11.9) In Hoffman Vol II

Consider the system of partial differential equations given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

Where ρ is constant

a) *Write the system in a vector form where the unknown vector Q :*

$$Q = \begin{bmatrix} u \\ v \\ p \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 0 & 0 \\ u & 0 & \frac{1}{\rho} \\ 0 & u & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u \\ v \\ p \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ v & 0 & 0 \\ 0 & v & \frac{1}{\rho} \end{bmatrix} \frac{\partial}{\partial y}\begin{bmatrix} u \\ v \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b) Recast the system of equations to a conservative form

Conservative Form System:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(u^2 + \frac{p}{\rho}\right) + \frac{\partial}{\partial y}(uv) = 0$$

$$\frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}\left(u^2 + \frac{p}{\rho}\right) = 0$$

Vector Form:

$$\frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\begin{bmatrix} u \\ u^2 + \frac{p}{\rho} \\ uv \end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} v \\ uv \\ v^2 + \frac{p}{\rho} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to get $[A]\&[B]$

we should put the system in the following form :

$$\frac{\partial E}{\partial Q}\frac{\partial Q}{\partial x}+\frac{\partial F}{\partial Q}\frac{\partial Q}{\partial y}=0$$

Where

$$[A]=\frac{\partial E}{\partial Q}\ ,\qquad [B]=\frac{\partial F}{\partial Q}\ ,\qquad [E]=\begin{bmatrix} u \\ u^2+\frac{p}{\rho} \\ uv \end{bmatrix},\qquad [F]=\begin{bmatrix} v \\ uv \\ v^2+\frac{p}{\rho} \end{bmatrix},\qquad [Q]=\begin{bmatrix} u \\ v \\ p \end{bmatrix}$$

$$\rightarrow [E]=\begin{bmatrix} Q_1 \\ Q_1^2+\frac{Q_3}{\rho} \\ Q_1Q_2 \end{bmatrix}$$

$$\rightarrow [F]=\begin{bmatrix} Q_2 \\ Q_1Q_2 \\ Q_2^2+\frac{Q_3}{\rho} \end{bmatrix}$$

$$[A]=\begin{bmatrix} \frac{\partial E_1}{\partial Q_1} & \frac{\partial E_1}{\partial Q_2} & \frac{\partial E_1}{\partial Q_3} \\ \frac{\partial E_2}{\partial Q_1} & \frac{\partial E_2}{\partial Q_2} & \frac{\partial E_2}{\partial Q_3} \\ \frac{\partial E_3}{\partial Q_1} & \frac{\partial E_3}{\partial Q_2} & \frac{\partial E_3}{\partial Q_3} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 2Q_1 & 0 & \frac{1}{\rho} \\ Q_2 & Q_1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 2u & 0 & \frac{1}{\rho} \\ v & u & 0 \end{bmatrix}$$

$$[B]=\begin{bmatrix} \frac{\partial F_1}{\partial Q_1} & \frac{\partial F_1}{\partial Q_2} & \frac{\partial F_1}{\partial Q_3} \\ \frac{\partial F_2}{\partial Q_1} & \frac{\partial F_2}{\partial Q_2} & \frac{\partial F_2}{\partial Q_3} \\ \frac{\partial F_3}{\partial Q_1} & \frac{\partial F_3}{\partial Q_2} & \frac{\partial F_3}{\partial Q_3} \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ Q_2 & Q_1 & 0 \\ 0 & 2Q_2 & \frac{1}{\rho} \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ v & u & 0 \\ 0 & 2v & \frac{1}{\rho} \end{bmatrix}$$

Conservative Form System:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2u & 0 & \frac{1}{\rho} \\ v & u & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u \\ v \\ p \end{bmatrix}+\begin{bmatrix} 0 & 1 & 0 \\ v & u & 0 \\ 0 & 2v & \frac{1}{\rho} \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} u \\ v \\ p \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

c) Transform the vector form obtained in (b) to (ξ, η) coordinates

$$\frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0$$

Substitute by:

$$\frac{\partial E}{\partial x} = \xi_x \frac{\partial E}{\partial \xi} + \eta_x \frac{\partial E}{\partial \eta}$$

$$\frac{\partial F}{\partial y} = \xi_y \frac{\partial F}{\partial \xi} + \eta_y \frac{\partial F}{\partial \eta}$$

We obtain:

$$\xi_x \frac{\partial E}{\partial \xi} + \eta_x \frac{\partial E}{\partial \eta} + \xi_y \frac{\partial F}{\partial \xi} + \eta_y \frac{\partial F}{\partial \eta} = 0$$

