Rafael Mejía Zuluaga

Simulaciones: https://github.com/Rmejiaz/ModeladoSimulacion/blob/main/Cuadernos/Cap3.ipynb

Capítulo 3 - Bifurcaciones

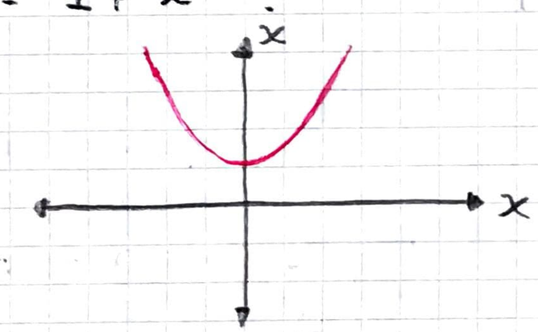
3.1 Saddle - Node Bifurcation

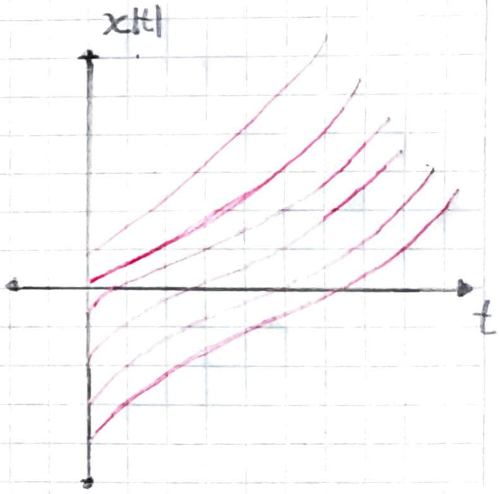
For each of the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

3.1.1 $\dot{x} = 1 + rx + x^2$

Para $r = 0$:

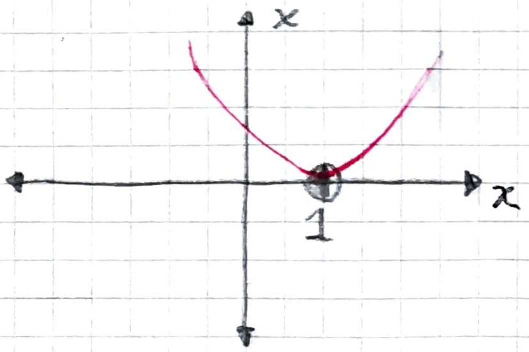
$\dot{x} = 1 + x^2$ → No hay equilibrios

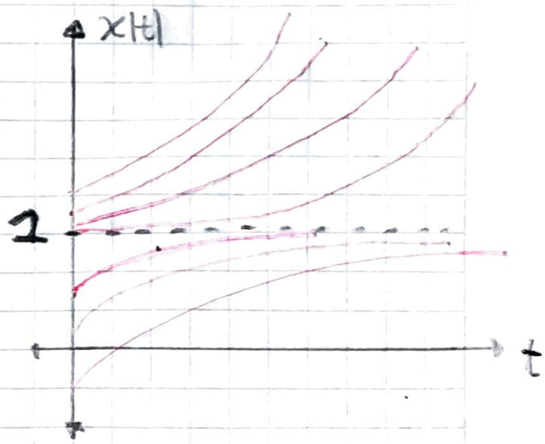




Para $r = -2$

$\dot{x} = x^2 - 2x + 1$ → Un punto de equilibrio semiestable



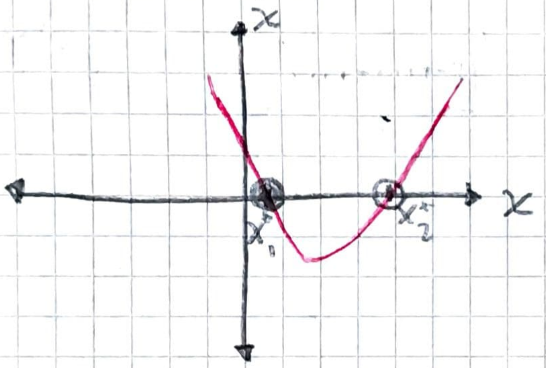


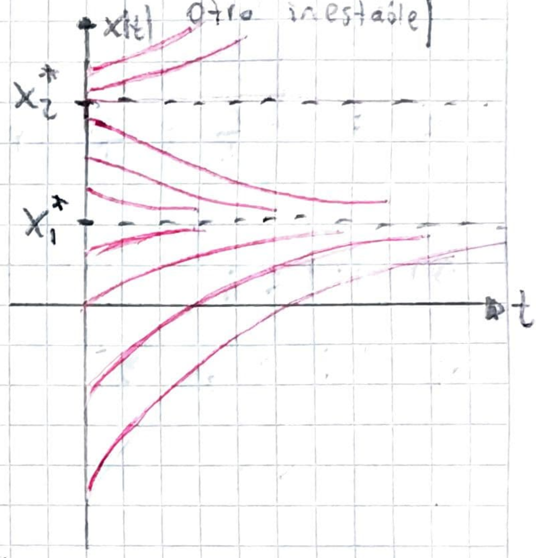
Para $r < -2$

$$\dot{x} = x^2 + rx + 1$$

$$x^2 + rx + 1 = 0$$

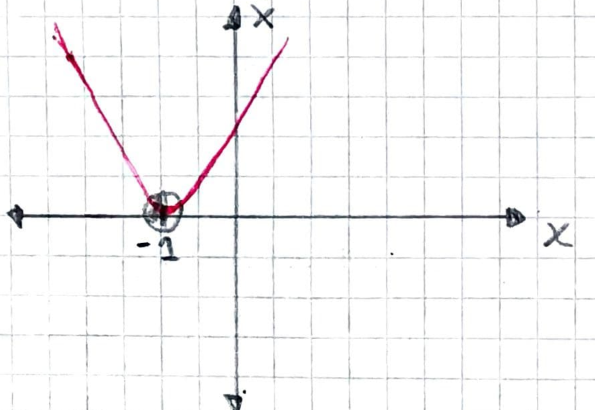
$$x^* = -\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - 4}$$ → Puntos de equilibrio (uno estable y otro inestable)

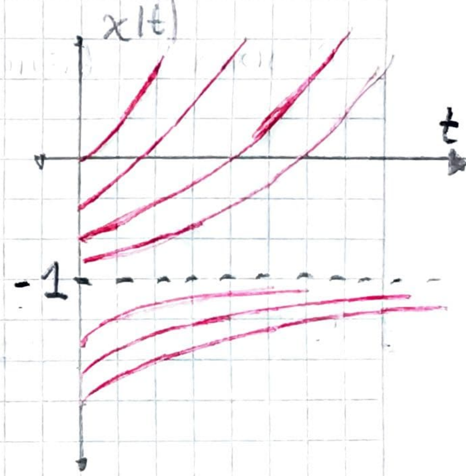




Para $r = 2$

$\boxed{x^* = -1}$ → semiestable





En resumen:

Para $r = \pm 2 \rightarrow$ Un punto de equilibrio semiestable

Para $|r| > 2 \rightarrow$ Dos puntos de equilibrio (uno estable y otro inestable)

Para $|r| < 2 \rightarrow$ No hay puntos de equilibrio

$$\dot{x} = x^2 + rx + 1$$

$$x^* = -\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - 4}$$

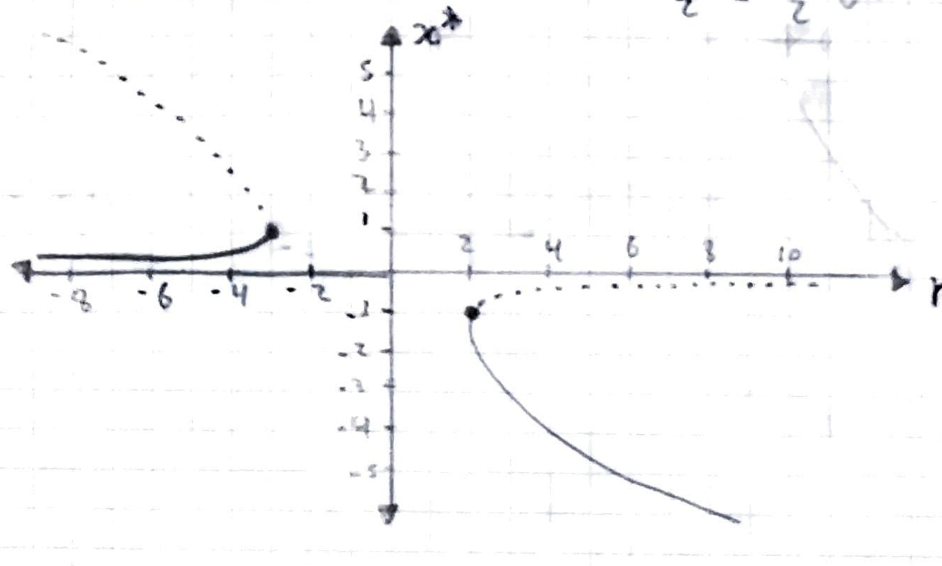
Si $|r| < 2$, x^* es complejo

Si $|r| = 2$, x^* es una solución real única

Si $|r| > 2$, x^* es real (doble)

Hay una bifurcación en $r = \pm 2$

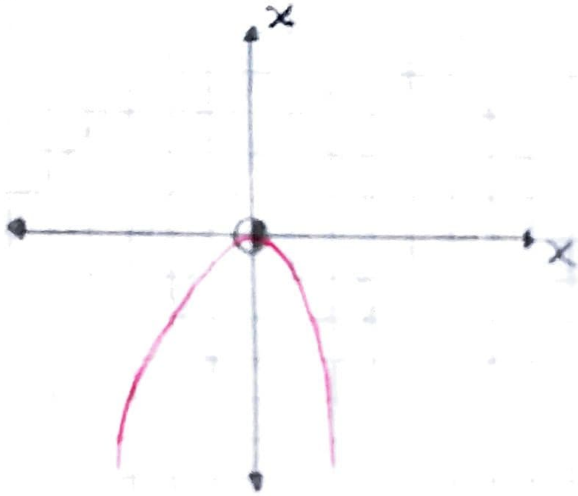
Diagrama de bifurcación: $\quad x^* = -\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - 4}$

