Chapter 5: Higher-Order Linear Differential Equations

Definitions

nth order nonhomogeneous linear equation: *takes the form*

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

nth order homogenous linear equation: *takes the form*

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

Wronskian: *a test for linear dependence based on the determinant of an $n \times n$ matrix of a function and its derivatives*

$$W(f_1, \cdots, f_n) = \det \begin{bmatrix} f_1 & \cdots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}$$

On the Wronskian of nth order linear equations

Take the homogenous nth order linear equation,

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

which has Wronskian $W = W(y_1,, \cdots y_n)$,

- $W = 0 \Longrightarrow y_1, \cdots, y_n$ are linearly dependent
- $W \neq 0 \Longrightarrow y_1, \cdots, y_n$ are linearly independent

Differential Equations and Solutions

General solution for homogenous equation (1) – principle of superposition

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n y = 0 \tag{1a}$$

$$y = c_1 y_1 + \cdots + c_n y_n \tag{1b}$$

General solution for nonhomogeneous equations (2)

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n y = f(x) \tag{2a}$$

$$y = y_c + y_p \tag{2bi}$$

$$y = (c_1 y_1 + \cdots + c_n y_n) + y_p \tag{2bii}$$

Characteristic equations (3)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{3ai}$$

$$a_n D_y^{(n)} + a_{n-1} D_y^{(n-1)} + \cdots + a_1 D_y + a_0 y = 0 \tag{3aii}$$

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0 \tag{3b}$$

$$y = c_1 e^{r_1 x} + \cdots + c_n e^{r_n x} \tag{3ci}$$

If $r = a \pm bi$

$$y = e^{ax}(c_1 \cos(bx) + c_2 \sin(bx)) \tag{3cii}$$

Variation of parameters (4)

$$y'' + P(x)y' + Q(x)y = f(x) \tag{4a}$$

Find y_c from associated homogenous equation of $(4a)$

$$y_c = c_1 y_1 + c_2 y_2 \tag{4b}$$

$$y_p = -y_1 \int \frac{y_2 f(x)}{W} dx + y_2 \int \frac{y_1 f(x)}{W} dx \tag{4c}$$

$$y = y_c + y_P \tag{4d}$$