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Simulaciones: https://github.com/Rmejiaz/ModeladoSimulacion/blob/main/Cuadernos/Cap2.ipynb

Nonlinear Dynamics And Chaos - Ejercicios

Capitulo 2

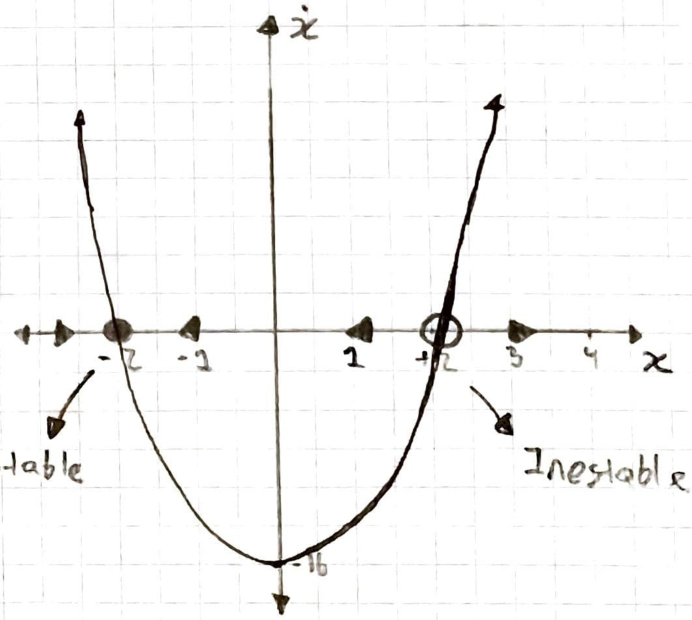
2.2 - Fixed Points and Stability

Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

2.2.1

$$\dot{x} = 4x^2 - 16$$

$$f(x) = 4x^2 - 16$$



$$4x^2 - 16 = 0$$

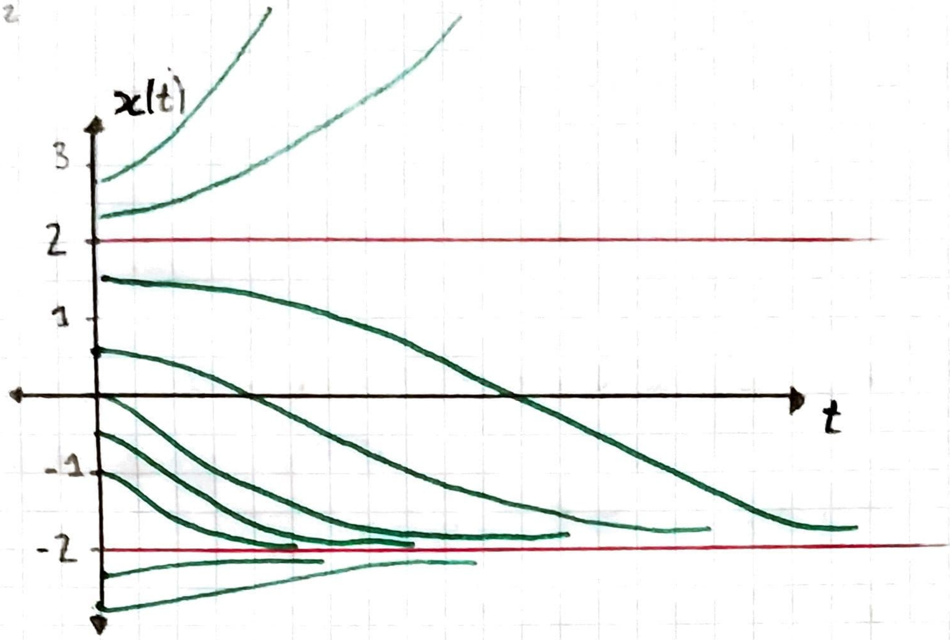
$$4x^2 = 16$$

$$x^2 = \frac{16}{4}$$

$$\boxed{x = \pm 2}$$

$$f_x = 8x \qquad f_x\Big|_{x=-2} = -16 \rightarrow \text{Estable!}$$

$$f_x\Big|_{x=2} = 16 \rightarrow \text{Inestable!}$$



$$x(t) = ?$$

$$\frac{\partial x}{\partial t} = 4x^2 - 16$$

$$\frac{\partial x}{4x^2-16} = \partial t$$

$$t = \int \frac{1}{4x^2-16}\,\partial x$$

$$= \frac{1}{4}\int \frac{1}{x^2-4}\,\partial x$$

$$x = 2\sec(\theta)$$

$$= \frac{1}{4}\int \frac{1}{4\sec^2(\theta)-4} \qquad \partial x = 2\sec(\theta)\tan(\theta)\,\partial\theta$$

$$= \frac{1}{16}\int \frac{1 \cdot 2\sec(\theta)\tan(\theta)}{\sec^2(\theta)-1}\,\partial\theta = \frac{1}{8}\int \frac{\sec(\theta)\cancel{\tan(\theta)}}{\tan^{\cancel{2}}(\theta)}\,\partial\theta$$

$$t = \frac{1}{8}\int \frac{\sec(\theta)}{\tan(\theta)}\, d\theta$$

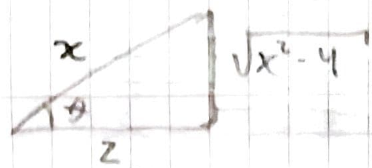
$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \longrightarrow \qquad \frac{\sec(\theta)}{\tan(\theta)} \quad \frac{\frac{1}{\cos(\theta)}}{\frac{\operatorname{sen}(\theta)}{\cos(\theta)}} \qquad \csc(\theta)$$

$$\tan(\theta) = \frac{\operatorname{sen}(\theta)}{\cos(\theta)}$$

$$t = \frac{1}{8}\int \csc(\theta)\, d\theta = -\frac{1}{8} \operatorname{Ln}|\csc(\theta) + \cot(\theta)| + C$$

$$\sec(\theta) = \frac{x}{2}$$

$$\cos(\theta) = \frac{2}{x}$$



$$\csc(\theta) = \frac{x}{\sqrt{x^2-4}}$$

$$\cot(\theta) = \frac{2}{\sqrt{x^2-4}}$$

$$t = -\frac{1}{8} \operatorname{Ln}\left| \frac{x}{\sqrt{x^2-4}} + \frac{2}{\sqrt{x^2-4}} \right|$$

$$= -\frac{1}{8} \operatorname{Ln}\left| \frac{x+2}{\sqrt{x^2-4}} \right|$$

$$t = -\frac{1}{8}\left(\operatorname{Ln}(x+2) - \operatorname{Ln}\left(\sqrt{x^2-2}\right)\right)$$

$$t = -\frac{1}{8}\operatorname{Ln}(x+2) + \frac{1}{16}\operatorname{Ln}(x^2-2) \longrightarrow \text{Forma implicita}$$