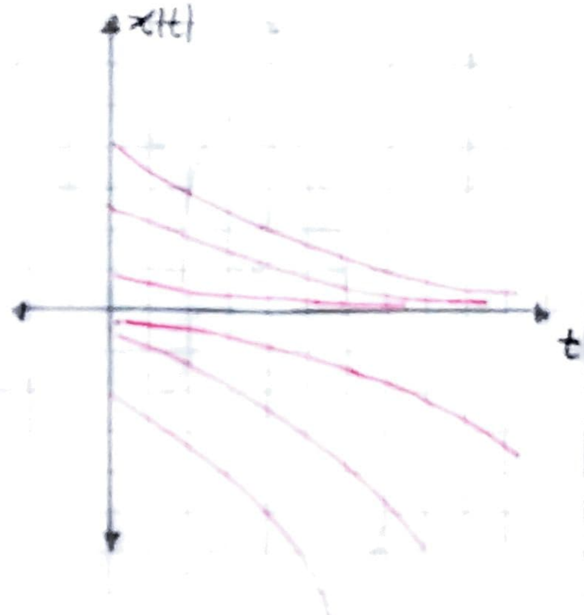


3.1.2

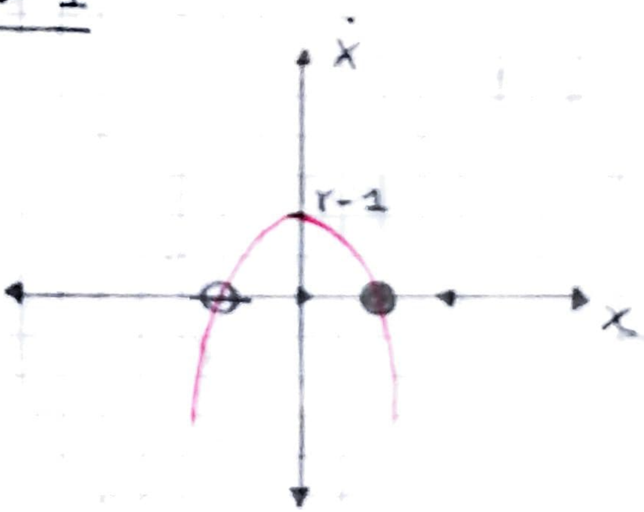
$\dot{x} = r - \cosh x$

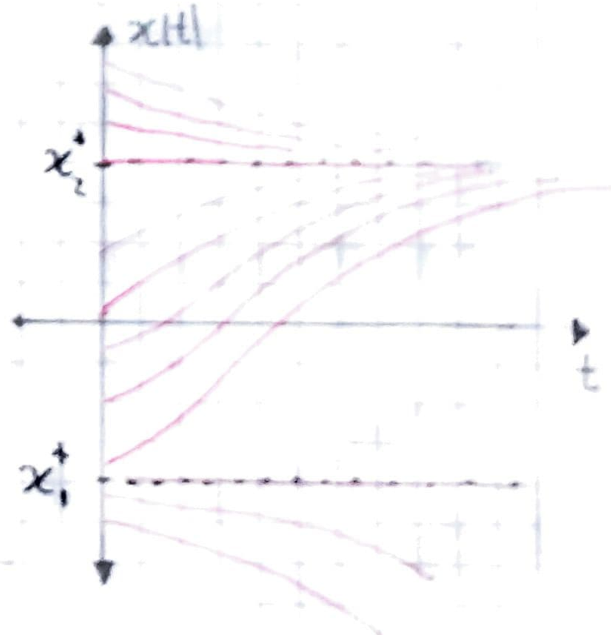
$r = 1$





$r > 1$



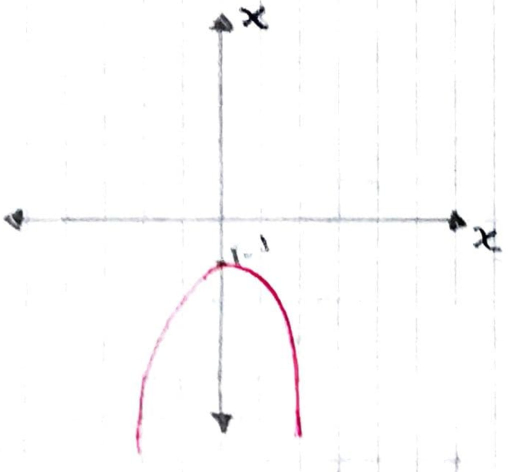


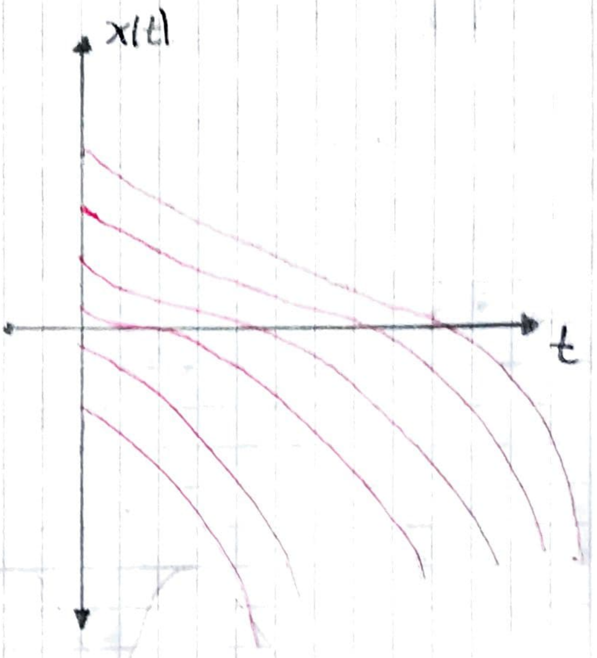
$\dot{x} = 0$

$r - \cosh(x) = 0$

$\cosh(x^*) = r$

$\underline{r < 1}$



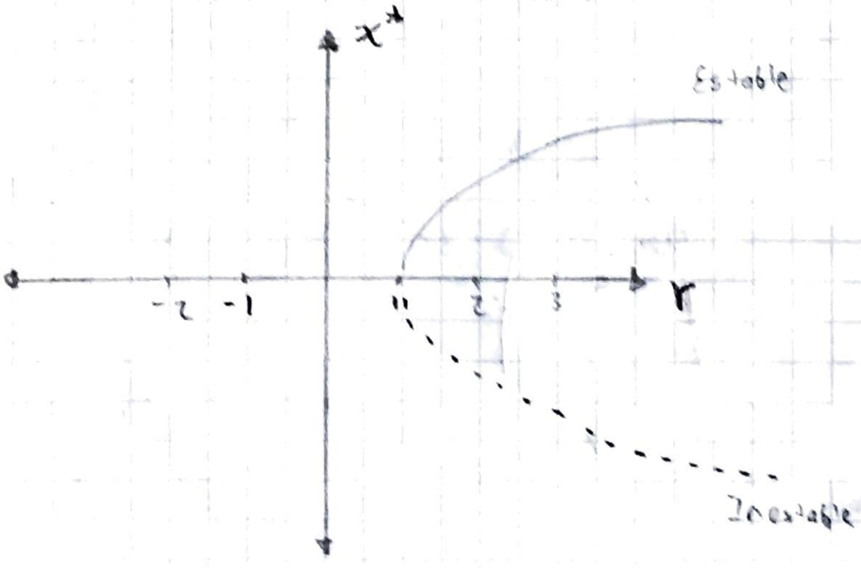


No hay puntos de equilibrio

En $r=1$ ocurre una bifurcación silla - nodo

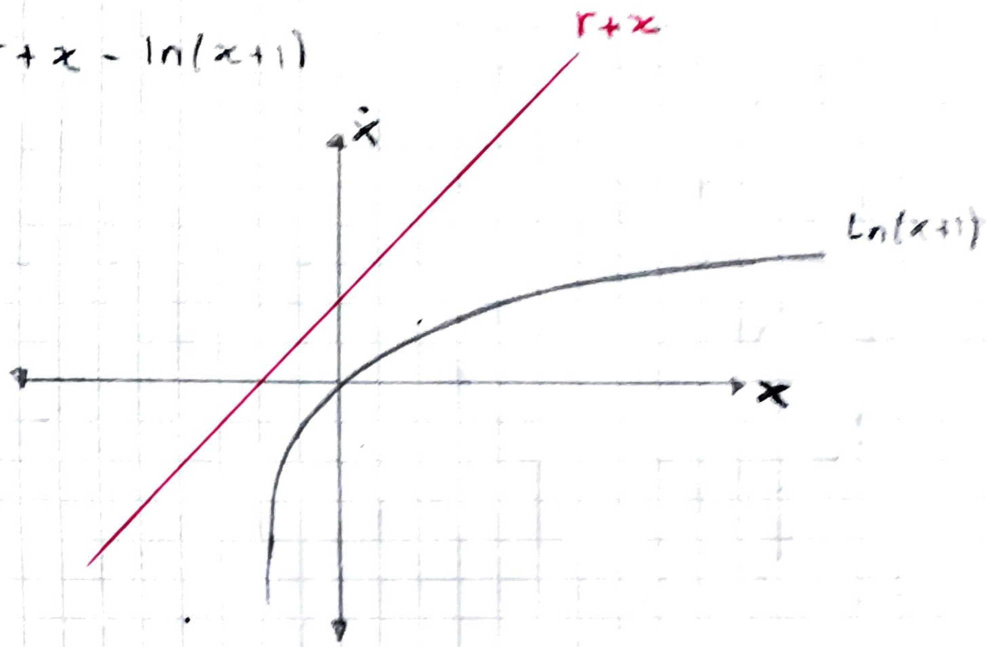
$r - \cosh x = 0$

$\boxed{r = \cosh x}$



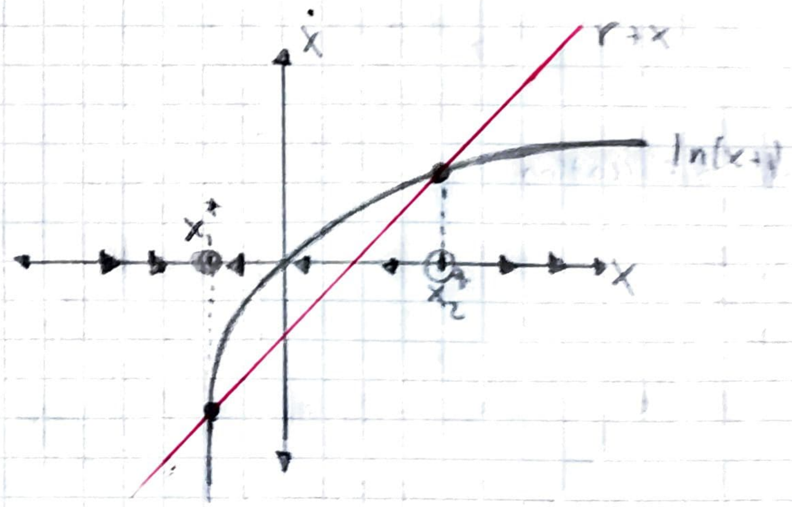
3.1.3

$\dot{x} = r + x - \ln(x+1)$



Puede verse que para $r > 0$ no hay ningún punto de equilibrio. Cuando $r = 0$ hay un único punto de equilibrio en $x = 0$.

