Assignment 1

ELG3106 – Fall 2025

Section: A00

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Due Date: Sunday September 27^{th}, 2025

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Introduction:

The polarization of electromagnetic (EM) waves is an important discussion in the study of wave propagation, as it defines the orientation and temporal evolution of the electric field vector in space. For a uniform plane wave propagating in the $+z$-direction, the phasor representation of the electric field is expressed as

$$\mathbf{E}(z) = \hat{x}E_x(z) + \hat{y}E_y(z), \qquad (1.1)$$

with the field components given by

$$E_x(z) = E_{x0}e^{-jkz}, E_y(z) = E_{y0}e^{-jkz}, \qquad (1.2)$$

where,

$$E_{x0} = a_x e^{j\delta_x} \text{ and } E_{y0} = a_y e^{j\delta_y}. \qquad (1.3)$$

This allows us to create the corresponding instantaneous field

$$\boldsymbol{E}(z,t) = Re\{\tilde{\mathrm{E}}(z)e^{j\omega t}\} = \hat{\boldsymbol{x}}a_x \cos(\omega t - kz + \delta_x) + \hat{\boldsymbol{y}}a_y \cos(\omega t - kz + \delta_y) \qquad (1.4)$$

The relative phase difference, $\delta = \delta_y - \delta_x$, together with the amplitudes a_x and a_y, allows us to investigate the resulting polarization state.

The electric field vector traces out a polarization ellipse, whose shape, orientation, and handedness are composed of two angles: the rotation angle γ, which specifies the tilt of the ellipse with respect to the Cartesian axes (x,y), and the ellipticity angle χ, which indicates the axial ratio between the major and minor axis which determines whether the polarization is linear, circular, or elliptical. Left-handed polarization corresponds to $\delta > 0$ while right-handed polarization corresponds to $\delta < 0$. The relationships

$$\tan(2\gamma) = \tan(2\psi_0)\cos\delta \qquad (1.5)$$
$$\sin(2\chi) = \sin(2\psi_0)\sin\delta \qquad (1.6)$$

link the polarization angles (γ, χ) to the field parameters a_x, a_y, and δ, where,

$$\psi_0 = \tan^{-1}(\frac{a_y}{a_x}) \qquad (1.7)$$

In addition, a separate equation for $\psi_0$can be derived through the relationship found in *Equations (1.5)* and *(1.6)* which can be found in the *Appendix*.

$$\psi_0 = \frac{1}{2}\arccos\left(\sqrt{\frac{1 - sin^2(2\chi)}{1 + tan^2(2\gamma)}}\right) \qquad (1.8)$$

The objective of this assignment is to determine the field parameters a_x, a_y, and δ corresponding to 25 combinations of polarization states defined by combinations of (γ, χ) demonstrated in *Figure 1.* below. These values will be computed, tabulated, and verified through simulation with the Module 7.3 polarization visualization tool. The analysis will demonstrate how variations in amplitude and phase produce different polarization states and will provide practical insight into interpreting the geometry and handedness of the polarization ellipse.

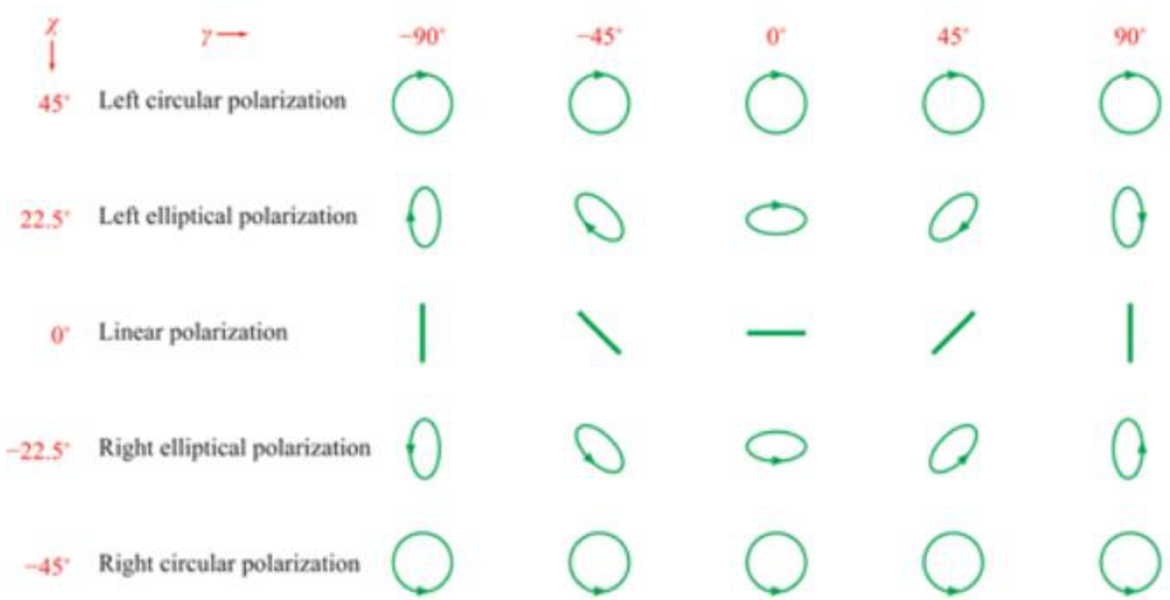

Figure 1: Polarization states for various combinations of the polarization angles (γ, χ) for a wave traveling out of the page.

Flow Diagram:

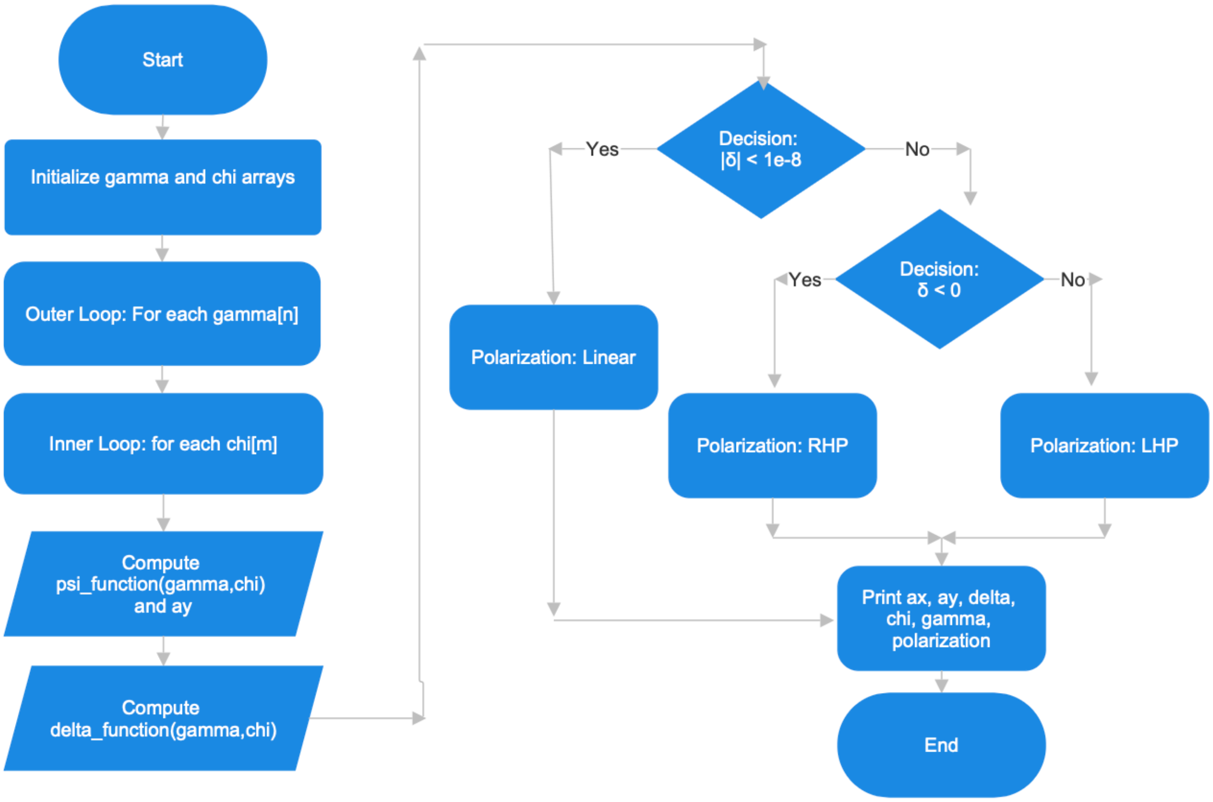

Figure 2: Flow diagram used to create code to tabulate values.

Code:

```python
import numpy as np

# Rotation and ellipticity angle defined
GAMMA_VALUES_DEG = [-90, -45, 0, 45, 90]
CHI_VALUES_DEG = [ 45,  22.5, 0, -22.5, -45]

1 usage
def calculate_polarization_parameters(gamma_deg: float, chi_deg: float) -> dict:

    # Radians for trig
    gamma = np.radians(gamma_deg)
    chi = np.radians(chi_deg)

    tiny = 1e-10  # trig values may result in small numbers that don't equal 0
    half_pi = np.pi / 2

    # Special cases first
    if abs(chi_deg) == 45:                # Circular → enforce ax = ay
        psi0 = np.pi / 4                  # ensures ax = ay = 1/sqrt(2) (normalized)
        delta = half_pi if chi_deg > 0 else -half_pi

    elif chi_deg == 0:                     # Linear
        psi0 = gamma
        delta = 0.0

    else:                                  # General elliptical
        tan2g = np.tan(2 * gamma)
        sin2c = np.sin(2 * chi)

        if abs(gamma_deg) == 90:           # tan(2γ) ~ 0
            delta = half_pi if chi_deg > 0 else -half_pi
            if abs(sin2c) > tiny:
                sin2p = np.clip(sin2c / np.sin(delta), -1.0,  a_max: 1.0)
                psi0  = 0.5 * np.arcsin(sin2p)
                if psi0 < 0: psi0 += half_pi
```

Figure 3: Code for simulation (1 of 3).

```python
                else:
                    psi0 = 0.0
            else:
                # tan^2(2ψ0) = tan^2(2γ) + sin^2(2x)
                tan2p = np.sqrt(tan2g**2 + sin2c**2)
                psi0  = 0.5 * np.arctan(tan2p)
                if psi0 < 0: psi0 += half_pi

                if abs(np.sin(2 * psi0)) > tiny:
                    sin_delta = np.clip(sin2c / np.sin(2 * psi0), -1.0, 1.0)
                    delta = np.arcsin(sin_delta)
                    if abs(tan2p) > tiny:
                        cos_delta_expect = tan2g / tan2p
                        if abs(cos_delta_expect) <= 1 and np.cos(delta) * cos_delta_expect < 0:
                            delta = (np.pi - delta) if delta > 0 else (-np.pi - delta)
                else:
                    delta = 0.0

    # Normalize psi0 to [0, π/2] for amplitudes
    psi0 = psi0 % (np.pi / 2)

    # Normalized amplitudes
    ax = np.cos(psi0)
    ay = np.sin(psi0)

    #Targeted fix: swap ax/ay for γ = ±90° and x = ±22.5° only
    if abs(gamma_deg) == 90 and abs(chi_deg) == 22.5:
        ax, ay = ay, ax

    # Handedness per sign of chi (consistent with sin δ sign)
    handedness = "Left" if chi_deg > 0 else ("Right" if chi_deg < 0 else "Linear")

    return {
        "gamma_deg": gamma_deg,
```

Figure 4: Code for simulation (2 of 3).

```python
        "chi_deg": chi_deg,
        "ax": float(ax),
        "ay": float(ay),
        "delta_deg": float(np.degrees(delta)),
        "handedness": handedness,
        "psi_0_deg": float(np.degrees(psi0)),
    }

1 usage
def wave_type(chi_deg: float) -> str:
    """Classify by chi."""
    if abs(chi_deg) == 45: return "Circular"
    if chi_deg == 0:       return "Linear"
    return "Elliptical"

if __name__ == "__main__":
    # Header
    print("gamma(deg), chi(deg), ax, ay, delta(deg), handedness, wave_type, psi_0(deg)")

    # All 25 combinations
    for chi in CHI_VALUES_DEG:
        for gamma in GAMMA_VALUES_DEG:
            r = calculate_polarization_parameters(gamma, chi)
            wt = wave_type(r["chi_deg"])
            print(f"{r['gamma_deg']:>4.0f}, "
                  f"{r['chi_deg']:>5.1f}, "
                  f"{r['ax']:>7.4f}, "
                  f"{r['ay']:>7.4f}, "
                  f"{r['delta_deg']:>9.2f}, "
                  f"{r['handedness']:<6}, "
                  f"{wt:<10}, "
                  f"{r['psi_0_deg']:>7.2f})")
```

Figure 5: Code for simulation (3 of 3).

