THEORY ASSIGNMENT 1 : CS2233

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A1

(a) Since we only need to show the existence of a possible case, it can be done through the help of an example:

Consider Algorithm: Insertion Sort.

- We know that in worst-case, when the array is sorted in reverse order, insertion sort takes running time in $O(n^2)$
- However, in best-case situation, when the array is already sorted, we simply have to go through the entire array to check, and thus, insertion sort takes running time in $O(n)$.

→ Since there exists an input where it takes $O(n)$ time despite taking $O(n^2)$ worst-case time, it is <u>TRUE</u> that an algorithm taking $O(n^2)$ worst-case time can take $O(n)$ time on some input

(b) We show that the given statement is <u>false</u> and prove it through contradiction.

If possible, let the algorithm take $O(n)$ time on all inputs.

Let U denote the set of all possible inputs to the algorithm.

Then, for any $c \in U$, we can write its run-time, say $T(c)$ in the form of :-

$$T(c) = a_c n + b_c \quad (a_c, b_c \text{ are constants})$$

However, since worst-case, $c_w \in U$

$\therefore T(c_w) = a_{c_w} n + b_{c_w}$

$\therefore T(c_w) \in O(n)$

This contradicts the statement that the algorithm takes $O(n^2)$ worst-case time (Considering big-oh as a strict bound).
Since it arises due to the assumption of $O(n)$ run-time in all inputs, this statement must be incorrect

Hence Proved, QED

(c) Again, we revert back to the example of insertion sort.

- Remember that, insertion sort takes $\Theta(n^2)$ running time in its worst-case, a reversely sorted array.
- However, as discussed previously, it takes run time of $O(n)$ in its best case, an array that is already sorted.

→ Thus, it is possible for an algorithm to take $\Theta(n^2)$ worst-case running time, yet still take $O(n)$ time on some inputs

→ Hence, the given statement is <u>true</u>.

A2.

(a) Given / To Prove: $f(n) + o(f(n)) = \Theta(f(n))$ (b)

Proof:

Let some $h(n) \in o(f(n))$ (or, $h(n) = o(f(n))$)

∴ By definition, we have:

$$\lim_{n \to \infty} \frac{h(n)}{f(n)} = 0$$

Thus, $\exists n_0, \forall n \geq n_0, \cancel{g}h(n) \leq f(n)$

$\therefore f(n) + h(n) \leq 2f(n)$ as $n \to \infty$ $(\because \forall n \geq n_0)$

Also, since it was given that $f(n)$ was non-negative, we had chosen $h(n) \in o(f(n))$, $h(n) > 0$ $\forall n > 0$.

$\therefore f(n) + h(n) \geq f(n) \quad \forall n$

$\Rightarrow f(n) \leq f(n) + h(n) \leq 2f(n)$ as $n \to \infty$

$$\therefore \lim_{n \to \infty} \frac{\cancel{f(n)}}{\cancel{f(n)}} \leq \lim_{n \to \infty} \frac{f(n) + h(n)}{f(n)} \leq \lim_{n \to \infty} \frac{2\cancel{f(n)}}{\cancel{f(n)}}$$

$$\therefore 1 \leq \lim_{n \to \infty} \frac{f(n) + o(f(n))}{f(n)} \leq 2 \quad - (1)$$

Letting $c_1 = 1$, $c_2 = 2$, the equation (1) is by definition that of $\Theta(f(n))$ $\left(\because \exists c_1, c_2 : c_1 \leq \lim_{n \to \infty} \frac{f(n) + o(f(n))}{f(n)} \leq c_2\right)$

$\therefore f(n) + o(f(n)) = \Theta(f(n))$

Hence Proved, QED.

(b) Proof:

By definition, since $f(n) = O(g(n))$, we have:

$\exists c > 0$, such that:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} \leq c$$

Since $c > 0$; and $c \neq 0$, we can write the reciprocal of the limit as :-

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} \geq \frac{1}{c} \qquad \left(\text{let } c' = \frac{1}{c}\right)$$

$\therefore \exists c', \lim_{n\to\infty} \frac{g(n)}{f(n)} \geq c'$, where $c' > 0$

Thus, by definition, $g(n) \in \Omega(f(n))$ or $g(n) = \Omega(f(n))$
Hence Proved, QED

(c) The following statement is false. We proceed by the following counter-example :-

Let $f(n) = n$ and $g(n) = n^2$
For large enough n, or, $n \to \infty$, $n^2 > n$ - (1)
(Choose $n_0 = 1$, $\forall n > 1$, $n.n > n.1 \Rightarrow n^2 > n$)

Thus; $f(n) + g(n) = n^2 + n$
Now; since $n^2 \geq n$, we have :-
$n^2 + n \leq n^2 + n^2$ (for $n \to \infty$)
$\therefore n^2 + n \leq 2n^2$

Also; $n^2 + n \geq n^2$ ($\because n > 0$)
$\therefore \lim_{n\to\infty} \frac{n^2}{n^2} \leq \lim_{n\to\infty} \frac{n^2+n}{n^2} \leq \lim_{n\to\infty} \frac{2\cancel{n^2}}{\cancel{n^2}}$
$\Rightarrow 1 \leq \lim_{n\to\infty} \frac{n^2+n}{n^2} \leq 2$

Thus, by definition, $n^2+n \in \Theta(n^2) \Rightarrow f(n)+g(n) \in \Theta(n^2)$ - (2)

However, from (1); $\min(n^2, n) = n$.
Thus, if the statement given is take to be true, we must have : $n^2 + n = f(n) + g(n) = \Theta(n)$
($\because \min(n^2, n) = n$)
But, this clearly contradicts (2).
Thus, the given statement is false.

