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Nonlinear Dynamics And Chaos - Ejercicios

Capitulo 2

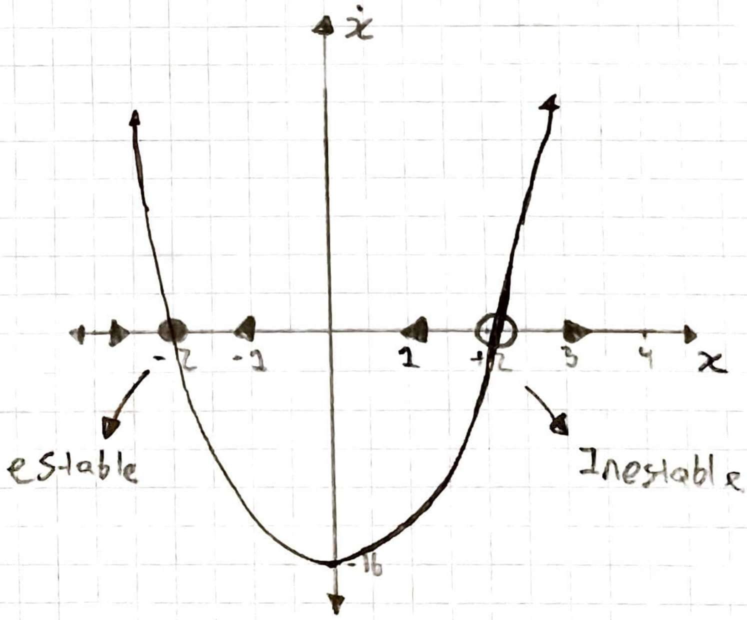
2.2 - Fixed Points and Stability

Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

2.2.1

$$\dot{x} = 4x^2 - 16$$

$$f(x) = 4x^2 - 16$$



$$4x^2 - 16 = 0$$

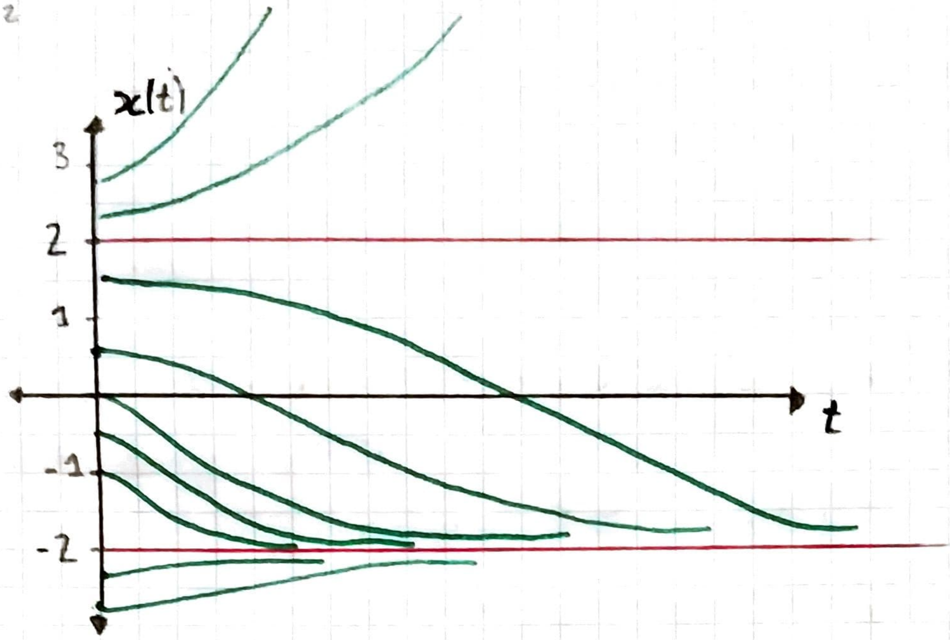
$$4x^2 = 16$$

$$x^2 = \frac{16}{4}$$

$$\boxed{x = \pm 2}$$

$$f_x = 8x \quad f_x\Big|_{x=-2} = -16 \rightarrow \text{Estable!}$$

$f_x\big|_{x=2} = 16 \to$ Inestable!



