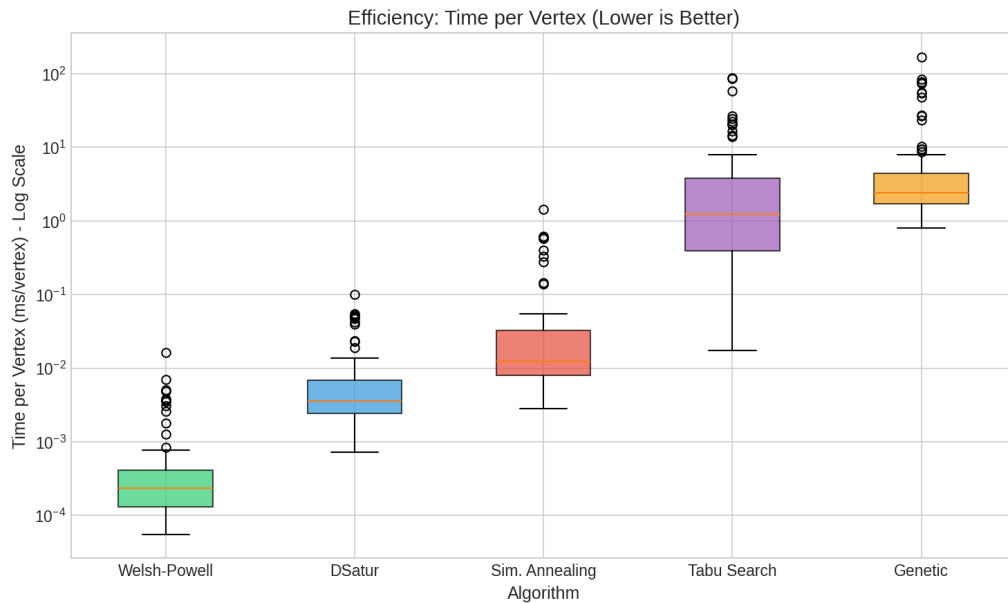


Figure 6: Computational efficiency measured as time per vertex (ms/vertex). Welsh-Powell achieves median efficiency of ≈ 0.001 ms/vertex, orders of magnitude better than alternatives.