2.2.2

$$\dot{x} = 1 - x^{14}$$

$$f(x) = 1 - x^{14}$$

$$1 - x^{14} = 0$$

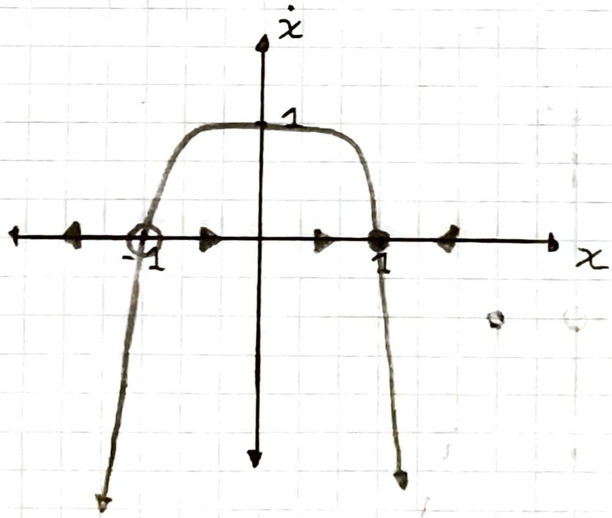
$$x^{14} = 1$$

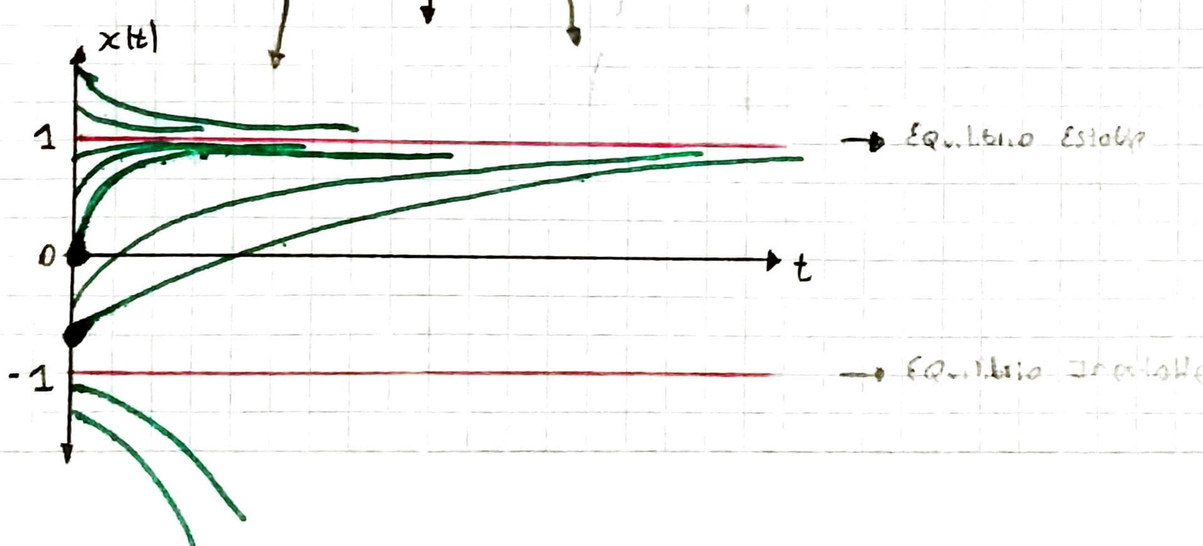
$\boxed{x = \pm 1}$ (soluciones Reales)

$$f_x = -14x^{13}$$

$f_x\big|_{x=-1} = 14$ → Inestable

$f_x\big|_{x=1} = -14$ → Estable





2.2.3

$\dot{x} = x - x^3$

$f(x) = x - x^3$

$x - x^3 = 0$

$-x^3 + x = 0$

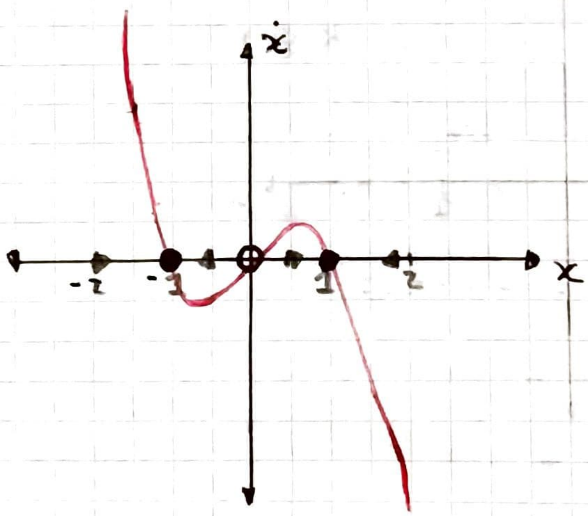
$x = \{-1, 0, 1\}$

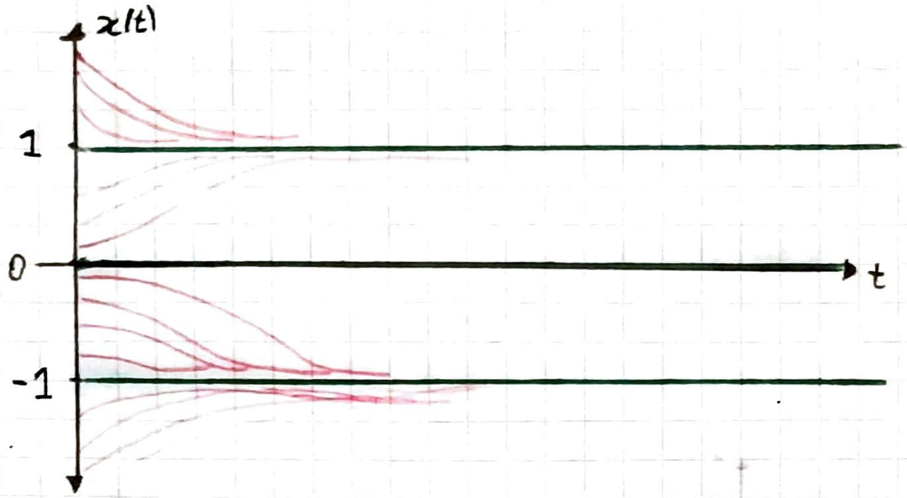
$f_x = 1 - 3x^2$

$f_x\big|_{x=-1} = -2 < 0 \rightarrow$ Estable.

