On Modular Prime Bias in Exponential Prime-Generating Functions

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1 Abstract

Prime numbers play a crucial role in number theory, cryptography, and computational mathematics. This paper explores a novel empirical discovery regarding a strong modular bias in prime-generating functions of the form:

$$f(a,b,x,c) = \frac{a^x + b}{c}$$

where a, b, c are integers, x is a positive integer exponent, and $f(a,b,x,c)$ is tested for primality. Computational analysis reveals that when $b \equiv 0 \mod 15$, the function is significantly more likely to produce prime numbers. This suggests an underlying structural property in modular arithmetic that influences prime generation. The findings have potential applications in number theory, probabilistic prime generation, and cryptographic algorithms.

2 Introduction

Prime-generating functions have been widely studied in mathematics, with classic examples like Euler's polynomial $x^2 + x + 41$, which produces a large number of primes for small values of x. However, exponential prime-generating functions and their modular properties remain less explored.

This research investigates a strong modular bias in prime generation within functions of the form $a^x + b$. Empirical testing has shown that when $b \equiv 0 \mod 15$, the likelihood of producing prime numbers increases significantly. This bias suggests deeper mathematical patterns worth further exploration.

3 Key Discovery

Computational analysis over a broad range of inputs has revealed a non-random bias in prime generation related to the modulus of b. The core finding is:

- When $b \equiv 0 \mod 15$, the function is significantly more likely to appear among the top prime-generating functions compared to random expectation.

This suggests that modular properties of b interact with exponential growth, influencing the density of primes.

4 Related Work

Several studies have explored prime-generating functions and modular arithmetic in number theory. Euler's famous quadratic polynomial $x^2 + x + 41$ is a well-documented example of a polynomial that produces many primes for small values of x [1]. More recent studies have investigated the role of modular constraints in prime distributions [2]. Unlike previous research, this work provides empirical evidence for a specific modular condition—$b \equiv 0 \mod 15$—that significantly increases the probability of prime generation.

Prior research has examined linear recurrence relations in prime sequences [3] and probabilistic methods for prime selection [4]. However, our approach focuses on exponential functions and their interaction with modular arithmetic, providing novel insights into structured prime generation.

5 Empirical Findings

Recent computational analysis examined the distribution of high-performing prime-generating functions in relation to $b \equiv 0 \mod 15$. The results showed:

- **Total top 50 prime-generating functions analyzed:** 50
- **Functions where $b \equiv 0 \mod 15$:** 30
- **Percentage of top functions with $b \equiv 0 \mod 15$:** 60%

While the probability of $b \equiv 0 \mod 15$ appearing in the top 50 performers is high, the percentage declines sharply in larger subsets:

- **Top 100:** 57 functions (57%)
- **Top 1% (1125 functions):** 230 functions (20.4%)
- **Top 5% (5625 functions):** 854 functions (15.2%)
- **Top 20% (22,500 functions):** 2,040 functions (9.1%)
- **Top 50% (56,251 functions):** 3,554 functions (6.3%)

5.1 Density Comparison to Random Distribution

Since the total dataset contains **112,503** prime-generating functions, the expected random occurrence of $b \equiv 0 \mod 15$ would be around:

$$\frac{1}{15} \approx 6.67\%$$

Comparing this to the observed values:

- **Top 50 functions:** $\sim 9\times$ more likely than random
- **Top 100 functions:** $\sim 8.5\times$ more likely than random
- **Top 1%:** $\sim 3\times$ more likely than random
- **Top 5%:** $\sim 2.3\times$ more likely than random
- **Top 20%:** $\sim 1.4\times$ more likely than random
- **Top 50%:** $\sim 0.94\times$ more likely than random

This demonstrates that while the modular condition greatly increases the chances of appearing in the **top prime-generating functions**, its influence diminishes as more functions are considered. The effect is highly concentrated at the **extreme high-end** but does not apply universally.

6 Significance and Applications

The discovery that certain modular values of b lead to increased prime generation has potential applications in several fields:

- **Number Theory**: Understanding modular prime biases could contribute to existing research on prime distributions.
- **Cryptography**: Prime numbers are essential in encryption algorithms, and structured biases in prime generation could impact cryptographic key generation.
- **Computational Mathematics**: Efficient prime-finding algorithms may benefit from leveraging modular biases to improve performance.

7 Future Work

This study raises important questions that warrant further exploration:

- Expanding the search range to $b \in [-5000, 5000]$ to test if the effect holds consistently.

- Investigating whether other modular conditions (e.g., $b \equiv 0 \mod 30$) exhibit similar biases.
- Developing a formal mathematical proof explaining why $b \equiv 0 \mod 15$ exhibits this effect.
- Further refining the dataset to confirm trends in prime generation bias across different modular conditions.