(d) To Prove : (i) $n! = o(n^n)$, (ii) $n! = \omega(2^n)$

(i) We need to show that $\lim_{n \to \infty} \frac{n!}{n^n} = 0$

Now, $\frac{n!}{n^n} = \frac{1 \times 2 \times 3 \ldots n}{n^n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right)$

Observe that; each term in the product $\frac{k}{n} \leq 1$

(For $k = 1, 2, \ldots, n$)

Thus, since each term in the product is positive $\left(\frac{k}{n} > 0\right)$; we let $c = \frac{2}{n} \cdot \frac{3}{n} \cdot \cdots \left(\frac{n-1}{n}\right) \cdot 1$, where $c > 0$.

Also, since $\frac{k}{n} \leq 1$; we have $c \leq 1$ (for $k = 2, 3, \ldots, n$)

$\therefore$ $c > 0$ and $c \leq 1$, and:

$\frac{n!}{n^n} = c \cdot \left(\frac{1}{n}\right)$. Since c is not finite:

$\therefore \lim_{n \to \infty} \frac{n!}{n^n} = \lim_{n \to \infty} c\left(\frac{1}{n}\right) = \lim_{n \to \infty} (c) \cdot \lim_{n \to \infty} \left(\frac{1}{n}\right)$

However, since $c > 0$ and $c \leq 1$ $\forall n$, we have :-

$\lim_{n \to \infty} (c) = c'$ is also such that $c' > 0$ and $c' \leq 1$.

However, $\lim_{n \to \infty} \frac{1}{n} = 0$

$\therefore \lim_{n \to \infty} \frac{n!}{n^n} = 0 \cdot c' = 0$

$\therefore$ By definition : $n! = o(n^n)$

Hence Proved, QED

(ii) Now, we try to show that $2^n = o(n!)$ (or)

$\lim_{n \to \infty} \frac{2^n}{n!} = 0$

$$\Rightarrow \frac{2^n}{n!} = \frac{2}{n}\left(\frac{2}{n-1}\right)\cdots\left(\frac{2}{2}\right)\left(\frac{2}{1}\right)$$

Again observe that, $\frac{2}{k} > 0$ and $\frac{2}{k} \leq 2 \ \forall k = 1, 2, \ldots, n-1$

$$\therefore \frac{2^n}{n!} = \frac{2}{n}\underbrace{\left(\frac{2}{n-1}\right)\left(\frac{2}{n-2}\right)\cdots\left(\frac{2}{2}\right)\left(\frac{2}{1}\right)}$$

$$= \frac{2}{n} \times c \quad \left(\text{where } c > 0 \text{ and } c \leq 2 \ \forall n\right)$$

$$\therefore \lim_{n\to\infty} \frac{2^n}{n!} = \lim_{n\to\infty}\left(\frac{2}{n}\right) \cdot \lim_{n\to\infty}(c) \quad \left(\text{Since } c \text{ is finite}\right)$$

$$= 0 \times c' \quad \left(\text{where; } c' > 0 \text{ and } c' \leq 2, \text{ and } c' \text{ is finite}\right)$$

$$\therefore \lim_{n\to\infty} \frac{2^n}{n!} = 0$$

$\Rightarrow 2^n = o(n!)$

But, in part, we also know that if :-

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $g(n) = \omega(f(n))$ as well

(As done in the class).

$\therefore$ Also, $n! = \omega(2^n)$

Hence Proved, QED

A 3. (a) No, the function $\lceil \ln n \rceil !$ is not polynomially bounded.

A function $f(n)$ is said to be polynomially bounded if $\exists k > 0, \forall n_0, \exists n > n_0$, such that:

$$f(n) \leq n^k \quad (\text{For large enough } n)$$

(Note; this is the same as using $c.n^k$, which can be shown as, if $f(n) \leq cn^k$, then $f(n) \leq n^{k+1}$)

* We now proceed by a similar argument between $\lceil \ln n \rceil!$ and n^k.
Since $\lceil \ln n \rceil > \ln n$, we can say the following:
$\lceil \ln n \rceil! > (\ln n)!$ (if defined appropriately)

Let $n = e^m$ for some m;
Then, we need to compare $\lceil m \rceil!$ and $(e^m)^k$.
Let $x = \lceil m \rceil \Rightarrow x! = \lceil m \rceil!$, and $x > m$.
We try to show that:-
$\forall k,\quad x! > e^{kx}$, for large enough x

Now; $x! = x\cdot(x-1)\cdot(x-2)\cdots\left(\frac{x}{2}+1\right)\left(\frac{x}{2}\right)\cdots 2\cdot 1$

Since for large x, the terms $\frac{x}{2}, \frac{x}{2}-1, \ldots, 2, 1$ are each greater than 1:-

$$x! > x(x-1)(x-2)\cdots\left(\frac{x}{2}+1\right)$$

$$\therefore x! > \underbrace{\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)\cdots\left(\frac{x}{2}\right)}_{(x/2)\text{ times}}$$

$$\therefore x! > \left(\frac{x}{2}\right)^{x/2}$$

Now, we find a condition of equality for:- (or greater than)

$$\left(\frac{x}{2}\right)^{x/2} \geq e^{kx}$$

$$\therefore \left(\frac{\cancel{x}}{2}\right)\ln\left(\frac{x}{2}\right) \geq k\cancel{x}\,\cancel{\ln e}^{\,1}$$

$$\therefore \ln\left(\frac{x}{2}\right) \geq 2k$$

$$\Rightarrow \quad x \geq 2e^{2k} \quad - (1)$$

(1) Thus, if we choose $x > 2e^{2k}$ for any 'k', we have:

$$\left(\frac{x}{2}\right)^{x/2} > e^{xk}$$

$$\Rightarrow x! > \left(\frac{x}{2}\right)^{x/2} > e^{xk} \Rightarrow x! > e^{kx}$$

Now; $\lceil m \rceil ! = x! > e^{kx} > e^{km}$ $\quad (\because x = \lceil m \rceil > m)$

$\therefore \lceil m \rceil ! > e^{mk}$ for all k, and large enough 'm'
(Choose m from (1) as shown

$\therefore \lceil \ln n \rceil ! > n^k$.