$f_x\big|_{x=0} = 1 > 0 \rightarrow$ Inestable

$f_x\big|_{x=1} \quad -2 < 0 \rightarrow$ Estable





2.2.4.

$\dot{x} = e^{-x} \sin x$

$f(x) = e^{-x} \sin(x)$

$e^{-x} \sin(x) = 0$

$x = K\pi \rightarrow$ fixed points ; $K \in \mathbb{Z}$

$f_x = -e^{-x} \sin(x) + e^{-x} \cos(x)$

$f_x = e^{-x} (\cos(x) - \sin(x))$

$f_x \Big|_{x = K\pi} = e^{-K\pi} \left[\cos(K\pi) - \sin(K\pi) \right]$

Como $e^{-K\pi} > 0$ Para todo K nos enfocamos en el Segundo termino

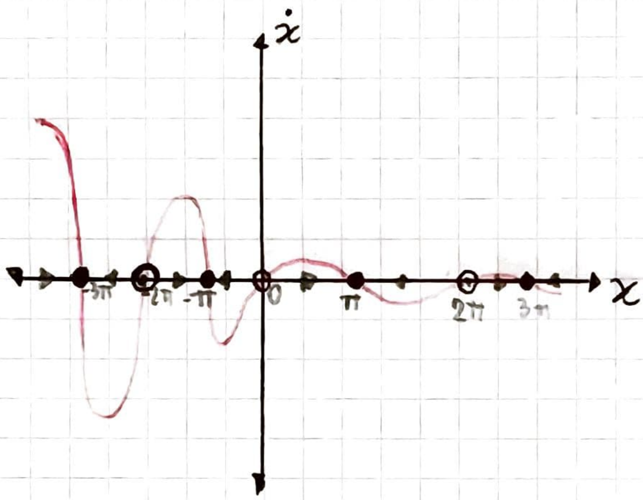
$\cos(K\pi) - \sin(K\pi)$

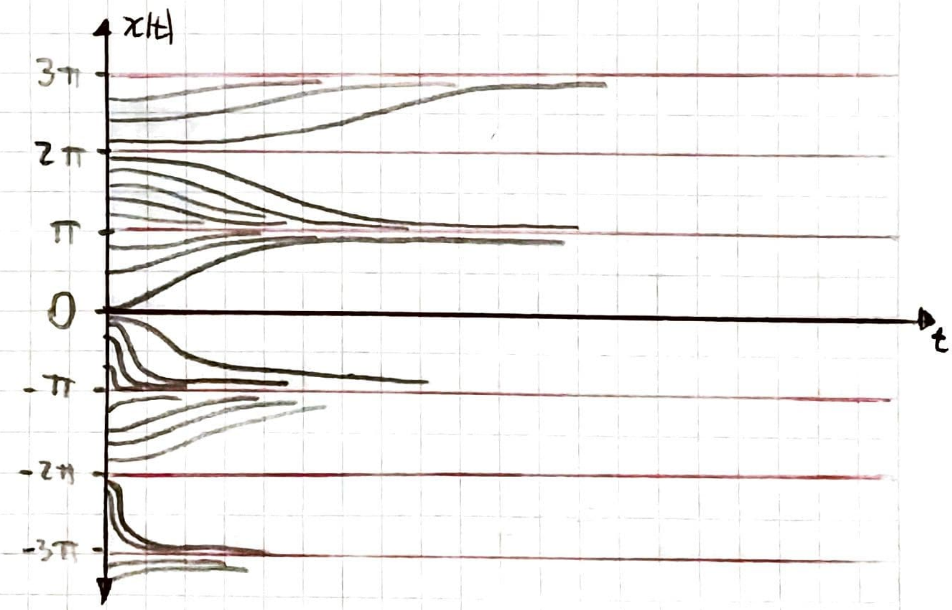
$\cos[K\pi] = 1$ Para K Par
$= -1$ Para K inPar

$\sin[K\pi] = 0$ Para cualquier K

Por lo tanto, $f_x > 0$ para K par y $f_x < 0$ para K impar

Puntos de equilibrio $= K\pi \begin{cases} K \text{ impar} \rightarrow \text{Equilibrio estable} \\ K \text{ par} \rightarrow \text{Equilibrio Inestable} \end{cases}$





2.2.5

$\dot{x} = 1 + \frac{1}{2}\cos x$

$f(x) = 1 + \frac{1}{2}\cos x$

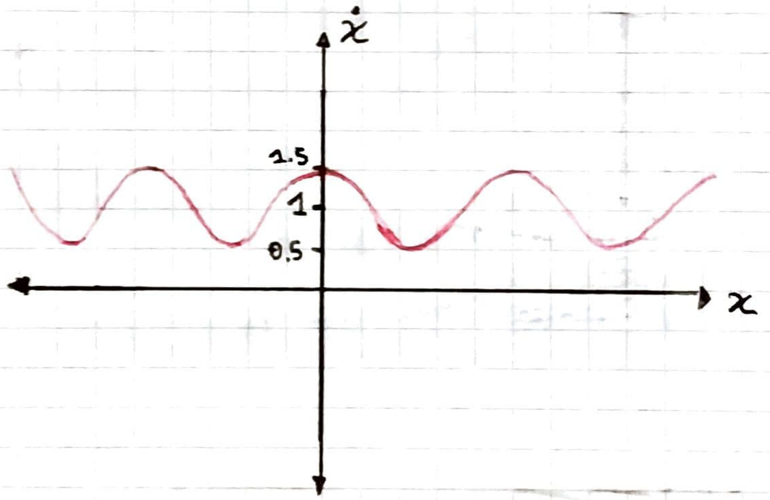
$1 + \frac{1}{2}\cos(x) = 0$

$\frac{1}{2}\cos(x) = -1$

$\cos(x) = -2$

x no tiene Solución en $\mathbb{R}$

No hay Puntos de eQuilibrio



2.2.6

$\dot{x} = 1 - 2\cos(x)$

$f(x) = 1 - 2\cos(x)$

$1 - 2\cos(x) = 0$

$2\cos(x) = 1$

$\cos(x) = \frac{1}{2}$

$x = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$x_1 = \frac{\pi}{3} + 2\pi K \quad , \quad K \in \mathbb{Z}$

