Partial derivatives & applications

Gradient vector of function $f(x, y, z)$…

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$$

Directional derivative of $f(x, y)$ in direction of $\vec{v} = \langle a, b \rangle$…

$$D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{\|v\|} = \frac{a}{\sqrt{a^2 + b^2}} f_x + \frac{b}{\sqrt{a^2 + b^2}} f_y$$

Tangent planes…

$$\vec{n} = \langle f_x, f_y, -1 \rangle$$

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

Multivariable chain rule: given $z = f(u, v)$: $u(s, t), v(s, t)$…

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t}$$

$$\frac{dz}{ds} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s}$$

The point (x_0, y_0) is a critical point on the surface $z = f(x, y)$ if…

$$f_x(x_0, y_0) = 0 \textit{ AND } f_y(x_0, y_0) = 0 \text{ or undefined}$$

Discriminant (second derivative test) of $f(x, y)$ at (x_0, y_0)…

$$D(x_0, y_0) = \left[f_{xx}(x_0, y_0) f_{yy}(x_0, y_0)\right] - \left[f_{xy}(x_0, y_0)\right]^2$$

Second derivative test…

$D(x_0, y_0) > 0$	$f_{xx}(x_0, y_0) > 0$	local min
$D(x_0, y_0) > 0$	$f_{xx}(x_0, y_0) < 0$	local max
$D(x_0, y_0) < 0$		saddle point
$D(x_0, y_0) = 0$		inconclusive

Lagrange multipliers: find extrema of $f(x, y)$ given constraint $g(x, y) = k$…

$$\nabla f = \lambda \nabla g$$

$$= \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = k \end{cases}$$

GIVEN FOR EACH STATEMENT…

In $\mathbb{R}^2$…

$$R = \{(x, y) \mid x_1 \leq x \leq x_2, y_1 \leq y \leq y_2\}$$

$$\mathrm{C} = \{(\mathrm{r}, \theta) \mid \mathrm{r}_1 \leq \mathrm{r} \leq \mathrm{r}_2, \theta_1 \leq \theta \leq \theta_2\}$$

In $\mathbb{R}^3$…

$$R = \{(x, y, z) \mid x_1 \leq x \leq x_2, y_1 \leq y \leq y_2, z_1 \leq z \leq z_2\}$$

$$\mathrm{C} = \{(\mathrm{r}, \theta, \mathrm{z}) \mid \mathrm{r}_1 \leq \mathrm{r} \leq \mathrm{r}_2, \theta_1 \leq \theta \leq \theta_2, z_1 \leq z \leq z_2\}$$

$$S = \{(\rho, \phi, \theta) \mid \rho_1 \leq \rho \leq \rho_2, \phi_1 \leq \phi \leq \phi_2, \theta_1 \leq \theta \leq \theta_2\}: \phi = [-\pi, \pi], \theta = [0, 2\pi]$$

TRANSFORMATIONS BETWEEN COORDINATE SYSTEMS…

General…

$$x^2 + y^2 + z^2 = r^2 \text{ or } x^2 + y^2 + z^2 = \rho^2$$

$$\tan\theta = \frac{y}{x}$$

Cartesian to polar…

$$dx\,dy \to \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \to r\,dr\,d\theta$$

Cartesian to cylindrical

$$dz\,dx\,dy \to \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \to r\,dz\,dr\,d\theta$$

Cartesian to spherical…

$$dz\,dx\,dy \to \begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ r = \rho\sin\phi \end{cases} \to \rho^2\sin\phi\;d\rho\,d\phi\,d\theta$$

Integrals & applications in different coordinate systems

Double integral of $f(x, y)$ over region R (cartesian)…

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Average value of $f(x, y)$ over region R…

$$f_{avg} = \frac{1}{area\ of\ R} \iint_R f(x,y)\,dA$$

Area of region R…

$$area\ of\ R = \iint_R 1\,dA$$

Volume of region R with height $f(x, y)$…

$$volume\ of\ R = \iint_R f(x,y)\,dA$$

Double integrals in polar coordinates…

$$\iint_R f(x,y)\,dA = \iint_C f(r,\theta)\cdot r\,dr\,d\theta$$

Volume of region R…

$$volume\ of\ region\ R = \iiint_R 1\,dV$$

Triple integrals in cartesian over R (hypervolume)…

$$\iiint_R f(x,y,z)\,dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx = \int_{y_1}^{y_2}\int_{x_1}^{x_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dx\,dy$$

Integral in cylindrical coordinates over region C…

$$\iiint_R f(x,y,z)\,dV = \iiint_C f(r,\theta,z)\,r\,dz\,dr\,d\theta = \iiint_C f(r,\theta,z)\,r\,dz\,d\theta\,dr$$

Integral in spherical coordinates over region S…

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(\rho,\phi,\theta)\,(\rho^2\sin\phi)d\rho\,d\phi\,d\theta = \iiint_S f(\rho,\phi,\theta)\,(\rho^2\sin\phi)d\rho\,d\theta\,d\phi$$