Cuando $r < 0$, existen dos puntos de equilibrio:



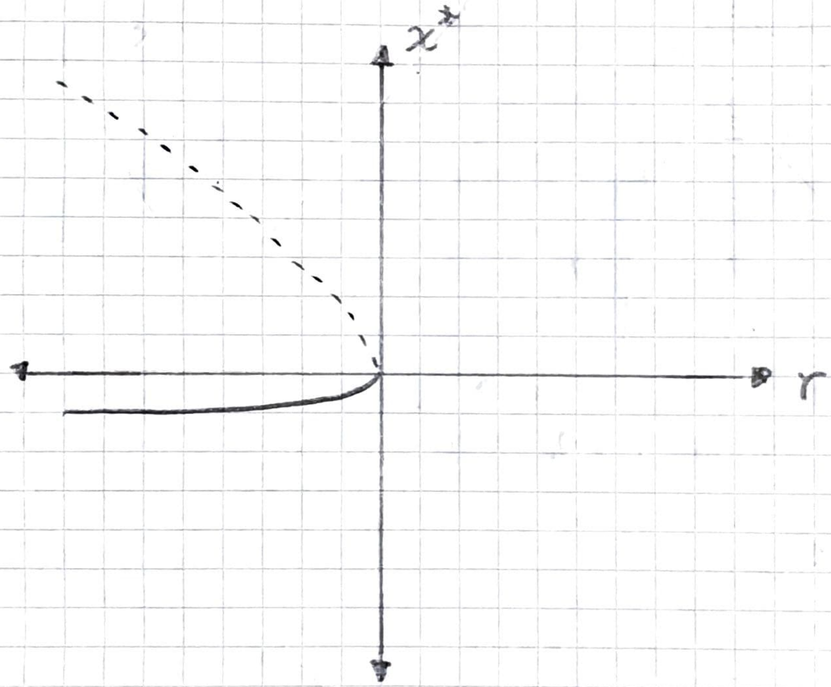
El punto de equilibrio menor a cero (x_1^*) es estable, mientras que el otro es inestable

Claramente hay una bifurcación Silla - nodo Para $r = 0$

$$\dot{x} = r + x - \ln(x+1)$$

$$r + x - \ln(x+1) = 0$$

$$r = \ln(x+1) - x$$



3.2 Transcritical Bifurcation

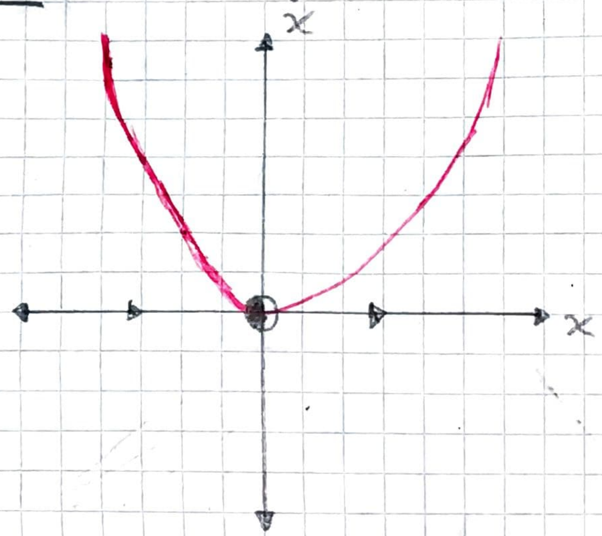
For each of the following exercises, Sketch all the Qualitatively different Vector Vields that occur at a Critical Value of r to be determined. Finally, Sketch the bifurcation diagram of Fixed Points x^* vs r

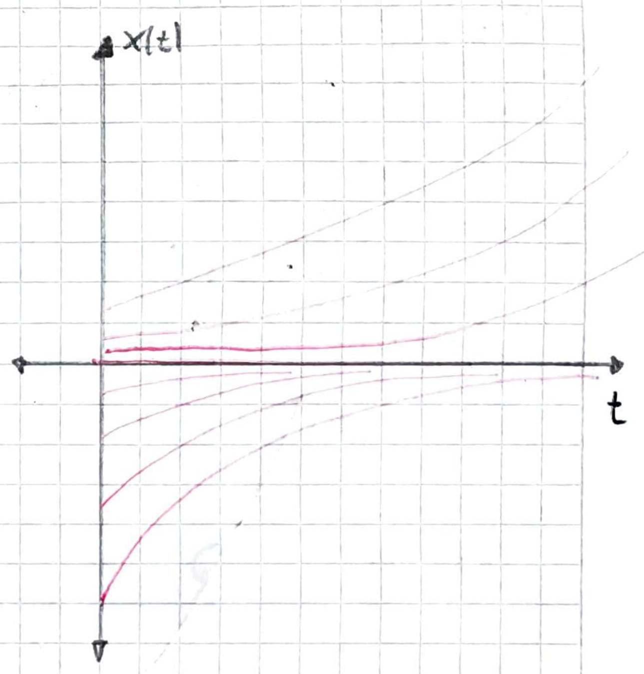
3.2.1

$$\dot{x} = rx + x^2$$

$r = 0$

$\dot{x} = x^2$





$r > 0$

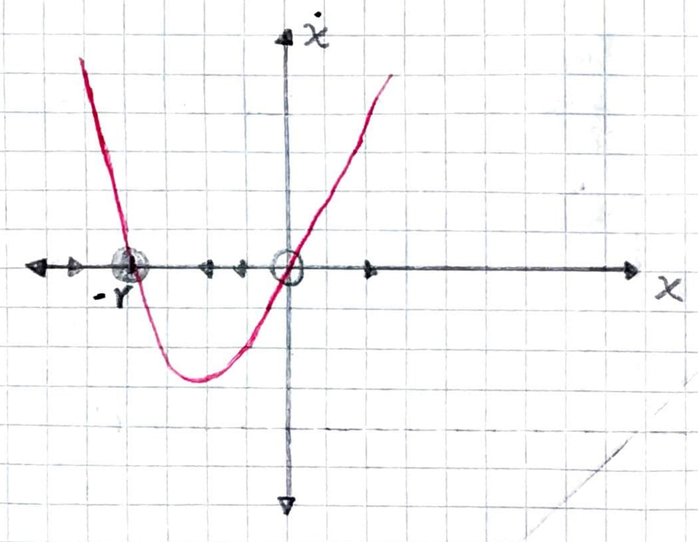
$\dot{x} = rx + x^2$

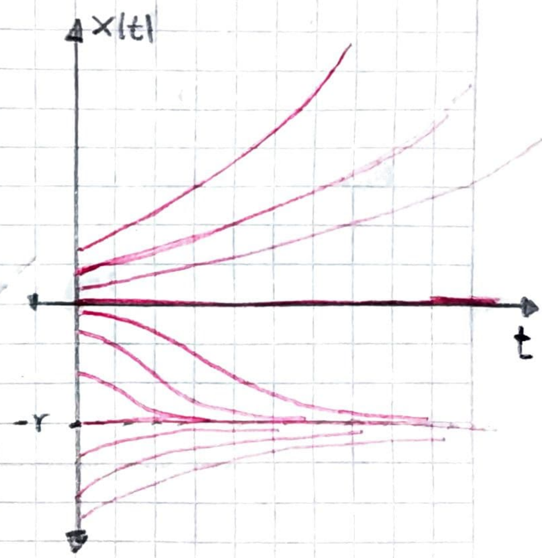
$rx + x^2 = 0$

$x(r + x) = 0$

$x_1^* = 0$

$x_2^* = -r \quad , \quad r > 0$

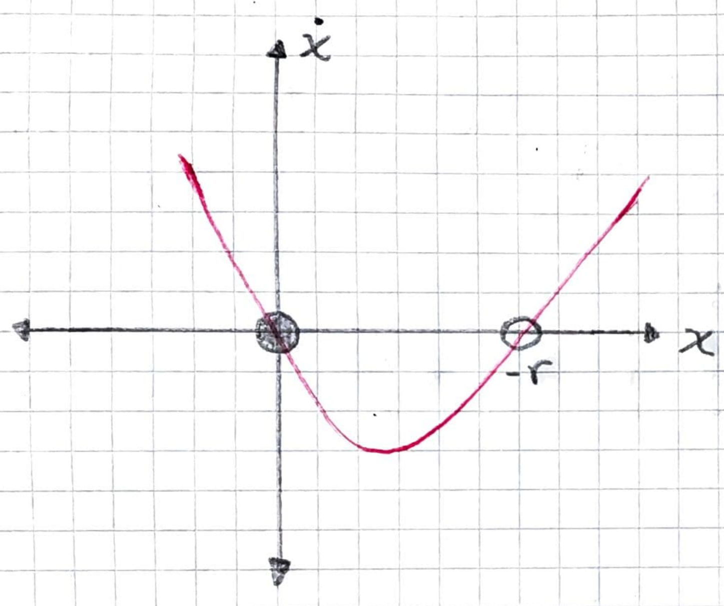


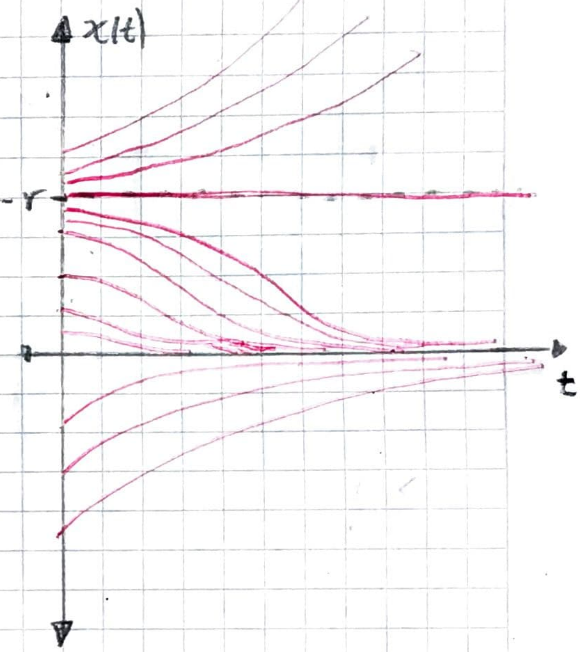


$r<0$

$x_1^* = 0$

$x_2^* = -r \quad , \quad r<0$



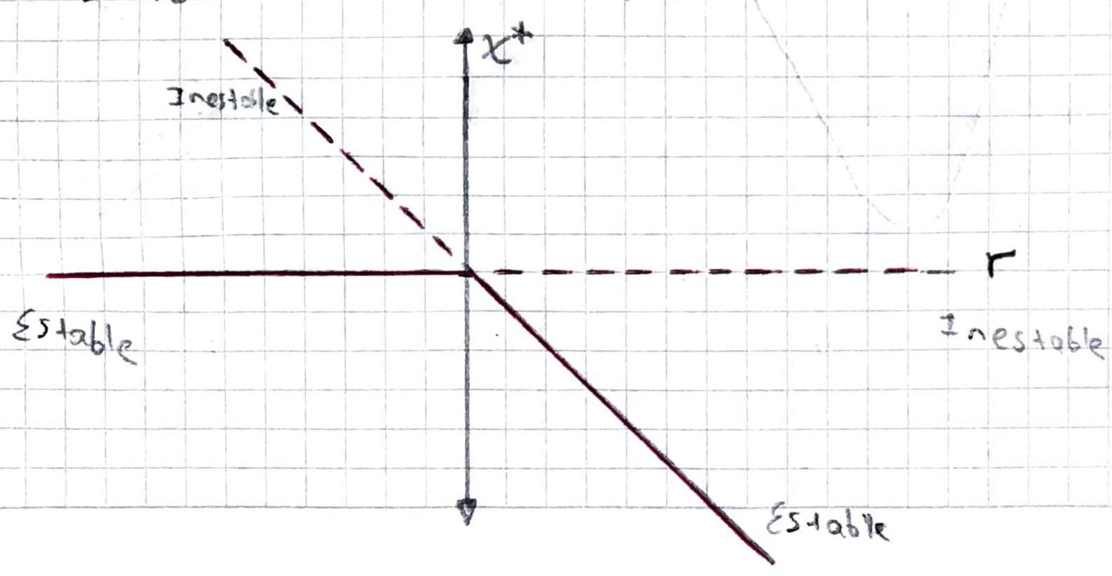


$x^* = 0$ es un punto de equilibrio para todos los valores de r. Sin embargo, su estabilidad cambia:

$r<0 \rightarrow x^*=0$ es estable

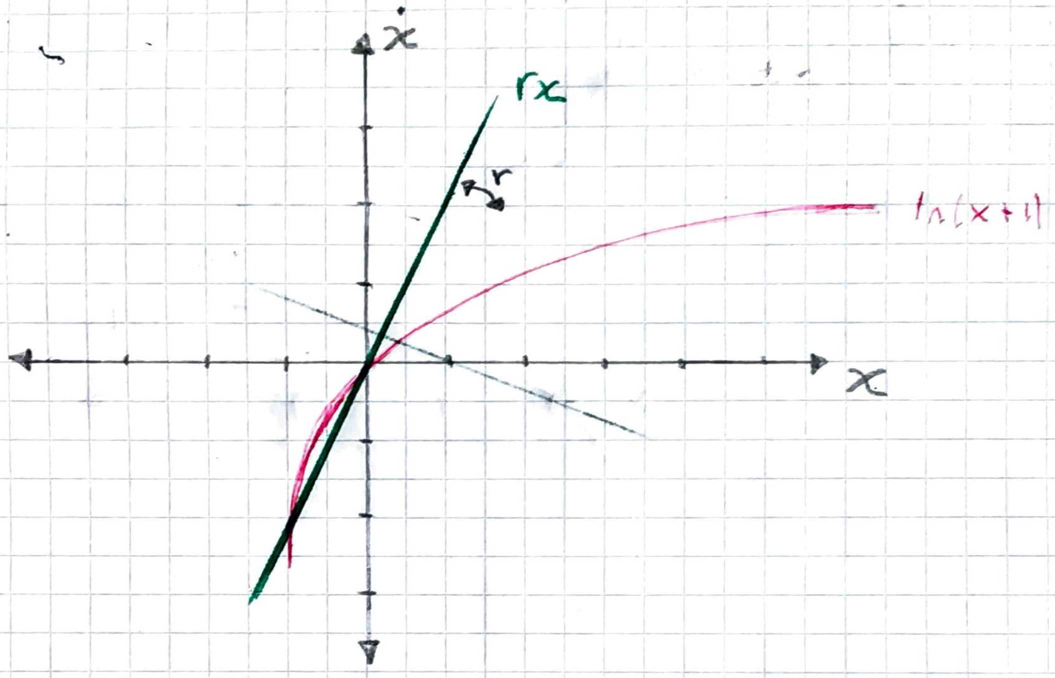
$r=0 \rightarrow x^*=0$ es semi estable

$r>0 \rightarrow x^*=0$ es inestable



3.2.2

$\dot{x} = rx - \ln(x+1)$



Es evidente que para cualquier r hay un punto de equilibrio $\boxed{x^* = 0}$

Dependiendo del valor de r existe otro punto de equilibrio.

Existe un valor de r que hace que solo exista el punto de equilibrio $x^* = 0$, este se da cuando la recta rx es tangente a la curva $\ln(x+1)$:

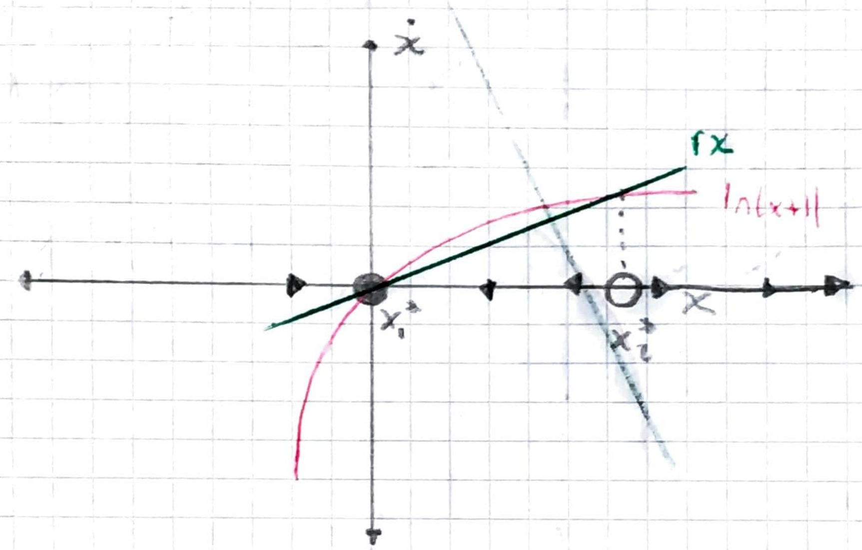
$$r_{cr} = \left.\frac{d(\ln(x+1))}{dx}\right|_{x=0}$$

$$r_{cr} = \left.\frac{1}{x+1}\right|_{x=0} = 1$$

$$\boxed{r_{cr} = 1}$$

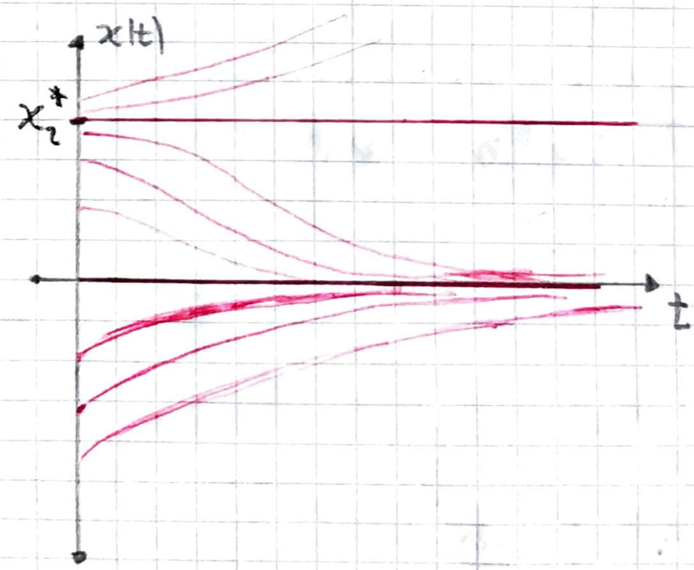
Existe otro $r<1$ a partir del cual el único punto de equilibrio es $\boxed{x^*=0}$. Este queda por determinar (r_{cr_2})