$x_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} + 2\pi n \quad n \in \mathbb{Z}$

$f_x = 2\,\mathrm{Sen}(x)$

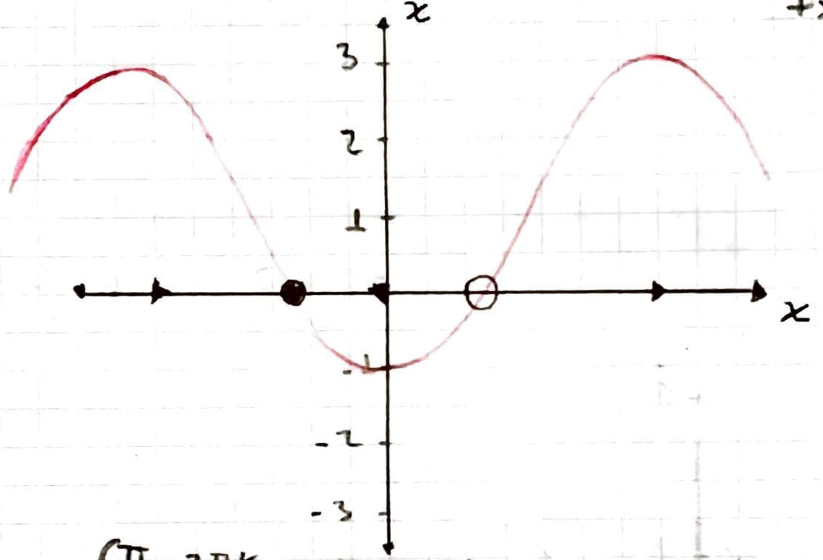
$$f_x \Big|_{x=\frac{\pi}{3}+2\pi K} = 2\,\mathrm{sen}\left(\frac{\pi}{3} + 2\pi K\right)$$

$$= 2\left[\mathrm{sen}\left(\frac{\pi}{3}\right) \overset{1}{\cancel{\cos(2\pi K)}} + \overset{0}{\cancel{\mathrm{sen}(2\pi K)\cos\left(\frac{\pi}{3}\right)}}\right]$$

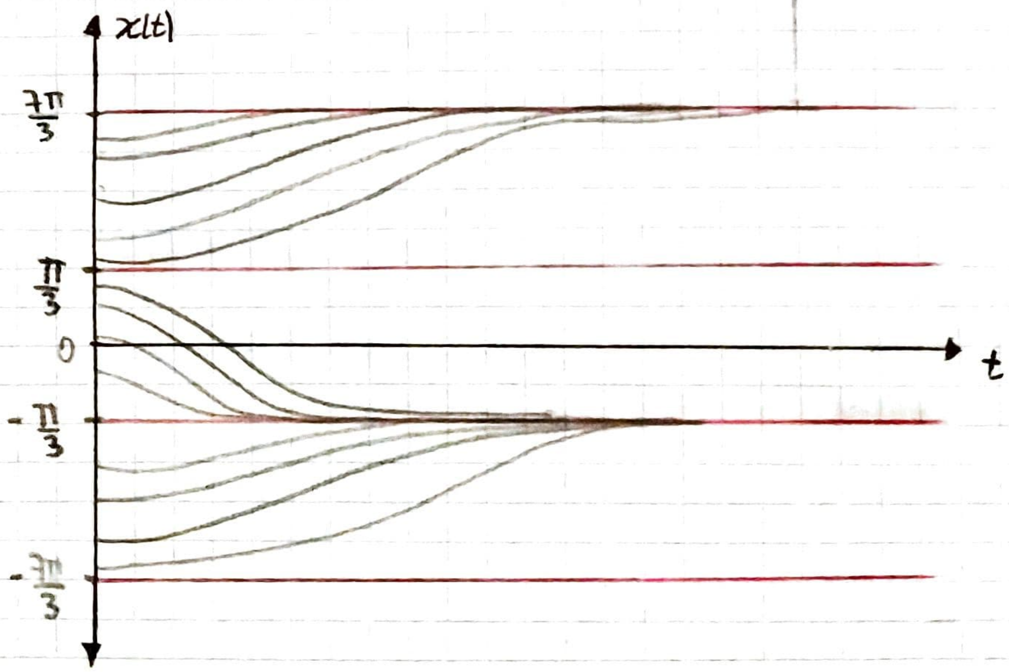
$$= 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} > 0$$

$$f_x \Big|_{x=\frac{5\pi}{3}+2\pi n} = 2\left[\mathrm{sen}\left(\frac{5\pi}{3}\right)\right]$$

$$f_x = -\sqrt{3} < 0$$

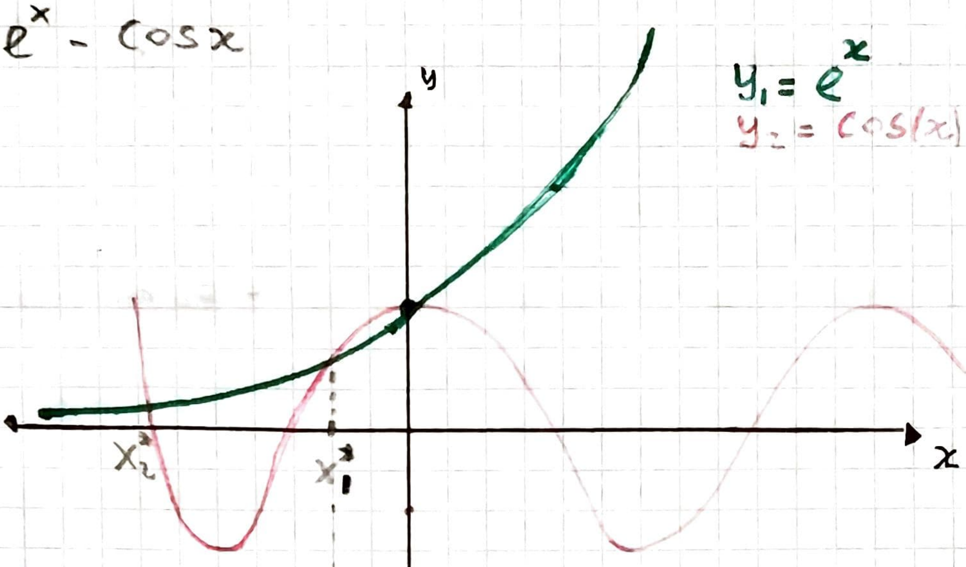


Puntos de Equilibrio $= \begin{cases} \frac{\pi}{3} + 2\pi K \rightarrow \text{Inestables para todo } K \\ \frac{5\pi}{3} + 2\pi n \rightarrow \text{Estables para todo } n \end{cases}$



