IE525 Final Project

Abstract: In this project, we consider four algorithms for time-discretization and Monte Carlo simulation of Heston stochastic volatility models. Tests on realistic model parameterizations are applied to compare the computational efficiency and robustness of the simulation schemes proposed in the project. We find that generalized Marsaglia method based on non-central chi-squared distribution has the most robust performance and a relatively low computational cost.

1 Introduction to Heston Model

The most standard way of option pricing is to use the Black-Scholes model. But one of the main drawbacks of the Black-Scholes model is the strong assumption that the stock returns' volatility is constant. The Heston model is an extension of the Black-Scholes model that tackles this volatility issue replacing the constant volatility with a stochastic process. This model was presented by Heston in 1993 and approaches the problem introducing an efficient closed form solution to compute vanilla options. The Heston model can be represented as below,

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t} S_t dW_{t,1} \quad (observed) \\
dV_t &= \kappa(\theta - V_t) + \beta\sqrt{V_t} S_t dW_{t,2} \quad (unobserved) \\
dW_{t,1} dW_{t,2} &= \rho dt
\end{aligned}
$$

The second equation, is the Cox-Ingersoll-Ross (CIR) model equation, a well-known short-rate model that describes the interest rate movements driven by one source of market risk. The first equation states the asset price process. r is the asset's rate of return, $dW_{t,1}$ and $dW_{t,2}$ are two correlated Brownian Motions with correlation coefficient of ρ. The parameters κ, θ and σ represent the speed of mean reversion, the long run mean variance and the volatility of the variance.

The Heston Model has analytical solution proposed by Heston[1] in 1993. We can use it as a benchmark to evaluate the performance of the numerical methods. The closed-form formula is shown as below.

[1] Heston, S.L. (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options," Review of Financial Studies, vol. 6, no. 2, pp. 327-343.

Heston Call Option Price

The solution of the PDE is given by

$$C(S_t, \sigma_t, t, T) = S_t P_1 - K e^{-r(T-t)} P_2$$

where

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} Re \left(\frac{e^{-i\phi \ln K} f_j(x, \sigma_t, T, \phi)}{i\phi} \right) d\phi$$

$$f_j(x, \sigma_t, T, \phi) = \exp\{c(T-t, \phi) + D(T-t, \phi)\sigma_t + i\phi x\}$$

$$c(T-t, \phi) = r\phi ir + \frac{\alpha}{\beta^2}\left((b_j - \rho\beta\phi i + d)(T-t) - 2\ln\left(\frac{1 - ge^{dr}}{1-g} \right) \right)$$

$$D(T-t, \phi) = \frac{b_j - \rho\beta\phi i + d}{\beta^2} \left(\frac{1 - e^{dr}}{1 - ge^{dr}} \right)$$

$$g = \frac{b_j - \rho\beta\phi i + d}{b_j - \rho\beta\phi i - d}, j = 1, 2$$

$$\alpha = \kappa\theta, \quad b_1 = \kappa + \lambda - \rho\sigma, \quad b_2 = \kappa - \lambda$$

2 Euler Scheme with Full Truncation Modification

The underlying asset is temporal dependent upon the solution of the SDE's volatility, so we need to simulate the volatility's path before the asset's.

We use S and V to denote the discrete-time approximations to the stock and variance stochastic process. We denote that $t_0 = 0 < t_1 < \cdot \cdot \cdot < t_N = T$ as partitions of a time interval of M equal segments of length Δt , A basic Euler scheme would be like

$$V_{t+1} = V_t + \kappa(\theta + V_t)\Delta t + \beta\sqrt{V_t} Z_V \sqrt{\Delta t}$$
$$S_{t+1} = S_t + rS_t + \sqrt{V_t} S_t Z_S \sqrt{\Delta t}$$
$$where \quad Z_S, Z_V \sim N(0,1); dZ_S dZ_V = \rho dt$$

To simulate the correlated Normal random variables we can first generate two independent Normal random variables Z_1 and Z_2. Then let Z_V= Z_1, and Z_S= $\rho Z_1 + \sqrt{1-\rho^2} Z_2$.