$x(t) = ?$

$$\frac{\partial x}{\partial t} = 4x^2 - 16$$

$$\frac{\partial x}{4x^2 - 16} = \partial t$$

$$t = \int \frac{1}{4x^2 - 16}\,\partial x$$

$$= \frac{1}{4}\int \frac{1}{x^2 - 4}\,\partial x$$

$x = 2\sec(\theta)$

$$= \frac{1}{4}\int \frac{1}{4\sec^2(\theta) - 4}$$

$\partial x = 2\sec(\theta)\tan(\theta)\,\partial\theta$

$$= \frac{1}{16}\int \frac{1 \cdot 2\sec(\theta)\tan(\theta)}{\sec^2(\theta) - 1}\,\partial\theta = \frac{1}{8}\int \frac{\sec(\theta)\cancel{\tan(\theta)}}{\tan^{\cancel{2}}(\theta)}\,\partial\theta$$

$$t = \frac{1}{8}\int \frac{\sec(\theta)}{\tan(\theta)}\, d\theta$$

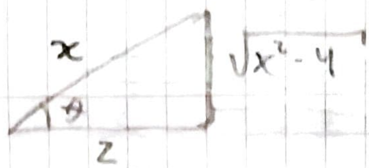
$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \rightarrow \qquad \frac{\sec(\theta)}{\tan(\theta)} \quad \frac{\frac{1}{\cos(\theta)}}{\frac{\operatorname{sen}(\theta)}{\cos(\theta)}} \quad \csc(\theta)$$

$$\tan(\theta) = \frac{\operatorname{sen}(\theta)}{\cos(\theta)}$$

$$t = \frac{1}{8}\int \csc(\theta)\, d\theta = -\frac{1}{8} \ln|\csc(\theta) + \cot(\theta)| + C$$

$$\sec(\theta) = \frac{x}{2}$$

$$\cos(\theta) = \frac{2}{x}$$



$$\csc(\theta) = \frac{x}{\sqrt{x^2-4}}$$

$$\cot(\theta) = \frac{2}{\sqrt{x^2-4}}$$

$$t = -\frac{1}{8} \ln\left|\frac{x}{\sqrt{x^2-4}} + \frac{2}{\sqrt{x^2-4}}\right|$$

$$= -\frac{1}{3} \ln\left|\frac{x+2}{\sqrt{x^2-4}}\right|$$

$$t = -\frac{1}{8}\left(\ln(x+2) - \ln\left(\sqrt{x^2-2}\right)\right)$$

$$t = -\frac{1}{8} \ln(x+2) + \frac{1}{16}\ln(x^2-2)$$ → Forma implicita

2.2.2

$\dot{x} = 1 - x^{14}$

$f(x) = 1 - x^{14}$

$1 - x^{14} = 0$

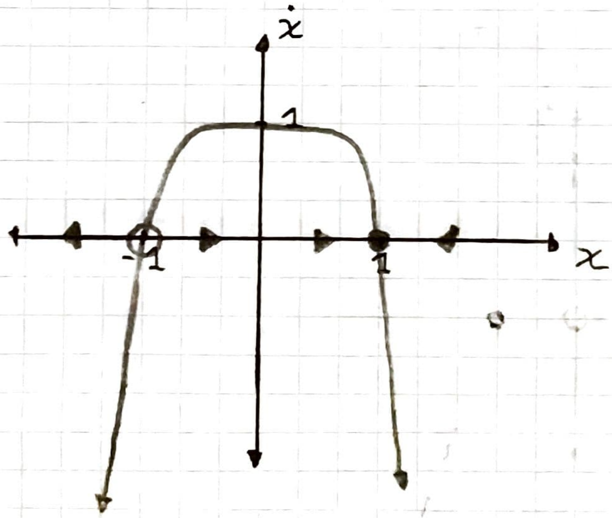
$x^{14} = 1$

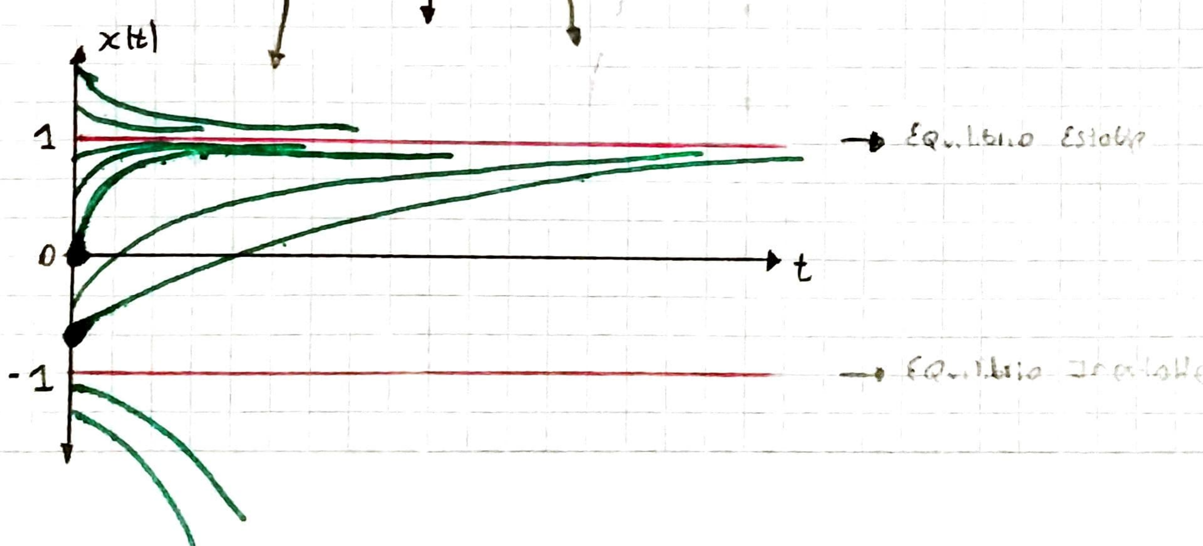
$\boxed{x = \pm 1}$ (soluciones Reales)