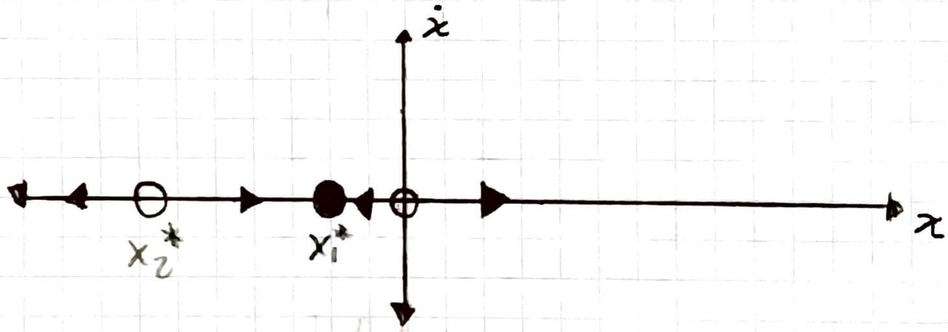
2.2.7

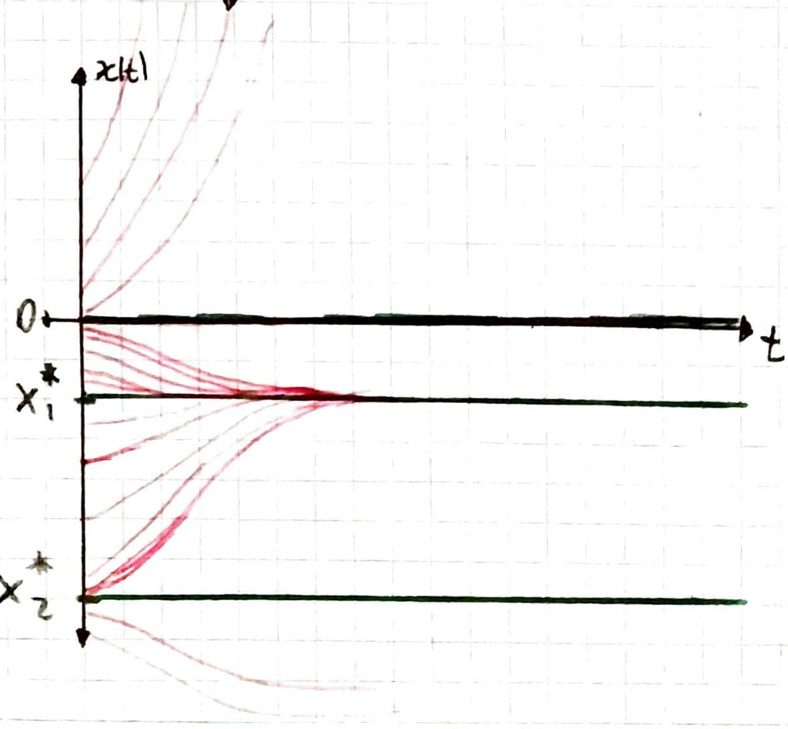
$\dot{x} = e^{x} - \cos x$



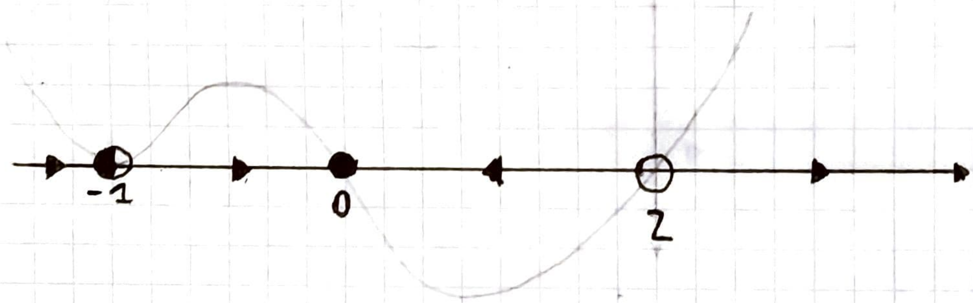
$f(x) = e^{x} - \cos(x)$

* Para $x > 0$ $\dot{x}$ siempre es > 0





2.2.8 (Working backwords, From flows to equations) Given an equation $\dot{x} = f(x)$, we know how to sketch the corresponding flow on the real line. Here you are asked to solve the opposite problem: For the phase portrait shown in Figure 1, find an equation that is consistent with it. (There are an infinite number of correct answers - and wrong ones two)



$f(x) > 0 \; ; \; x > 2$

$f(x) < 0 \; ; \; 0 < x < 2$

$f(x) > 0 \; ; \; x < 0$

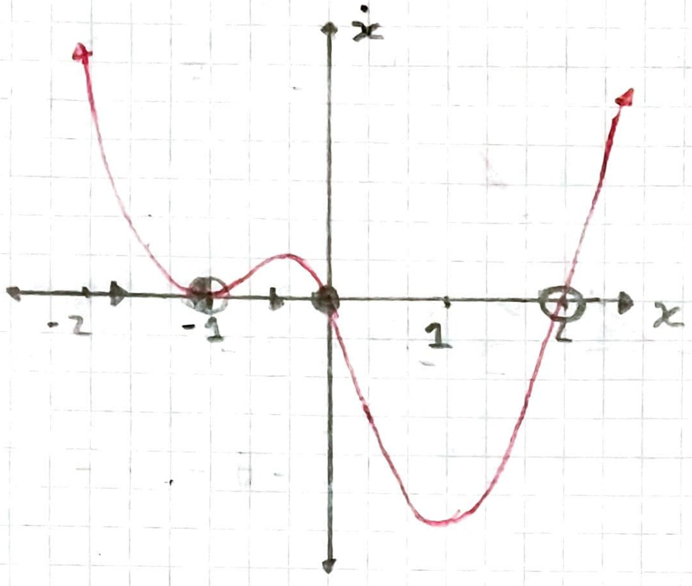
$f(x) = 0 \; ; \; x = \{-1, 0, 2\}$

$f_x \big|_{x=-1} = 0$

Se puede ajustar un polinomio de cuarto grado: que satisface los condiciones:

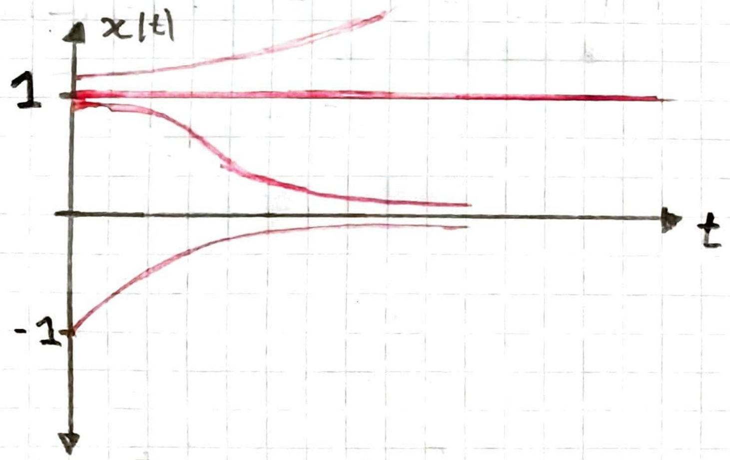
$$\boxed{f(x) = (x+1)^2 x (x-2)}$$

Al graficarlo se obtiene:



2.2.9 (Backwards again, now from solutions to equations)

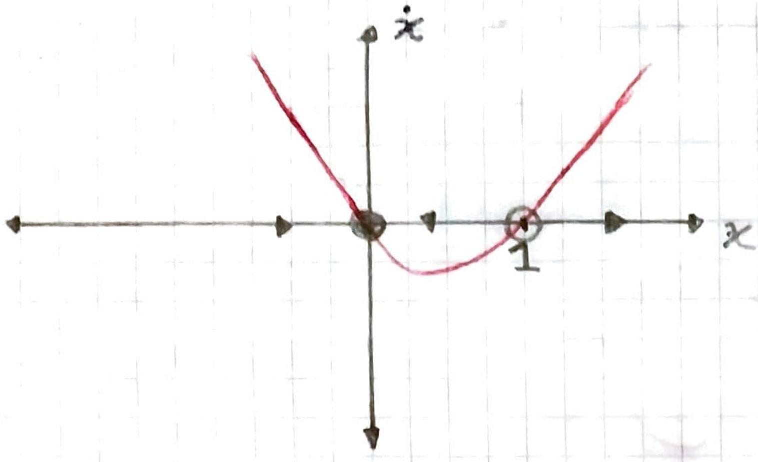
Find an equation $\dot{x} = f(x)$ whose solutions $x(t)$ are consistent with those shown in the following figure



Se puede ver que existe un equilibrio estable en 0, uno inestable en 1, y posiblemente un cambio de concavidad alrededor de 0.5

$f(x) = x(x-1)$

Cumple estos condiciones



[Ver Simulaciones]

https://github.com/Rmejiaz/ModeladoSimulacion/blob/main/Cuadernos/Cap2.ipynb

2.4 Linear Stability Analysis

Use linea Stability analysis to classify the fixed Points of the following Systems. If linear stability analysis Fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

2.4.1

$$\dot{x} = x(1-x)$$

$$f(x) = x(1-x)$$

$$x(1-x) = 0$$

$$x^* = \{0, 1\} \rightarrow \text{Fixed Points}$$

$$f(x) = x - x^2$$

$$f_x = 1 - 2x$$

$$f_x\big|_{x=0} = 1 > 0 \rightarrow \text{Inestable}$$

$$f_x\big|_{x=1} = -1 < 0 \rightarrow \text{Estable}$$

2.4.2

$$\dot{x} = x(1-x)(2-x)$$

$$f(x) = x(1-x)(2-x)$$

fixed Points:

$x = \{0, 1, 2\}$

Estabilidad:

$f(x) = (x - x^2)(2 - x)$

$f(x) = 2x - 2x^2 - x^2 + x^3$

$f(x) = x^3 - 3x^2 + 2x$

$$\boxed{f_x = 3x^2 - 6x + 2}$$

$f_x\big|_{x=0} = 2 > 0 \rightarrow$ Inestable

$f_x\big|_{x=1} = -1 < 0 \rightarrow$ Estable

$f_x\big|_{x=2} = 2 > 0 \rightarrow$ Inestable

2.4.3

$\dot{x} = \tan(x)$

$f(x) = \tan(x)$

$\tan(x) = 0$

$x = \tan^{-1}(0)$

$x^* = 2\pi n \quad n \in \mathbb{Z}$

Estabilidad:

$f(x) = \tan(x)$

$f_x = \sec^2(x)$

$$f_x\Big|_{x=2\pi n} = \sec^2(2\pi n) = \frac{1}{\cos^2(2\pi n)} = 1 \text{ Para cualquier } n$$