8 Computational Methodology

The following Python code was used to conduct the analysis:

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Listing 1: Prime-Generating Function Analysis

```python
import math
import random
import sys
import time

# Miller-Rabin probabilistic prime test
def is_probable_prime(n, k=5):
    if n < 2:
        return False
    # check small primes
    for p in [2, 3, 5, 7, 11, 13, 17, 19, 23, 29]:
        if n % p == 0:
            return n == p

    # write n-1 as 2^s * d
    s, d = 0, n - 1
    while d % 2 == 0:
        s += 1
        d //= 2

    for _ in range(k):
        a = random.randrange(2, n - 1)
        x = pow(a, d, n)
        if x == 1 or x == n - 1:
            continue
        for __ in range(s - 1):
            x = pow(x, 2, n)
            if x == n - 1:
                break
        else:
            return False
    return True

def valid_and_count_primes(a, b, c):
    """Evaluate f(x) = (a^x + b) / c for x in 1..28.
       Returns (prime_count, list_of_values) if every evaluation is
            an integer,
       otherwise returns None.
    """
    prime_count = 0
    values = []
    for x in range(1, 29):
        numerator = pow(a, x) + b
        if numerator % c != 0:
            return None  # not an integer output
        value = numerator // c
        values.append(value)
        if is_probable_prime(value):
            prime_count += 1
    return prime_count, values

def print_progress(progress, total, bar_length=40):
    percent = progress / total
    arrow = '-' * int(round(percent * bar_length) - 1) + '>' if
        percent > 0 else ''
    spaces = '␣' * (bar_length - len(arrow))
    sys.stdout.write(f"\rProgress:␣[{arrow}{spaces}]␣{int(round(
        percent␣*␣100))}%")
    sys.stdout.flush()

def main():
    start_time = time.time()
    candidates = []  # will hold tuples: (prime_count, a, b, c,
        values)
    total_checks = (8 - 2 + 1) * (4000 + 4000 + 1) * 8  # a in
        [2,8], b in [-4000,4000], c in [1,8]
    check_count = 0

    # Loop over candidates (a from 2 to 8)
    for a in range(2, 9):
        for b in range(-4000, 4001):
            for c in range(1, 9):
                check_count += 1
                # update progress every 5000 iterations
                if check_count % 5000 == 0:
                    print_progress(check_count, total_checks)
                result = valid_and_count_primes(a, b, c)
                if result is not None:
                    prime_count, values = result
                    candidates.append((prime_count, a, b, c, values
                        ))

    # Final progress update
    print_progress(total_checks, total_checks)
    print("\n\nSearch␣complete!")

    # Sort candidates by prime_count first and then by the sum of
        values (secondary measure)
    candidates.sort(key=lambda x: (x[0], sum(x[4])), reverse=True)

    total_candidates = len(candidates)
    print(f"Total␣candidate␣functions␣found:␣{total_candidates}\n")

    # Print the top 50 functions in a nice format
    top_50 = candidates[:50]
    print("===␣Top␣50␣Prime-Generating␣Functions␣===")
    for idx, (prime_count, a, b, c, values) in enumerate(top_50,
        start=1):
        print(f"Rank␣{idx}:␣f(x)␣=␣(␣{a}^x␣+␣({b})␣)␣/␣{c}")
        print(f"␣␣␣Prime␣count:␣{prime_count}/28")
        print(f"␣␣␣Values:␣{values}\n")

    # Now compute counts for various ranking tiers where b mod 15
        == 0.
    def count_b_mod_15(candidates_list):
        return sum(1 for candidate in candidates_list if candidate
            [2] % 15 == 0)

    def report_tier(name, num_candidates):
        subset = candidates[:num_candidates] if num_candidates <=
            total_candidates else candidates
        count_mod15 = count_b_mod_15(subset)
        print(f"{name}:␣In␣top␣{num_candidates}␣functions,␣{
            count_mod15}␣have␣b␣mod␣15␣==␣0.")

    print("===␣b␣mod␣15␣Statistics␣===")
    # Top 50
    report_tier("Top␣50", 50)
    # Top 100
    report_tier("Top␣100", 100)
    # Top 1%
    tier_1_percent = max(1, int(total_candidates * 0.01))
    report_tier("Top␣1%", tier_1_percent)
    # Top 5%
    tier_5_percent = max(1, int(total_candidates * 0.05))
    report_tier("Top␣5%", tier_5_percent)
    # Top 20%
    tier_20_percent = max(1, int(total_candidates * 0.20))
    report_tier("Top␣20%", tier_20_percent)
    # Top 50%
    tier_50_percent = max(1, int(total_candidates * 0.50))
    report_tier("Top␣50%", tier_50_percent)

    end_time = time.time()
    print(f"\nCompleted␣in␣{end_time␣-␣start_time:.2f}␣seconds.")

if __name__ == "__main__":
    main()
```

This script efficiently evaluates prime-generating functions and examines the modular bias in prime generation.

10 Conclusion

This paper presents empirical evidence of a **strong modular prime bias in exponential prime-generating functions**, particularly when $b \equiv 0 \mod 15$.

The observed increase in prime density suggests an underlying structure in modular arithmetic influencing prime formation. The effect is highly concentrated among **top-performing prime-generating functions**, but its significance declines as broader function sets are analyzed. Future research will focus on theoretical justifications and extended computational testing.

11 Acknowledgments

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12 Bibliography

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