Extremal Combinatorics - Turán Type Problems

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1 Introduction

One of the most important problem in mathematics is concerned with the study of structure that describes mathematical objects. When looking at graphs, one might ask what is the structure that describes them. One approach in describing these structures is trying to look at substructures. Here, the question at hand is the following: given a graph G that does not contain a subgraph F, what is the maximum number of edges that G can have? Turán answered this question in the case of complete subgraphs K_p. However, even if a lot of progres has been done since Turán's results, there are still a lot of unsolved problems in this area.

2 Definitions and Preliminaries

Definition 1. A graph $G = (V, E)$ is defined by a set of vertices V and a set of edges E between them. For our purposes, we're only going to work with simple graphs that have no loops (edges that leave and arrive at the same vertex) or paralel edges (more than one edge between two different points).

Definition 2. A complete graph K_p on p vertices is a graph that has all the possible edges between the vertices. Such a K_p graph is called a p-clique. Note that such a complete graph has $p(p-1)/2$ edges.

What we're interested in is finding $ex(n, K_p)$, the maximum number of edges in a graph on n vertices that doesn't contain K_p as a subgraph. Our goal will be to look at Turán theorem and give three different proofs of the result.

But let's first look at a construction example. Given n vertices, divide V into $p-1$ pairwise disjoint subsets such that $V = V_1 \cup ... \cup V_{p-1}$. We then insert edges between 2 vertices if and only if they lie in different subsets. We denote this graph by $K_{n_1,n_2,...n_{p-1}}$ where $|V_i| = n_i$ for all $i \in \{1, ..., p-1\}$. It's not hard to see that in order to maximize the number of edges in this graph we need the sizes of vertex subsets to be as equal as possible, that is $|n_i - n_j| \leq 1$ for all i, j. That's because if we would have two distict indeces say 1 and 2 such that $n_1 \geq n_2 + 1$. We can then move one vertex from V_1 to V_2 and get a graph $K_{n_1-1,n_2+1,...,n_{p-1}}$ that will have $(n_1-1)(n_2+1) - n_1n_2 \geq 1$ more edges than the initial graph. So we want graphs $K_{n_1,n_2,...n_{p-1}}$ with $|n_i - n_j| \leq 1$ for all i, j. We call such graphs the Turán graphs. Notice that if n is divisible by $p-1$, we're going to have

$$\binom{p-1}{2}\left(\frac{n}{p-1}\right)^2 = \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}$$

edges. So we've looked at a constructive example that builds a complete $p-1$-partite graph on n vertices that has no p-clique and $\left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}$ edges. What we're going to show next is a result of Turán's that states that this is actually the upper bound for the maximum number of edges that a graph on n not containing a p-clique can have, that is $ex(n, K_p) = \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}$.

3 The Theorem and Three Different Proofs

Theorem 3. *Let $G = (V, E)$ be a graph on n vertices that contains no p-clique, where $p \geq 2$. Then*

$$|E| \leq \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}.$$

In the following, we're going to look at three different proofs of this result. The first one due to Turán (1941) is based on an inductive argument, while the other two use ideas from probability theory and a constructive reasoning, respectively.

First Proof. We're going to do induction on the number of vertices n. It is easy to see that the statement is true for $n < p$ because we always have $|E| \leq \binom{n}{2}$ which will be less than $\left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}$ in this case.

We then now do the inductive step for $n \geq p$. Let G be our graph on n vertices, with no p-clique and maximum number of edges. We then know G must contain a $p-1$-clique A; if this hadn't been the case, we could just insert more edges until we get a $p-1$-clique, but this would contradict the fact that our graph already has a maximum number of edges. We're now going to use the fact that A is a subgraph. Let B be the set of edges of the graph not contained in A and let $e(A)$, $e(B)$ be the sets of egdes of G with both ends in A, and B respectively. We're going to denote the rest of the edges that have an end in A and one in B by $e(A, B)$.