$f_x = -14x^{13}$

$f_x \big|_{x=-1} = 14 \rightarrow$ Inestable

$f_x \big|_{x=1} = -14 \rightarrow$ Estable





2.2.3

$\dot{x} = x - x^3$

$f(x) = x - x^3$

$x - x^3 = 0$

$-x^3 + x = 0$

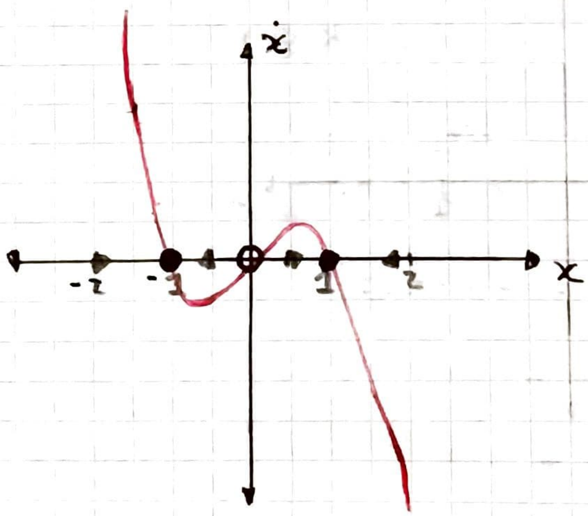
$x = \{-1, 0, 1\}$

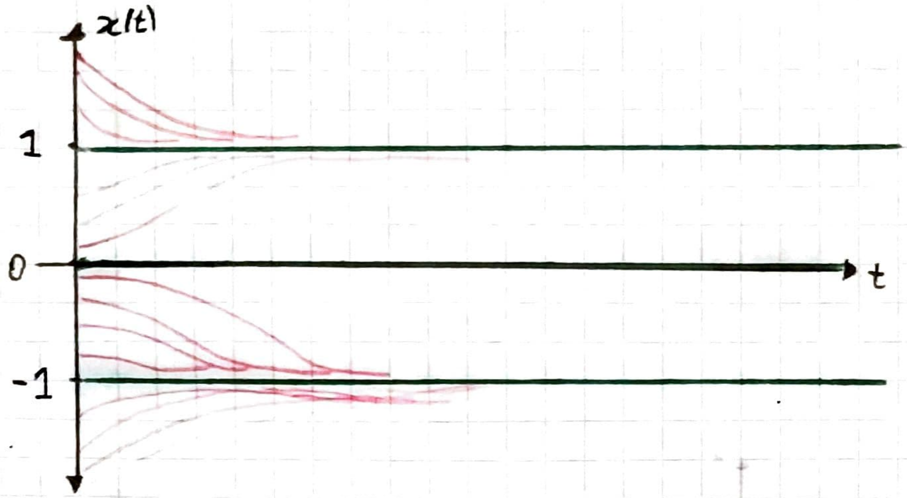
$f_x = 1 - 3x^2$

$f_x\big|_{x=-1} = -2 < 0 \rightarrow$ Estable.

$f_x\big|_{x=0} = 1 > 0 \rightarrow$ Inestable

$f_x\big|_{x=1} \; -2 < 0 \rightarrow$ Estable





2.2.4.

