Exercises

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E9: a)

$$PDF(\Delta\psi) = \begin{cases} N \cdot e^{-|\Delta\psi| k} &, \Delta\psi \in [-\pi, \pi) \\ 0 &, \text{sonst} \end{cases}$$

$$1 \overset{!}{=} \int_{-\pi}^{\pi} N \underbrace{\exp(-|\Delta\psi| k)\, d(\Delta\psi)}_{\text{gerade in } \Delta\psi} = 2N \int_0^{\pi} \exp(-\Delta\psi \cdot k)\, d(\Delta\psi) = \frac{-2N}{k} \left[\exp(-\Delta\psi \cdot k)\right]_0^{\pi} = \frac{-2N}{k} \left(e^{-\pi k} - 1\right)$$

$$\Rightarrow N = \frac{-k}{2(e^{-\pi k} - 1)} = \underline{\underline{\frac{k}{2(1-e^{-\pi k})}}}$$

b)

$$CDF(x) = \int_a^x PDF(\tilde{x})\, d\tilde{x}$$

$$= N \int_{-\pi}^{x} e^{-|\Delta\psi| k}\, d(\Delta\psi) \longrightarrow \text{Fallunterscheidung}$$

$$x<0: \quad CDF(x) = N \int_{-\pi}^{x} e^{\Delta\psi k}\, d(\Delta\psi) = \frac{N}{k}\left(e^{\Delta\psi k}\right)\Big|_{-\pi}^{x} = \frac{N}{k}\left(e^{xk} - e^{-\pi k}\right) = \frac{e^{xk} - e^{-\pi k}}{2(1-e^{-\pi k})}$$

Insbesondere: $x = 0 \Rightarrow CDF(0) = \frac{1}{2}$

$$x>0: \quad CDF(x) = \frac{1}{2} + N \int_0^x e^{-\Delta\psi k}\, d(\Delta\psi) = \frac{1}{2} - \frac{N}{k} e^{-\Delta\psi k}\Big|_0^x = \frac{1}{2} - \frac{e^{-xk} - 1}{2(1-e^{-\pi k})} = \frac{1 - e^{-\pi k} - e^{-xk} + 1}{2(1-e^{-\pi k})} = \underline{\underline{\frac{2 - e^{-\pi k} - e^{-xk}}{2(1-e^{-\pi k})}}}$$

Probe: $x = \pi$: $CDF(\pi) = \frac{2(1-e^{-\pi k})}{2(1-e^{-\pi k})} = 1$ passt

$$\Rightarrow CDF(x) = \underbrace{\frac{1}{2(1-\exp(-\pi k))}}_{=:\alpha} \begin{cases} \exp(xk) - \exp(-\pi k) &, x \le 0 \\ 2 - \exp(-\pi k) - \exp(-xk) &, x > 0 \end{cases}$$

c)

$$PPF(\underbrace{CDF(x)}_{=:p}) = CDF^{-1}(p) = \min\{x : CDF \geq p\}$$

Also CDF(x) nach x auflösen:

$x \le 0$: $p = CDF(x) = \alpha\left(\exp(xk) - \exp(-\pi k)\right) \Leftrightarrow \exp(-\pi k) + \frac{p}{\alpha} = \exp(xk)$

$$\Leftrightarrow \frac{1}{k} \ln\left(\exp(-\pi k) + 2p\,(1 - \exp(-\pi k))\right) = x \quad , p \le \frac{1}{2}$$

$x > 0$: $p = CDF(x) = \alpha\left(2 - \exp(-\pi k) - \exp(-xk)\right) \Leftrightarrow \exp(-\pi k) + \frac{p}{\alpha} - 2 = \exp(-xk)$

$$\Leftrightarrow \frac{-1}{k} \ln\left(\exp(-\pi k) + 2p\,(1 - \exp(-\pi k)) - 2\right) = x \quad , p > \frac{1}{2}$$