Clearly, $e(A) = \binom{p-1}{2}$. Also, there can't be a vertex in B connected to all vertices in A because our graph can't have a p-clique. Therefore, all the vertices in B are linked to at most $p-2$ vertices in A, so $e(A, B) \leq (p-2)(n+1-p)$. For $e(B)$ we can use the inductive hypothesis since the subgraph on the vertices in B clearly doesn't contain a p-clique , so $e(B) \leq \left(1 - \frac{1}{p-1}\right)\frac{(n+1-p)^2}{2}$. After putting all these together and doing the computations, we'll get exactly what we need

$$|E| = e(A) + e(B) + e(A, B) \leq \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}.$$

□

Second Proof. The idea here is to consider a probability distribution $w = (w_1, \ldots. w_n)$ that distributes non-negative weights to the vertices of the graph. So, $w_i \geq 0$ for all i and $\sum_{i=1}^{n} w_i = 1$. We're going to define the following function $f(w) = \sum_{v_i v_j \in E} w_i w_j$ and try to maximize its value.

Firstly, let w be any distribution and assume there exist two disjoint, non-adjacent vertices v_i, v_j. Let s_i, s_j be the sum of the weights of vertices adjacent to v_i, and v_j. Without loss of generality, assume $s_i \geq s_j$. We're going to define another distribution w' by moving the weight w_j from the vertex v_j to the vertex v_i. The weight on v_j will then be 0, while the weight on v_i will be $w_i + w_j$, so looking at the value of our function we'll now have $f(w') = f(w) + w_j s_i - w_j s_j \geq f(w)$. We can thus repeat this process all over again until there are no non-adjacent vertices of positive weights. Therefore, there exists some optimal distribution whose nonzero weights will be distributed on the vertices of a complete graph on, say k vertices, that is, a k-clique.

So we know that having all nonzero weights distributed on a k-clique subgraph maximizes our function. Let's see how we can maximize it even further. Assume there exists two different positive weights w_1 and w_2 that are distinct, say $w_1 > w_2$. Clearly, there will be some $\epsilon > 0$ such that $\epsilon < w_1 - w_2$. This time we're going to modify the probability distibution by moving a weight of ϵ from vertex v_1 to v_2, this giving us a weight of $w_1 - \epsilon$ on v_1 and $w_2 + \epsilon$ on v_2. Therefore, keeping in mind that v_1 and v_2 share an edge, $f(w') = f(w) + (w_1 - \epsilon)(w_2 + \epsilon) - w_1 w_2 = f(w) + \epsilon(w_1 - w_2) - \epsilon^2 > f(w)$. Thus the function gets maximized when the weights on the k-clique have the same value $1/k$.

So, the maximal value of the function is attained when $w_i = 1/k$ if w_i on the k-clique and $w_i = 0$ otherwise. In this case $f(w) = \frac{1}{k^2}\frac{k(k-1)}{2} = \frac{1}{2}\left(1 - \frac{1}{k}\right)$ which is maximized when k is maximum and since we can't have a p-clique, the largest value k can have is $p - 1$. Thus, whatever probability distribution w, we have

$$f(w) \leq \frac{1}{2}\left(1 - \frac{1}{p-1}\right).$$

If we let w be the uniform distribution, $f(w) = \sum_{v_i v_j \in E} w_i w_j = \frac{1}{n^2}|E|$ which will be less or equal then the maximal value $\frac{1}{2}\left(1 - \frac{1}{p-1}\right)$. Thus $|E| \leq \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}$.

□

Third Proof. Let G be a our graph on n vertices with maximum number of edges that doesn't contain a p-clique.

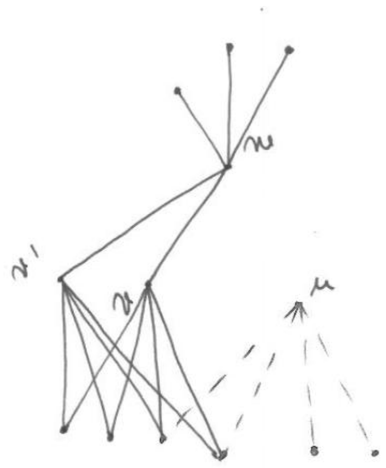
Claim. G does not contain three vertices u, v, w such that $vw \in E$, but $uv \notin E$, $uw \notin E$.