The basic scheme has an obvious problem, that is when the Variance Process V become negative, the calculation of $\sqrt{V_t}$ becomes impossible. There are several ways to tackle this problem. In this project, we employed the "Full Truncation Scheme" [2], to ensure that we can take square root on V.

Another weakness of the above scheme is that it has both some discretization error of the stock price and variance process. We can simply take log on BHS of stock price process to obtain log-Euler scheme[3].

The Modified Log-Euler Full Truncation scheme can be written on the form

$$
\begin{aligned}
&V_{t+1} = V_t + \kappa(\theta + V_t^+)\Delta t + \beta\sqrt{V_t^+} Z_V \sqrt{\Delta t} \\
&\ln S_{t+1} = \ln S_t + (r - 1/2V_t^+)\Delta t + \sqrt{V_t^+} Z_S \sqrt{\Delta t} \\
&where \quad Z_S, Z_V \sim N(0,1); dZ_S dZ_V = \rho dt; V_t^+ = \max(V_t, 0)
\end{aligned}
$$

3 Milstein Scheme with Full Truncation Modification

We can simply modify our variance discretization scheme from Euler to Milstein to give a better approximation of variance process V. The scheme can be written on the form

$$
\begin{aligned}
&V_{t+1} = V_t + \kappa(\theta + V_t^+)\Delta t + \beta\sqrt{V_t^+} Z_V \sqrt{\Delta t} + \frac{1}{4}\beta^2 (Z_v^2 - 1)\Delta t \\
&\ln S_{t+1} = \ln S_t + (r - \frac{1}{2}V_t^+)\Delta t + \sqrt{V_t^+} Z_S \sqrt{\Delta t} \\
&where \quad Z_S, Z_V \sim N(0,1); dZ_S dZ_V = \rho dt; V_t^+ = \max(V_t, 0)
\end{aligned}
$$

4 Kahl-Jackel Scheme

Suggested in Kahl and Jackel (2006)[4], a generic implicit Milstein scheme for the variance process V in combination with an alternative discretization called "IJK" for the stock price process S can be used to simulate the Heston Model. The IJK scheme include both the information of V_{t+1} and V_t. The scheme can be written on the form

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[2] Lord, R., R. Koekkoek and D. van Dijk (2006), "A Comparison of biased simulation schemes for stochastic volatility models," Working Paper, Tinbergen Institute.

[3] Lord, R., R. Koekkoek and D. van Dijk (2006), "A Comparison of biased simulation schemes for stochastic volatility models," Working Paper, Tinbergen Institute.

[4] Kahl, C. and P. Jackel (2005), "Fast strong approximation Monte-Carlo schemes for stochastic volatility models," Working Paper, ABN AMRO and University of Wuppertal.

$$V_{t+1} = \frac{V_t + \kappa\theta\Delta t + \beta\sqrt{V_t} Z_V \sqrt{\Delta t} + \frac{1}{4}\beta^2 (Z_v^2 - 1)\Delta t}{1 + \kappa\Delta t}$$

$$\ln S_{t+1} = \ln S_t + (r - \frac{V_t + V_{t+1}}{4})\Delta t + \rho\sqrt{V_t} Z_V \sqrt{\Delta t}$$

$$+ \frac{1}{2}(\sqrt{V_{t+1}} + \sqrt{V_t})(Z_S\sqrt{\Delta t} - \rho Z_V \sqrt{\Delta t}) + \frac{1}{4}(Z_V^2 - 1)\beta\rho\Delta t$$

$$where \quad Z_S, Z_V \sim N(0,1); dZ_S dZ_V = \rho dt; V_t^+ = \max(V_t, 0)$$

This discretization scheme will result in positive paths for the V process if $4\kappa\theta > \beta^2$, this restriction may not be always satisfied in practice, which may cause the simulation of variance process encountering negative values. Kahl and Jackel do not provide a solution for this problem, but it is reasonable to use a truncation scheme similar to log-Euler full truncation scheme.[5] That is, whenever the V drops below zero, we use

$$V_{t+1} = V_t + \kappa(\theta + V_t^+)\Delta t + \beta\sqrt{V_t^+} Z_V \sqrt{\Delta t}$$

and we need to use V_t^+ and V_{t+1}^+ rather than V_{t+1} and V_t in stock process S.

