Notes of Lecture3: Set Functions

Measure continouity

We start by a set of definitions.

DEFINITION Let $\mathcal{C} \subseteq \mathcal{S}(\Omega)$, where $\mathcal{S}(\Omega)$ denote a collection of subests of Ω, define a measure μ on $\mathcal{C}$ whose values are on $\mathbb{R}_+ + \{+\infty\}$.

- $\forall E \in \mathcal{C}$, μ is said to be continous form below at E, if $\forall (E_n)_{n\geq 1}, E_n \in \mathcal{C}, E_n \subseteq E_{n+1}, \bigcup_{n\geq 1} E_n = E, s.t. \lim_{n\to+\infty} \mu(E_n) = \mu(E)$, where the conditions satisfied by E_n is denoted by $E_n \uparrow E$.
- $\forall E \in \mathcal{C}$, μ is said to be continous form above at E, if $\forall (E_n)_{n\geq 1}, E_n \in \mathcal{C}, E_n \supseteq E_{n+1}, \exists \eta_0, \mu(E_{\eta_0}) < \infty, \bigcap_{n\geq 1} E_n = E, s.t. \lim_{n\to+\infty} \mu(E_n) = \mu(E)$, where the conditions satisfied by E_n is denoted by $E_n \downarrow E$.
- μ is said to be continous at E if and only if μ is both continous from above and below at E.

LEMMA Let a be an algebra and $a \subseteq \mathcal{S}(\Omega)$, measure μ is defined on a whose values are on $\mathbb{R}_+ + \{+\infty\}$ and assuming that μ is additive. We have the following conclusions:

- (1) μ is σ-additive $\Rightarrow \forall E \in a$, μ is continous at E.
- (2) μ is continous from below $\Rightarrow \mu$ is σ-additive.
- (3) μ is continous from above at $\varnothing$ and μ is finite $\Rightarrow \mu$ is σ-additive.

$PROOF$

- (1.1) We first prove that μ is σ-additive $\Rightarrow \mu$ is continous from below $\forall E \in a$. Suppose $E \in a$ and a sequence of sets $E_n \uparrow E$. Since E_n is increasing, we construct a new sequence $\{F_n\}$ based on $\{E_n\}$ where $F_n = E_n | E_{n-1}$ and $E_0 = \varnothing$. Hence, $F_n's$ are disjoint pairwise i.e. $F_i \cap F_j = \varnothing, i \neq j$. It is obvious that $\bigcup_{i\geq 1}^n F_i = \bigcup_{i\geq 1}^n E_i$ and $\sum_{i=1}^n F_i = E_n$. Since μ is σ additive, $\mu(E) = \sum \mu(E_i) = \sum \mu(F_i) = \lim_{n\to\infty} \sum_{i=1}^n \mu(F_i) = \lim_{n\to\infty} \mu(\sum_{i=1}^n F_i) = \lim_{n\to\infty} \mu(E_n)$. This proves that μ is continous from below at any $E \in a$.
- (1.2) Now we prove that μ is continous from above. Suppose $E \in a$ and a sequence of sets $E_n \downarrow E$ where $\exists n_0, s.t. \forall n > n_0, \mu(E_n) < \infty$. Considering the following set sequence G_n where $G_1 = E_{n_0} | E_{n_0+1}, ..., G_n = E_{n_0} | E_{n_0+n}$, E_n is decreasing thus G_n is increasing and $G_n \uparrow E_{n_0} | E$. Since all $E_n's$ are in the algebra a, then $G_n \in a$. By the conclusion we draw on (1), we have that $\lim_{n\to\infty} \mu(G_n) = \mu(E_{n_0} | E)$. Since $E \subseteq E_n$. Then

$$\mu(E_{n_0}) - \mu(E) = \mu(E_{n_0}|E) = \lim_{n\to\infty} \mu(G_n) = \lim_{n\to\infty} (\mu(E_{n_0}) - \mu(E_{n_0+n})) = \mu(E_{n_0}) - \lim_{n\to\infty} \mu(E_{n_0+n})$$

, which leads to $\mu(E) = \lim_{n\to\infty} \mu(E_{n_0+n})$. This proves that μ is continous from above. Thus we concludes the proof. *The statement here is not as useful as the statements below because we it is derived from μ is σ-*

additive. In reality, we usually don't have any idea if a measure μ is σ additive and instead we usually do the opposite: prove that a mearsure is σ-additive.

