Project 5
Motor Vehicle Design
Automotive Engineering
AA 2017-2018
Rigid Body Model, Stability analysis

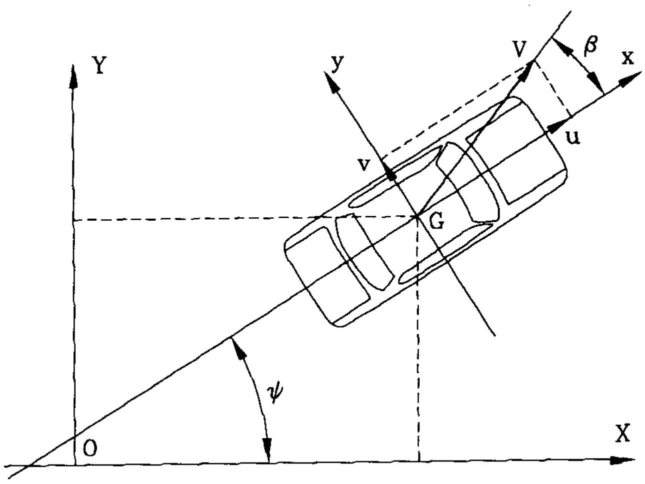

Figure 1: Layout of the vehicle and reference system

1 Stability of the vehicle: rigid body model, open loop control

The present project is devoted to study the lateral stability of a two axle vehicle with open loop control. A 3 dof model is adopted to this end. It derives that:

- the motion of the vehicle can be described only in the XY plane,
- the vehicle is an hyperstatic structure. The vertical tire - ground forces are undefined if the structure compliance is not accounted for. This is bypassed if the so called "bicycle" model is adopted.

The reference frame is reported in figure 1. The motion of the vehicle is described by the X, Y coordinates of the center of gravity G and by the yaw angle ψ. With reference to the $(OXYZ)$ frame, the equations of motion can be written as:

$$\left\{\begin{array}{l} m\,\ddot{X} = F_X \\ m\,\ddot{Y} = F_Y \\ J_z\ddot{\psi} = M_z \end{array}\right. . \tag{1}$$

It is more convenient to write the equations of motion with reference to the frame $(Gxyz)$ as

- the generalized forces applied to the vehicle $(F_x, F_y, F_z, M_x, M_y, M_z)$ can be written quite simply with reference to that frame.
- except for the yaw angle ψ, with reference to such a frame it is possible to linearize the equations of motion.

Therefore, considering that

$$\left\{ \begin{array}{c} F_X \\ F_Y \end{array} \right\} = \left[\begin{array}{cc} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{array} \right] \left\{ \begin{array}{c} F_x \\ F_y \end{array} \right\},$$

and that

$$\left\{ \begin{array}{c} \dot{X} \\ \dot{Y} \end{array} \right\} = \left[\begin{array}{cc} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{array} \right] \left\{ \begin{array}{c} u \\ v \end{array} \right\},$$

the equations 1 can be written as:

$$\left\{ \begin{array}{l} m\ (\dot{u} - rv) = F_x \\ m\ (\dot{v} + ru) = F_y \\ J_z \dot{r} = M_z. \end{array} \right. \tag{2}$$

where $r = \dot{\psi}$.

To solve the problem it is requested to define

- the relation between the vehicle slip angle β and the side slip angle α_i of each tire,
- the relation between the side force F_{yi} and the self aligning torque M_{zi} of each tire with respect to the corresponding side slip angle α_i.

It is worth to note that the problem is non linear as:

- are present the products of the variables,
- the forces developed by the tires are non linear with respect to the side slip angle.

It is possible to linearize the equations of motion if:

- the side slip angles of the tires are small,
- the vehicle side slip angle β and the steering angle δ are small.

Considering β small equation 2 simplifies as:

$$\left\{ \begin{array}{l} m\ (\dot{V} - rV\beta) = F_x \\ mV\ (\dot{\beta} + r) + m\beta\dot{V} = F_y \\ J_z\ \dot{r} = M_z. \end{array} \right. \tag{3}$$

where the longitudinal velocity V is a known function of the time. Therefore, it derives that the unknown variables are β, r and F_x. Neglecting the interactions of the tire force component F_{xi} and F_{yi}, the first of the three equations in 3 can be uncoupled from the last two. Therefore, the equations of motion describing the lateral dynamics of the vehicle are:

$$\left\{ \begin{array}{l} mV\ (\dot{\beta} + r) + m\beta\dot{V} = F_y \\ J_z\ \dot{r} = M_z. \end{array} \right. \tag{4}$$

