Satisfiability

- Satisfiability (SAT) is the problem of determining, given some Boolean formula, if there exists some truthiness assignment of variables which would result in the formula evaluating to true.
- Satisfiability is NP-hard in the general case.

- 2-satisfiability (2SAT) is the problem of determining, given a set of constraints on pairs of Boolean variables, if there exists some truthiness assignment of variables which would result in the conjunction of all the constraints evaluating to true.
- Unlike general satisfiability, 2-satisfiability can be solved in linear time

2SAT

- The inputs to a 2SAT problem are a set of constraints on Boolean variables with standard Boolean operators.
- In this context, only these make sense:
  - $x_1 \vee x_2$ (disjunction)
  - $\neg x_1$ (negation)

2SAT

- We can write these in equivalent implicative normal form:
  - $x_1 \vee x_2 \equiv (\neg x_1 \rightarrow x_2) \wedge (\neg x_2 \rightarrow x_1)$
  - $\neg x_1 \equiv (x_1 \rightarrow \neg x_1)$

- These implications now form a directed graph with the Boolean variables (and their negations) as vertices, called the implication graph.
- What does it mean when some variable x can reach some other variable y in this implication graph?
  - If x can reach y in this graph, then $x \rightarrow y$.

2SAT

- When do we have a valid solution to our 2SAT instance?
- As long as we don't have any contradictions (i.e. $x \to \neg x$ and $\neg x \to x$, we can solve our 2SAT instance.
- In our implication graph, this is exactly the same as checking to see if x and $\neg x$ are in the same strongly connected component!

- We can then easily construct an actual solution to our 2SAT instance after computing the strongly connected components by assigning to each variable whichever truthiness value comes second in the topological ordering of the SCC condensed graph.
- If x comes after $\neg x$ in the topological ordering of the condensed implication graph, then we say x is true. Otherwise, we say it's false.