- (2) Assume that a measure μ is continous from below and $E = \sum_{k\geq 1} E_k, E, E_k \in a, E_i \cap E_j = \varnothing, i \neq j$. We know that $\forall n \geq 1, \sum_{k=1}^n E_k \subseteq E$, thus $\mu(\sum_{k=1}^n E_k) \leq \mu(E)$. Since μ is additive then we have $\sum_{k=1}^n \mu(E_k) \leq \mu(E)$. By the limit inequality, $lim_{n\to\infty} \sum_{k=1}^n \mu(E_k) = \sum_{k\geq 1} \mu(E_k) \leq \mu(E)$. Let $F_n = \sum_{k=1}^n E_k \in a$, thus $F_n \uparrow E$. Since μ is continous from below, thus

$$\lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \sum_{k=1}^n \mu(E_k) = \sum_{k\geq 1} \mu(E_k) = \mu(E).$$

  This proves that μ is σ additive.
- (3) Suppose that μ is continous from above at $\varnothing$ and $\mu(\Omega) < \infty$. Let $E, E_k \in a, E = \sum_{k\geq 1} E_k$, then $F_n = \sum_{k\geq n} E_k \in a$, thus $F_n = (E | \sum_{j=1}^{n-1} E_j)$ and it is obvious that $F_n \downarrow \varnothing$, we also have that $\mu(F_1) < \infty$. By the continouity above $\varnothing$, then $\mu(F_n) \to \mu(\varnothing) = 0$. Then $\mu(E) = \mu(\sum_{k=1}^n \cup \sum_{k\geq n} E_k) = \sum_{i=1}^n \mu E_k + \mu(F_{n+1})$. By taking the limit on both side of the equation, we hanve $\mu(E) = \mu(\sum_{k\geq 1} E_k) = \sum_{k\geq 1} \mu(E_k) + 0 = \sum_{k\geq 1} \mu(E_k)$, this proves that μ is σ-additive. Thus we conclude the lemma. $\square$

EXAMPLE Suppose $\Omega = (0,1)$ and $\mathcal{C}$ be the collection of intervals in which the intervals has the form of $(a,b], 0 \leq a < b < 1$. Define the measure μ to be as follow:

$$\mu((a,b]) = \begin{cases} b-a, \text{ if } a > 0 \\ +\infty, \text{ if } a = 0 \end{cases}.$$

Suppose $E_1 = (a_1, b_1] \in \mathcal{C}, E_2 = (a_2, b_2] \in \mathcal{C}, E_1 \cap E_2 = \varnothing$, where a_1, a_2 are not zero, then by definition of μ, we have $\mu(E_1 \cup E_2) = b_1 - a_1 + b_2 - a_2 = \mu((a_1, b_1]) + \mu((a_2, b_2])$, the same conclusion holds for a_1, a_2 are 0. Thus μ is additive. However, μ is *NOT* σ-additive. $\square$

Extension of measure

Theorem 1 Suppose $\mathcal{F} \subseteq \Omega$ be a semi-algebra and μ be a additive measure defined on $\mathcal{F}$. Now we show that there exists a extension measure ν, s.t. (1) ν is a measure defined on the algebra generated by $\mathcal{F}$, which is denoted by $a(\mathcal{F})$; (2) $\forall A \in \mathcal{F}, \nu(A) = \mu(A)$; (3) if μ_1, μ_2 are defined on $a(\mathcal{F})$, then $\forall A \in \mathcal{F}, \mu_1(A) = \mu_2(A)$ implies $\mu_1(E) = \mu_2(E), \forall E \in a(\mathcal{F})$

$PROOF$ Recall the important property of an algebra generated by a semi-algabra: $A \in a(\mathcal{F})$ implies that there exists a finite sequence $E_i, s.t. A = \sum_{i=1}^n E_i, E_i \in \mathcal{F}$. Then we propose

$$\nu(A) = \sum_{j=1}^n \mu(E_j).$$

Next we prove the following properties that ν has: (1) ν is well-define;(2) ν is additive;(3) the uniqueness of ν.