5 Generalized Marsaglia method

The methods above all used normal random variables to approximate the process. We can also use some other method based on the fact that the transition probability of a Cox–Ingersoll–Ross process can be represented by a non-central chi-square density.[6] This method is based on simulating the known transition probability density for the Cox–Ingersoll–Ross process. Denote that $\eta(\Delta t) = 4\kappa \exp(-\kappa\Delta t) / (\beta(1 - \exp(-\kappa\Delta t)))$ It have been proved that that we can set non-central parameter λ as $\eta(\Delta t)V_t$ and degree of freedom f as $4\kappa\theta / \beta^2$. Then we have that conditional on V_t, V_{t+1} is distributed as $\exp(\kappa\Delta t) / \eta(\Delta t)$ times a non-central chi-squared distribution with f degrees of freedom and non-centrality parameter λ. Thus, we can approximate our

[5] L. Andersen. Efficient simulation of the Heston stochastic volatility model. http://papers.ssrn.com/sol3/papers.cfm?abstract_id=946405, 2007.

[6] Simon Malham and Anke Wiese. Chi-square simulation of the CIR process and the Heston model. International journal of theoretical and applied finance, 16:1–38, 2013.

variance process V as

$$V_{t+1} = \chi^2 \exp(\kappa \Delta t) / \eta(\Delta t)$$
$$where \quad \chi^2 \sim \chi_f^2(\lambda)$$

Then based on our information of V_t and V_{t+1}, we can estimate the stock price process S as following. We denote that

$$K_1 = \Delta t(\kappa\rho / \beta - 1/2)/2 - \rho / \beta; K_2 = \Delta t(\kappa\rho / \beta - 1/2) + \beta / 2$$
$$K_3 = \Delta t(1-\rho^2)/2; s = K_2 + K_3 / 2; s^\star = s \exp(\kappa \Delta t) / \eta(h)$$
$$K_0 = -\frac{\lambda s^\star}{1-2s^\star} + (f/2)\ln(1-2s^\star) - (K_1 + K_3/2)V_t$$

The approximate price process is computed as follows

$$S_{t+1} = S_t \exp(K_0 + K_1 V_t + K_2 V_{t+1} + Z\sqrt{K_3(V_t + V_{t+1})})$$
$$where \quad Z \sim N(0,1)$$

6 Numerical Tests

To test our schemes, we turn to price European call options in the Heston model and compare it to the analytical result. Specifically, if we denote simulation times as N, we can write our estimated option price as

$$\hat{C}(0) = \exp(-rT)E[(S_T - K)^+] = \frac{1}{N}\sum_{i=1}^{N}(S_T^i - K)^+$$

Then we can denote the analytical option price as C(0), the error term e can be written as

$$e = C(0) - \hat{C}(0)$$

We can also define the standard error of Monte Carlo simulation as

$$\sigma_{MC} = \sigma(S_T^i - K)^+ / \sqrt{N}$$

If we use a sufficiently high number N of samples, we can keep the standard deviation σ_{MC} low and then obtain a high accuracy estimate for $\hat{C}(0)$. However, due to computational power limitation, in this project, we only choose N=100000 and time step Δt =1/32 for each scheme.

The next step is to set up the parameters for the specific test case, following the lead of

Anderson (2007)[7], we set up two different set of parameters. The first one is expected to be straightforward to handle numerically, which may be encountered in equity option markets. While the second is representative of the market for long-dated FX options and harder for numerical methods to compute. We use strike price K at K=60, 100, 140 and initial stock price S_0=100 for two sets. The other parameters of the two sets are listed as follows

Parameters	Set I	Set II
β	1	1
κ	1	0.5
ρ	-0.3	-0.9
T	5	10
θ, V_0	0.09	0.04
r	0.05	0

We would compare the price, standard deviation and CPU time for each sets of parameters.

6.1 Result for Set I

The average price and standard deviation of different schemes is listed as below. We can see that all methods work well in this sets of parameters. We can see that the Euler scheme performs slightly worse than the other schemes. Given the fact that Euler scheme is the simplest methods, it is reasonable that it would slightly underperform. But in general, all schemes have a relative stable estimate of the call option price.