The equations 4 are already non linear as

- the forces F_{yi} and the torques M_{zi} of the tires are non linear functions of the tires' side slip angles α_i,
- the tire side slip angles α_i are non linear functions of the variables β, r. Note that the angles α_i can be expressed as,

$$\alpha_i = \arctan\left(\frac{v + rx_i}{u - ry_i}\right) \simeq \arctan\left(\frac{V\beta + rx_i}{V - ry_i}\right).$$

Considering that the term $y_i r$ is negligible with respect to V, it derives that:

$$\begin{cases} \alpha_1 = \beta + \dfrac{a}{V}r - \delta_1 \\ \alpha_2 = \beta - \dfrac{b}{V}r \end{cases}, \tag{5}$$

where the suffix 1 and 2 are referred to the front and rear axles. Eliminating the term y_i in fact, the term y disappears completely from the equations of motion and the side slip forces are referred only to the axles and not to the single tires.

On the base of the assumptions above described, linearizing F_y and M_z about their null values and considering that the aerodynamic forces can be expressed as a linear function of the side angle β of the vehicle, it is possible to obtain a complete linearization of the equations of motion. In details, the total lateral force F_y acting on the vehicle can be expressed as:

$$F_y = Y_\beta \beta + Y_r r + Y_\delta \delta, \tag{6}$$

where:

$$\begin{cases} Y_\beta = -C_1 - C_2 + \dfrac{1}{2}\rho V^2 S (C_y)_\beta \\ Y_r = \dfrac{1}{V}(-aC_1 + bC_2) \\ Y_\delta = C_1 + F_{x1_p} \end{cases}. \tag{7}$$

The terms C_1 and C_2 indicate the side slip stiffness of the front axle and of the rear axle while $(C_y)_\beta$ is referred to the aerodynamic coefficient ($C_y = (C_y)_\beta \beta$, $\rho =$ air density, $S=$ front surface).

The self aligning torque M_z can be expressed by a similar linearized equation 8:

$$M_z = N_\beta \beta + N_r r + N_\delta \delta \tag{8}$$

where:

$$\begin{cases} N_\beta = -aC_1 + bC_2 + (M_{z1})_\alpha + (M_{z2})_\alpha + \dfrac{1}{2}\rho V^2 S l (C_{M_z})_\beta \\ N_r = \dfrac{1}{V}(-a^2 C_1 - b^2 C_2 + a(M_{z1})_\alpha - b(M_{z2})_\alpha \\ N_\delta = aC_1 - (M_{z1})_\alpha + \alpha F_{x1_p} \end{cases}. \tag{9}$$

The term $(C_{M_z})_\beta$, multiplied by the side slip angle β, has that same meaning of F_y with $(C_y)_\beta$. Additionally, the terms $(M_{z1})_\alpha$ and $(M_{z2})_\alpha$ indicate the aligning stiffness coefficient. The terms

f_0 and K describe the rolling resistance of the tires. The term F_{x1_p}, indicating the traction force on the front axle, can be neglected.

On the base of the equations above mentioned, the equations of motion can be written as

$$\begin{cases} mV\ (\dot{\beta} + r) + m\dot{V}\beta = Y_\beta \beta + Y_r r + Y_\delta \delta + F_{ye} \\ J_z\ \dot{r} = N_\beta \beta + N_r r + N_\delta \delta + M_{ze}. \end{cases} . \tag{10}$$

The terms F_{ye} and M_{ze} indicate the generalized external forces applied in y direction and about axis z that cannot be expressed as a function of β and r.

At steady state, considering that

$$r = \frac{V}{R}, \tag{11}$$

it is possible to write the following useful transfer functions:

$$\begin{aligned} \frac{1}{R\delta} &= \frac{Y_\delta N_\beta - N_\delta Y_\beta}{V\left(N_\beta\left(mV - Y_r\right) + N_r Y_\beta\right)}, \\ \frac{V^2}{R\delta} &= \frac{V\left(Y_\delta N_\beta - N_\delta Y_\beta\right)}{N_\beta\left(mV - Y_r\right) + N_r Y_\beta}, \\ \frac{\beta}{\delta} &= \frac{-N_\delta\left(mV - Y_r\right) - N_r Y_\delta}{N_\beta\left(mV - Y_r\right) + N_r Y_\beta}. \end{aligned}$$

that indicate the curvature gain, the lateral acceleration gain and the gain of the side slip angle.

On the base of the considerations above reported, **it is requested to compute**:

- the curvature gain, the lateral acceleration gain and the side slip angle gain as a function of the vehicle speed V.

  Make reference to the vehicle data used in Project 1 and to the tire data in Project 2. Select the most adequate tire for the vehicle.