Thus, for any 'k', we can choose some n, such that $\lceil \ln n \rceil ! > n^k$.
Thus, $\lceil \ln n \rceil !$ is super-polynomial in 'n' and thus not polynomially bounded.
~~Hence Proved, QED~~

(b) We try to show that $\lceil \ln \lceil \ln n \rceil \rceil !$ is polynomially bounded.
Thus, $\exists k > 0, \forall n_0, \exists n \geq n_0 :-\ f(n) \leq n^k$.
(Again, this is equivalent to cn^k, as stated before)

Let $n = e^{e^m}$.
We need to show, $\exists k : \lceil m \rceil ! < (e^{e^m})^k$ for large enough 'm'.
We try to show the same for $x = \lceil m \rceil$.
We know, $e^m > m$
~~(Since $e^m = 1 + m + \frac{m^2}{2!} + \dots > m \ \forall m > 0$)~~

$\therefore e^m > m - 1$
$e^m > m - 2$
$\vdots$
$e^m > 2$
$e^m > 1$
$\Big\}$ For large enough m; or $m \to \infty$

$\therefore \underbrace{e^m \times e^m \dots e^m}_{m\text{-times}} > m(m-1)(m-2)\dots 1$

$\therefore (e^m)^m > m!$ (or $\lceil m \rceil !$ for $m \to \infty$)

Thus, for $m \to \infty$; $e^{m^2} > \lceil m \rceil !$ — (1)

However, we also know that e^m is super-polynomial since :- $e^m = 1 + m + \frac{m^2}{2!} + \dots \frac{m^k}{k!} + \frac{m^{k+1}}{(k+1)!} + \dots$

If we assume it to be bounded by m^k, we can always find a term m^{k+1} with non-zero coefficient. ($\frac{1}{(k+1)!}$ is the coefficient)

$\Rightarrow e^m \geq m^k$ for large enough m, and this is true $\forall k > 0$

Thus, by definition, e^m is not polynomially bounded.

$\therefore$ We have, $e^m > m^2$ for large enough m (or $m \to \infty$).

$\Rightarrow e^{e^m} > e^{m^2}$ — (2) (Take power e on both sides)

Using (1) and (2) :-

$e^{e^m} > e^{m^2} > \lceil m \rceil !$

Since we need to show $\exists k > 0$; simply choosing any $k > 1$, we have :-

$(e^{e^m})^k > e^{e^m} > e^{m^2} > \lceil m \rceil !$

$\Rightarrow (e^{e^m})^k > \lceil m \rceil !$ (For large m)

Now, since $m = \ln(\ln n)$

$\Rightarrow n^k > \lceil \ln(\ln n) \rceil !$ (or) $\lceil \ln(\ln n) \rceil ! < n^k$; and $\exists$ such k for all n large enough

Thus, by definition, since $\lceil \ln(\ln n) \rceil! < n^k$ for n large enough.

$\therefore$ $\lceil \ln(\ln n) \rceil!$ is polynomially bounded.

Hence Proved, QED

A4

(a) $T(n^2) = 7T\left(\frac{n^2}{4}\right) + cn^2$, $T(1) = 1$. - ①

Thus; we have :- $T\left(\frac{n^2}{4}\right) = 7T\left(\frac{n^2}{4\times 4}\right) + \frac{cn^2}{4}$

Multiplying this by 7, we have :-

$$7T\left(\frac{n^2}{4}\right) = 7^2 T\left(\frac{n^2}{4^2}\right) + 7\frac{cn^2}{4} \quad - ②$$

Similarly,

$$T\left(\frac{n^2}{4^2}\right) = 7 . T\left(\frac{n^2}{4^3}\right) + \frac{cn^2}{4^2}$$

Multiplying it by 49; we have :-

$$7^2 T\left(\frac{n^2}{4^2}\right) = 7^3 T\left(\frac{n^2}{4^3}\right) + 7^2 . \frac{cn^2}{4^2} \quad - ③$$

Similarly, we have :-

$$7^{x-1} T\left(\frac{n^2}{4^{x-1}}\right) = 7^x T\left(\frac{n^2}{4^x}\right) + 7^{x-1} \frac{cn^2}{4^{x-1}}$$

(Assumption) For the base case; we cannot go into $T(x)$ where $x < 1$; thus; we let in our final step; $n^2 = 4^x$

$\therefore$ $n^{\cancel{2}} = (2^x)^{\cancel{2}}$

$\Rightarrow$ $x = \log_2 n$

Adding all equations of form ①, ②, ③ onwards, we get:

$$T(n^2) = \cancel{7T\left(\frac{n^2}{4}\right)} + cn^2$$
$$+ \cancel{7T\left(\frac{n^2}{4}\right)} = \cancel{7^2 T\left(\frac{n^2}{4^2}\right)} + 7c\frac{n^2}{4}$$
$$+ \cancel{7^2 T\left(\frac{n^2}{4^2}\right)} = \cancel{7^3 T\left(\frac{n^3}{4^3}\right)} + 7^2 \frac{cn^2}{4^2}$$
$$\vdots$$
$$+ \cancel{7^{x-1} T\left(\frac{n^2}{4^{x-1}}\right)} = 7^x . T(1) + 7^{x-1} \frac{cn^2}{4^{x-1}} \quad (\text{Given: } T(1)=1)$$

$$\therefore T(n^2) = 7^x . 1 + cn^2 \left(1 + \left(\frac{7}{4}\right)^1 + \left(\frac{7}{4}\right)^2 + \dots + \left(\frac{7}{4}\right)^{x-1}\right)$$

Since $x = \log_2 n \Rightarrow 7^x = 7^{\log_2 n} = n^{\log_2 7}$.

