Chapter 4: Vector Spaces

Definitions

Vectors: *has direction and magnitude*

$$\boldsymbol{v} = \vec{v} = (x_1, x_2, \cdots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R}$$

Real vector space: *a set of elements on which the two operations of vector addition and scalar multiplication are defined*

$$(V, +, \cdot)$$

Subspace: *a smaller nonempty set of vectors containing some of a larger vector space*

$$W \subseteq V : W \neq \emptyset$$

Linear combination: *a combination of different vectors by scalar multiplication*

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \sum a_i\vec{v}_i$$

Null space, solution set: *a subspace of the set of solutions that satisfy the following…*

$$W = \{x \in \mathbb{R}^n | A\vec{x} = \vec{0}\}$$

Span: *the set of vectors that make up the linear combination of the vectors in a vector space*

$$S = \text{span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\} \subseteq V$$

$$\sum x_i\vec{v}_n = \vec{s}$$

Linear dependence: *when a vector in a set can be written as a linear combination of the other vectors*

$$\sum a_i\vec{v}_i = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0}$$

$$detA = 0$$

Linear independence: *when the linear combination can only be written using the trivial solution*

$$a_1 = a_2 = \cdots = a_n = 0$$

$$detA \neq 0$$

Standard basis vector: *an n vector $\vec{e}_i$ where the ith entry is one and zeros everywhere else*

Properties of a real vector space

- Take vector space $(V, +, \cdot)$

I. V is closed under addition if $\vec{u}, \vec{v} \in V \Longrightarrow \vec{u} + \vec{v} \in V$

  a. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for all $\vec{u}, \vec{v} \in V$

  b. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ for all $\vec{u}, \vec{v}, \vec{w} \in V$

  c. There exists $\vec{0} \in V: \vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ for any $\vec{u} \in V$

  d. For each $\vec{u} \in V$, there exists $-\vec{u} \in V: \vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$

II. V is closed under scalar multiplication if $\vec{u} \in V, c \in \mathbb{R} \Longrightarrow c\vec{v} \in V$

  a. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ for any $\vec{u}, \vec{v} \in V$ and $c \in \mathbb{R}$

  b. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$ for any $\vec{u} \in V$ and $c, d \in \mathbb{R}$

  c. $c(d\vec{u}) = (cd)\vec{u}$ for any $\vec{u} \in V$ and $c, d \in \mathbb{R}$

  d. $(1)\vec{u} = \vec{u}$ for any $\vec{u} \in V$

Basis vectors and matrix characteristics

- If a vector space V has a basis of n vectors, then every linearly independent set of vectors in V has at most n vectors
- Take $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\}$ and $T = \{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_n\}$
  - If S, T are both bases for U, $m = n$
  - If S is a basis, but T is linearly independent $\Rightarrow m \geq n$
  - If S is linearly independent, but T is a basis $\Rightarrow m \leq n$
- The dimension of a vector space $\dim V$ is the number of basis vectors n
  - Ex) $\dim\{\mathbb{R}^n\} = n$
- Take n-dimensional vector space V and set a set of vectors $S: S \subseteq V$
  - If S has n vectors and is linearly independent → S is a basis for V
  - If S has n vectors and spans V → S is a basis for V
  - If S spans V → S contains a basis for V
- Basis for null space (solution space) of a matrix A
  (1) Compute $RREF(A)$
  (2) Set one parameter for each free variable → parameterized set
  (3) Form the solution vector into linear combination
  (4) The vectors in the linear combination represent the basis
  - $\dim[null(A)] = \#\, of\ free\ variables$

Column spaces

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$col(A) = span\,\{columns\ of\ A\} = \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \cdots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}$$

$$\text{rank}\,\{A\} = \dim\{col(A)\} = \#\text{ pivot colums}$$

$$\text{nullity}\,\{A\} = \dim\{\text{null}\,A\} = n - \text{rank}\,A$$

$$\text{rank}\,\{A\} + \text{nullity}\,\{A\} = n$$