Virtual Simulations:

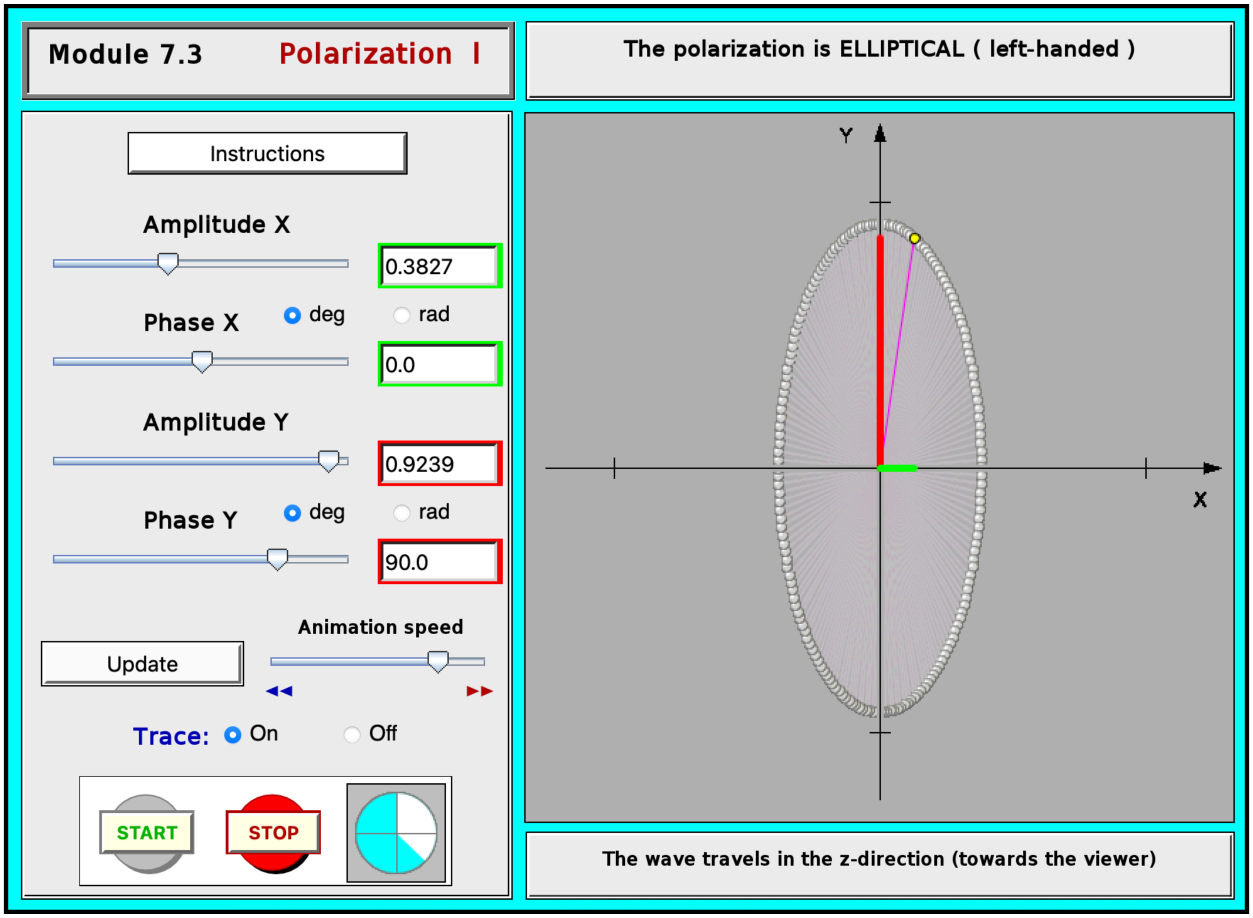

Figure 6: Virtual simulation result for $\gamma = -90^{\circ}$ $and\ \chi = 22.5^{\circ}$.

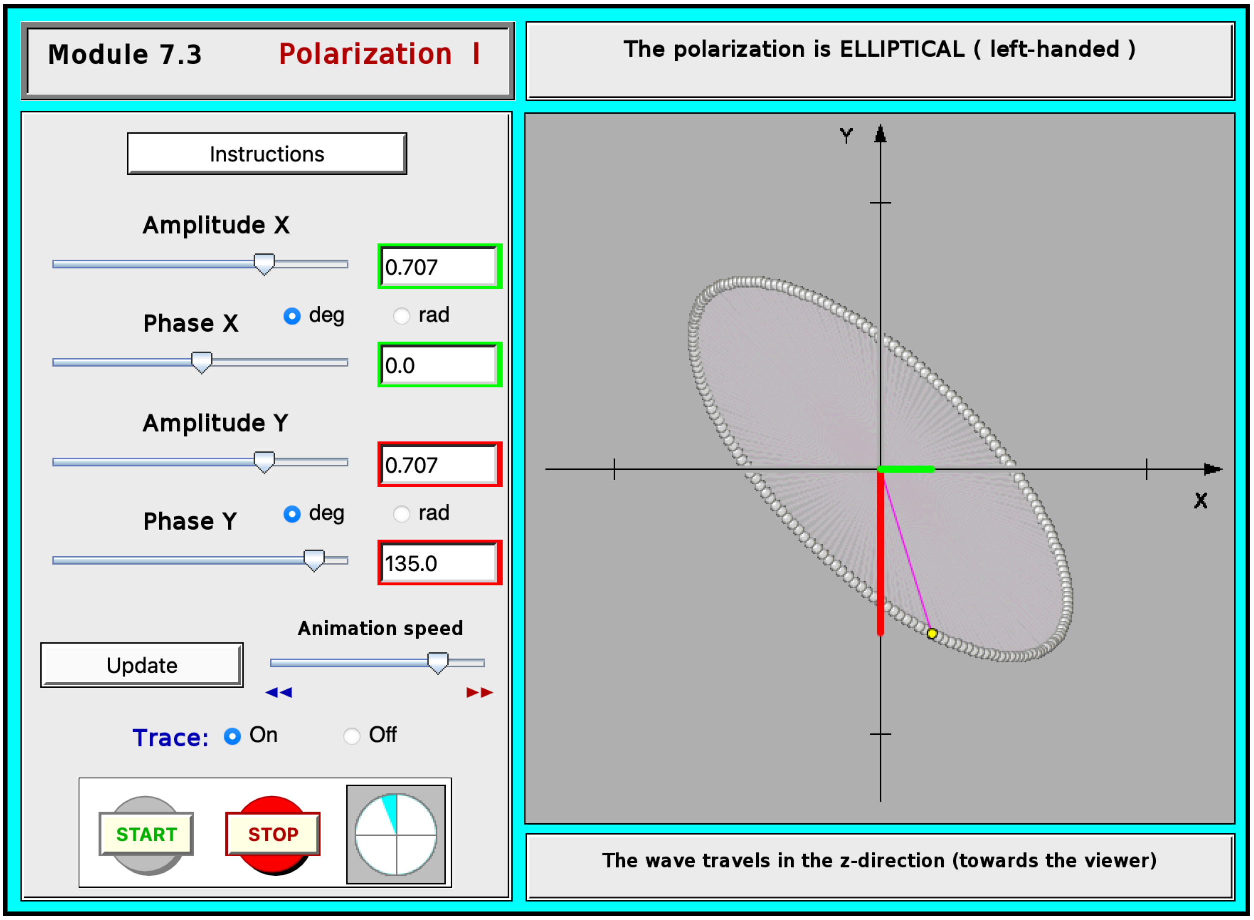

Figure 7: Virtual simulation result for $\gamma = -45° \; and \; \chi = 22.5°$.