We're going to assume this can happen and give a proof by contradiction. In order to do that, let $d(v)$, the degree of the vertex v, count the number of neighbours of v. We then distinguish two cases:

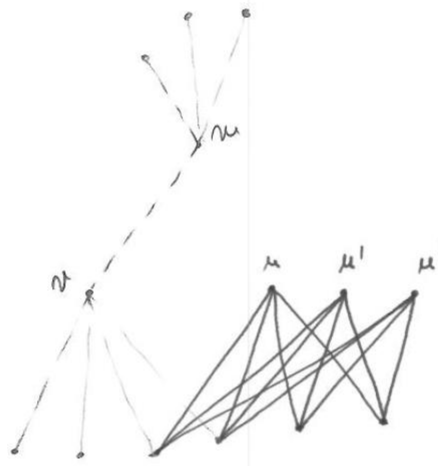
Case 1. We either have $d(u) < d(v)$ or $d(u) < d(w)$. Without loss of generality, assume $d(u) < d(v)$. We say we delete a vertex if we remove it from the vertex set of the graph and also remove the edges adjacent to it. We say we copy a vertex v if we introduce a new vertex v' with the same neighbours as v (but no edge between v and v'). Let's then construct a graph G' by deleting u and copying the vertex v into v'. It is easy to see that G' can't contain a p-clique. On the other hand,

$$|E(G')| = |E(G)| + d(v) - d(u) > |E(G)|$$

which contradicts the assumption that G has maximum number of edges.



Case 1



Case 2

Case 2. When $d(u) \geq d(v)$ and $d(u) \geq d(w)$, we construct a new graph G' by deleting v and w and make two copies of u into u' and u". The graph G' is again a graph on n vertices that doesn't contain a p-clique (we leave it as an exercise to check this). However,

$$|E(G')| = |E(G)| + 2d(u) - d(v) - d(w) + 1 > |E(G)|$$

which gives us a contradiction once again.

We got contradictions in both cases, which proves the claim. So we know that we can't have a configuration of three vertices u, v, w for which uv is an edge, but uw and vw are not. This actually defines the following equivalence relation:

$$u \sim v \text{ iff } uv \notin E.$$

In order to prove this, one has to check reflexivity, symmetry and transitivity. The first two are rather trivial, so we're only going to look at the third one. If $u \sim v$ and $u \sim v'$ then by definition the edges uv and uv' don't exist. On the other hand, we know there exists an impossible configuration of three points as shown in the proof of the above claim, so it is impossible to have an edge between v and v' which proves the transitivity property.

So we know that two vertices are in the same equivalence class if and only if they don't share an edge. If we consider the equivalence classes as being partitions on the set of vertices of our graph G, the equivalence relation tells us that G must be a complete multipartite graph. Since we want to maximize the number of edges, we want as many subsets as possible and since our graph doesn't contain a p-clique we can only have at most $p-1$ subsets. Finally, by the same reasoning ideas as shown in the construction example from the beginning of the section, we can see that our graph G must be the Turán graph $K_{n_1,\ldots n_{p-1}}$, which clealy proves the boundary needed on the number of edges.

□

Remark. One can notice that the first and the last proof actually imply the stronger result that the Turán graph is actually the unique example that gives the maximum number of edges.