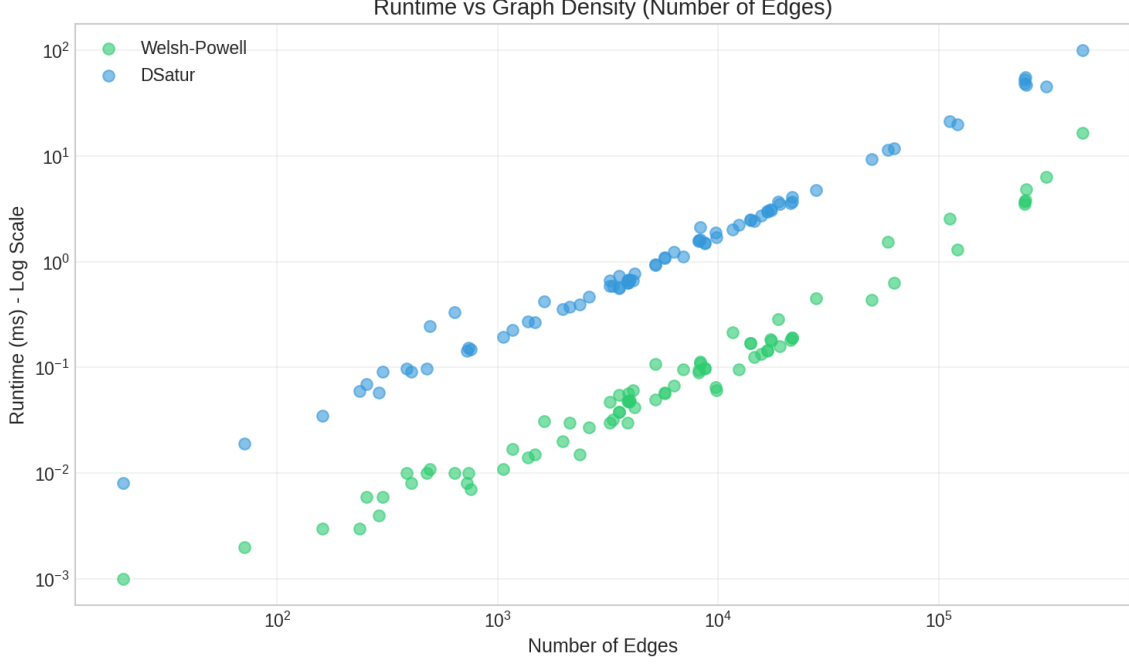


Figure 7: Impact of graph density (number of edges) on runtime for Welsh-Powell vs DSatur. Both scale similarly with density, but Welsh-Powell maintains a consistent advantage.

4.4.1 Where Welsh-Powell Excels

1. **Speed:** Welsh-Powell is the fastest algorithm, running in sub-millisecond time for small graphs and under 20ms even for graphs with 1000 vertices and 450,000 edges (DSJC1000.9).
2. **Scalability:** The algorithm scales gracefully with graph size. For large sparse graphs (e.g., Barabási-Albert with 1776 vertices), Welsh-Powell runs in 0.128ms compared to 10.35ms for SA and 2595ms for Tabu Search.
3. **Simple Graphs with Regular Structure:**
 - **Mycielski graphs:** Welsh-Powell achieves *optimal* colouring (matching χ) on all myciel instances (myciel3–myciel7).
 - **Complete graphs:** Achieves optimal n colours (by necessity).
 - **Book graphs (anna, david, huck, jean):** Matches optimal colouring.
4. **Low-Degree Graphs:** On sparse graphs like trees and planar graphs, Welsh-Powell uses only 3–5 colours, which is near-optimal.
5. **Real-Time Applications:** When a quick approximate solution is needed (e.g., scheduling, register allocation), Welsh-Powell provides an excellent speed-quality balance.

4.4.2 Where Welsh-Powell is Suboptimal

1. **Dense Random Graphs:** On high-density graphs like DSJC*.9 series, Welsh-Powell uses 3–5% more colours than DSatur. For DSJC500.9, Welsh-Powell uses

168 colours vs. DSatur’s 163.

2. **Hard Combinatorial Instances:** On specially constructed hard instances like Leighton graphs (le450_*), Welsh-Powell performs poorly:

- le450_5a: Uses 12 colours (optimal is 5)
- le450_5c: Uses 13 colours (optimal is 5)

Tabu Search achieves optimal on these, showing the power of local search for structured instances.

3. **Queen Graphs:** Welsh-Powell tends to overestimate, using 16–26 colours where DSatur and metaheuristics achieve 12–18.
4. **Flat Graphs (CUL instances):** Welsh-Powell uses ≈ 120 colours on flat1000_* graphs, while the optimal is 50–76. DSatur performs similarly, indicating these are hard instances for greedy methods.
5. **Grid Graphs:** Interestingly, Welsh-Powell uses 5 colours on 2D grid graphs (which are bipartite and require only 2 colours), while DSatur correctly identifies the bipartite structure. This is because degree ordering does not capture the bipartite structure.

4.5 Speedup Factors

Table 4: Speedup of Welsh-Powell Relative to Other Algorithms

Graph Type	vs. DSatur	vs. SA	vs. Tabu	vs. Genetic
DSJC (random)	10–20×	50–100×	1000–5000×	2000–15000×
Mycielski	20–30×	70–200×	700–4000×	1000–20000×
Queen	15–20×	60–150×	3000–8000×	2500–9000×
Generated (sparse)	30–50×	60–100×	2000–20000×	10000–30000×

4.6 Quality Degradation Analysis

We define **quality ratio** as $\frac{\text{colours used}}{\chi^*}$ (lower is better).