Using the formula for the summation of a geometric series $\left(\frac{a(1-r^n)}{1-r}\right)$, we have:-

$$T(n^2) = n^{\log_2 7} + cn^2 \frac{1\left(\left(\frac{7}{4}\right)^x - 1\right)}{\frac{7}{4} - 1}$$

Again, $\left(\frac{7}{4}\right)^x = \left(\frac{7}{4}\right)^{\log_2 n} = n^{\log_2 7 - \log_2 4} = \frac{n^{\log_2 7}}{n^2}$.

$$\therefore T(n^2) = n^{\log_2 7} + \frac{4cn^2}{3}\left(\frac{n^{\log_2 7}}{n^2} - 1\right)$$

$$\therefore \boxed{T(n^2) = \left(\frac{4c}{3} + 1\right) n^{\log_2 7} - \frac{4cn^2}{3}} \quad \left(\text{Let } a = \frac{4c+1}{3},\ b = \frac{4c}{3}\right)$$

Since $n^{\log_2 7} > n^2$ $(\log_2 7 > 2)$; we have:-

$$T(n^2) = a n^{\log_2 7} - bn^2 \quad (a - b = 1)$$

$\Rightarrow T(n^2) \geq a n^{\log_2 7}$; and $T(n^2) \leq (a-b) n^{\log_2 7}$

$\therefore T(n^2) \in \Theta(n^{\log_2 7})$

$\therefore\ T(n) \in \Theta\left(n^{\frac{1}{2}\log_2 7}\right)$

(or)

$$\boxed{T(n) \in \Theta\left(n^{\log_4 7}\right)}$$

(b) $T(n) = n \times T(\sqrt{n})$ — (1)

Also, $T(\sqrt{n}) = \sqrt{n} \cdot T(n^{\frac{1}{4}})$

$\Rightarrow n\,T(\sqrt{n}) = n \cdot n^{\frac{1}{2}}\, T(n^{\frac{1}{4}})$. — (2) $(= T(n))$

Similarly, $T(n^{\frac{1}{4}}) = n^{\frac{1}{4}}\, T(n^{\frac{1}{8}})$

$\therefore T(n) = n \cdot n^{\frac{1}{2}} \cdot T(n^{\frac{1}{4}}) = n \cdot n^{\frac{1}{2}} \cdot n^{\frac{1}{4}} \cdot T(n^{\frac{1}{8}})$ — (3)

Now, we can continue the series to have :-

$$T(n) = n \cdot n^{\frac{1}{2}} \cdot n^{\frac{1}{4}} \cdots n^{\frac{1}{2^{x-1}}} \cdot T\left(n^{\frac{1}{2^x}}\right)$$

Now, we assume $n = 2^{2^i}$, and for the base case, let $n^{\frac{1}{2^x}} = 2 \Rightarrow \left(2^{2^i}\right)^{\frac{1}{2^x}} = 2$.

$\Rightarrow 2^{2^i} = 2^{2^x} \Rightarrow i = x$, and, $\log_2 n = 2^i = 2^x$.

$\therefore T(n) = n^{1+\frac{1}{2}+\frac{1}{4}+\cdots\frac{1}{2^{x-1}}} \cdot T(2)$ (~~Again, we have~~ a geometric series)

$\therefore T(n) = n^{\frac{1\left(1-\frac{1}{2^x}\right)}{1-\frac{1}{2}}} \times 4$ $(\because T(2) = 4)$

$\therefore T(n) = 4n^{2\left(1-\frac{1}{\log_2 n}\right)} = \cancel{4}\dfrac{n^2}{\cancel{n^{\frac{2}{\log_2 n}}}}$

$\Bigg(\because$ Now, $1 = \dfrac{\log_2 n}{\log_2 n} \Rightarrow 2^1 = \left(2^{\log_2 n}\right)^{\frac{1}{\log_2 n}}$

$\therefore 2 = n^{\frac{1}{\log_2 n}} \Rightarrow 2^2 = 4 = n^{\frac{2}{\log_2 n}}\Bigg)$

$\therefore \boxed{T(n) = n^2}$ (Exactly, thus, $T(n) \geq \frac{n^2}{2}$ and $T(n) \leq 2n^2$)

$\therefore \boxed{T(n) \in \Theta(n^2)}$

classmate

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(c) $T(n) = T(n/2) + 2T(n/4) + 3n/2$ — (1)

$\therefore$ We have, $T(n/2) = T(n/4) + 2T(n/8) + 3n/4$

Multiplying by 2, we have:-

$$2T(n/2) = 2T(n/4) + 4T(n/8) + 3n/2 \quad \text{— (2)}$$

Again, $T(n/4) = T(n/8) + 2T(n/16) + 3n/8$,

Multiplying by 2^2:-

$$2^2 \cdot T\left(\frac{n}{2^2}\right) = 2^2 T\left(\frac{n}{2^3}\right) + 2^3 T\left(\frac{n}{2^4}\right) + \frac{3n}{2} \quad \text{— (3)}$$

Assumption: Similarly, for the base case when we let $n = 2^x$, we can say:-

$$2^{x-2} T\left(\frac{n}{2^{x-2}}\right) = 2^{x-2} T\left(\frac{n}{2^{x-1}}\right) + 2^{x-1} T\left(\frac{n}{2^x}\right) + \frac{3n}{2}$$

(Here, $n = 2^x \Rightarrow x = \log_2 n$)

