17) | Lecture 39.5-41 | Computing SSSP for Graph with -ve Edge|

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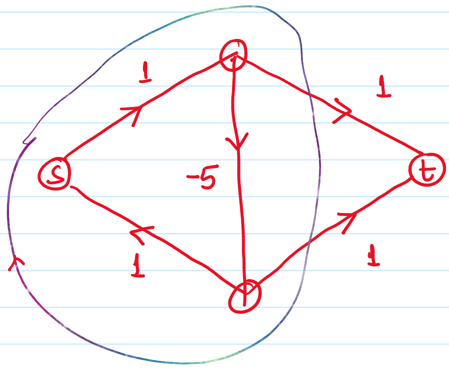
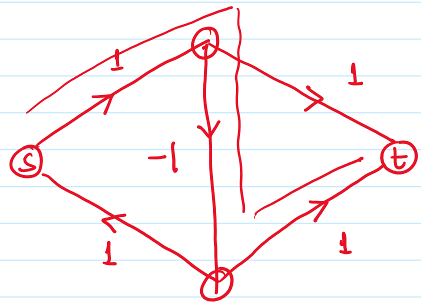
Problem: compute shortest path from s to t $t \in V/s$

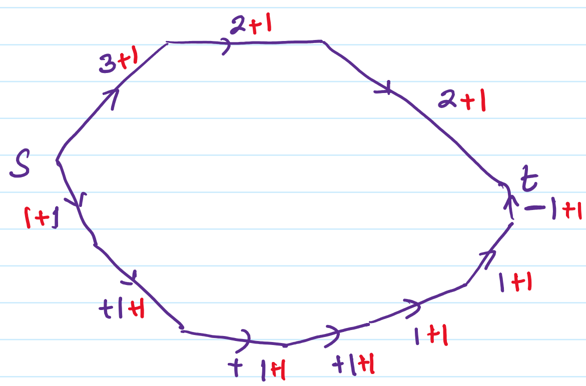
Problem -ve cycle, the shortest path is not defined

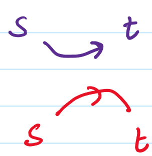




shortest path from s to t is 1

Assume that the graph doesn't have -ve weight cycle.





is shortestpath

we set as shortest path

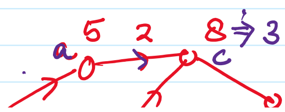
Since -ve cycle are not allowed path from s to v shouldn't contain cycle

Therefore all the path from s to v should be simple path

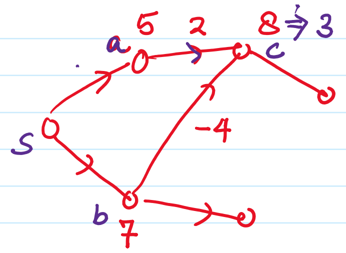
$\min\{8, 5+2, 7+(-4)\}$

Idea: propagate the gossip (consider each vertex as a person)

Each node (person) propagate the information they have to



Each node (person) propogate the information they have to their adjacent nodes. Information they propogate is the length of shortest path.



1+(−4)}

Bellman-Ford Algorithm 1956

label of u

$d[u]$:= upper bound on length of shortest path from s to u

$\delta(s,u)$:= exact length of shortest path from s to u

```
bellman-ford (G, s)
{
    for each  u ∈ V
    {
        d[u] = +∞
    }
    d[s] = 0

    repeat {
        converged = true
        for each (u,v) ∈ E {
            if  d[u] + l(u,v) < d[v]
            {
                d[v] = d[u] + l(u,v)
                converged = false
            }
        }
    } until (converged)
```

converge in at most n−1 iteration

Time complexity = $O(nm)$

(z, s)

(t, x)

min{∞, ∞+5}

(t, x)

