11.8) In Hoffman Vol II

- Transform the following to (ξ, η) coordinates

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substitute by:

$$\frac{\partial u}{\partial x} = \xi_x \frac{\partial u}{\partial \xi} + \eta_x \frac{\partial u}{\partial \eta}$$

$$\frac{\partial v}{\partial y} = \xi_y \frac{\partial v}{\partial \xi} + \eta_y \frac{\partial v}{\partial \eta}$$

We obtain:

$$\xi_x \frac{\partial u}{\partial \xi} + \eta_x \frac{\partial u}{\partial \eta} + \xi_y \frac{\partial v}{\partial \xi} + \eta_y \frac{\partial v}{\partial \eta} = 0$$

Divide by Jacobian $J = \frac{1}{x_\xi\, y_\eta - x_\eta\, y_\xi}$ and add & subtract some terms

$$\frac{1}{J}\xi_x \frac{\partial u}{\partial \xi} + \frac{1}{J}\eta_x \frac{\partial u}{\partial \eta} + \left[u\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right) - u\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right)\right] + \left[u\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right) - u\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right]$$

$$+\frac{1}{J}\xi_y \frac{\partial v}{\partial \xi} + \frac{1}{J}\eta_y \frac{\partial v}{\partial \eta} + \left[v\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right) - v\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right)\right] + \left[v\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right) - v\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right] = 0$$

The added terms would help us to obtain a conservative form

Rearrange the equation

$$\left[\frac{1}{J}\xi_x \frac{\partial u}{\partial \xi} + u\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right)\right] + \left[\frac{1}{J}\eta_x \frac{\partial u}{\partial \eta} + u\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right] - u\left[\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right]$$

$$+\left[\frac{1}{J}\xi_y \frac{\partial v}{\partial \xi} + v\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right)\right] + \left[\frac{1}{J}\eta_y \frac{\partial v}{\partial \eta} + v\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right] - v\left[\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right] = 0$$

Substitute by the following 4 relations in the red brackets:

$$\xi_x = J y_\eta$$

$$\eta_x = -J y_\xi$$

$$\xi_y = -J x_\eta$$

$$\eta_y = J x_\xi$$

$$\left[\frac{1}{J}\xi_x\frac{\partial u}{\partial \xi}+u\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right)\right]+\left[\frac{1}{J}\eta_x\frac{\partial u}{\partial \eta}+u\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right]-u\left[\frac{\partial}{\partial \xi}(y_\eta)+\frac{\partial}{\partial \eta}(-y_\xi)\right]$$

$$+\left[\frac{1}{J}\xi_y\frac{\partial v}{\partial \xi}+v\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right)\right]+\left[\frac{1}{J}\eta_y\frac{\partial v}{\partial \eta}+v\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right]-v\left[\frac{\partial}{\partial \xi}(-x_\eta)+\frac{\partial}{\partial \eta}(x_\xi)\right]=0$$

$$\left[\frac{1}{J}\xi_x\frac{\partial u}{\partial \xi}+u\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right)\right]+\left[\frac{1}{J}\eta_x\frac{\partial u}{\partial \eta}+u\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right]-u(y_{\eta\xi}-y_{\eta\xi})$$

$$+\left[\frac{1}{J}\xi_y\frac{\partial v}{\partial \xi}+v\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right)\right]+\left[\frac{1}{J}\eta_y\frac{\partial v}{\partial \eta}+v\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right]-v(-x_{\xi\eta}+x_{\xi\eta})=0$$

$$\left[\frac{1}{J}\xi_x\frac{\partial u}{\partial \xi}+u\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right)\right]+\left[\frac{1}{J}\eta_x\frac{\partial u}{\partial \eta}+u\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right]$$

$$+\left[\frac{1}{J}\xi_y\frac{\partial v}{\partial \xi}+v\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right)\right]+\left[\frac{1}{J}\eta_y\frac{\partial v}{\partial \eta}+v\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right]=0$$

Now Each bracket can be considered as one derivative

$$\frac{\partial}{\partial \xi}\left(\xi_x\frac{u}{J}\right)+\frac{\partial}{\partial \eta}\left(\eta_x\frac{u}{J}\right)+\frac{\partial}{\partial \xi}\left(\xi_y\frac{v}{J}\right)+\frac{\partial}{\partial \eta}\left(\eta_y\frac{v}{J}\right)=0$$

Rearranging:

$$\frac{\partial}{\partial \xi}\left(\xi_x\frac{u}{J}+\xi_y\frac{v}{J}\right)+\frac{\partial}{\partial \eta}\left(\eta_x\frac{u}{J}+\eta_y\frac{v}{J}\right)=0$$

Simplify:

$$\frac{\partial \bar{U}}{\partial \xi}+\frac{\partial V}{\partial \eta}=0$$

Where

$$\bar{U}=\frac{1}{J}(\xi_x u+\xi_y v)$$

$$\bar{V}=\frac{1}{J}(\eta_x u+\eta_y v)$$

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