K	60	100	140
Real	56.58233	33.60832	18.16897
Eul	56.73264	33.6089	18.35825
Mil	56.36025	33.76375	18.08684

[7] L. Andersen. Efficient simulation of the Heston stochastic volatility model. http://papers. ssrn.com/sol3/papers.cfm?abstract_id=946405, 2007.

KJ	56.47815	33.55623	18.0541
GM	56.56414	33.75816	17.85594

K	60	100	140
Eul	0.200624	0.179587	0.18911
Mil	0.195319	0.200858	0.153814
KJ	0.197996	0.183634	0.155655
GM	0.198272	0.180917	0.152832

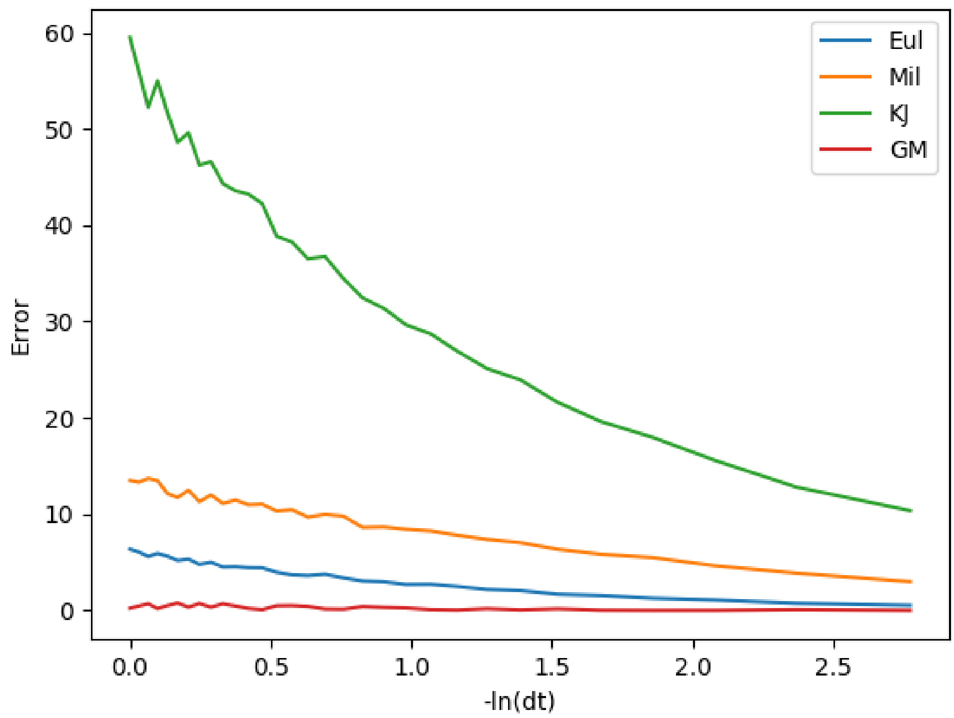
6.2 Result for Set II

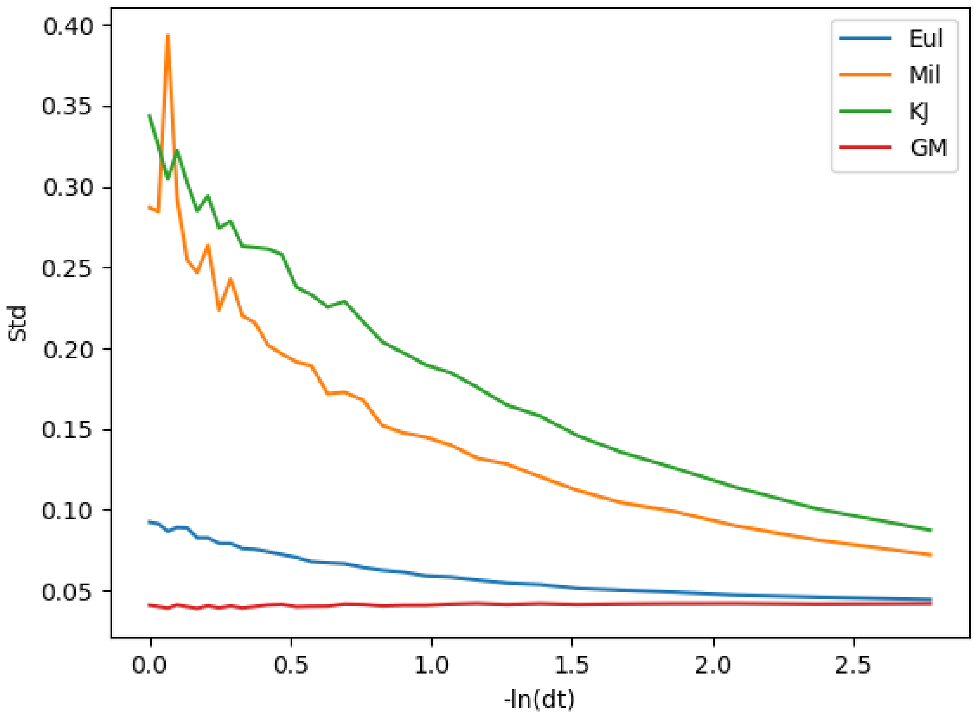
Set II is an extreme case presented in Anderson's paper (2007), because the mean-reversion parameter is reduced. The average price and the standard deviation of different schemes is listed as below. We can see that the Milstein and Kahl-Jackel scheme have a terrible performance in this extreme case. It is a surprising result, since Milstein Scheme is in theory giving a better approximation of the variance process than the Euler Scheme. However, the discretization bias can be magnified in approximating the stock price process S, and the magnification is not monotone. In this case, we can see that stock price process S can be approximated more stable with Euler Scheme than Milstein Scheme.

K	60	100	140
Real	44.33289	13.09114	0.296602
Eul	44.40438	13.35298	0.349432
Mil	45.42205	14.80711	1.937238
KJ	47.92047	19.80038	2.967954
GM	44.36995	13.13926	0.283919

K	60	100	140
Eul	0.080968	0.043164	0.008656
Mil	0.094601	0.060405	0.026146
KJ	0.111088	0.07	0.025507
GM	0.079174	0.042145	0.007814

Besides, given the fact that Kahl-Jackel performed even worse than the Milstein Scheme, we can consider that the unique "IJK" scheme of stock price S should be responsible for that. From the research of Anderson (2007), we know that "IJK" scheme is not robust. We can further explore the issue and try to see if the poor performance is related to time discretization, i.e. we change the time step Δt from 1 to 1/32 and plot the convergence with K=100 and standard error corresponding to it. The graph is shown as below.