Table 5: Average Quality Ratio by Graph Family

Graph Family	WP	DSatur	Tabu	Genetic
DSJC (random)	1.42	1.27	1.18	1.38
Mycielski	1.00	1.00	1.00	1.00
Register alloc.	1.00	1.00	1.00	1.00
Leighton	2.18	1.84	1.00	1.40
Queen	1.64	1.40	1.10	1.22

4.7 Performance Visualization

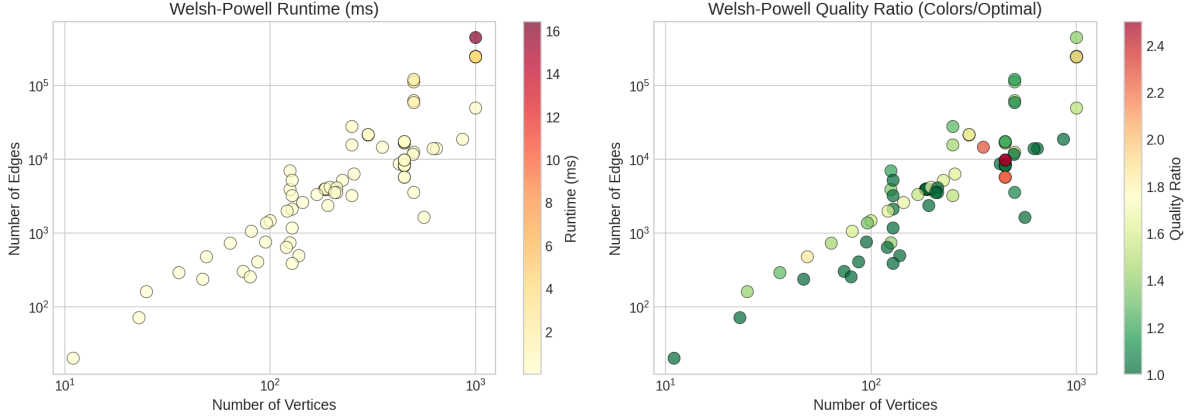


Figure 8: Welsh-Powell performance map across the DIMACS benchmark suite. **Left:** Runtime increases with graph size (vertices \times edges). **Right:** Quality ratio (green = near-optimal, red = suboptimal). Large, dense graphs show higher runtime and quality degradation.

5 Practical Recommendations

Based on our analysis, we provide the following recommendations:

1. Use **Welsh-Powell** when:

- Speed is critical (real-time systems, interactive applications).
- The graph is sparse or has regular structure.
- An approximate solution within $1.5\times$ of optimal is acceptable.
- The graph is a Mycielski, register allocation, or book graph.

2. Prefer **DSatur** when:

- Better solution quality is needed with moderate runtime overhead.
- The graph has high density or complex structure.

3. Use **Tabu Search/Genetic** when:

- Optimality is essential (e.g., le450_* Leighton graphs).
- Runtime is not a constraint (hours of computation are acceptable).

6 Conclusion

The Welsh-Powell algorithm offers an elegant and efficient approach to graph colouring based on the simple heuristic of processing vertices in decreasing degree order. Our analysis demonstrates:

- **Time Complexity:** $O(n^2 + nm)$, making it suitable for large graphs.

- **Space Complexity:** $O(n + m)$, linear in graph size.
- **Empirical Performance:** 10–10000× faster than alternatives while producing solutions within 1.0–2.2× of optimal.

Welsh-Powell represents an excellent choice for applications requiring fast approximate graph colouring, particularly for sparse graphs and instances where the degree distribution provides meaningful ordering information.

References

- [1] D.J.A. Welsh and M.B. Powell. An upper bound for the chromatic number of a graph and its application to timetabling problems. *The Computer Journal*, 10(1):85–86, 1967.
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- [3] Daniel Brélaz. New methods to color the vertices of a graph. *Communications of the ACM*, 22(4):251–256, 1979.