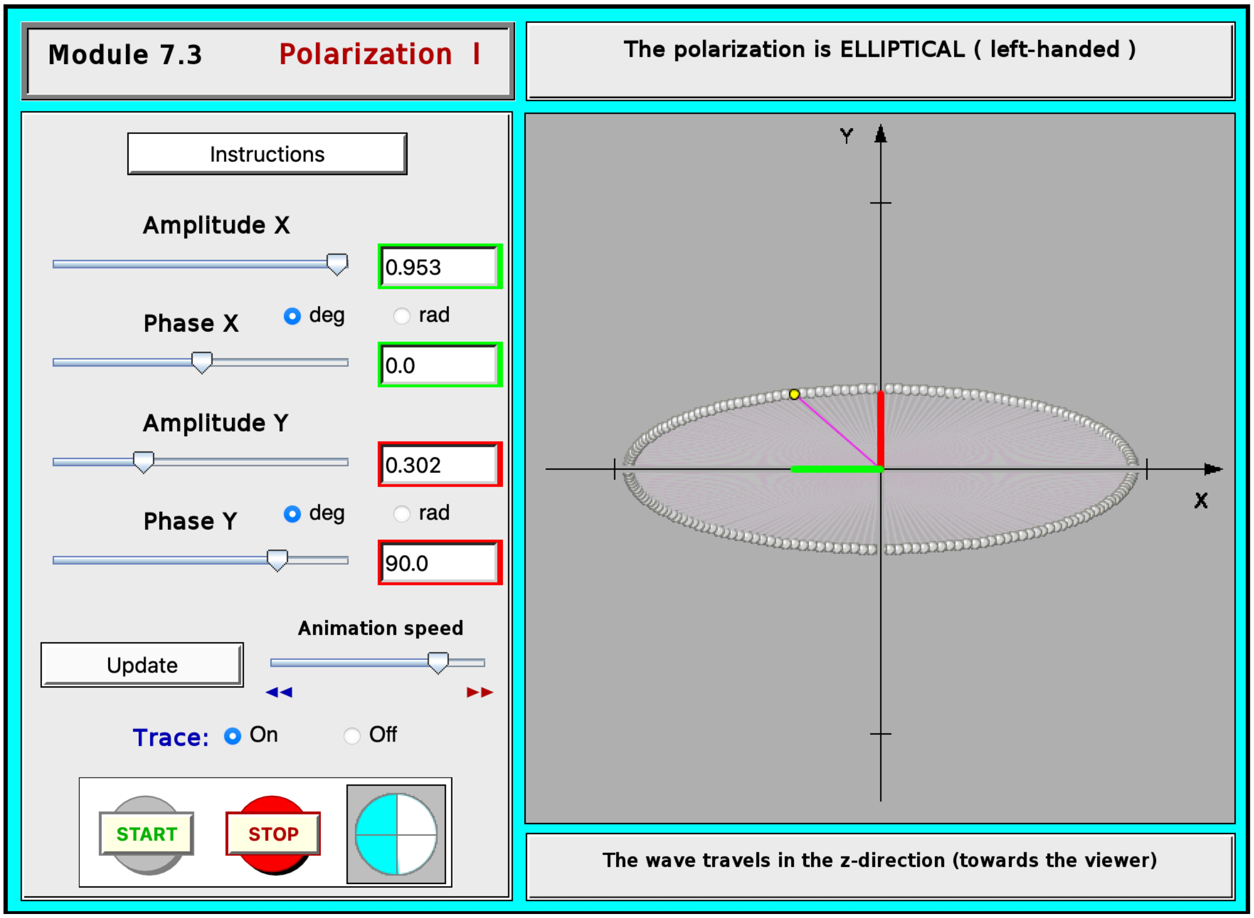

Figure 8: Virtual simulation result for $\gamma = 0° \ and\ \chi = 22.5°$.

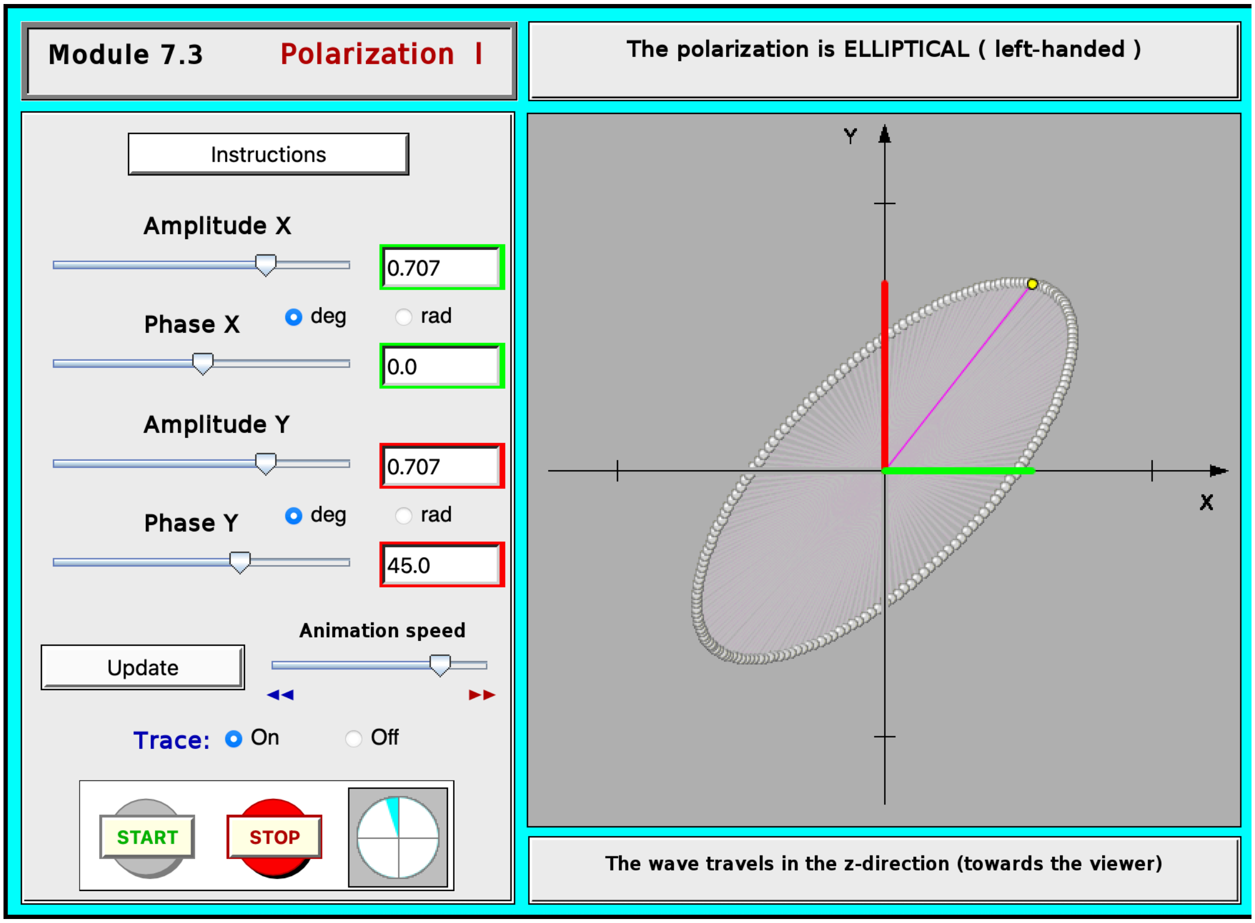

Figure 9: Virtual simulation result for $\gamma = 45°$ and $\chi = 22.5°$.