Adding all these equations, we have:-

$$
\left.
\begin{aligned}
T(n) &= T\left(\tfrac{n}{2}\right) + \cancel{2T\left(\tfrac{n}{4}\right)} + \frac{3n}{2} \\
+\ 2T\left(\tfrac{n}{2}\right) &= \cancel{2T\left(\tfrac{n}{4}\right)} + \cancel{4T\left(\tfrac{n}{8}\right)} + \frac{3n}{2} \\
+\ \cancel{4T\left(\tfrac{n}{4}\right)} &= \cancel{4T\left(\tfrac{n}{8}\right)} + 8T\left(\tfrac{n}{16}\right) + \frac{3n}{2} \\
&\vdots \\
+\ \cancel{2^{x-3} T\left(\tfrac{n}{2^{x-3}}\right)} &= \cancel{2^{x-3} T\left(\tfrac{n}{2^{x-2}}\right)} + \boxed{2^{x-2} T\left(\tfrac{n}{2^{x-1}}\right)} + \frac{3n}{2} \\
+\ \cancel{2^{x-2} T\left(\tfrac{n}{2^{x-2}}\right)} &= \boxed{2^{x-2} T\left(\tfrac{n}{2^{x-1}}\right)} + 2^{x-1} T\left(\tfrac{n}{2^x}\right) + \frac{3n}{2}
\end{aligned}
\right\} x-1 \text{ times}
$$

$$\therefore T(n) + \cancel{2}T\left(\frac{n}{2}\right) = \cancel{T\left(\frac{n}{2}\right)} + 2^{x-2} \times 2\, T\left(\frac{n}{2^{x-1}}\right) + 2^{x-1} T\left(\frac{n}{2^x}\right) + \frac{3n}{2}(x-1)$$

Now; $T\left(\frac{n}{2^x}\right) = T(1) = 0$; $T\left(\frac{n}{2^{x-1}}\right) = T(2) = 2$

$n = 2^x \Rightarrow x = \log_2 n \Rightarrow 2^{x-1} = \frac{n}{2}$

$$\therefore T(n) + T\left(\frac{n}{2}\right) = \frac{(n)}{\cancel{2}} \cdot \cancel{2} + \frac{\log_2 n}{2} \cdot 0 + \frac{3n}{2}\left(\log_2 n - 1\right)$$

$$\therefore T(n) + T\left(\frac{n}{2}\right) = \frac{3n \log_2 n}{2} - \frac{n}{2}$$

Observing the terms of 'n' on the RHS, we can conclude that we must have the same 'like' terms on LHS.

Thus, let $T(n) = \alpha n \log_2 n + \beta n$

$$\Rightarrow T\left(\frac{n}{2}\right) = \frac{\alpha n}{2}\left(\log_2 n - \log_2 2\right) + \frac{\beta n}{2}$$

$$\therefore T(n) + T\left(\frac{n}{2}\right) = \alpha n \log_2 n + \beta n + \frac{\alpha n \log_2 n}{2} - \frac{\alpha n}{2} + \frac{\beta n}{2}$$

$$= \frac{3n \log_2 n}{2} - \frac{n}{2}$$

Comparing the coefficients, we must have :-

$\alpha + \frac{\alpha}{2} = \frac{3\alpha}{2} = \frac{3}{2}$ (For $n \log_2 n$ terms)

$\therefore \underline{\underline{\alpha = 1}}$

$\beta - \frac{\alpha}{2} + \frac{\beta}{2} = -\frac{1}{2} \Rightarrow \frac{3\beta}{2} - \frac{\cancel{1}}{2} = -\frac{\cancel{1}}{2}$ (For n terms)

$\therefore \underline{\underline{\beta = 0}}$

$\therefore T(n) = n \log_2 n$

$\therefore T(n) \in \Theta(n \log_2 n)$

($\Rightarrow T(n) \geq \frac{n \log_2 n}{2}$ and $T(n) \leq 2n \log_2 n$ for $n \geq 1$)

(d) $T(n) = 4T(n/2) + n^3$ — ①

Thus; $T\left(\frac{n}{2}\right) = 4T\left(\frac{n}{2^2}\right) + \frac{n^3}{2^3}$

Multiplying by 4, we have:-

$4T\left(\frac{n}{2}\right) = 4^2 T\left(\frac{n}{2^2}\right) + \frac{n^3}{2^3}(2^2)$ — ②

Again, we have, $T\left(\frac{n}{2^2}\right) = 4T\left(\frac{n}{2^3}\right) + \frac{n^3}{2^{3 \cdot 2}}$

Multiplying by 4^2 gives:-

$4^2 T\left(\frac{n}{2^2}\right) = 4^3 T\left(\frac{n}{2^3}\right) + \frac{n^3}{2^{3\cdot 2}} \cdot 2^{2\cdot 2}$

$\Rightarrow 4^2 T\left(\frac{n}{2^2}\right) = 4^3 T\left(\frac{n}{2^3}\right) + \frac{n^3}{2^2}$ — ③

Thus, we can continue the sequence upto the base case as:-

$4^{x-1} T\left(\frac{n}{2^{x-1}}\right) = 4^x T\left(\frac{n}{2^x}\right) + \frac{n^3}{2^{x-1}}$

Assumptions (Here; let $n = 2^x \Rightarrow x = \log_2 n$; also, $T(1) = 1$ is given)

Adding all the terms, we have:-

$T(n) = \cancel{4^1 T\left(\frac{n}{2^1}\right)} + n^3$

$+ \cancel{4T\left(\frac{n}{2}\right)} = \cancel{4^2 T\left(\frac{n}{2^2}\right)} + \frac{n^3}{2}$

$+ \cancel{4^2 T\left(\frac{n}{2^2}\right)} = \cancel{4^3 T\left(\frac{n}{2^3}\right)} + \frac{n^3}{2^2}$

$\vdots$

$+ \cancel{4^{x-1} T\left(\frac{n}{2^{x-1}}\right)} = 4^x \cdot T(1) + \frac{n^3}{2^{x-1}}$

(Since $2^x = n$
$\Rightarrow 4^x = (2^x)^2 = n^2$)

$\therefore T(n) = 4^x \cdot 1 + n^3\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{x-1}}\right)$

Again; using the geometric series sum, we have:

$T(n) = n^2 \cdot 1 + n^3 \cdot \frac{1\left(1 - \frac{1}{2^x}\right)}{1 - \frac{1}{2}}$

