Chapter 3: Linear Systems and Matrices

Definitions

Linear equation: *takes the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Linear system: *a set of linear equations; takes the form*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \quad (*)$$

Inconsistent: *when a linear system has no solution*

Consistent: *when a linear system has a solution*

Homogenous system: *solution vector is zero*

Equivalent: *when two linear systems have the same solutions*

Coefficient matrix: *matrix of coefficients from linear system*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Solution vector: *column matrix with all solutions from the linear system*

$$\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Augmented matrix: *combined coefficient matrix and solution vector*

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

Leading entry: *the first nonzero element in each nonzero row*

Gaussian elimination: *the process of putting a matrix in row echelon form*

Gauss-Jordan elimination: *process of putting a matrix in reduced row from*

Homogenous system: *a linear system such that each equation has a solution of zero*

Nonsingular: *a property of square matrices demonstrating invertibility*

Elementary Operations of Linear Systems & Matrices

I. Multiply an equation by a nonzero constant
II. Interchange/swap the row equations
III. Replace equations with added ones

- A homogenous system of m linear equations with n unknows always has a non-trivial solution of $n > m$

Methods of Matrices

Echelon form

I. Each row that consists entirely of zeros lies beneath every row that contains a nonzero element
II. Staircase rule; each row that contains a nonzero element, the first nonzero element lies strictly to the right of the first nonzero element in the preceding row

Reduced row form

I. Each leading entry is a one
II. Each leading entry is the only nonzero element in its leading column

- To solve, must already be in echelon form

Types of Linear Systems & Matrices

Homogenous systems

- m linear equations and n unknowns has a non-trivial solution if $n > m$
- Non trivial solutions → unknowns exceed the number of equations
- For $n \times n$ homogenous systems, if there are ∞ many solutions, then the reduced form must have a row of zeros

Identity matrix

$$I_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Triangular matrices

- Types of square matrices
- Upper triangular

$$a_{ij} = 0 \text{ for } i > j$$

- Lower triangular

$$a_{ij} = 0 \text{ for } i < j$$

- Determinant
  - Product of main diagonal

$$\det A = a_{11}a_{22} \cdots a_{nn}$$

Matrix Operations

Addition

- All matrices must have same size $m \times n$
- Add each ith and jth element

$$A + B = B + A$$

$$A + (B + C) = (A + B) + C$$

$$A + O = A \Rightarrow O = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

Scalar multiplication

$$C = rA$$

$$c_{ij} = ra_{ij}$$

Row and column vector multiplication

$$[a_1 \quad \cdots \quad a_n]\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum a_n b_n$$

General matrix multiplication

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

- If $A = m \times p, B = p \times n, AB = m \times n$ but BA does not exist
- If $A = n \times p, B = p \times n, AB = n \times n = BA$
- AB and BA may not always be the same size
- The following also work when appropriate sizes…

$$A(BC) = (AB)C$$
$$(A + B)C = AC + BC$$
$$C(A + B) = CA + CB$$

Matrix-vector products

$$A\vec{c} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

- Linear combination of matrix and vector of scalars
- Can also be used to define linear systems:

$$A\vec{x} = \vec{b} \Rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Matrix powers

$$A^p = AA \cdots A$$

$$A^0 = I_n$$

$$A^p A^q = A^{p+q}$$

$$(A^p)^q = A^{pq}$$

$$(A^p)^{-1} = (A^{-1})^p$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Minor and cofactor

- For 3×3 matrix A $= \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \ldots$
  - The minor of A is the matrix when deleting the i row and j column: $\text{minor}(A)_{ij}$

$$\text{minor}(\text{A})_{11} = \begin{bmatrix} e & f \\ h & i \end{bmatrix} \qquad (example)$$

$$\text{cofactor}(A) = (-1)^{i+j}\ \text{minor}(A)_{ij}$$

Transpose

- Obtained by interchanging the rows and columns of a matrix
- $m \times n$ becomes $n \times m$
- For square matrices $\det A = \det A^T$

Inverse of Matrices

General process

$$A^{-1} = \prod_i E_i : E_i = \text{elementry row operation}$$

$$[A|I_n] \xrightarrow{\text{row operations}} [I_n|A^{-1}]$$

Rules[1]

I. A must be an $n \times n$ matrix
II. A is nonsingular
III. $A\vec{x} = \vec{0}$ has the only trivial solution
IV. $A\vec{x} = \vec{b}$ has a unique solution, and is consistent for every $n \times 1$ matrix $\vec{b}$
V. A is the product of elementary matrices E
VI. $\det A \neq 0$

Determinants

General calculation

- For 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$…

$$|A| = \det A = ad - bc$$

- For all $n \times n$ matrices…

$$\det A = \sum_{j=1}^{n} a_{ij} \operatorname{cof}(A)_{ij} = \sum_{i=1}^{m} a_{ij} \operatorname{cof}(A)_{ij}$$

[1] Copied directly from notes

<u>Properties</u>

- If two rows/columns are equal $\Longrightarrow \det A = 0$
- If a row/column consists entirely of zeros $\Longrightarrow \det A = 0$

$$\det kA = k^n \det A$$

$$\det AB = \det A \det B$$

$$\det A^{-1} = \frac{1}{\det A}$$

$$\det A_{kr_i \to r_i} = k \det A$$

$$\det A_{r_i \leftrightarrow r_j} = -\det A : i \neq j$$

$$\det A_{kr_i + r_j \to r_j} = \det A : i \neq j$$