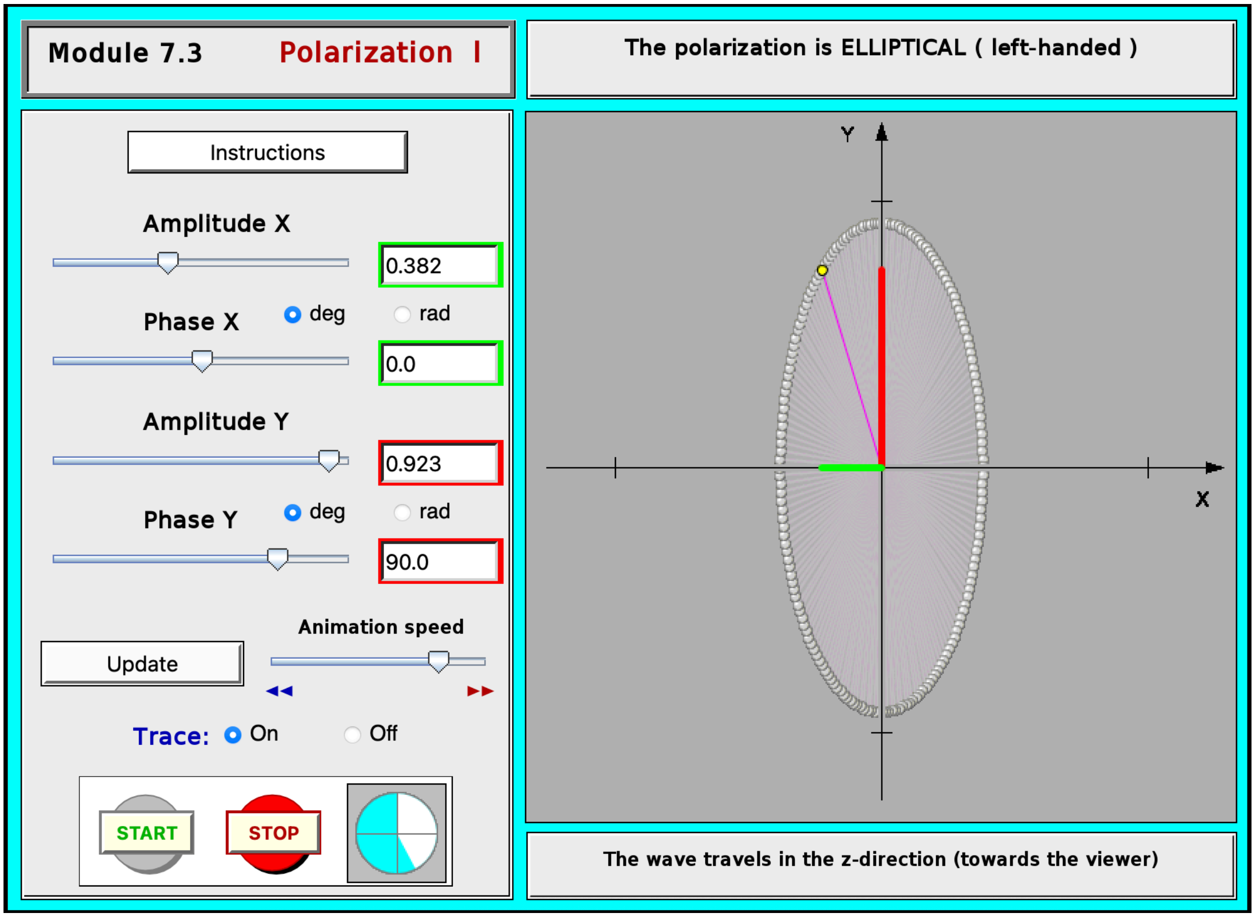

Figure 10: Virtual simulation result for $\gamma = 90°$ and $\chi = 22.5°$.