From the graph above, we can find that the Kahl-Jackel has the lowest convergence with respect to time step Δt, meaning that the "IJK" scheme introduce huge bias in this problem. On the other hand, the Generalized Marsaglia has the best convergence speed and the lowest standard deviation, means that this method has the best performance among those four methods.

6.3 CPU Time Comparison

The computation time for each set are listed as below. The extreme case will cost us more time to compute. We can see Kahl-Jackel Scheme has the biggest computation cost and the worse performance in the extreme case. So, this method is not practical in application. The best method, Generalized Marsaglia, has the second the most computation cost. But given the fact that it has faster convergence speed and better accuracy, in practice we can set the time step larger and number of simulation fewer and can still yield a result with high accuracy. So we consider it not a huge problem for this method to be applicable.

Set II:

K	60	100	140
Eul	2.952827	2.912949	2.888711
Mil	4.323505	4.374727	4.353581
KJ	12.66932	12.35687	12.42906
GM	6.437124	6.398004	6.365934

Set I

K	60	100	140
Eul	1.505979	1.665431	1.436823
Mil	2.201859	2.411415	2.190861
KJ	6.281738	6.337277	6.233699
GM	3.684212	3.770026	3.59356

7 Conclusion

We have compared four different schemes of numerically solving Heston Model. Three methods, Euler, Milstein and Kahl-Jackel is based on generating normal random variables, while the Generalized Marsaglia method is based on non-central chi square random variables. Those four methods all work well in gentle sets of parameters. However, in extreme parameters, Milstein and Kahl-Jackel methods shows huge instability. On the other hand, Generalized Marsaglia has the fastest convergence speed and robustness. Though it is slightly computational costly, we still recommend it as the best way to simulate Heston Model.