$\dot{x} = e^{-x} \sin x$

$f(x) = e^{-x} \sin(x)$

$e^{-x} \sin(x) = 0$

$x = K\pi \rightarrow$ fixed points ; $K \in \mathbb{Z}$

$f_x = -e^{-x} \sin(x) + e^{-x} \cos(x)$

$f_x = e^{-x} (\cos(x) - \sin(x))$

$f_x \Big|_{x = K\pi} = e^{-K\pi} [\cos(K\pi) - \sin(K\pi)]$

Como $e^{-K\pi} > 0$ Para todo K nos enfocamos en el segundo termino

$\cos(K\pi) - \sin(K\pi)$

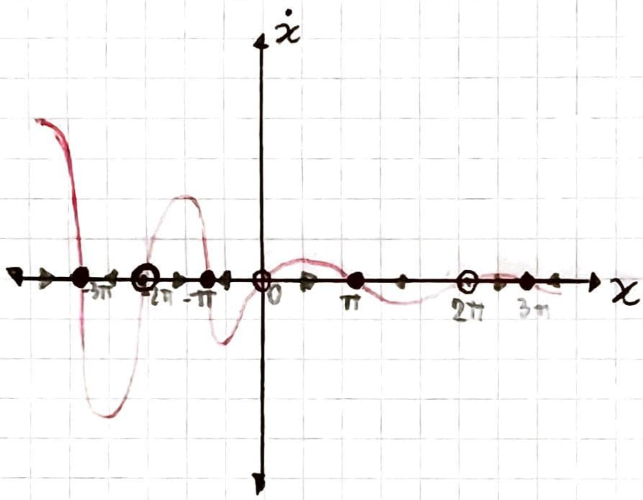
$\cos[K\pi] = 1$ Para K Par

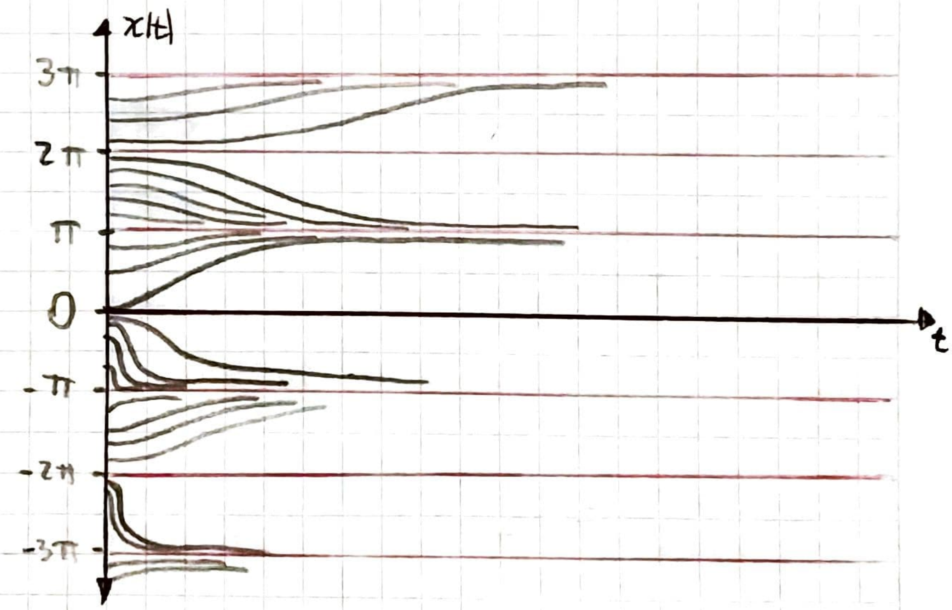
$= -1$ Para K inPar

$\sin[K\pi] = 0$ Para Cualquier K

Por lo tanto, $f_x > 0$ para K par y $f_x < 0$ para K impar

Puntos de equilibrio $= K\pi$ $\begin{cases} K \text{ impar} \rightarrow \text{Equilibrio estable} \\ K \text{ par} \rightarrow \text{Equilibrio Inestable} \end{cases}$