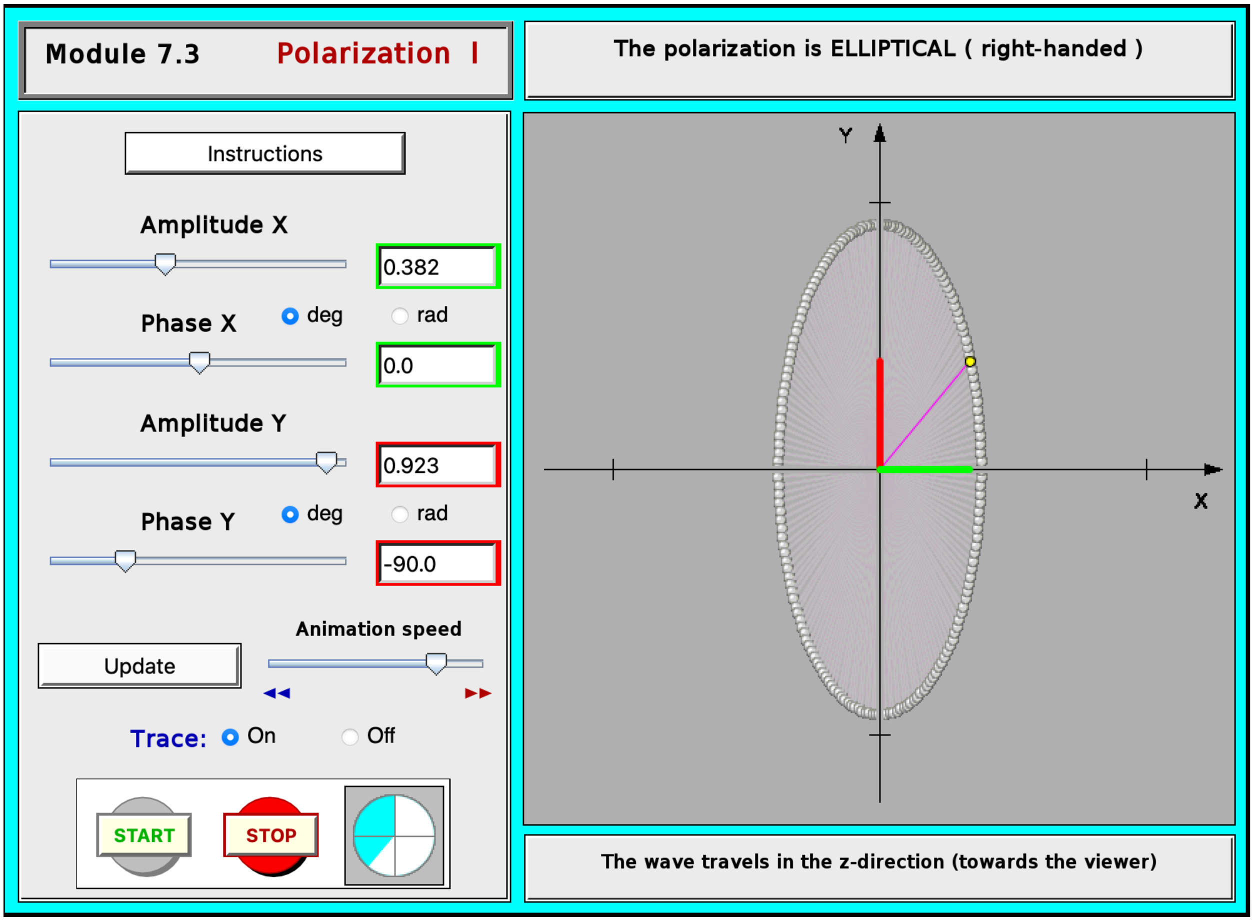

Figure 11: Virtual simulation result for $\gamma = -90° \ and\ \chi = -22.5°$.

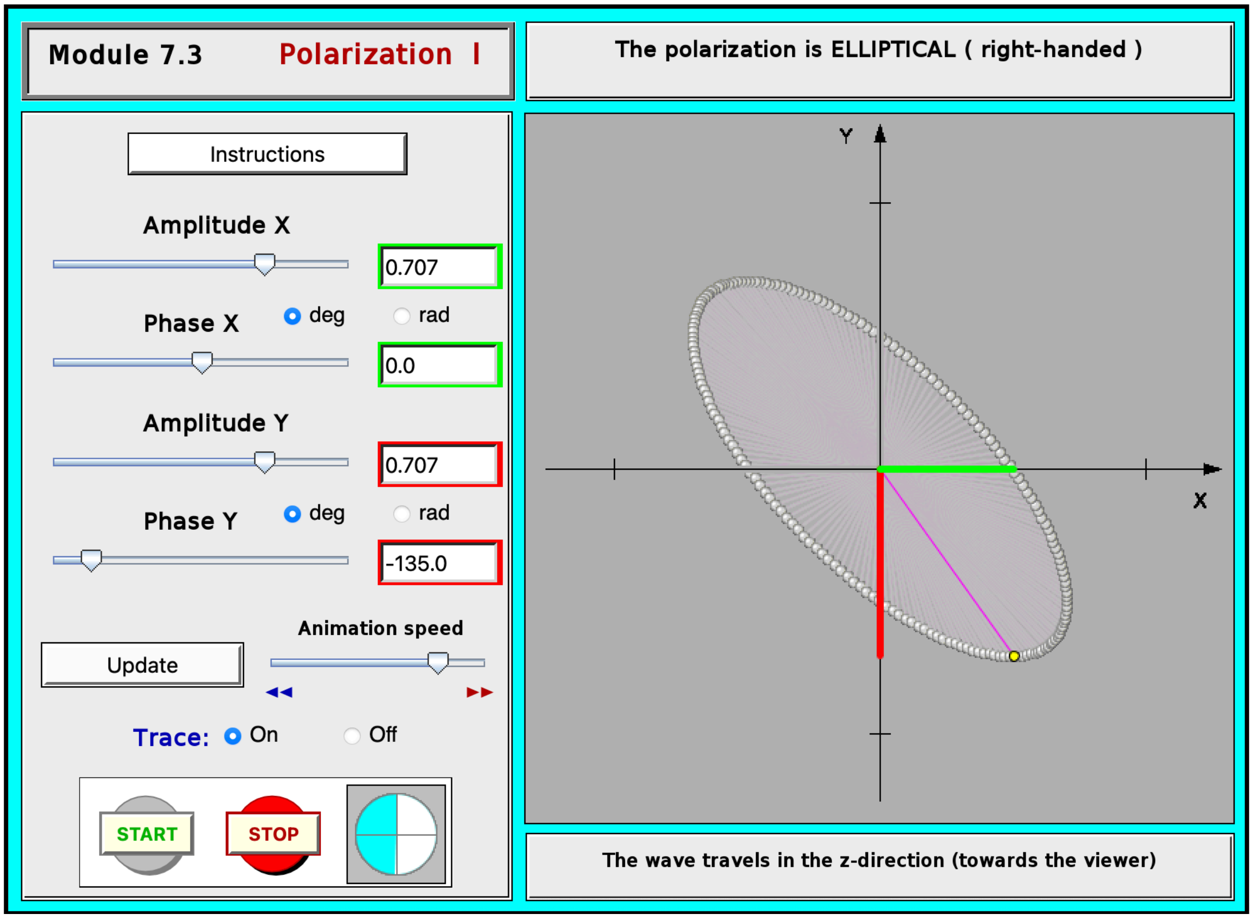

Figure 12: Virtual simulation result for $\gamma = -45° \ and \ \chi = -22.5°$.

