Computational Physics SuSe 2023
0. Exercise Sheet

Handing out: 26.05.2023 Prof. H. Jelger Risselada
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Exercise 0: Comprehension questions 0 Points

1) Hier könnte Ihre Verständnisfrage stehen

2) Hier könnte Ihre Verständnisfrage stehen.

Exercise 1: Bifurcation diagrams 10 Points

The Banach fixed point theorem predicts under certain conditions the existence of a unique fixed point of an iteration rule. While a numerical solution in these cases is usually unproblematic, it is all the more difficult to deal with problems where the theorem does not hold, i.e., those with unstable fixed points, orbits, and "chaos". Bifurcation diagrams are an illustrative way to capture these phenomena using a single parameter.

Calculate and plot bifurcation diagrams for

a) the logistic map ($x_n \in [0, 1]$):
$$x_{n+1} = rx_n(1 - x_n),$$

b) the cubic map ($x_n \in [-\sqrt{1+r}, \sqrt{1+r}]$):
$$x_{n+1} = rx_n - x_n^3,$$

by numerical iteration. Proceed as follows:

1. Write a routine that, given r and x_0, executes the respective map N times. This "warm-up" is to guarantee that one has converged to a fixed point/orbit, if this exists. Consider which values $r > 0$ may assume at most, so that x_n remains within of the given intervals in each step.

2. Now write a routine that continues to iterate after the warm-up. Count the orbits on which the points are distributed for the different r.

3. Vary r in steps $\Delta r = 1 \cdot 10^{-3}$ and, thus, generate a bifurcation diagram for the respective map. Test different starting values x_0, because certain fixed points/orbits can only be reached from certain starting values. Store those values of r for which the number of orbits doubles. Use those to determine the Feigenbaum constant. Furthermore determine in an r_∞, starting from which no more "real" orbits occur and the bifurcation diagram becomes chaotic. Since the orbit size from a certain point on grows very rapidly with r (keyword Feigenbaum constant), it is a good approximation to assume chaos already from an orbit size of $G > 64$ to determine r_∞.

Note: Often the bifurcation diagrams are also drawn in the chaotic area, even if there are no more fixed points/orbits. This is mainly for the purpose of illustration, because a map becomes chaotic exactly at the point, where discrete lines in the bifurcation diagram change into a continuum.

Exercise 2: Der Wetterfrosch — 10 Points

With the help of the so-called Poincaré cut, the continuous dynamics of a system can be mapped onto a discrete system. Thus, it is a useful tool to analyze deterministic chaos in several dimensions. In this task, we consider a simple DGL system that exhibits chaotic behavior in certain cases. To analyze this, among other things a Poincaré cut of the phase space is to be performed.

Predicting the weather is a complicated matter. A simple approach to the model-based investigation of the dynamics of complex weather phenomena is the chaotic Lorenz model, given by

$$\begin{aligned}\dot{X} &= -\sigma X + \sigma Y \\ \dot{Y} &= -XZ + rX - Y \\ \dot{Z} &= XY - bZ.\end{aligned}$$

Here σ and b are dimensionless constants describing material properties of the system, and r is an external parameter that depends on the temperature difference in the system.

a) Integrate the differential equations using the fourth order Runge-Kutta method. Choose $\sigma = 10$ and $b = 8/3$. In particular, investigate the cases $r = 20$ and $r = 28$. What is the role of the initial conditions?

b) Plot the result graphically

1. as a projection of the continuous motion onto the X-Y plane.
2. as a Poincaré cut to $Z = \text{const} = 20$ and $\dot{Z} < 0$. To determine the intersection on the plane $Z = \text{const} = 20$ more precisely, one should linearly interpolate between the last value above 20 and the first one below 20.
3. as a 3D plot.