2.2.5

$\dot{x} = 1 + \frac{1}{2}\cos x$

$f(x) = 1 + \frac{1}{2}\cos x$

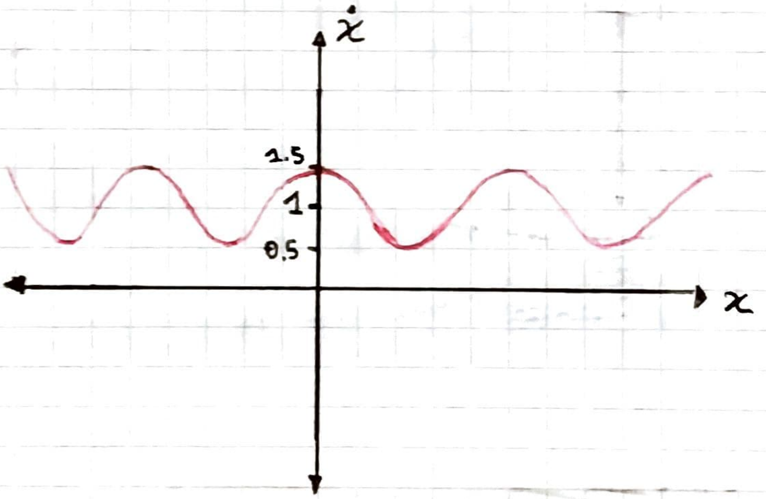
$1 + \frac{1}{2}\cos(x) = 0$

$\frac{1}{2}\cos(x) = -1$

$\cos(x) = -2$

x no tiene Solución en $\mathbb{R}$

No hay Puntos de eQuilibrio



2.2.6

$\dot{x} = 1 - 2\cos(x)$

$f(x) = 1 - 2\cos(x)$

$1 - 2\cos(x) = 0$

$2\cos(x) = 1$

$\cos(x) = \frac{1}{2}$

$x = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$x_1 = \frac{\pi}{3} + 2\pi K \quad , \quad K \in \mathbb{Z}$

$x_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} + 2\pi n \quad n \in \mathbb{Z}$

$f_x = 2\,\mathrm{Sen}(x)$

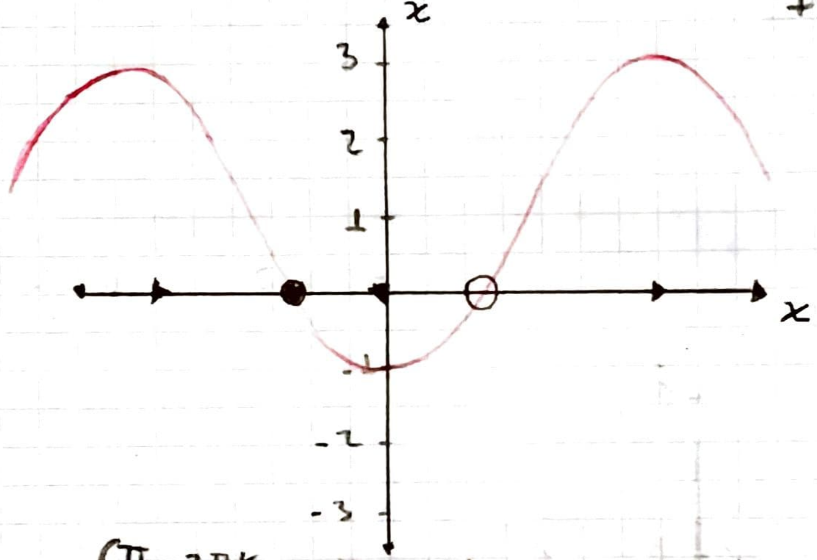
$$f_x \Big|_{x=\frac{\pi}{3}+2\pi K} = 2\,\mathrm{sen}\left(\frac{\pi}{3} + 2\pi K\right)$$

$$= 2\left[\mathrm{sen}\left(\frac{\pi}{3}\right)\overset{1}{\cancel{\cos(2\pi K)}} + \overset{0}{\cancel{\mathrm{sen}(2\pi K)}}\cos\left(\frac{\pi}{3}\right)\right]$$

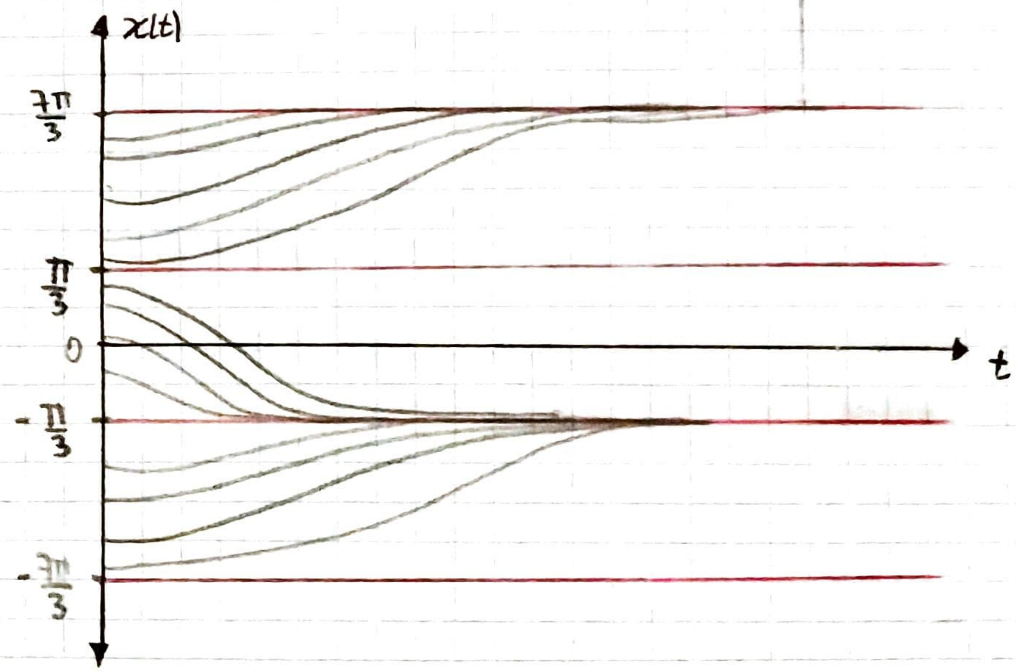
$$= 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} > 0$$