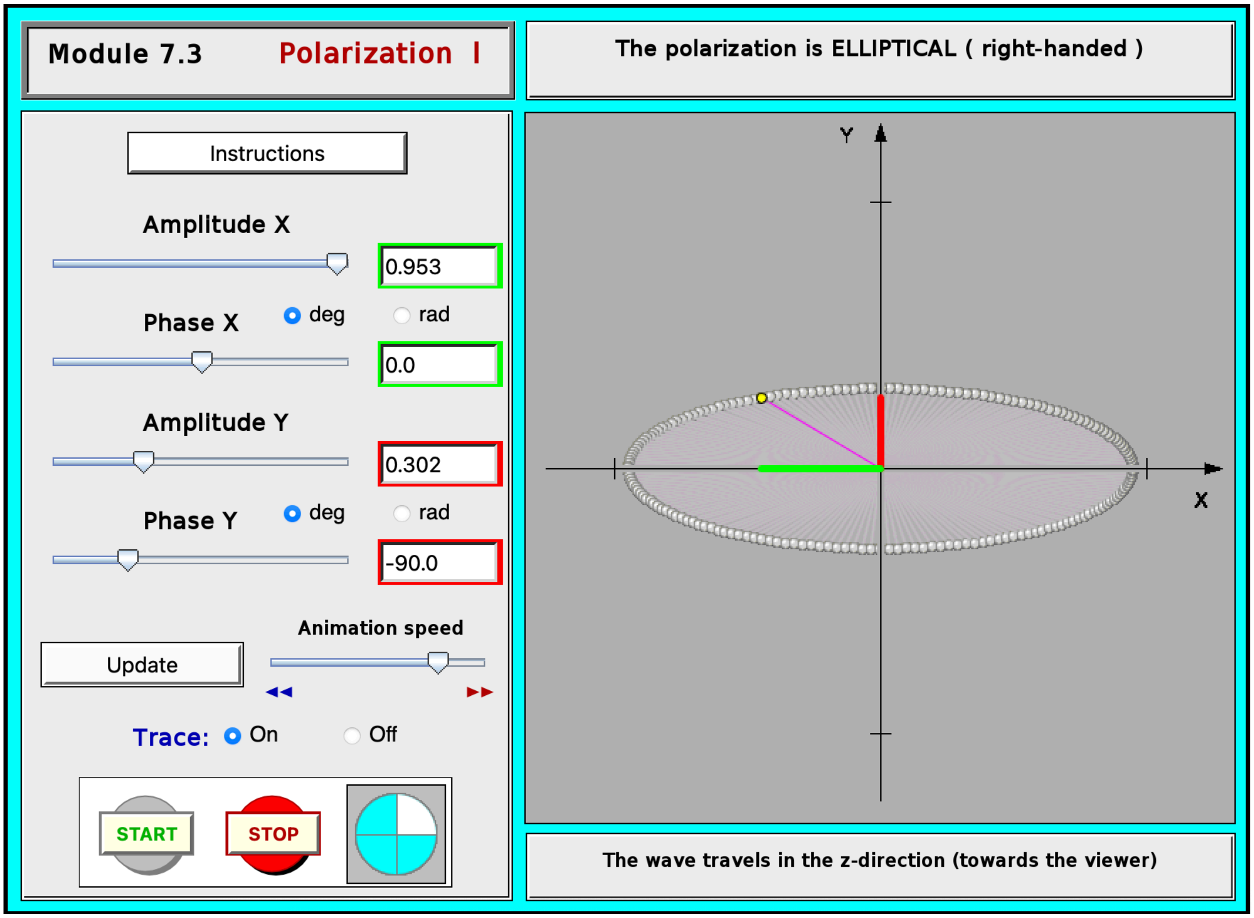

Figure 13: Virtual simulation result for $\gamma = 0°$ and $\chi = -22.5°$.

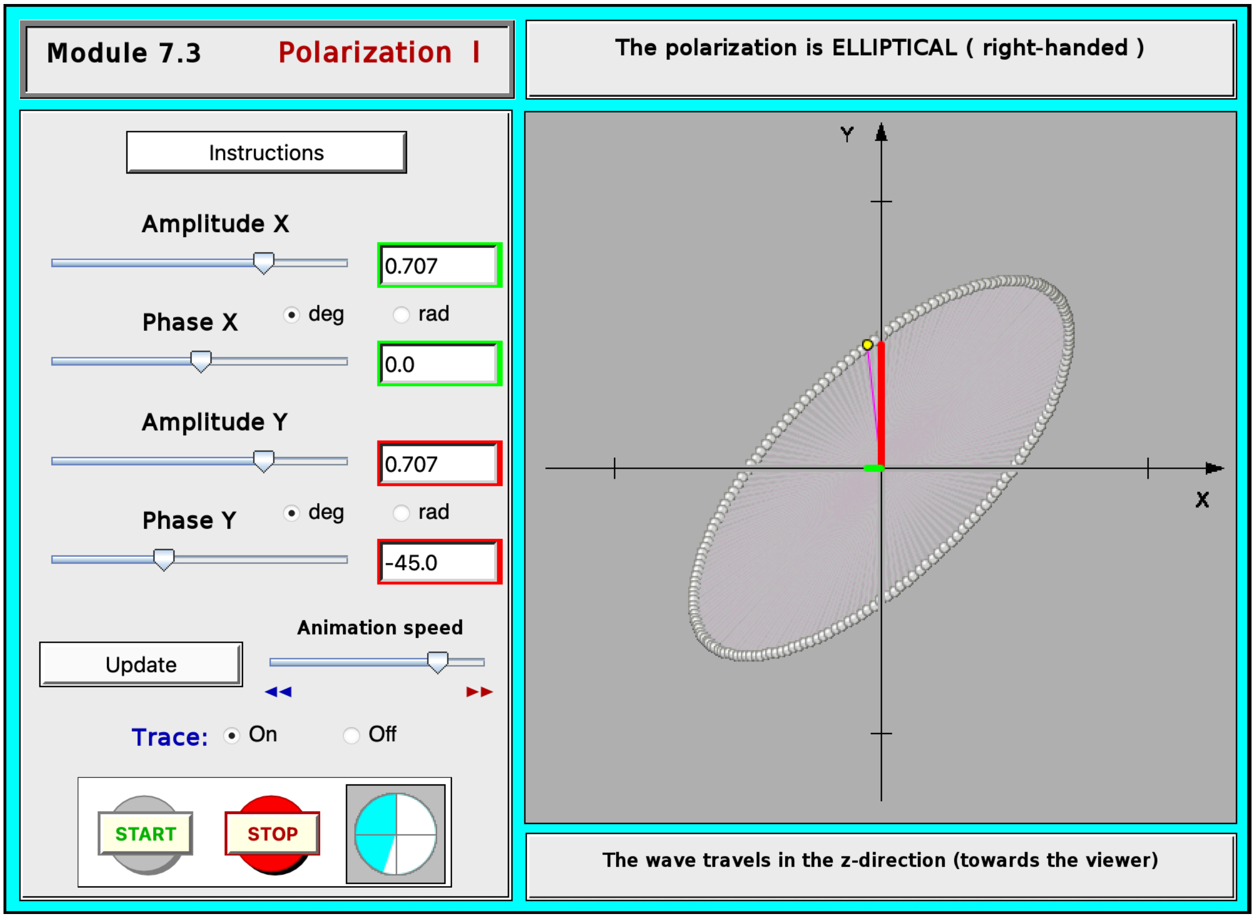

Figure 14: Virtual simulation result for $\gamma = 45° \ and\ \chi = -22.5°$.

