MINIMUM BACKGROUND TEST

VECTORS AND MATRICES

① $x = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ $\quad y = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad z = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Inner product $\quad y^T z = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = (1)(2) + (3)(3)$

$= 11$

② $x\,y = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$

③ $\det(x) = 2(3) - 4(1) = 2 \neq 0$

$\therefore$ x is invertible

$x^{-1} = \frac{1}{\det(x)} \operatorname{adj}(x) = \frac{1}{\det(x)} \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$

④ $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \xrightarrow[\text{to echelon form}]{\text{convert}} \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$

$\therefore$ The no. of non-zero rows = 2

$\therefore$ Rank = 2

CALCULUS

① $y = x^3 + x - 5$

$$\frac{dy}{dx} = 3x^2 + 1$$

② $f(x_1, x_2) = x_1 (\sin x_2) e^{-x_1}$

$$\nabla f(x) = \begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \end{pmatrix} = \begin{pmatrix} e^{-x_1}(1 - x_1) \sin x_2 \\ x_1 (\cos x_2) e^{-x_1} \end{pmatrix}$$

PROBABILITY AND STATISTICS

① Sample mean $\bar{x} = \frac{\sum x_i}{N} = \frac{3}{5}$

② Sample Variance $= \frac{\sum (x_i - \bar{x})^2}{N} = \frac{1}{5}\left(\frac{2}{5}^2 + \frac{2}{5}^2 + \left(\frac{-3}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{-3}{5}\right)^2 \right)$

$$= \frac{6}{25}$$

③ $P(5 \text{ independent events}) = (P(\text{each event}))^5$

$$= \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

④ $p \rightarrow$ probability that it is heads

$n \rightarrow$ number of heads in 5 tosses

The probability of getting n heads in 5 tosses is

$$P = {}^5C_n \, p^n (1-p)^{5-n}$$

In our sample data, $n = 3$

$$\therefore \quad P(p) = {}^5C_3\, p^3 (1-p)^2$$

We need to maximize $P(p)$

$$\frac{dP}{dp} = {}^5C_3 \left(3p^2(1-p)^2 - 2p^3(1-p)\right) = 0$$

$$\Rightarrow \quad 3(1-p) = 2p$$

$$\Rightarrow \quad p = \frac{3}{5}$$

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a) From the table, it is clear that

$$P(z=T \;\&\; y=b) = 0.1$$

b)

$$P(z=T \mid y=b) = \frac{P(z=T \;\&\; y=b)}{P(y=b)}$$ → Bayes theorem

$$= \frac{0.1}{0.1 + 0.15} = 0.4$$

BIG O - NOTATION

1 $\quad f(n) = \ln(n) \qquad g(n) = \log(n) = \frac{\ln(n)}{\log_e 10}$ → constant

$$\therefore \quad \cancel{O(f(n))} \Rightarrow g(n) = c f(n)$$

$\therefore$ both are true

$$f(n) = O(g(n)) \qquad g(n) = O(f(n))$$

(2) $f(n) = 3^n$ $\quad g(n) = n^{100}$

$\exists\, n_0$ s.t. $\forall\, n > n_0$

$$n^{100} < 3^n$$

$$\therefore\ g(n) = O(f(n))$$

But the other case is not true

(3) $f(n) = 3^n$ $\quad g(n) = 2^n$

$\exists\, n_0$ s.t. $\forall\, n > n_0$

$$2^n < 3^n$$

$$\Rightarrow\ g(n) = O(f(n))$$

But the other way around is not true.

(4) $f(n) = 1000\, n^2 + 2000\, n + 4000$ $\quad g(n) = 3n^3 + 1$

$\exists\, n_0$ s.t $\forall\, n > n_0$

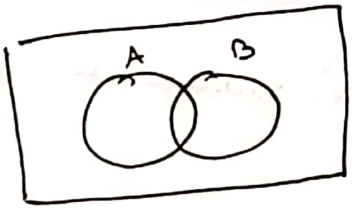
$$f(n) < g(n)$$

$$\Rightarrow\ f(n) = O(g(n))$$

ALGORITHMS

PROBABILITY AND RANDOM VARIABLES

c) $P(A \cap (B \cap A^c)) = \varphi \neq P(A \cup B)$



$\therefore$ False

d) $LHS = \dfrac{P(A \cap B)}{P(B)}$ $\quad RHS = \dfrac{P(A \cap B)}{P(A)}$

only true if $P(A) = P(B) = 0.5$

otherwise false

e) $P(A_3 \mid (A_2 \cap A_1))\ P(A_2 \mid A_1)\ P(A_1) = RHS$

$$= \frac{P(A_3 \cap A_2 \cap A_1)}{P(A_2 \cap A_1)} \frac{P(A_2 \cap A_1)}{P(A_1)} P(A_1) = P(A_3 \cap A_2 \cap A_1) = LHS$$