$$f_x \Big|_{x=\frac{5\pi}{3}+2\pi n} = 2\left[\mathrm{sen}\left(\frac{5\pi}{3}\right)\right]$$

$$f_x = -\sqrt{3} < 0$$

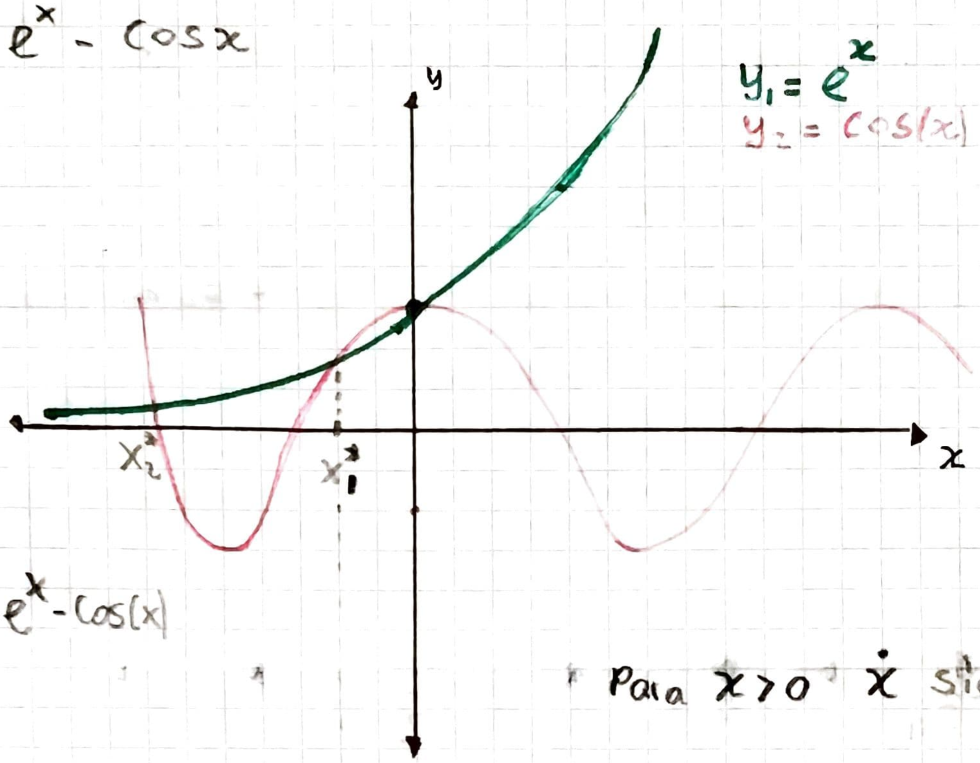


Puntos de Equilibrio $= \begin{cases} \frac{\pi}{3} + 2\pi K \rightarrow \text{Inestables para todo } K \\ \frac{5\pi}{3} + 2\pi n \rightarrow \text{Estables para todo } n \end{cases}$