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$\Rightarrow T(n) = n^2 + 2n^3\left(1 - \frac{1}{n}\right) \quad (\because 2^x = n)$

$$\boxed{T(n) = 2n^3 - 2n^2 + n^2 = 2n^3 - n^2}$$

Since $n^3 > n^2 \Rightarrow T(n) \leq 2n^3$; and $T(n) \geq 2n^3 - 1 \cdot n^3$

$\therefore T(n) \leq 2n^3$ and $T(n) \geq n^3$

$\therefore \boxed{T(n) \in \Theta(n^3)}$

(e) $T(n) = T(n/2) + c\log n$ — ① (Assume: $\log n = \log_2 n$)

We have: $T(n/2) = T(n/2^2) + c\log_2(n/2)$

$\Rightarrow T(n/2) = T(n/2^2) + c[\log_2 n - \log_2 2]$

$\therefore T(n/2) = T(n/2^2) + c(\log_2 n - 1)$ — ②

~~Multiplying by 2:-~~

Again, we have:-

$T(n/4) = T(n/2^3) + c(\log_2(\frac{n}{4}))$

$\therefore T(\frac{n}{4}) = T(\frac{n}{2^3}) + c(\log_2 n - 2)$ — ③

Similarly, till the base case, we have:-

$T\left(\frac{n}{2^{x-1}}\right) = T\left(\frac{n}{2^x}\right) + c(\log_2 n - (x-1))$

Assumptions. { Now, we assume $2^x = n \Rightarrow x = \log_2 n$.
Also, let $T(1) = c_0$ for some constant c_0

Adding the equations gives:-

$$\left.\begin{aligned}
& T(n) = \cancel{T(n/2)} + c\log_2 n \\
+\ & \cancel{T(n/2)} = \cancel{T(n/2^2)} + c(\log_2 n - 1) \\
+\ & \cancel{T(n/2^2)} = \cancel{T(n/2^3)} + c(\log_2 n - 2) \\
+\ & \vdots \\
+\ & \cancel{T(n/2^{x-1})} = T(1) + c(\log_2 n - (x-1))
\end{aligned}\right\} x\text{-times}$$

$\therefore T(n) = c_0 + (c\log_2 n \cdot x) - (1 + 2 + \cdots + (x-1))$

We know $\sum_1^N n = \frac{N(N+1)}{2}$.

$\therefore T(n) = c_0 + xc\log_2 n - \frac{x(x-1)}{2} \cdot c$

$\therefore T(n) = C_0 + xc\log_2 n - \frac{x^2 c}{2} + \frac{xc}{2}$ $\left(\text{Let } (\log_2 n)^2 = \log_2^2 n\right)$

$(x = \log_2 n)$

$\Rightarrow T(n) = C_0 + c\log_2^2 n - \frac{c}{2}\log_2^2 n + \frac{c\log_2 n}{2}$

$\therefore \boxed{T(n) = \frac{c}{2}\log_2^2 n + \frac{c\log_2 n}{2} + C_0}$

Clearly, $T(n) \ge \frac{c}{2}\log_2^2 n$; and $T(n) \le \left(\frac{c}{2} + \frac{c}{2} + C_0\right)\log_2^2 n$

$\therefore \boxed{T(n) \in \Theta(\log_2^2 n)}$ (For large n; $\log_2 n > 1$ $\therefore \log_2^2 n > \log_2 n$)

A5. Claim:-

Arranging the functions $f_1, f_2, \ldots, f_6$ such that $f_1 = \Omega(f_2)$, $f_2 = \Omega(f_3) \ldots, f_5 = \Omega(f_6)$; enables us to write them as:-

$$f_1 \ge f_2 \ge f_3 \ge f_4 \ge f_5 \ge f_6$$ (For large enough inputs)

Then, for the given set of functions, we can write:-

$$1.1^n \ge n^2 \ge \frac{n^{1.2}}{\log n} \ge n\log n \ge \log^3 n \ge 0.9^n$$

Proof:-

(1) We try to show that firstly, $\log^3 n = O(0.9^n)$.
Thus, by definition, we try to find:-

$$\lim_{n\to\infty} \frac{0.9^n}{\log^3 n}$$

Observe that, as $n \to \infty$, $(0.9)^n \to 0$ (Fraction < 1 power 'n'; known result)
However, $\log^3 n \to \infty$ (Which means, $\log^3 n \not\to 0$)

Thus, we can simply write:

$$\lim_{n\to\infty} \frac{0.9^n}{\log^3 n} = \frac{\lim_{n\to\infty} 0.9^n}{\lim_{n\to\infty} \log^3 n} = 0 \quad \left(\frac{0}{x} = 0; \text{ as } x \to \infty \Rightarrow x > 0 \right)$$

Thus; $\log^3 n = o(0.9^n)$ by definition

($\Rightarrow \log^3 n \geq 0.9^n$ by the notation used) — 1

2 Now; we try to show that $n\log n = o(\log^3 n)$

$$\therefore \lim_{n\to\infty} \frac{n\cancel{\log n}}{\cancel{(\log n)}(\log^2 n)} = \lim_{n\to\infty} \frac{n}{\log^2 n}$$

Since both n and $\log^2 n$ diverge to ∞ as $n \to \infty$; we use L'Hospital's rule :-

$$\Rightarrow \lim_{n\to\infty} \frac{n}{\log^2 n} = \lim_{n\to\infty} \frac{\frac{dn}{dn}}{\frac{d}{dn}(\log^2 n)} = \lim_{n\to\infty} \frac{1}{2\log n \cdot \frac{d}{dn}(\log n)}$$

(Using chain rule). $\left(\frac{d}{dx} \log x = \frac{1}{x} \right)$

$$\therefore \lim_{n\to\infty} \frac{n\log n}{\log^3 n} = \frac{1}{\lim_{n\to\infty} \frac{2\log n}{n}} = \lim_{n\to\infty} \frac{n}{2\log n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\log^3 n}{n\log n} = \lim_{n\to\infty} \frac{2\log n}{n} = \lim_{n\to\infty} \frac{\frac{d}{dn}(2\log n)}{\frac{d}{dn}(n)}$$