$\therefore$ True

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

True

c) $P(A \cap B) + P(A^c \cap B) = P(A) + P(B) - P(A \cup B) + P(A^c) + P(B)$

$$= P(B) \neq P(A)$$

$\therefore$ False

Discreet & continuous distribution

a) Multivariate Gaussian $\rightarrow \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left(-\frac{1}{2} - (x-\mu)^T \Sigma^{-1} (x-\mu)\right)$

b) Bernoullie $\rightarrow p^x (1-p)^{1-x}$

d) Binomial $\rightarrow {}^n C_x\, p^x (1-p)^{n-x}$

c) Uniform $\rightarrow \frac{1}{b-a}$ $a \leq x \leq b$ 0 otherwise

Mean, Variance, Entropy

a) $Var(x) = E\left((x - E(x))^2\right)$

$= E\left(x^2 + (E(x))^2 - 2xE(x)\right)$

$= E(x^2) + (E(x))^2 - 2E(xE(x))$

$= E(x^2) - (E(x))^2$

b)
mean $= E(x) = \sum x\, p(x) = 1(p) + 0(1-p)$

$= p$

Variance $= \sum x^2 p(x) - p^2$

$= 1(p) + 0(1-p) - p^2$

$= p - p^2 = p(1-p)$

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Entropy $= -\sum_p p \ln p = -\left[p \ln p + (1-p)\ln(1-p) \right]$

Law of large numbers

a) As N increases, the binomial probability distribution turns into a gaussian. Hence it becomes more & more likely that the random variable takes the value at the peak

b) $\bar{x} = np = \frac{n}{2} \Rightarrow \bar{x} - \frac{1}{2} = \frac{1}{2}(n-1)$

From central limit theorem, the binomial distribution tends to gaussian as $n \to \infty$ with

$E(x) = 0$ & $\sigma = p(1-p) = \frac{1}{4}$

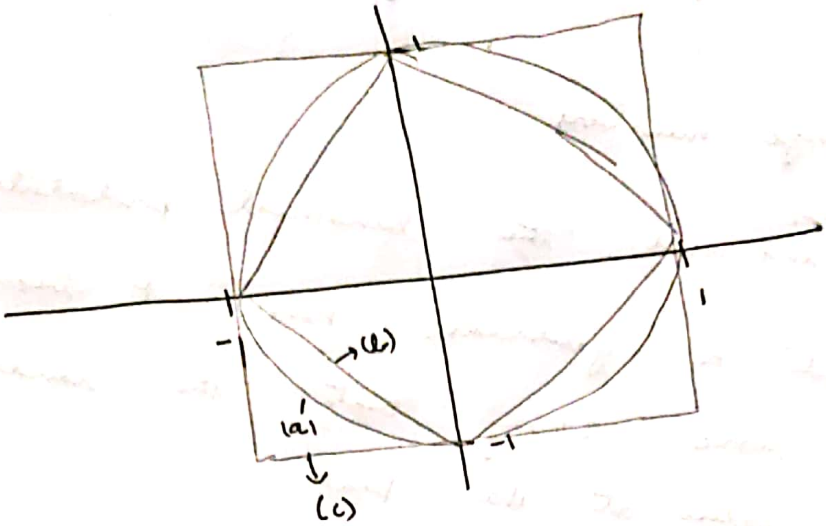
$N(0, 1/4)$

This would have been $\frac{1}{2}$ but we have to take

limit $\sqrt{n}\left[\left(\bar{x} - \frac{1}{2}\right)\right]$

LINEAR ALGEBRA

a)



$\|x\|_2 \leq 1 \Rightarrow x_1^2 + x_2^2 \leq 1$ → represents circle

b) $|x_1| + |x_2| \leq 1$ → set of four lines

c) $\max(x_1, x_2) \leq 1$

$\frac{(x_1 + x_2)}{2} + \frac{|x_1 - x_2|}{2} \leq 1$ → set of ~~four~~ two lines

Geometry

(a) First we choose a vector parallel to the line. We take two points on the line x_1 & x_2. Now $x_1 - x_2$ is a vector ∥ to the line. Now if W is ⊥ to this vector we should get $W^T(x_1 - x_2) = 0$

$$LHS = w^T(x_1 - x_2)$$

$$= w^T x_1 - w^T x_2$$

$$= (-b) - (-b) = 0 = RHS$$

$\therefore$ w^T is $\perp$ to the line

b) Let x be a point on the line

Then the perpendicular distance is given by the projection $\dfrac{\| w^T(x - 0) \|}{\| w \|} = \dfrac{\| w^T x \|}{\| w \|} = \dfrac{\| b \|}{\| w \|}$

ALGORITHMS

$(0, 0 \dots 0, 1, 1\ 1 \dots)$

Divide the array recursively into two until you find the transition point. Compare the end points of the divided arrays

This is a divide & conquer algorithm which is proven to be $\cancel{O(n)}$ $O(\log n)$