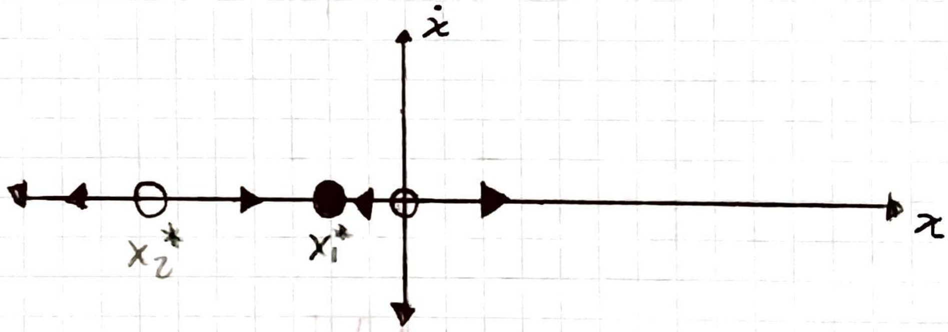
2.2.7

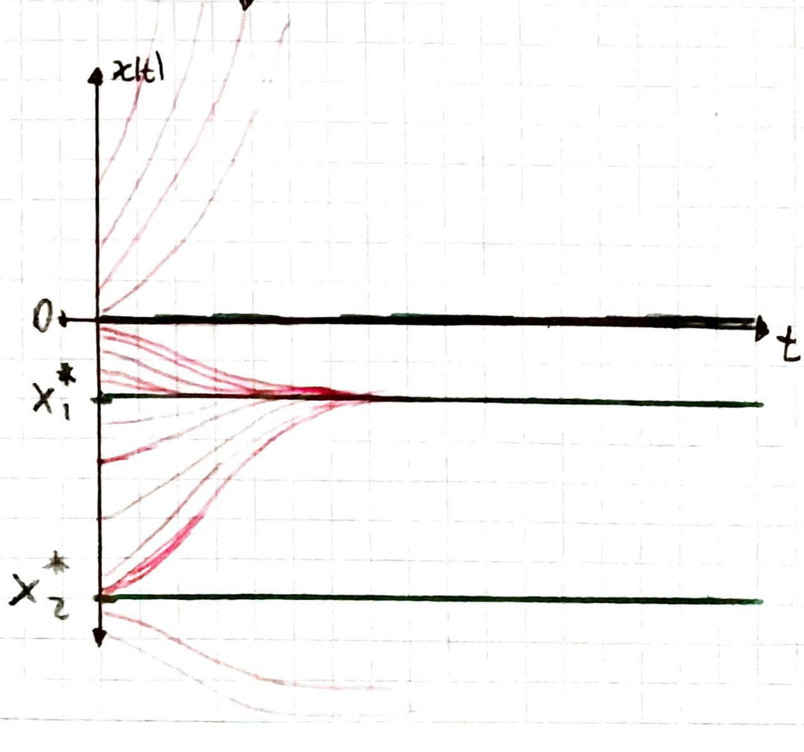
$\dot{x} = e^x - \cos x$



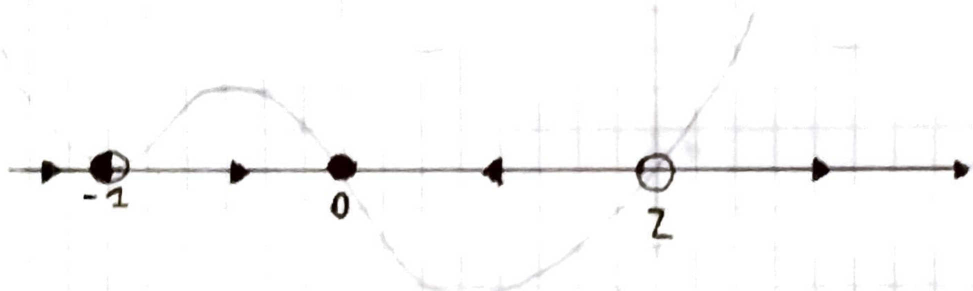
$f(x) = e^x - \cos(x)$

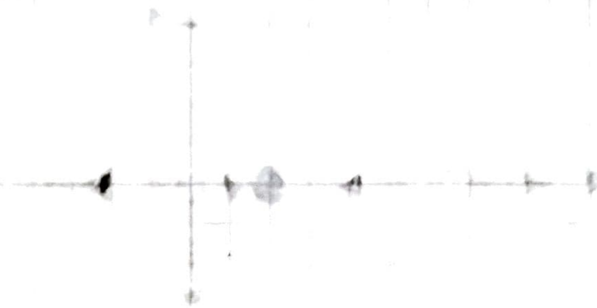
* Para $x > 0$ $\dot{x}$ siempre es > 0





2.2.8 (Working backwords, from flows to equations) Given an equation $\dot{x} = f(x)$, we know how to sketch the corresponding flow on the real line. Here you are asked to solve the opposite problem: For the phase portrait shown in Figure 1, find an equation that is consistent with it (there are an infinite number of correct answers - and wrong ones two)





$f(x) > 0 \; ; \; x > 2$

$f(x) < 0 \; ; \; 0 < x < 2$

$f(x) > 0 \; ; \; x < 0$

$f(x) = 0 \; ; \; x = \{-1, 0, 2\}$

$f_x \big|_{x=-1} = 0$

Probamos ajustando una función polinomial de tercer grado

$f(x) = ax^3 + bx^2 + cx + d$

$f_x = 3ax^2 + 2bx + c$

$f_x\big|_{x=-1} = 3a - 2b + c = 0$

$\boxed{3a - 2b + c = 0}$ (1)