$x^* = 2\pi n$ son los únicos puntos de equilibrio y son inestables todos

2.4.4

$\dot{x} = x^2(6-x)$

$f(x) = x^2(6-x)$

$f(x) = 0$

$x^2(6-x) = 0$

Fixed Points

$x^* = 0$

$x^* = 6$

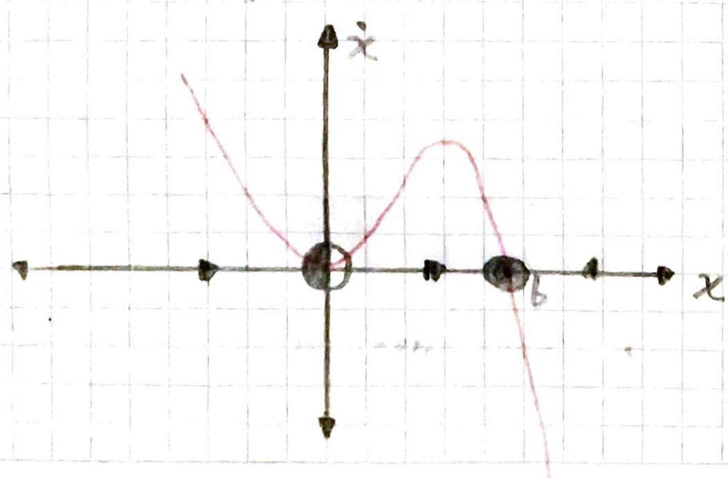
Stability

$f(x) = 6x^2 - x^3$

$f_x = 12x - 3x^2$

$f_x|_{x=0} = 0 \rightarrow$ No se puede establecer

$f_x|_{x=6} = 72 - 108 = -36 \rightarrow$ Estable



$x^* = 0$ es semiestable, pues por la izquierda atrae, pero por la derecha repele

2.4.5

$\dot{x} = 1 - e^{-x^2}$

$f(x) = 1 - e^{-x^2}$

Fixed Points

$1 - e^{-x^2} = 0$

$e^{-x^2} = 1$

$\ln(e^{-x^2}) = \ln(1)$

$-x^2 = 0$

$\boxed{x^* = 0}$

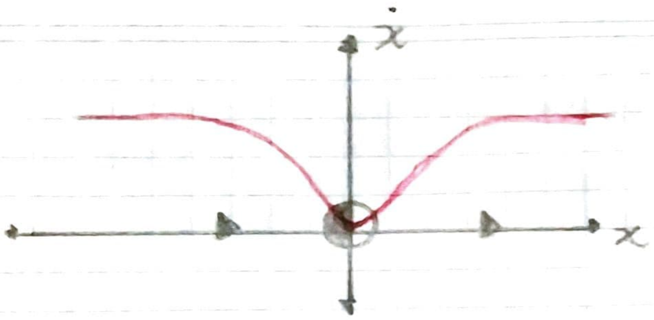
Stability

$f_x = -e^{-x^2}(-2x)$

$f_x = 2x e^{-x^2}$

Para $x^* = 0$

$f_x \big|_{x=0} = 0 \rightarrow$ No sirve el criterio



$x^* = 0$ es semiestable, pues por la izquierda es atractor y por la derecha repulsor

2.4.6

$\dot{x} = \ln(x)$

$f(x) = \ln(x)$

$\ln(x) = 0$

$\boxed{x^* = 1}$ → Fixed point

<u>Stability</u>

$f_x = \frac{1}{x}$

$f_x|_{x=1} = 1$ → Inestable

2.4.7

$\dot{x} = ax - x^3$, where a can be positive, negative, or zero. Discuss all three cases.

$f(x) = ax - x^3$.

Fixed Points

$ax - x^3 = 0$

$x(a - x^2) = 0$

$\boxed{x_1^* = 0}$

$a - x^2 = 0$

$x^2 = a$

$\boxed{x^* = \pm\sqrt{a}}$

Si $a > 0$.

$\boxed{x^* = \{0, -\sqrt{a}, +\sqrt{a}\}}$

Si $a < 0$

$\boxed{x^* = 0}$

Si $a = 0$

$\boxed{x^* = 0}$

Stability

$f_x = a - 3x^2$

Para $a < 0$:

$f_x\big|_{x=0} = a \rightarrow$ Estable $(a<0)$

Para $a = 0$

$f_x\big|_{x=0} = a \rightarrow$ No se puede decidir $(a=0)$

Para $a > 0$

$f_x\big|_{x=0} = a \rightarrow$ Inestable

$f_x\big|_{x=-\sqrt{a}} = a - 3a = -2a \rightarrow$ Estable

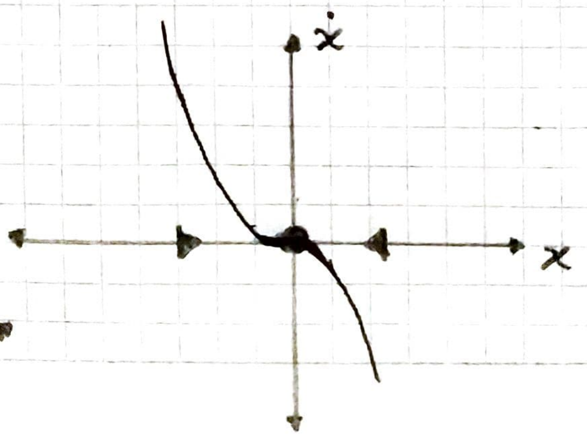
$f_x\big|_{x=\sqrt{a}} = a - 3a = -2a \rightarrow$ Estable

Para el único caso que no se puede decidir es en $a = 0$

$f(x) = -x^3$

$\boxed{x^* = 0}$

↳ Estable →



Graficamente se comprueba que es estable.

[Ver simulaciones]

https://github.com/Rmejiaz/ModeladoSimulacion/blob/main/Cuadernos/Cap2.ipynb

2.8. Solving Equations on the Computer

2.8.1 (Slope field) The slope is constant along horizontal lines in Figure 2.8.2. Why should we have expected this?

R/ Esto era de esperarse, pues $\frac{dx}{dt}$ depende únicamente de x y no de t, por lo que la pendiente varía según los cambios en x pero es constante a lo largo del tiempo para un x fijo

2.8.3 (Calibrating the Euler method) The goal of this problem is to test the Euler method on the initial value problem $\dot{x} = -x$, $x(0) = 1$.

a) Solve the problem analytically. What is the exact value of $x(1)$?

b) Using the Euler method with step size $\Delta t = 1$, estimate $x(1)$ numerically - call the result $\hat{x}(1)$. Then repeat, using $\Delta t = 10^{-n}$, for $n = 1, 2, 3, 4$.

c) Plot the Error $E = |\hat{x}(1) - x(1)|$ as a function of Δt. Then plot $\ln(E)$ vs $\ln(t)$.