4 A Taste of a More General Picture and Final Remarks

The idea of finding the maximum number of edges that a graph not containing a forbidden configuration can have can be generalized to hypergraphs. A hypergraph $\mathcal{H}$ is a generalized version of a graph, being again defined by a set of vertices $\mathcal{V}(\mathcal{H})$ and an edge set $\mathcal{E}(\mathcal{H})$ that is a collection of subsets of the vertex set (this means that an edge doesn't have to be defined by only two vertices as before, it can be defined by any subset of $\mathcal{V}(\mathcal{H})$). In the setting on hypergraphs we're interested in r-uniform hypergraphs, which are hypergraphs where all edges have r elements (the number of edges in a complete r-uniform hypergraph will thus equal $\begin{pmatrix} \mathcal{V}(\mathcal{H}) \\ r \end{pmatrix}$).

The problem now becomes the following: given an r-uniform hypergraph $\mathcal{F}$, the Turán number $\mathcal{F}$ is the maximum number of edges in an r-uniform hypergraph on n vertices that does not contain a copy of $\mathcal{F}$. As before, we denote this number by $ex(n, \mathcal{F})$.

In the case r is equal to 2, we deal with the "normal" graphs we defined at the beginning. As we've already seen, the question of finding $ex(n, F)$ (or its order of magnitude) has been solved for complete graphs, as well as for many other instances such as cycles $\mathcal{C}(2k)$ of even length for example (for $k = 2$ we have $ex(n, \mathcal{C}(4)) = \frac{1}{2}n^{3/2} + O(n^{4/3})$). More than that, asymptotic results are known for all non-bipartite graphs. These results are given by the following claim: $\pi(F) := \lim_{n\to\infty} ex(n, F)/\binom{n}{2}$ exists and equals $1 - \frac{1}{r}$ where $r + 1 = \chi(F)$ ($\chi(F)$ is the chromatic number of the graph F; notice that if F is bipartite, we're going to have $\chi(F) = 2$ which will give us a limit that goes to 0). The $\pi(F)$ we have defined is called the *coefficient of saturation*.

However, when $r > 2$ little is known. But we know that the coefficient of saturation $\pi(\mathcal{F}) := \lim_{n\to\infty} ex(n, \mathcal{F})/\binom{n}{r}$ still exists, even if its values are not known. And actually this coefficient of saturation, or the so called Turán density, is really important in trying to find the desired bound on the number of edges when we have some forbidden sub-hypergraph.

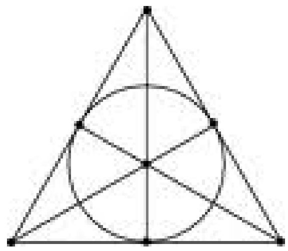
One of the most famous examples in this sense is Turán's conjecture claiming that $\pi_3(4) = \frac{5}{9}$ where $\pi_3(4) = \lim_{n\to\infty} ex(n, \mathcal{K}_3(4))/\binom{n}{3}$, $\mathcal{K}_3(4)$ being the complete 3-uniform hypergraph on 4 vertices.

Another beautiful example is the Turán number of the Fano plane $PG_2(2)$ which is known to be given by

$$ex(n, PG_2(2)) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}$$

for n sufficiently large.

The details of the proof are not the scope of this paper, but a sketch of the general framework of how the proof goes will give us a glimpse of how the above coefficient of saturation can be used in order to determine the Turán number. We first notice that the Fano plane is a 3-uniform hypergraph on 7 vertices and 7 edges (see figure below). The idea of the proof is then based on three steps. First, we know from the density theorem that the coeficient of saturation of the Fano plane exists. Moreover, it has been proved to be exactly 3/4. The second tool needed is a stability result. In general, the idea is to prove an approximate stucture theorem for hypergraphs with density close to the maximum possible one. In our case the stability result roughly states that a 3-uniform hypergraph with density close to 3/4 is approximately 2-colorable. This brings us to the last step which consists in smoothing out the details and finding the exact structure that has the maximum size among the approximate structures. This method provides a nice tool that can give us exact results for the Turán number once we know the coefficient of saturation.



As a closing remark, this last section is mainly meant to entice the reader by only offering a glimpse of what is actually going on. For a more detailed discussion on the subject, one is invited to look up the given references.

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References

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