Para $r_{cr} < r < 1$

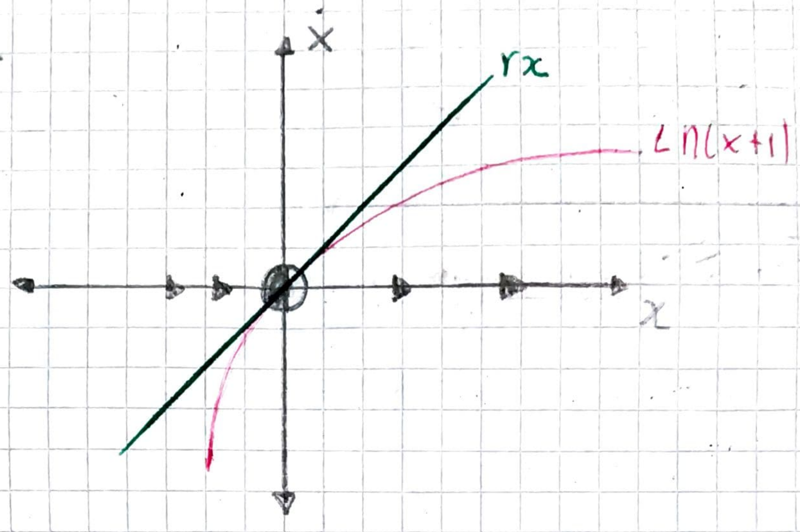


$x_1^* = 0$

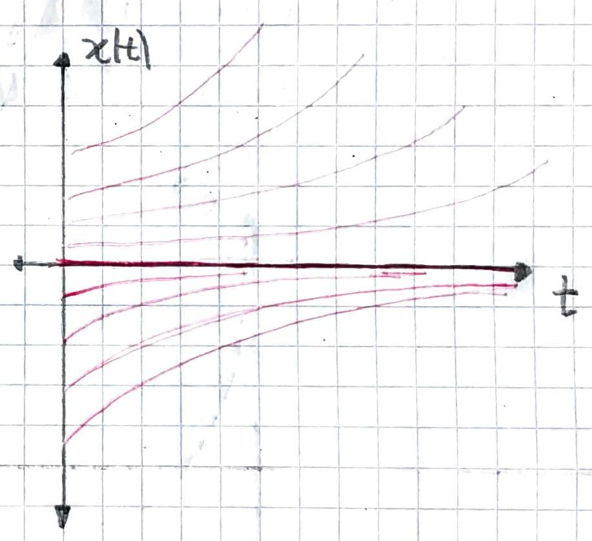
$rx^* = \ln(x^*+1)$



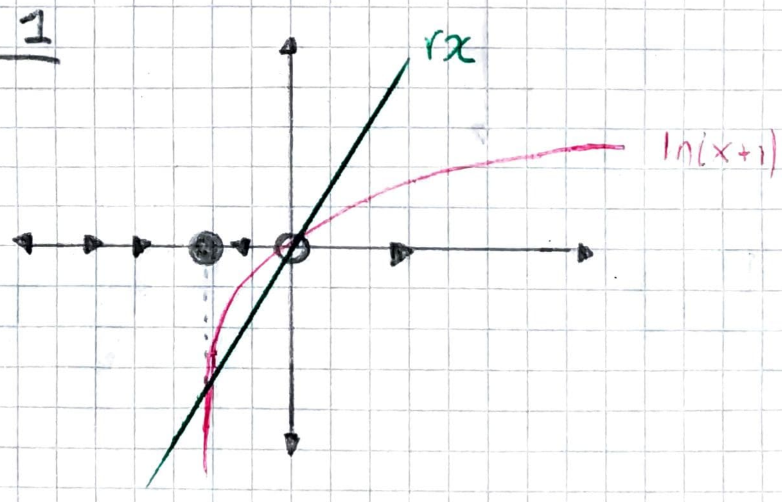
Para $r = 1$



$x^* = 0 \longrightarrow$ Semi estable

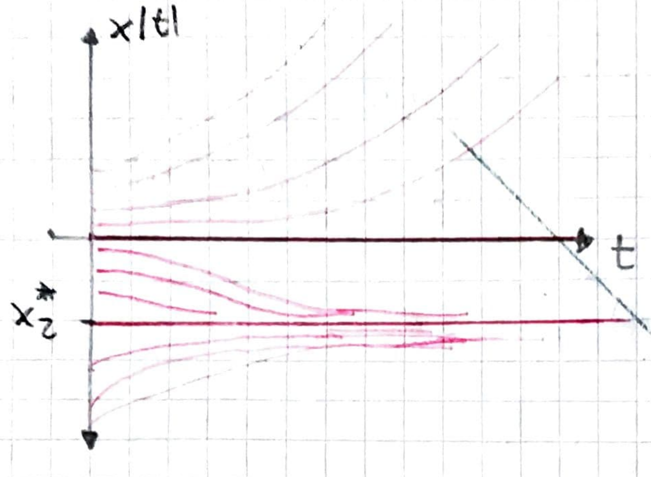


Para $r > 1$



$x_1^* = 0 \longrightarrow$ Inestable

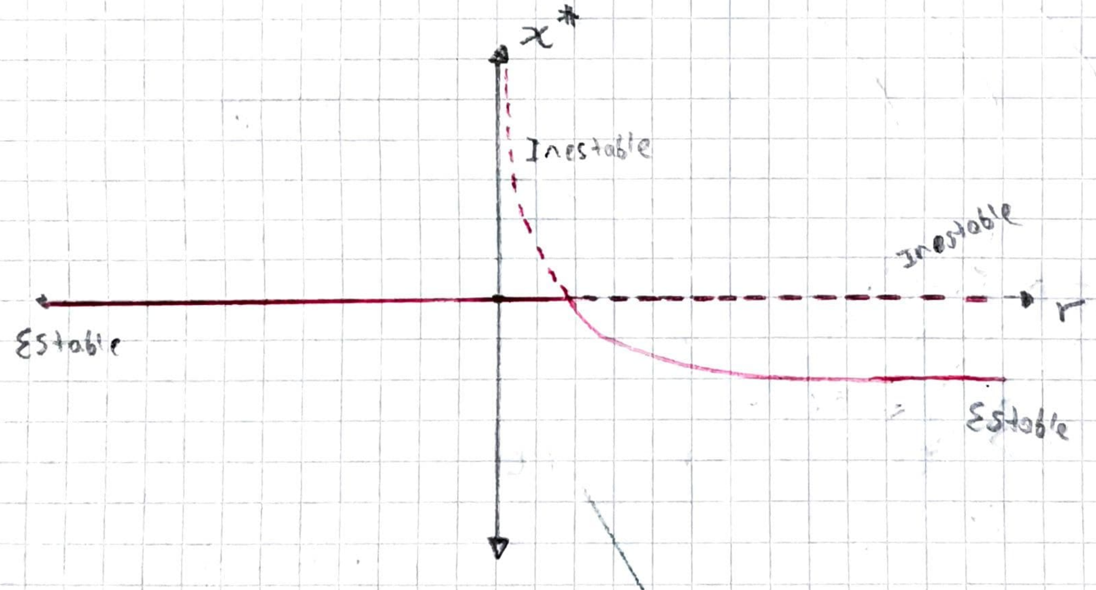
$x_2^* \longrightarrow$ Estable



$x^* = 0$ es: inestable para $r > 1$

semiestable para $r = 1$

estable para $r < 1$

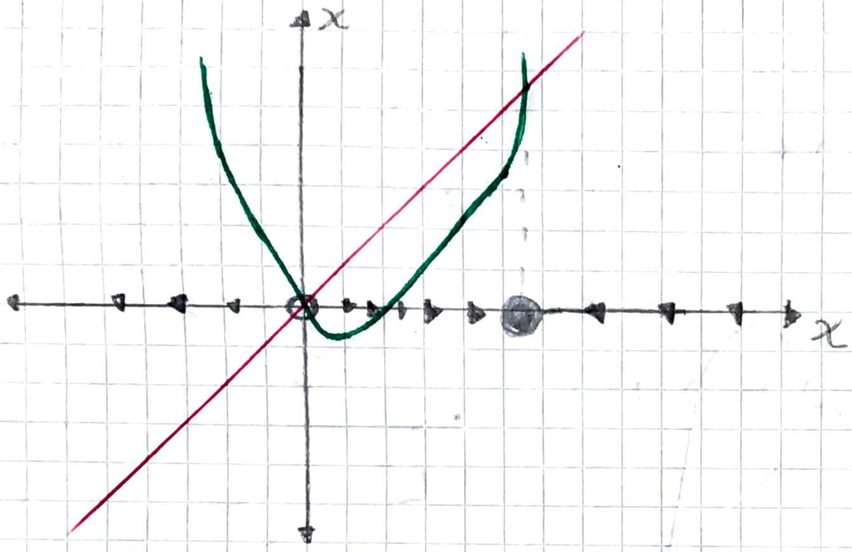


$rx = \ln(x+1)$

$r = \dfrac{\ln(x+1)}{x}$

3.2.3

$$\dot{x} = x - rx(1-x)$$



$x^* = 0$ siempre es un punto de equilibrio para cualquier r

$$\dot{x} = x - rx(1-x)$$

$$\dot{x} = x[1 - r(1-x)]$$

$$x[1-r(1-x)] = 0$$

$$\boxed{x^* = 0}$$

$$1 - r(1-x) = 0$$

$$r(1-x) = 1$$

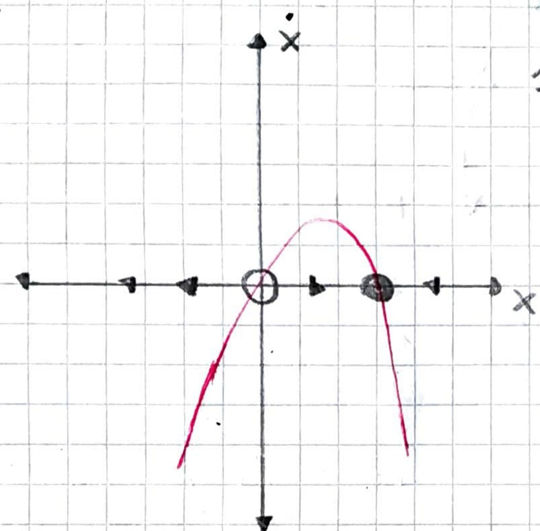
$$1 - x = \frac{1}{r}$$

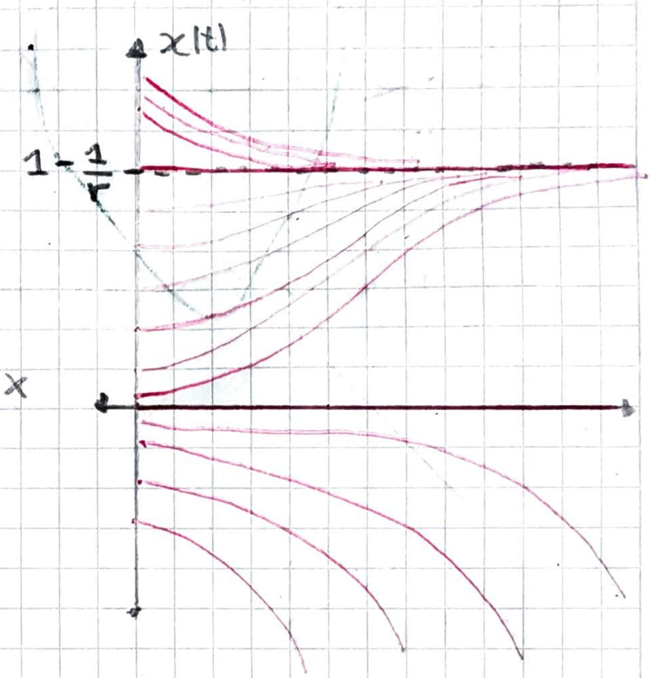
$$\boxed{x^* = 1 - \frac{1}{r}}$$ → Cuando $r = 1$ este punto de equilibrio se encuentra con el otro

Para $r = 1$ solo hay un punto de equilibrio semiestable

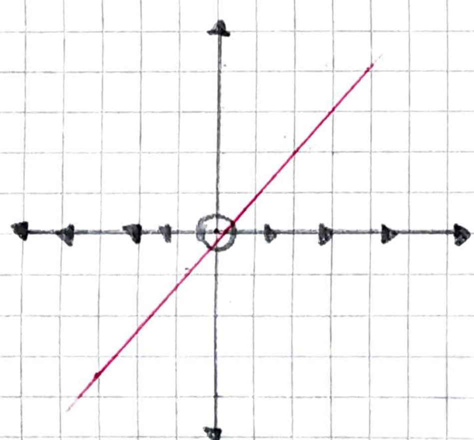
Para $r = 0$ hay un punto de equilibrio inestable