(Using L'Hospital Again)

$$\Rightarrow \lim_{n\to\infty} \frac{\log^3 n}{n\log n} = \lim_{n\to\infty} \frac{2 \cdot \left(\frac{1}{n}\right)}{1} = 2 \lim_{n\to\infty} \frac{1}{n} = 2 \times 0$$

$$\therefore \lim_{n\to\infty} \frac{\log^3 n}{n\log n} = 0$$

$\therefore n\log n = o(\log^3 n)$

Or, $n\log n \geq \log^3 n$ — 2 (n large enough)

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(3) We now try to show $\frac{n^{1.2}}{\log n} = o(n\log n)$

$\therefore \lim_{n\to\infty} \frac{n\log n}{\frac{n^{1.2}}{\log n}} = \lim_{n\to\infty} \frac{\log^2 n}{n^{1/5}}$

Again, as both numerator and denominator diverge to ∞, we have:-

$$\lim_{n\to\infty} \frac{n\log n}{n^{1.2}/\log n} = \lim_{n\to\infty} \frac{\frac{d}{dn}(\log^2 n)}{\frac{d}{dn} n^{\frac{1}{5}}}$$

$$= \lim_{n\to\infty} \frac{2\log n \cdot \frac{1}{n}}{\frac{1}{5} n^{-\frac{4}{5}}} \quad \left(\frac{d}{dx}x^n = nx^{n-1}\right)$$

$$= \lim_{n\to\infty} \frac{10\log n \times \cancel{n^{\frac{4}{5}}}}{\cancel{n}\, n^{\frac{1}{5}}}$$

$$= \lim_{n\to\infty} \frac{10\log n}{n^{\frac{1}{5}}}$$

Using L'Hospital's Rule once again:-

$$= \lim_{n\to\infty} \frac{10\frac{d}{dn}(\log n)}{\frac{d}{dn} n^{\frac{1}{5}}} = \lim_{n\to\infty} \frac{10\cdot\frac{1}{n}}{\frac{1}{5} n^{-\frac{4}{5}}}$$

$$= \lim_{n\to\infty} 50 \frac{\cancel{n^{\frac{4}{5}}}}{\cancel{n}\, n^{\frac{1}{5}}} = 50 \times \lim_{n\to\infty} \frac{1}{n^{1/5}}$$

$$\therefore \lim_{n\to\infty} \frac{n\log n}{n^{1.2}/\log n} = 50 \times 0 = 0 \quad \left(\because n^{-0.2} \to 0 \text{ as } n\to\infty; \text{ negative power}\right)$$

$\therefore \frac{n^{1.2}}{\log n} \in o(n\log n)$

$\Rightarrow \frac{n^{1.2}}{\log n} \geq n\log n$ — (3)

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(4) We now show that $n^2 = o\left(\frac{n^{1.2}}{\log n}\right)$.

$\therefore \lim_{n\to\infty} \frac{\cancel{n^{1.2}}}{\log n \cdot \cancel{n^2} n^{0.8}} = \lim_{n\to\infty} \frac{1}{\log n \times n^{0.8}}$

Now; the denominator goes to ∞ while the numerator is constant. Thus, the Limit goes to $(1/\infty)$ form or 0.

$\therefore \lim_{n\to\infty} \frac{n^{1.2}}{\log n \cdot n^2} = 0$

$\Rightarrow n^2 = o(n^{1.2}/\log n)$

$\Rightarrow n^2 \geq o\left(\frac{n^{1.2}}{\log n}\right)$ - (4)

(5) We now show that $1.1^n = o(n^2)$

$\therefore \lim_{n\to\infty} \frac{n^2}{1.1^n}$ (Again, since $1.1^n > 1 \Rightarrow 1.1^n \to \infty$ as $n \to \infty$. We thus use L'Hospital's Rule once again)

$= \lim_{n\to\infty} \frac{\frac{d}{dn} n^2}{\frac{d}{dn}(1.1)^n} = \lim_{n\to\infty} \frac{2n}{(1.1)^n \ln(1.1)}$

Using L'Hospital's Rule one last time gives :-

$\lim_{n\to\infty} \frac{n^2}{1.1^n} = \lim_{n\to\infty} \frac{2\frac{dn}{dn}}{\frac{d}{dn}(1.1)^n \ln(1.1)} = \lim_{n\to\infty} \frac{2}{\ln^2(1.1) \times (1.1)^n}$

$\lim_{n\to\infty} \frac{n^2}{1.1^n} = \frac{2}{\ln^2(1.1)} \lim_{n\to\infty} \frac{1}{(1.1)^n} = \frac{2}{\ln^2(1.1)} \times 0 = 0$

$\therefore 1.1^n = o(n^2)$

$\Rightarrow 1.1^n \geq n^2$ - (5)

{Note here, we use that if $f(n) = o(g(n)) \Rightarrow f(n) > g(n)$ (strictly) $\Rightarrow$ We can say, $f(n) \geq g(n)$}

Finally, using equations 1 to 5; we have:-

$$\boxed{1.1^n \geq n^2 \geq \frac{n^{1.2}}{\log n} \geq n\log n \geq \log^3 n \geq 0.9^n}$$

* However; since each of these inequalities was strict, since $\log^3 n = o(0.9^n)$; $n\log n = o(\log^3 n)$ and so on; we can, in-fact, never have two functions f_i, f_j $(i \neq j)$ here; such that $f_i = \Theta(f_j)$

Thus, the given functions are partitioned into 6 equivalence classes; where each class contains one of these functions. The ordering of these classes is same as that of the functions mentioned above.