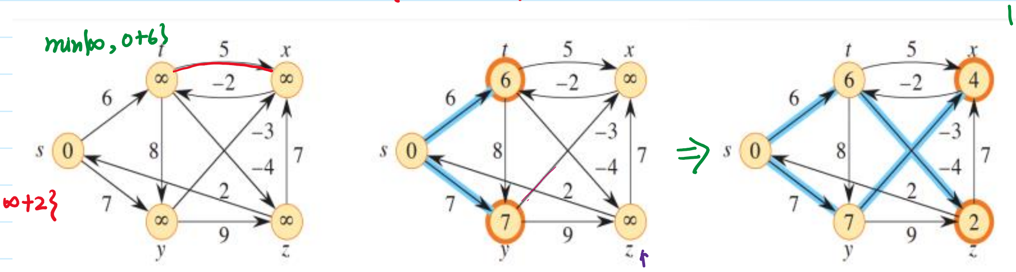
min{∞, 6+5}

11

(s, t)

min{∞, 0+6}

min{0, ∞+2}



(a) (b) (c)

(y, x)

min{11, 7+(−3)}

= 4

(t, z)

min{∞, 6+(−4)}

(a) (b)

(c) (t,z)

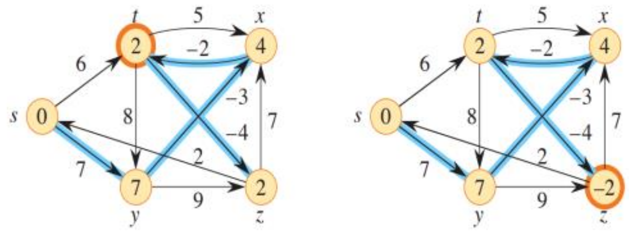
$\min\{\infty, 6+(-4)\}$

(y,x)

$\min\{\infty, 7+(-3)\} = 4$

(t,y)

$\min\{7, 6+8\}$



(d) (e)

Figure 22.4 The execution of the Bellman-Ford algorithm. The source is vertex s. The d values appear within the vertices, and blue edges indicate predecessor values: if edge (u, v) is blue, then $v.\pi = u$. In this particular example, each pass relaxes the edges in the order $(t,x), (t,y), (t,z), (x,t), (y,x), (y,z), (z,x), (z,s), (s,t), (s,y)$. **(a)** The situation just before the first pass over the edges. **(b)–(e)** The situation after each successive pass over the edges. Vertices whose shortest-path estimates and predecessors have changed due to a pass are highlighted in orange.

$m = |E|$, $n = |V|$

SSSP Problem (i) +ve $O(m \log n)$ (Dijkstra) $O(m + n\log n)$

(ii) -ve $\underbrace{O(mn) \text{ Bellman Ford}}_{\text{covered in class}}$ $\underbrace{O(mn)}_{\text{Best known}}$

All pair shortest path: Run same algo for all vertices

(i) +ve $O(mn\log n)$ $O(n(m+n\log n))$

(ii) -ve $\underbrace{O(n^2 m)}_{\text{covered in class}}$ $\underbrace{O(mn + n^2\log n)}_{\text{Best known}}$

<u>Proof of correctness</u> $\delta(s,v)$:= exact length of shortest path from s to v

$d[v]$:= upper bound on length of shortest path " "

We know $d[v] \geq \delta(s,v)$ — (I) $\forall\, v \in V$

Need to prove $d[v] = \delta(s,v)$ — (II) "

(II) require showing that $d[v] \leq \delta(s,v)$

Prove it via induction on # of edges in shortest path $(:= i)$

<u>Base case</u> $i = 0$ (Need to show that label of all vertices has correct shortest path)

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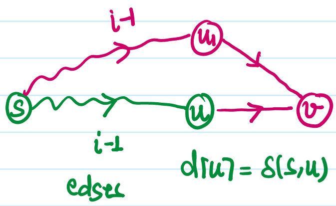
$d[s]=0$

$d[v]=\infty \quad \forall v \in V/s$

<u>Inductive case</u>: We assume that $\forall\, v \in V$, $d[v]$ correspond to correct shortest path when $i-1$ edge in shortest path



$d[v] \leftarrow \min\{d[v],\ d[u]+\ell(u,v)\}$

$\Rightarrow d[v] \leqslant d[u] + \ell(u,v)$

$\Rightarrow d[v] \leqslant \delta(s,u) + \ell(u,v)$

$\Rightarrow d[v] \leqslant \delta(s,v) \quad —$ (iii)

(i) & (iii) $\Rightarrow \quad d[v] = \delta(s,v)$