- (1) Assume that $A = \sum_{i=1}^m F_k$, then we want to show $\nu(A) = \sum_{j=1}^n \mu(E_j) = \sum_{k=1}^m \mu(F_k)$. We know that $E_j \subseteq A = \sum_{k=1}^m F_k$, thus $E_j = E_j \cap \sum_{k=1}^m F_k = \sum_{k=1}^m E_j \cap F_k$, by the property of semi-algebra the element $E_j \cap F_k \in \mathcal{F}$. By additivity of semi-algebra, we have

$$\mu(E_j) = \mu(\sum_{k=1}^{m} E_j \cap F_k)$$

. Therefore,$\nu(A) = \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{n} \sum_{k=1}^{m} \mu(E_j \cap F_k)$. The same argument holds for $\nu(A) = \sum_{k=1}^{m} \mu(F_k)$ and hence ν is well define.

- (2) We next prove the additivity of ν. Let $B = \sum_{k=1}^{m} F_k$ where $A \cap B = \varnothing, F_k \in \mathcal{F}$. Thus $A \cup B = \sum_{j=1}^{n} E_j + \sum_{k=1}^{m} F_k$. Thus, by definition of a semi-algebra, $\nu(A \cup B) = \sum_{j=1}^{n} \mu(E_j) + \sum_{k=1}^{m} \mu(F_k) = \nu(A) + \nu(B)$. This shows that ν is additive.
- (3) Now we show the uniqueness. We would like to show that for μ_1, μ_2 define on $a(\mathcal{F})$ and $\mu_1(A) = \mu_2(A), \forall A \in \mathcal{F}$ and μ_1, μ_2 are additive, $\mu_1(B) = \mu_2(B), \forall B \in a(\mathcal{F})$. Let $B \in a(\mathcal{F})$, then $B = \sum_{j=1}^{n} E_j, E_j \in \mathcal{F}$, then $\mu_1(B) = \sum_{j=1}^{n} \mu_1(E_j) = \sum_{j=1}^{n} \mu_2(E_j) = \mu_2(B)$. This proves the uniqueness. $\square$

We now show that if μ is σ-additive then ν is also σ-additive.

$PROOF$

Suppose $A = \sum_{k\geq 1} A_j$ where $A, A_j \in a(\mathcal{F})$, we would like to show that $\nu(A) = \sum_{k\geq 1} \nu(A_k)$. Since $A \in a(\mathcal{F})$, then $A = \sum_{j=1}^{n} E_j, E_j \in \mathcal{F}$. In addition, each $A_k \in a(\mathcal{F})$, thus $A_k = \sum_{l=1}^{n_k} E_{k,l}, E_{k,l} \in \mathcal{F}$. By definition, we have $\nu(A) = \sum_{j=1}^{n} \mu(E_j)$. Observe that $E_j = E_j \cap A = E_j \cap (\sum_{k\geq 1} A_k) = E_j \cap (\sum_{k\geq 1} \sum_{l=1}^{n_k} E_{k,l}) = \sum_{k\geq 1} \sum_{l=1}^{n_k} E_{k,l} \cap E_j$. Since μ is σ-additive, $\mu(E_j) = \sum_{k\geq 1} \sum_{l=1}^{n_k} \mu(E_{k,l} \cap E_j)$. Hence $\nu(A) = \sum_{j=1}^{n} \sum_{k\geq 1} \sum_{l=1}^{n_k} \mu(E_{k,l} \cap E_j)$. Moreover, $E_{k,l} = A \cap E_{k,l} = \sum_{j=1}^{n} E_{k,l} \cap E_j$, then $\mu(E_{k,l}) = \sum_{j=1}^{n} \mu(E_{k,l} \cap E_j)$. Recall that $\nu(A_k) = \sum_{l=1}^{n_k} \mu(E_{k,l})$, thus $\sum_{k\geq 1} \nu(A_k) = \sum_{k\geq 1} \sum_{l=1}^{n_k} \mu(E_{k,l}) = \sum_{k\geq 1} \sum_{l=1}^{n_k} \sum_{j=1}^{n} \mu(E_{k,l} \cap E_j) = \sum_{j=1}^{n} \sum_{k\geq 1} \sum_{l=1}^{n_k} \mu(E_{k,l} \cap E_j) = \nu(A)$. Therefore, $\nu(A) = \sum_{k\geq 1} \nu(A_k)$, this proves that the extension ν is also σ-additive. $\square$