11.4.

Tuesday, 11. April 2023 10:24

$f'(x) = -\lambda f(x)$

Euler

right

$\left.\frac{f(x+h)-f(x)}{h}\right|_{h\to 0} = -\lambda f(x) \to$ discretized $\frac{f_{n+1}-f_n}{h} = -\lambda f_n(x)$

$f_{n+1} = f_n - \lambda h f_n$

left

$\left.\frac{f(x)-f(x-h)}{h}\right|_{h\to 0} = -\lambda f(x) \to \frac{f_n - f_{n-1}}{h} = -\lambda f_n \iff f_n = f_{n-1} - \lambda h f_n$ → implicit scheme (1st order scheme)

$\hookrightarrow \underline{f_{n+1}} = f_n - \lambda h \underbrace{f_{n+1}}_{\text{unknown}}$ → usually inefficient. Useful for stiff diff' equations

central

$\left.\frac{f(x+h)-f(x-h)}{2h}\right|_{h\to 0} = -\lambda f(x)$

Differentiation and Integrations

$$f'(x) = \frac{f(x+h)-f(x)}{h} + \mathcal{O}(h) \quad h \to 0$$

Rounding error: (Taylor Expansion)

$$f(x+h) = f(x) + f'(x)h + \tfrac{1}{2}f''(x)h^2 \xrightarrow{\frac{1}{h}} \frac{f(x+h)-f(x)}{h} = f'(x) + \underbrace{\tfrac{1}{2}f''(x)h}_{\mathcal{O}(h)}$$

$$\frac{f(x+h)-f(x-h)}{2h} = f'(x) + \frac{h^2}{6}f'''(x) + \dots \quad \mathcal{O}(h^2)$$

Higher order scheme (five point stencil)

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + \mathcal{O}(h^5)$$

Second derivative:

$$f''(x) = \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h} = \frac{f(x+h)-2f(x)+f(x-h)}{h^2} + \mathcal{O}(h^2)$$

Numerical Integration:

$$\int_a^b f(x)\,dx = \int_a^{a+h} f(x)\,dx + \int_{a+h}^{a+2h} f(x)\,dx + \dots + \int_{b-h}^{b} f(x)\,dx$$

Riemann Sum

$$\sum_{n=0}^{n=N-1} f(x+nh)h \quad , \quad N = \frac{b-a}{h}$$

Basic Improvements/Rules/Methods

Newton trapezoidal: $\sim \frac{f(x+nh)+f(x+nh+h)}{2}$, $\mathcal{O}(h^3)$

Midpoint rule: $\sim f(x+nh+\frac{h}{2})h$, $\mathcal{O}(h^3)$

Simpson (parabolic): 3-points , $\mathcal{O}(h^5)$ harder to program → many if-else

$$\int_a^b f(x)\,dx = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_{n-1}) + f(x_n)\right]$$

$\Delta x = \frac{b-a}{n}$; $x_i = a + i\Delta x$, n even

Iterative:

Romberg-Integration (hybrid) ← show

Iterative trapezoidal

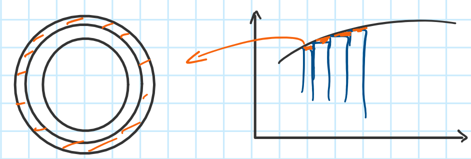
Euler-Mclaurin (error estimation integration method)

$$\sum_{k=1}^{N-1} g(k) = \int_0^N g(k)\,dk - \tfrac{1}{2}[g(0)-g(N)] + \tfrac{1}{12}[g'(N)-g'(0)] - \tfrac{1}{720}[g'''(N)-g'''(0)] + \dots$$

Trapezoidal rule → chose $g(k) = f(x_k)$, $dx = h\,dk$

$$\int_{x_0}^{x_N} f(x)\,dx = \underbrace{h\sum_{k=1}^{N-1} f(x_k) + \frac{h}{2}[f(x_0) + f(x_N)]}_{\text{trapezoidal rule}} - \frac{h^2}{12}\left[f'(x_N) - f'(x_0)\right] + \dots$$

$\hookrightarrow \mathcal{O}(h^2) \to h^d h^2 = h^3 \ (d=1)$ → dimensional dependent



→ most area at the surface

→ in higher dimensions: error blows up

For high dimensions: monte-carlo integration $\left(\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\right)$

↳ dimension independent