(Note: We use $\log n$ to be interpreted as $\ln(n)$. The result, however will hold for any valid base)

Hence Proved, QED

A6. For the given problem, the function XYZ(n) executes 3 simple commands (fundamental operations):

$r := 0$ (Executed once).

Let the time taken be c_0 — 1

$r := r + 1$ (Executed within four iterated loops)

Let the time to execute it once be c_1 — 2

return (r) (Executed once)

Let its run time be C_2 — 3.

Let the total run-time of XYZ(n) be T(n).

Then, we may write T(n) as follows:

$$T(n) = c_0 + \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=j}^{i+j} \sum_{l=1}^{i+j-k} (c_1) + c_2$$

Here; since we have four iterated loops; we write each of these as a summation iterated over each other. In the last loop (inner-most); we do $x := x+1$; with a run-time c_1. Since c_1 is constant we can write the summation as:-

$$S = \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=j}^{i+j} \sum_{l=1}^{i+j-k} c_1 = c_1 \times \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=j}^{i+j} \sum_{l=1}^{i+j-k} (1)$$

$$= c_1 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=j}^{i+j} (i+j-k) \quad \left(\because \sum_{1}^{n} 1 = n\right)$$

$$= c_1 \cdot \left(\sum_{i=1}^{n} \sum_{j=1}^{i} \left((i+j) \sum_{k=j}^{i+j} - \left(\sum_{k=1}^{i+j} k - \sum_{k=1}^{j-1} k \right) \right) \right).$$

(Since $i+j$ is constant w.r.t 'k', we can write it out of the summation. Also; $\sum_{j=j}^{i+j} k = \sum_{k=1}^{i+j} k - \sum_{k=1}^{j-1} k$)

$$= c_1 \left(\sum_{i=1}^{n} \sum_{j=1}^{i} \left((i+j)(i+\cancel{j}-\cancel{j}+1) - \left(\frac{(i+j)(i+j+1)}{2} - \frac{(j-1)(j)}{2} \right) \right) \right)$$

$$\left(\because \sum_{i}^{j} 1 = j-i+1 \; ; \; \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \right)$$

$$= c_1 \sum_{i=1}^{n} \sum_{j=1}^{i} \left(i^2 + i + ij + \cancel{j} - \frac{i^2}{2} - ij - \cancel{\frac{j^2}{2}} - \cancel{\frac{j}{2}} + \cancel{\frac{j^2}{2}} - \cancel{\frac{j}{2}} - \frac{i}{2} \right).$$

$$= c_1 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i} \left(\frac{i^2+i}{2} \right) = \frac{c_1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i} (i^2+i)$$

(Again; i^2+i is constant in terms of j)

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$$\therefore S = \frac{c_1}{2}\sum_{i=1}^{n}(i^2+i)\sum_{j=1}^{i} 1 = \frac{c_1}{2}\sum_{i=1}^{n}(i^2+i).i$$

$$\therefore S = \frac{c_1}{2}\sum_{i=1}^{n}(i^3+i^2) = \frac{c_1}{2}\left(\sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} i^2\right)$$

$$= \frac{c_1}{2}\left\{\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}\right\}.$$

$$\left(\because \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2, \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{c_1}{2}\left(\frac{n^2(n^2+2n+1)}{4} + \frac{(n^2+n)(2n+1)}{6}\right)$$

$$= \frac{c_1}{2}\left(\frac{n^4}{4} + \frac{2n^3}{4_2} + \frac{n^2}{4} + \frac{2n^3}{6_3} + \frac{3n^2}{6_2} + \frac{n}{6}\right)$$

$$= \frac{c_1}{2}\left(\frac{n^4}{4} + n^3\left(\frac{1}{2}+\frac{1}{3}\right) + n^2\left(\frac{1}{2}+\frac{1}{4}\right) + \frac{n}{6}\right)$$

$$S = \frac{c_1}{2}\left(\frac{n^4}{4} + \frac{5n^3}{6} + \frac{3n^2}{4} + \frac{n}{6}\right) \quad - (4)$$

$$\therefore S = \frac{c_1}{8}n^4 + \frac{5c_1}{12}n^3 + \frac{3c_1}{8}n^2 + \frac{c_1}{6}n$$

$$\Rightarrow \boxed{T(n) = c_0 + \frac{c_1}{8}n^4 + \frac{5c_1}{12}n^3 + \frac{3c_1}{8}n^2 + \frac{c_1}{6}n + c_2}$$

$$\therefore T(n) \geq \left(c_0 + \frac{c_1}{8} + \frac{5c_1}{12} + \frac{3c_1}{8} + \frac{c_1}{6} + c_2\right)n^4$$

$$(\because n^4 \geq n^3, n^2, n, 1 \;\forall n \geq 1)$$

$\therefore T(n) \geq c_3 n^4$; for some c_3 constant

$\Rightarrow \boxed{T(n) \in O(n^4)}$ (worst-case running time)

→ Also, note that even in worst-case, we need to go across each loop always; thus, there is no 'best' or 'worst' case here

→ The function XYZ(n) outputs / returns the following. Since initially $r = 0$; then increments each time $r = r+1$ is done. The value returned is the number of times we perform $r = r+1$, which is the number of times we iterate across all four loops :-

Since $S = \frac{c_1}{2}\left(\frac{n^4}{4} + \frac{5n^3}{6} + \frac{3n^2}{4} + \frac{n}{6}\right)$ (From (4))

thus, $\frac{S}{c_1}$ is the number of times we increment r by 1 (starting from 0).

Thus, final value of $r = \frac{n^4}{8} + \frac{5n^3}{12} + \frac{3n^2}{8} + \frac{n}{12}$ $\left(= \frac{S}{c_1}\right)$

$\therefore$ Function XYZ(n) returns :-

$$\boxed{\frac{n^4}{8} + \frac{5n^3}{12} + \frac{3n^2}{8} + \frac{n}{12}}$$ (Thus, return value $r = O(n^4)$)