Divide by Jacobian $J = \frac{1}{x_\xi \, y_\eta - x_\eta \, y_\xi}$ and add & subtract some terms

$$\frac{1}{J}\xi_x \frac{\partial E}{\partial \xi} + \frac{1}{J}\eta_x \frac{\partial E}{\partial \eta} + \left[E\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right) - E\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right)\right] + \left[E\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right) - E\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right]$$
$$+\frac{1}{J}\xi_y \frac{\partial F}{\partial \xi} + \frac{1}{J}\eta_y \frac{\partial F}{\partial \eta} + \left[F\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right) - F\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right)\right] + \left[F\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right) - F\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right] = 0$$

The added terms would help us to obtain a conservative form

Rearrange the equation

$$\left[\frac{1}{J}\xi_x \frac{\partial E}{\partial \xi} + E\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right)\right] + \left[\frac{1}{J}\eta_x \frac{\partial E}{\partial \eta} + E\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right] - E\left[\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right]$$
$$+\left[\frac{1}{J}\xi_y \frac{\partial F}{\partial \xi} + F\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right)\right] + \left[\frac{1}{J}\eta_y \frac{\partial F}{\partial \eta} + F\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right] - F\left[\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right] = 0$$

Substitute by the following 4 relations in the red brackets:

$$\xi_x = J y_\eta$$

$$\eta_x = -J y_\xi$$

$$\xi_y = -J x_\eta$$

$$\eta_y = J x_\xi$$

$$\left[\frac{1}{J}\xi_x\frac{\partial E}{\partial \xi} + E\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right)\right] + \left[\frac{1}{J}\eta_x\frac{\partial E}{\partial \eta} + E\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right] - E\left[\frac{\partial}{\partial \xi}(y_\eta) + \frac{\partial}{\partial \eta}(-y_\xi)\right]$$

$$+\left[\frac{1}{J}\xi_y\frac{\partial F}{\partial \xi} + F\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right)\right] + \left[\frac{1}{J}\eta_y\frac{\partial F}{\partial \eta} + F\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right] - F\left[\frac{\partial}{\partial \xi}(-x_\eta) + \frac{\partial}{\partial \eta}(x_\xi)\right] = 0$$

$$\left[\frac{1}{J}\xi_x\frac{\partial E}{\partial \xi} + E\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right)\right] + \left[\frac{1}{J}\eta_x\frac{\partial E}{\partial \eta} + E\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right] - E\left(y_{\eta\xi} - y_{\eta\xi}\right)$$

$$+\left[\frac{1}{J}\xi_y\frac{\partial F}{\partial \xi} + F\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right)\right] + \left[\frac{1}{J}\eta_y\frac{\partial F}{\partial \eta} + F\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right] - F\left(-x_{\xi\eta} + x_{\xi\eta}\right) = 0$$

$$\left[\frac{1}{J}\xi_x\frac{\partial E}{\partial \xi} + E\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right)\right] + \left[\frac{1}{J}\eta_x\frac{\partial E}{\partial \eta} + E\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right]$$

$$+\left[\frac{1}{J}\xi_y\frac{\partial F}{\partial \xi} + F\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right)\right] + \left[\frac{1}{J}\eta_y\frac{\partial F}{\partial \eta} + F\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right] = 0$$

Now Each bracket can be considered as one derivative

$$\frac{\partial}{\partial \xi}\left(\xi_x\frac{E}{J}\right) + \frac{\partial}{\partial \eta}\left(\eta_x\frac{E}{J}\right) + \frac{\partial}{\partial \xi}\left(\xi_y\frac{F}{J}\right) + \frac{\partial}{\partial \eta}\left(\eta_y\frac{F}{J}\right) = 0$$

Rearranging:

$$\frac{\partial}{\partial \xi}\left(\xi_x\frac{E}{J} + \xi_y\frac{F}{J}\right) + \frac{\partial}{\partial \eta}\left(\eta_x\frac{E}{J} + \eta_y\frac{F}{J}\right) = 0$$

Simplify:

$$\frac{\partial \bar{E}}{\partial \xi} + \frac{\partial \bar{F}}{\partial \eta} = 0$$

Where

$$\bar{E} = \frac{1}{J}\left(\xi_x E + \xi_y F\right)$$

$$\bar{F} = \frac{1}{J}\left(\eta_x E + \eta_y F\right)$$

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