Chapter 7: Linear Systems of Differential Equations

Definitions

First-order linear system: *a set of linear equations; takes the form*

$$\begin{aligned} x_1{}' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t) \\ &\vdots \\ x_n{}' &= p_{m1}(t)x_1 + p_{m2}(t)x_2 + \cdots + p_{mn}(t)x_n + f_n(t) \end{aligned} \qquad (*)$$

Homogenous first-order linear system: *all functions f are zero*

$$f_1(t) = \cdots = f_n(t) = 0$$

Complete: *property of eigenvalues with multiplicity k having k linearly independent eigenvectors*

Defective: *not complete; less than k linearly independent eigenvectors*

Defect: *d; the number of defective eigenvectors*

$$d = k - p$$

On the Wronskian of nth order linear equations

Take the n vector functions,

$$[\vec{x}_1 \quad \cdots \quad \vec{x}_n] = \begin{bmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & \ddots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{bmatrix}$$

which has Wronskian $W = W(\vec{x}_1, \cdots, \vec{x}_n)$,

- $W = 0 \Longrightarrow \vec{x}_1, \cdots, \vec{x}_n$ are linearly dependent
- $W \neq 0 \Longrightarrow \vec{x}_1, \cdots, \vec{x}_n$ are linearly independent

Differential Equations and Solutions

General form for linear systems (1)

$$\vec{x}'(t) = P(t)\vec{x} + \vec{f}(t) \tag{1i}$$

$$\begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} p_{11}(t) & \cdots & p_{m1}(t) \\ \vdots & \ddots & \vdots \\ p_{1n}(t) & \cdots & p_{mn}(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix} \tag{1ii}$$

General solution to homogenous equations (2)

$$\vec{x}'(t) = P(t)\vec{x} \tag{2a}$$

$$\vec{x}(t) = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n \tag{2b}$$

General solution to nonhomogenous systems (3)

$$\vec{x}' = P(t)\vec{x} + \vec{f}(t) \tag{3a}$$

$$\vec{x}' = \vec{x}_c + \vec{x}_p \tag{3bi}$$

$$\vec{x}' = (c_1\vec{x}_1 + \cdots + c_n\vec{x}_n) + \vec{x}_p \tag{3bii}$$

Eigenvalue method - general (4)

$$\vec{x}' = A\vec{x} \tag{4a}$$

$$\det(A - \lambda I_n) = 0 \tag{4b}$$

Find n linearly independent eigenvectors

$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda_1 t}, \cdots, \vec{x}_n(t) = \vec{v}_n e^{\lambda_n t} \tag{4c}$$

$$\vec{x} = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n \tag{4di}$$

$$\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t} \tag{4dii}$$

For complex eigenvalue $\lambda = a \pm bi$, find $\bar{\lambda}$; if needed expand using,

$$e^{\lambda t} = e^{(a+bi)t} = e^{at}e^{bti} = e^{at}(\cos(bt) + i\sin(bt)) \tag{4e}$$

Eigenvalue method – defective, $k = 2$ (5)

Find all defective eigenvalues,

$$\det(A - \lambda I_n) = 0 \tag{5a}$$

For each defective eigenvalue find a generalized eigenvector $\vec{v}_1$,

$$A - \lambda I_n = \vec{0} \tag{5b}$$

Show the $(A - \lambda I_n)^2 = 0$

Pick some $\vec{v}_2 \neq r\vec{v}_1$ where r is any number

$$(A - \lambda I_n)\vec{v}_2 = r\vec{v}_1 \tag{5c}$$

$$\vec{x}_1 = v_1 e^{\lambda t}, \vec{x}_2 = e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \tag{5d}$$

$$\vec{x} = c_1 v_1 e^{\lambda t} + c_2 e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \tag{5e}$$