$f(-1) = 0 = -a + b - c + d$

$\boxed{-a + b - c + d = 0}$ (2)

$f(0) = d = 0$

$\boxed{d = 0}$

$f(2) = 8a + 4b + 2c + d = 0$

$\boxed{8a + 4b + 2c + d = 0}$ (3)

Reemplazando $d = 0$

$$\left.\begin{array}{l} 3a - 2b + c = 0 \\ -a + b - c = 0 \\ 8a + 4b + 2c = 0 \end{array}\right\}$$

Lo cual resultaría en la solución trivial [0, 0, 0]

Por lo que no sirve este acercamiento

Ahora Pioremos con las otras condiciones

2.4 Linear Stability Analysis

Use linea Stability analysis to classify the fixed Points of the following Systems. If linear Stability analysis Fails because $f'(x^*)=0$, use a graphical argument to decide the stability.

2.4.1

$$\dot{x} = x(1-x)$$

$$f(x) = x(1-x)$$

$$x(1-x) = 0$$

$x^* = \{0, 1\}$ → Fixed Points

$$f(x) = x - x^2$$

$$f_x = 1 - 2x$$

$\left.f_x\right|_{x=0} = 1 > 0$ → Inestable

$\left.f_x\right|_{x=1} = -1 < 0$ → Estable

2.4.2

$$\dot{x} = x(1-x)(2-x)$$

$$f(x) = x(1-x)(2-x)$$

Fixed Points:

$$x = \{0, 1, 2\}$$

Estabilidad:

$$f(x) = (x - x^2)(2 - x)$$

$$f(x) = 2x - 2x^2 - x^2 + x^3$$

$$f(x) = x^3 - 3x^2 + 2x$$

$$\boxed{f_x = 3x^2 - 6x + 2}$$

$$f_x\Big|_{x=0} = 2 > 0 \rightarrow \text{Inestable}$$

$$f_x\Big|_{x=1} = -1 < 0 \rightarrow \text{Estable}$$

$$f_x\Big|_{x=2} = 2 > 0 \rightarrow \text{Inestable}$$

2.4.3

$$\dot{x} = \tan(x)$$

$$f(x) = \tan(x)$$

$$\tan(x) = 0$$

$$x = \tan^{-1}(0)$$

$$x^* = 2\pi n \quad n \in \mathbb{Z}$$

Estabilidad:

$f(x) = \tan(x)$

$f_x = \sec^2(x)$

$$f_x\big|_{x=2\pi n} = \sec^2(2\pi n) = \frac{1}{\cos^2(2\pi n)} = 1 \text{ Para cualquier } n$$

$x^* = 2\pi n$ son los únicos puntos de equilibrio y son inestables todos

2.8. Solving Equations on the Computer

2.8.1 (Slope field) The slope is constant along horizontal lines in Figure 2.8.2. Why should we have expected this?

R/ Esto era de esperarse, pues $\frac{dx}{dt}$ depende únicamente de x y no de t, por lo que la pendiente varía según los cambios en x pero es constante a lo largo del tiempo para un x fijo

2.8.3 (Calibrating the Euler method) The goal of this problem is to test the Euler method on the initial value problem $\dot{x} = -x$, $x(0) = 1$.

a) Solve the problem analytically. What is the exact value of $x(1)$?

b) Using the Euler method with step size $\Delta t = 1$, estimate $x(1)$ numerically - call the result $\hat{x}(1)$. Then repeat, using $\Delta t = 10^{-n}$, for $n = 1, 2, 3, 4$.

c) Plot the Error $E = |\hat{x}(1) - x(1)|$ as a function of Δt. Then plot $Ln(E)$ vs $Ln(t)$.

Explain the results

Solution:

a)

$$\dot{x} = -x \quad , \quad x(0) = 1$$

$$\frac{\partial x}{\partial t} = -x$$

$$-\frac{\partial x}{x} = dt$$

$$t = -\int \frac{\partial x}{x}$$

$$t = -\ln(x) + C$$

$$\ln(x) = -t + c$$

$$x = e^{(-t+c)}$$

$$x = e^{-t} \underbrace{e^{c}}_{k}$$

$$x(t) = k e^{-t}$$

$$x(0) = k = 1$$

$$\boxed{x(t) = e^{-t}}$$

$$x(1) = e^{-1} = 0,3678$$

b) El resto del ejercicio se encuentra simulado, ver en: https://github.com/Rmejiaz/ModeladoSimulacion/blob/main/Cuadernos/
Cap2.ipynb

2.8.4 y 2.8.5: Ver Simulaciones

https://github.com/Rmejiaz/ModeladoSimulacion/blob/main/Cuadernos/Cap2.ipynb