Explain the results

Solution:

a)

$$\dot{x} = -x \quad , \quad x(0) = 1$$

$$\frac{dx}{dt} = -x$$

$$-\frac{dx}{x} = dt$$

$$t = -\int \frac{dx}{x}$$

$$t = -\ln(x) + C$$

$$\ln(x) = -t + c$$

$$x = e^{(-t+c)}$$

$$x = e^{-t} \underbrace{e^{c}}_{k}$$

$$x(t) = k e^{-t}$$

$$x(0) = k = 1$$

$$\boxed{x(t) = e^{-t}}$$

$$x(1) = e^{-1} = 0,3678$$

b) El resto del ejercicio se encuentra simulado, ver en: https://github.com/Rmejiaz/ModeladoSimulacion/blob/main/Cuadernos/Cap2.ipynb

2.8.4 y 2.8.5 : ver Simulaciones

2.8.6 (Analytically intractable problem) Consider the initial value problem $\dot{x} = x + e^{-x}$, $x(0) = 0$. In contrast to Exercise 2.8.3, this problem can't be solved analytically

a) Sketch the solution $x(t)$ for $t \geq 0$

b) Using some analytical arguments, obtain rigorous bounds on the value of x at $t = 1$. In other words, prove that $a < x(1) < b$ for a, b to be determined. By being clever, try to make a and b as close as possible. (Hint: Bound the given vector field by approximate vector fields that can be integrated analytically.)

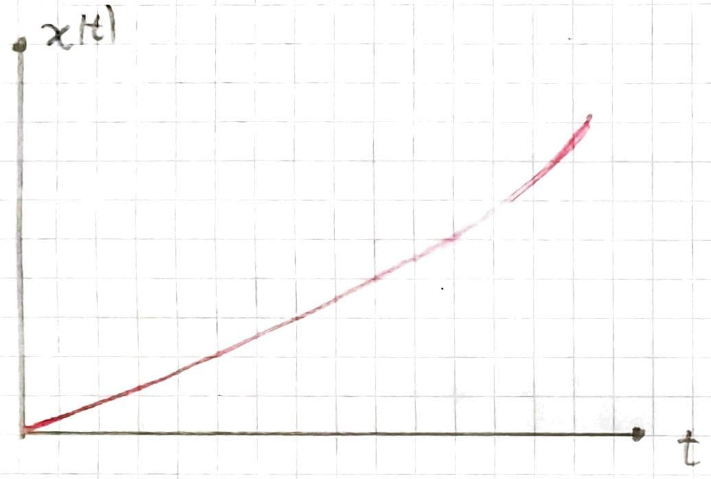
c) Now for the numerical part: Using the Euler method, compute x at $t = 1$, correct to three decimal places. How small does the stepsize need to be to obtain the desired accuracy? (Give the order of magnitude, not the exact number.)

d) Repeat part (b), now using the Runge - Kutta method. Compare the results for stepsizes $\Delta t = 1$, $\Delta t = 0.1$ and $\Delta t = 0.01$

a)

El flujo no tiene equilibrios, y para $x \geq 0$, $\dot{x}$ crece de forma monótona, por lo que podemos deducir que no hay cambios en la concavidad del flujo

La solución sería algo del estilo:



b)

El campo vectorial dado es:

$$\dot{x} = x + e^{-x}$$

Este campo puede acotarse con otros, tales como:

$$\dot{x} = 1 + x$$
$$\dot{x} = 0.9 + 0.4x$$

Se puede demostrar que:

$$0.9 + 0.4x < x + e^{-x} < 1 + x$$

En algún intervalo cerrado

$x + e^{-x} < 1 + x$

$e^{-x} < 1 \rightarrow$ Cierto para todo $x > 0$

$\boxed{x > 0}$

$0.9 + 0.4x < x + e^{-x}$

Se cumple para todo $\mathbb{R}$

Podemos tomar estos dos campos vectoriales como límites. Ahora, se halla el flujo de cada uno:

$\dot{x} = x + 1$

$\frac{dx}{dt} = x + 1$

$\frac{dx}{x+1} = dt$

$t = \int \frac{dx}{x+1}$

$u = x + 1$
$du = dx$

$t = \int \frac{du}{u}$

$t = \ln(x+1) + c$

$e^{t} = e^{\ln(x+1) + c}$

$e^{t} = e^{\ln(x+1)} e^{c}$

$e^{t} = (x+1) e^{c} \rightarrow c_2$

$(x+1) c_2 = e^{t}$

$x + 1 = \frac{e^{t}}{c_2}$ $\quad c_3 = \frac{1}{c_2}$

$x = c_3 e^{t} - 1$

$x(0) = 0$

$x(0) = c_3 - 1 = 0$

$c_3 = 1$

$\boxed{x(t) = e^{t} - 1}$

$x(1) = e - 1$

$\boxed{b = e - 1} \rightarrow$ Límite superior

Ahora para el límite inferior:

$\dot{x} = 0.9 + 0.4x$

$$dt = \frac{dx}{0.9 + 0.4x}$$

$$t = \int \frac{1}{0.9 + 0.4x}\, dx$$

$u = 0.9 + 0.4x$

$du = 0.4\, dx \qquad dx = \frac{du}{0.4}$

$$t = \int \frac{1}{0.4} \cdot \frac{1}{u}\, du$$

$$t = \frac{1}{0.4} \ln(0.9 + 0.4x) + c$$

$$0.4t = \ln(0.9 + 0.4x) + c$$

$$e^{0.4t} = e^{c}(0.9 + 0.4x) \qquad c_2 = e^{c}$$

$$c_2(0.9 + 0.4x) = e^{0.4t}$$

$$0.4x = \frac{e^{0.4t}}{c_2} - 0.9 \qquad c_3 = \frac{1}{c_2}$$

$$x = \frac{c_3 e^{0.4t} - 0.9}{0.4}$$

$$x = \frac{c_3}{0.4} e^{0.4t} - \frac{0.9}{0.4} \qquad c_4 = \frac{c_3}{0.4}$$

$$x(t) = c_4 e^{0.4t} - \frac{9}{4}$$

$$x(0) = c_4 - \frac{9}{4} = 0$$

$$\boxed{c_4 = \frac{9}{4}}$$

$$\boxed{x(t) = \frac{9}{4}\left(e^{0.4t} - 1\right)}$$

$$\boxed{x(1) = \frac{9}{4}\left(e^{0.4} - 1\right) \approx 1,1066}$$

$$\boxed{a \approx 1,1066}$$

$$\boxed{b \approx 1,718}$$

$$\boxed{1,1066 < x(1) < 1,718}$$

(Ver simulación):
https://github.com/Rmejiaz/ModeladoSimulacion/blob/main/Cuadernos/Cap2.ipynb