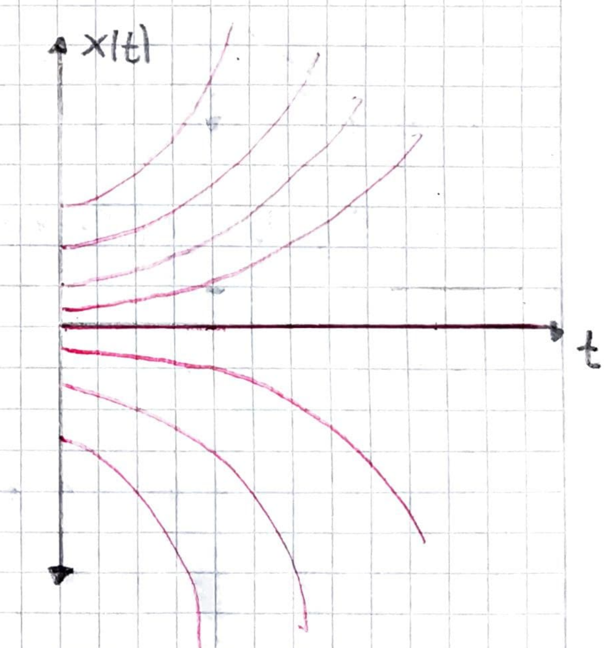
$r < 0$



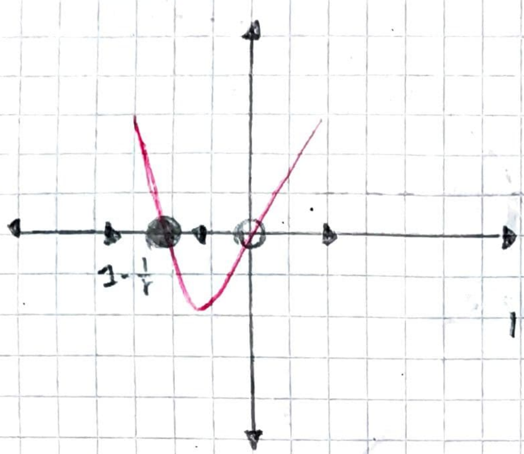


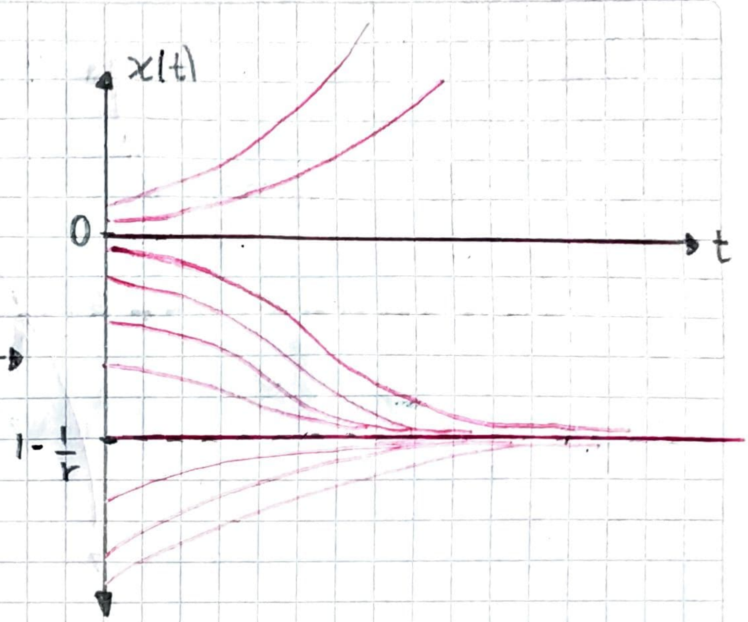
$r = 0$



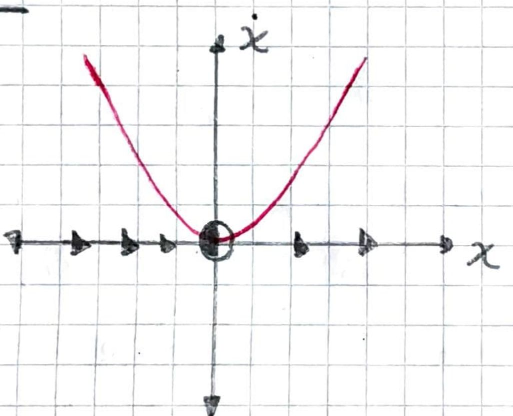


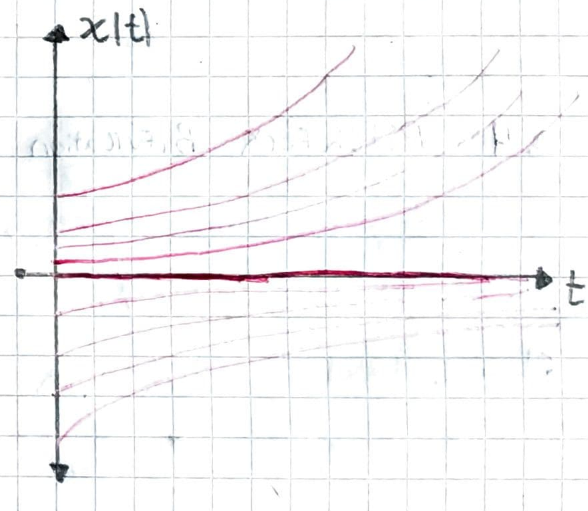
$0 < r < 1$



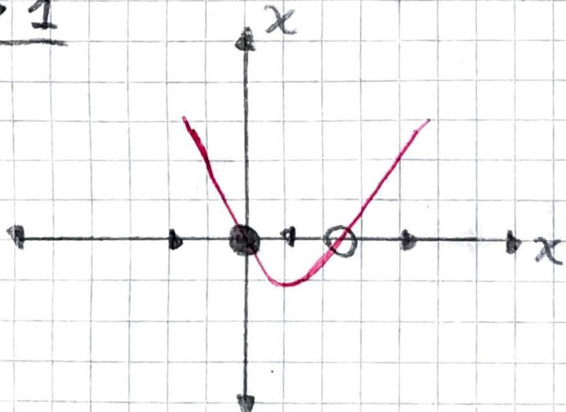


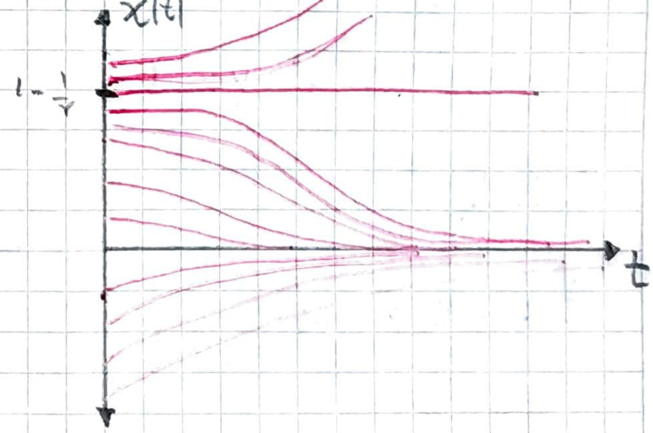
$r = 1$

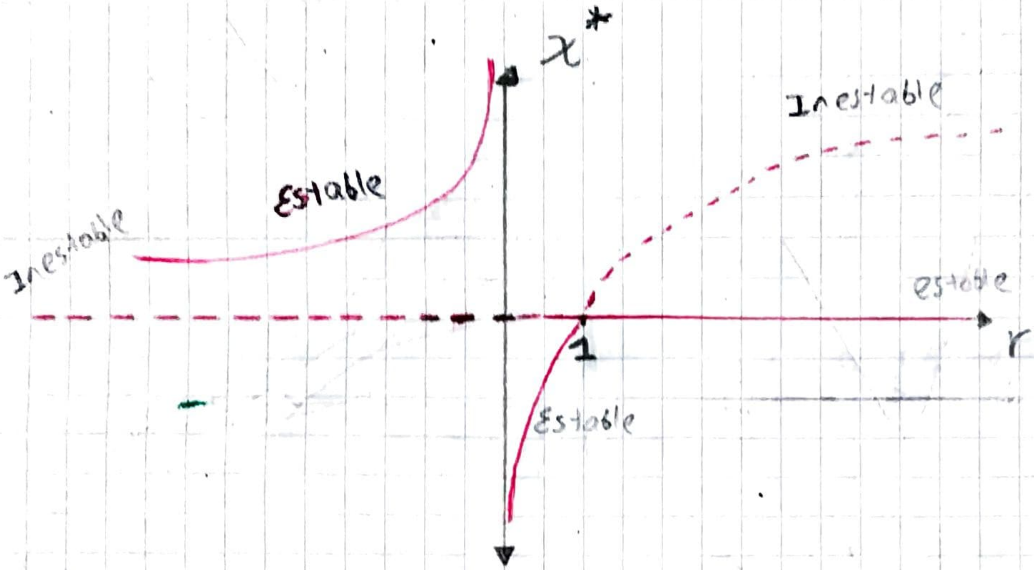




$r > 1$







$$x^* = 1 - \frac{1}{r}$$

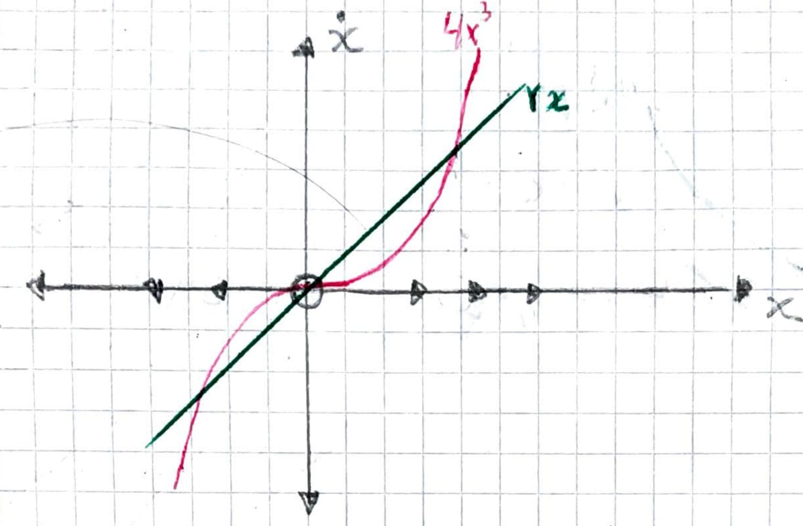
3.4 Pitchfork Bifurcation

In the following exercises, sketch all the Qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r (to be determined) and classify the bifurcation as supercritical or subcritical. Finally, sketch the bifurcation diagram of x^* vs r^*

3.4.1

$$\dot{x} = rx + 4x^3$$

$\dot{x} = rx + 4x^3$



$x^* = 0$

$rx + 4x^3 = 0$

$x(r + 4x^2) = 0$

$\boxed{x_1^* = 0}$

$r + 4x^2 = 0$

$4x^2 = -r$

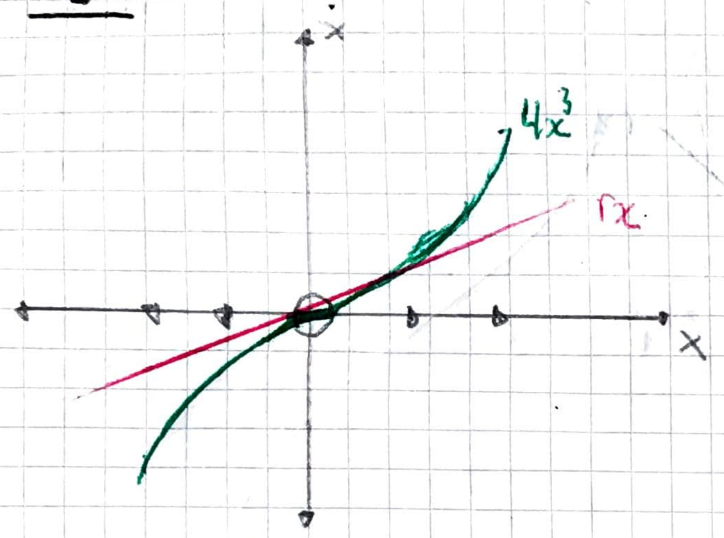
$x^2 = -\frac{r}{4}$ → Para $r > 0$ no existen más puntos de equilibrio

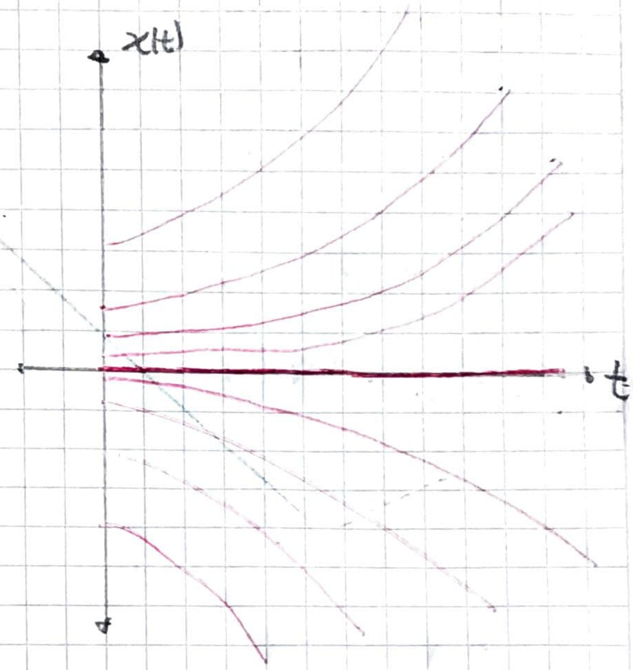
Si $r < 0$:

$x^2 = -\frac{r}{4}$

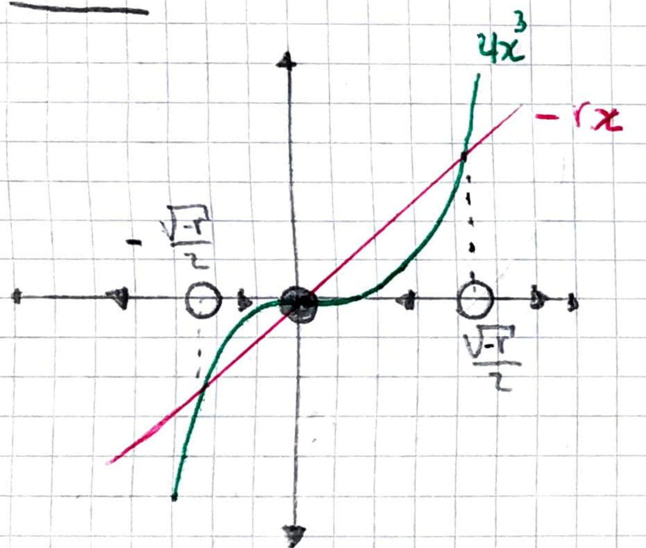
$x = \frac{\sqrt{|r|}}{2}$ $\boxed{r < 0}$

$r \geq 0$





$r < 0$



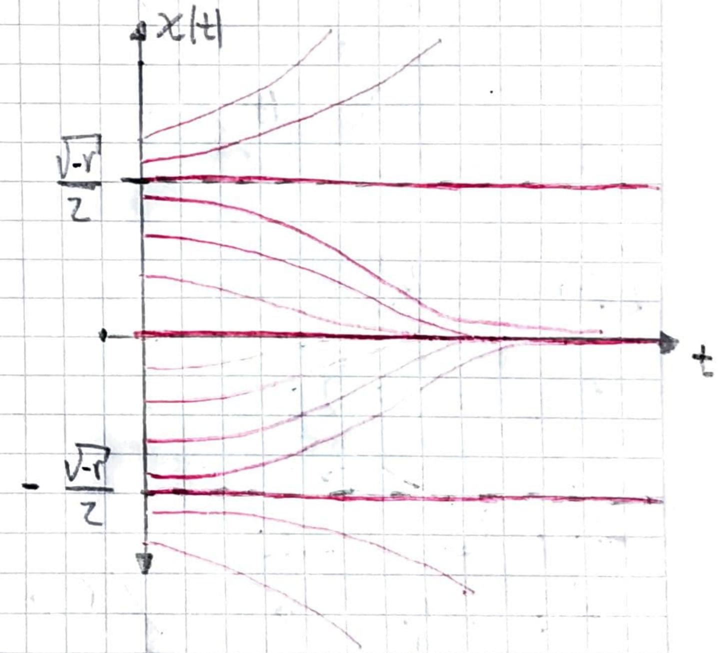
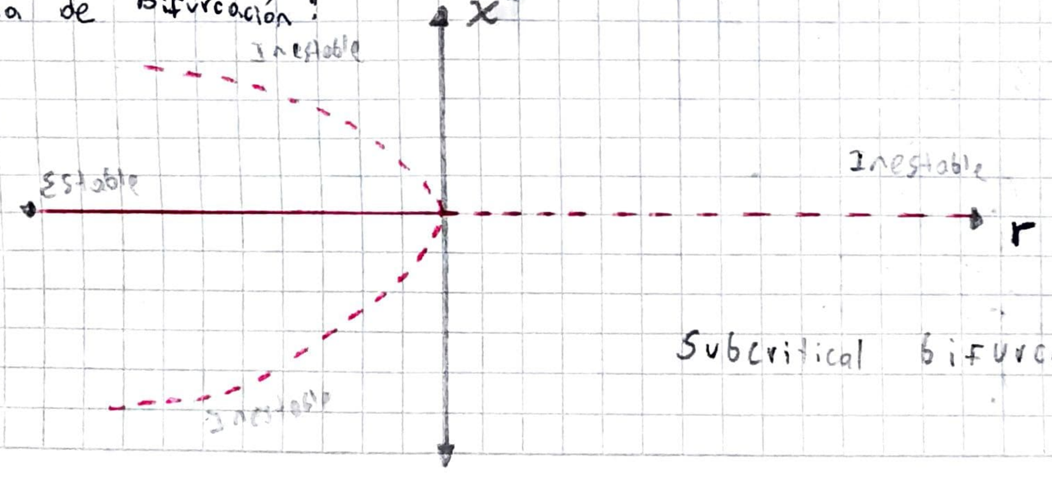


Diagrama de Bifurcación:



Subcritical bifurcation

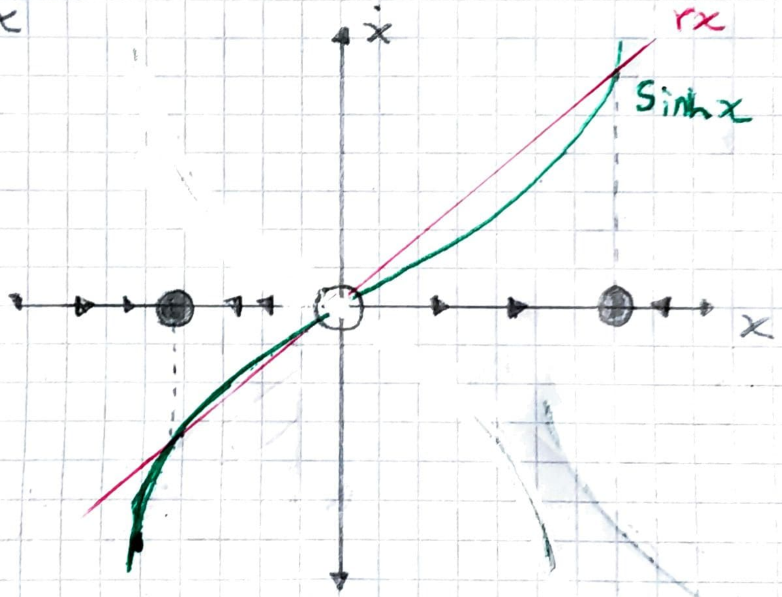
3.4.2

$\dot{x} = rx - \sinh x$

$rx - \sinh x = 0$

$rx = \sinh x$

$r = \dfrac{\sinh x}{x}$



$x^* = 0 \rightarrow$ Punto de equilibrio para cualquier r. Su estabilidad varía

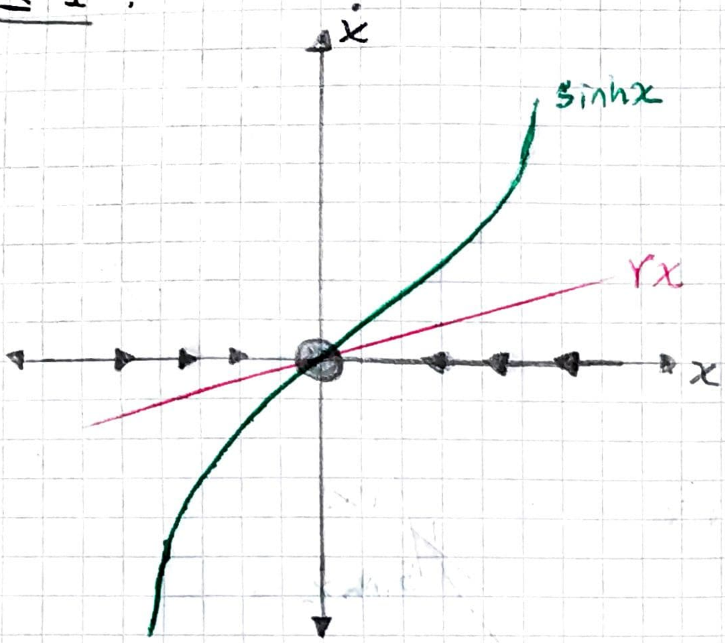
Cuando rx es tangente a $\sinh x$ en el origen, es el punto crítico en el cual aparecen los nuevos puntos de equilibrio.

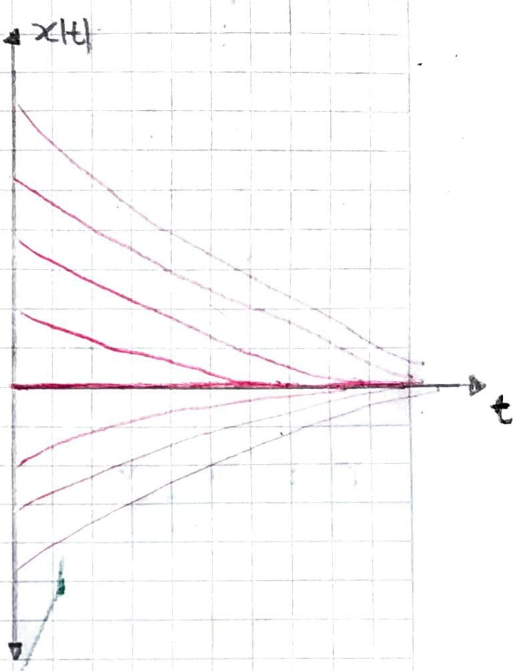
$r = \left.\dfrac{d \sinh x}{dx}\right|_{x=0}$

$r = \cosh x \big|_{x=0}$

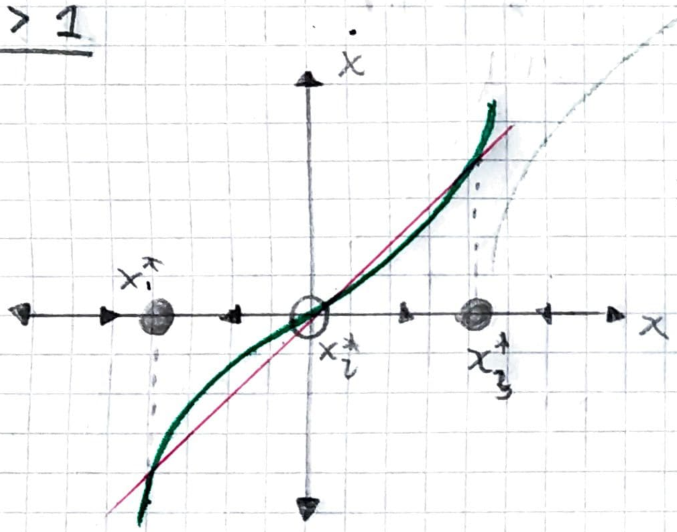
$\boxed{r = 1}$

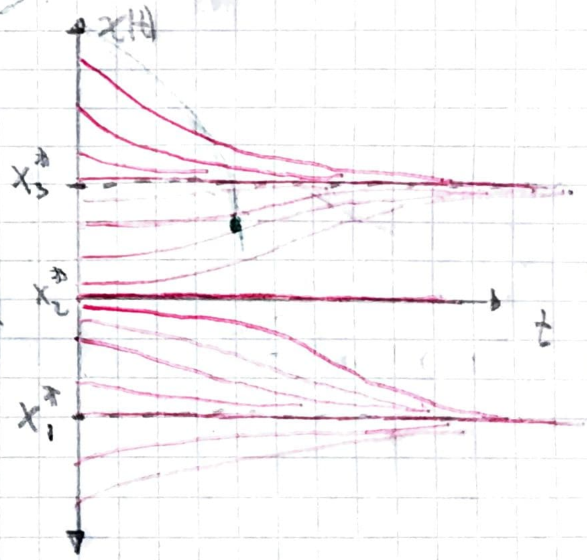
$r \leq 1$:





$r > 1$



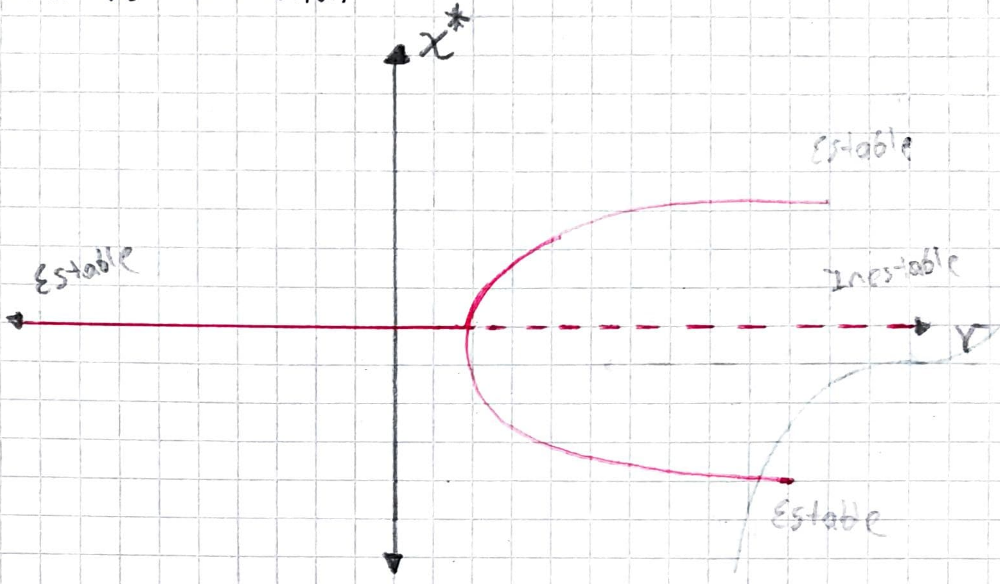


$rx - \sinh x = 0$

$rx = \sinh x$

$\boxed{r = \dfrac{\sinh x^*}{x^*}}$ → Puntos de equilibrio (forma implícita)

Diagrama de Bifurcación



Supercritical bifurcation

3.4.3

$\dot{x} = rx - 4x^3$

$\dot{x} = x(r - 4x^2)$

Puntos de equilibrio

$x(r - 4x^2) = 0$

$\boxed{x^* = 0}$ → Para cualquier r

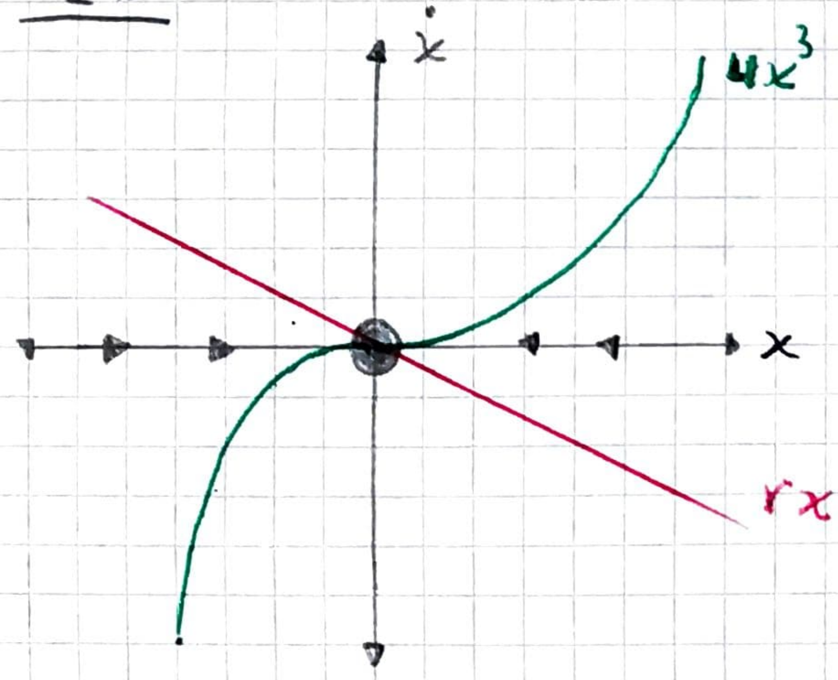
$r - x^2 = 0$

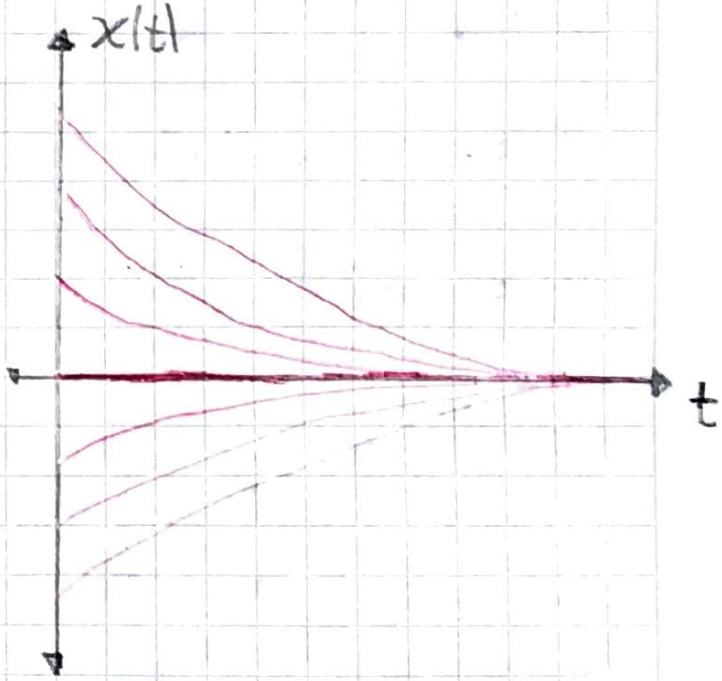
$4x^2 = r$

$x^2 = \frac{r}{4}$

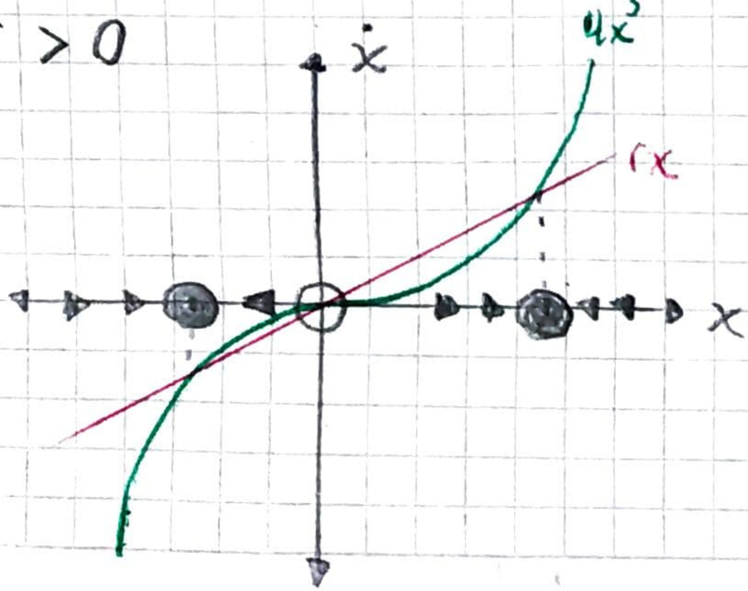
$x^* = \frac{\sqrt{r}}{2}$ → Solo existe para $r > 0$

$r \leq 0$





$r > 0$



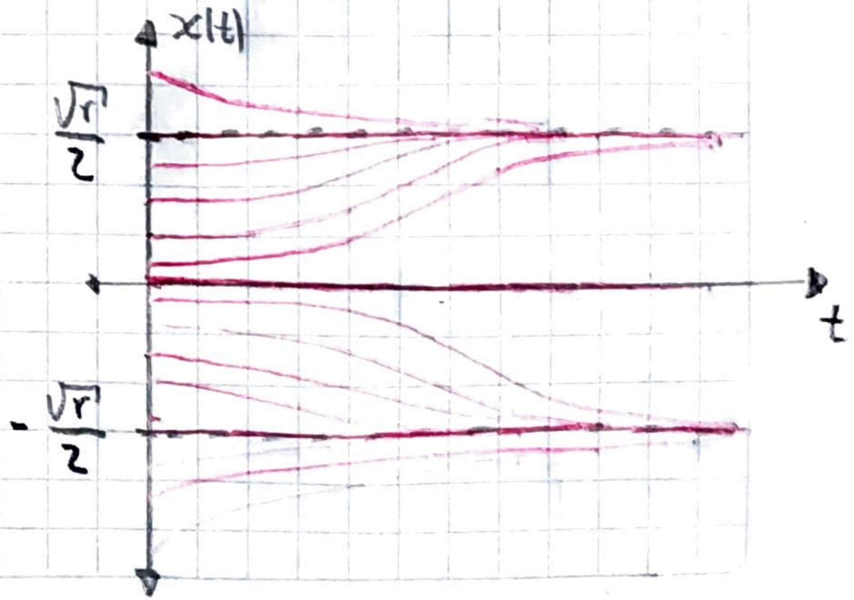
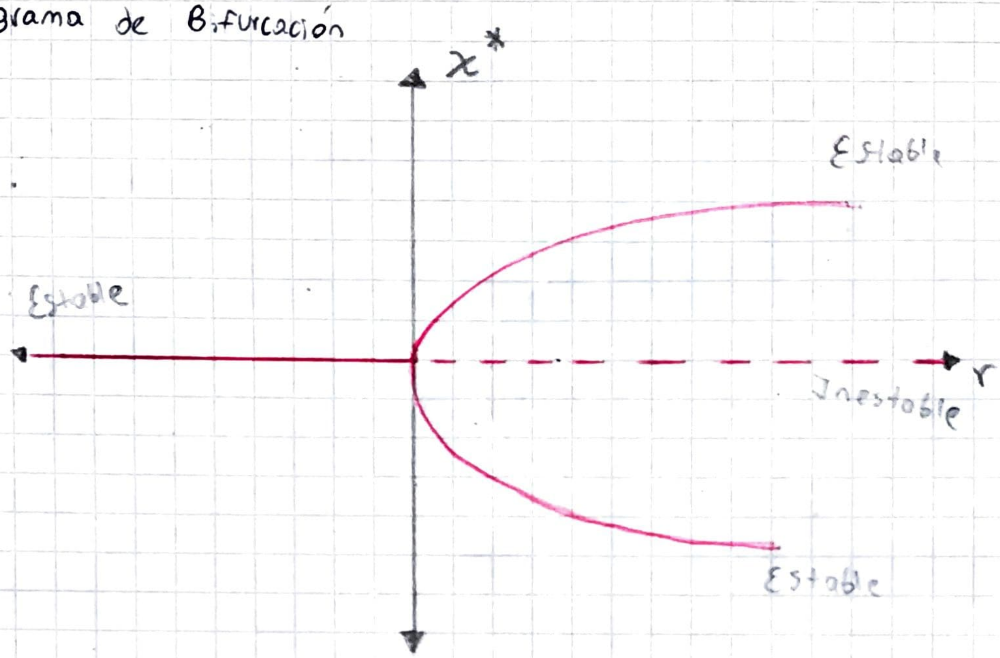


Diagrama de Bifurcación



Supercritical bifurcation