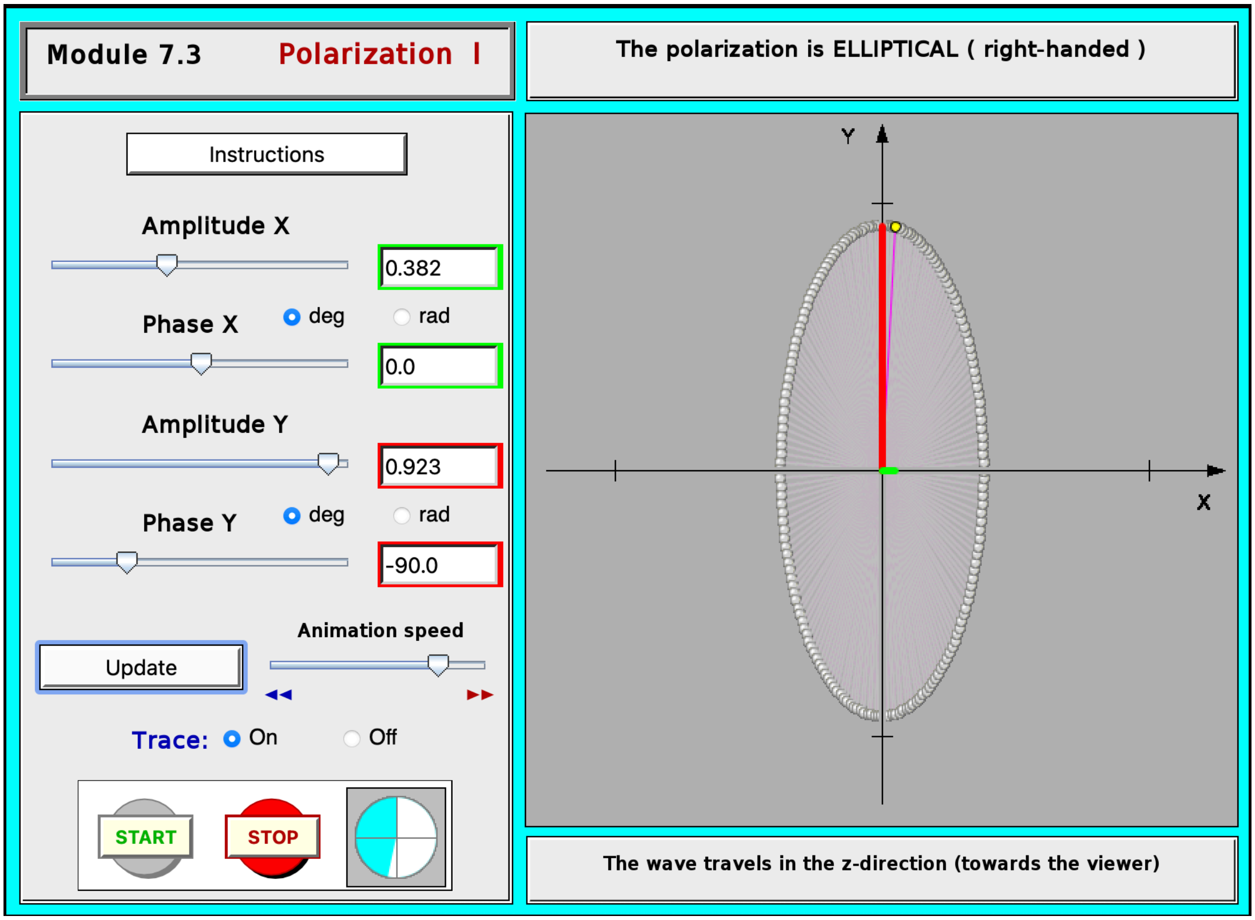

Figure 15: Virtual simulation result for $\gamma = 90°$ and $\chi = -22.5°$.

Results:

```
gamma(deg), chi(deg), ax, ay, delta(deg), handedness, wave_type, psi_0(deg)
 -90,  45.0,  0.7071,  0.7071,     90.00, Left  , Circular  ,   45.00)
 -45,  45.0,  0.7071,  0.7071,     90.00, Left  , Circular  ,   45.00)
   0,  45.0,  0.7071,  0.7071,     90.00, Left  , Circular  ,   45.00)
  45,  45.0,  0.7071,  0.7071,     90.00, Left  , Circular  ,   45.00)
  90,  45.0,  0.7071,  0.7071,     90.00, Left  , Circular  ,   45.00)
 -90,  22.5,  0.3827,  0.9239,     90.00, Left  , Elliptical,   22.50)
 -45,  22.5,  0.7071,  0.7071,    135.00, Left  , Elliptical,   45.00)
   0,  22.5,  0.9530,  0.3029,     90.00, Left  , Elliptical,   17.63)
  45,  22.5,  0.7071,  0.7071,     45.00, Left  , Elliptical,   45.00)
  90,  22.5,  0.3827,  0.9239,     90.00, Left  , Elliptical,   22.50)
 -90,   0.0,  1.0000,  0.0000,      0.00, Linear, Linear    ,    0.00)
 -45,   0.0,  0.7071,  0.7071,      0.00, Linear, Linear    ,   45.00)
   0,   0.0,  1.0000,  0.0000,      0.00, Linear, Linear    ,    0.00)
  45,   0.0,  0.7071,  0.7071,      0.00, Linear, Linear    ,   45.00)
  90,   0.0,  1.0000,  0.0000,      0.00, Linear, Linear    ,    0.00)
 -90, -22.5,  0.3827,  0.9239,    -90.00, Right , Elliptical,   22.50)
 -45, -22.5,  0.7071,  0.7071,   -135.00, Right , Elliptical,   45.00)
   0, -22.5,  0.9530,  0.3029,    -90.00, Right , Elliptical,   17.63)
  45, -22.5,  0.7071,  0.7071,    -45.00, Right , Elliptical,   45.00)
  90, -22.5,  0.3827,  0.9239,    -90.00, Right , Elliptical,   22.50)
 -90, -45.0,  0.7071,  0.7071,    -90.00, Right , Circular  ,   45.00)
 -45, -45.0,  0.7071,  0.7071,    -90.00, Right , Circular  ,   45.00)
   0, -45.0,  0.7071,  0.7071,    -90.00, Right , Circular  ,   45.00)
  45, -45.0,  0.7071,  0.7071,    -90.00, Right , Circular  ,   45.00)
  90, -45.0,  0.7071,  0.7071,    -90.00, Right , Circular  ,   45.00)
```

Figure 16: Tabulated results of all 25 combinations.

Discussion and Conclusion:

Tabulated Results:

- Effect of χ (3 distinct sections):
  - **Circular** ($|\chi| = 45°$): the phase difference is $|\delta| = 90°$ and the ellipse transforms into a circle. The handedness follows the sign of δ and χ:

    $$\delta \; and \; \chi > 0 \rightarrow \text{LHP}$$

    $$\delta \; and \; \chi < 0 \rightarrow \text{RHP}$$

    That physically requires equal amplitudes, so we pin ψ_0 to 45° to make $a_x = a_y$, while $\chi's$ sign still decides the rotation sense via $\delta = \pm 90°$.

  - **Linear** $\chi = 0°$: $\delta = 0°$ and neither field leads nor lags the other. The ellipse transforms into a line. The angle for γ sets the line's orientation but not handedness (there is none).

  - **Elliptical** $0° < |\chi| < 45°$: The angle δ receives values between (0° < |δ| < 90°). As $|\chi|$ increases toward 45°, the $|\delta|$ values in the table approach 90°, and the ax/ay ratio tends toward 1 (more circular). As $|\chi| \rightarrow 0°$, $|\delta| \rightarrow 0°$ and ax/ay becomes more inclined (begins taking shape of a linear function).

- Effect of γ:
  - The angle γ rotates the ellipse as seen in *Figure 17*. Changing the sign of γ flips the orientation by 90° in the 2γ relations, but it doesn't change the handedness (which is determined by the sign of $\sin(\delta)$. Rows with the same value of χ and opposite value of γ show the same |ax|, |ay|, $|\delta|$ but the ellipse angle changes.

- Symmetries (referenced with tabulated results):
  - For $\pm\chi$ with the same γ, δ changes sign but $|\delta|$ and amplitudes stay the same. This is why the left/right elliptical results are mirror images with opposite handedness.

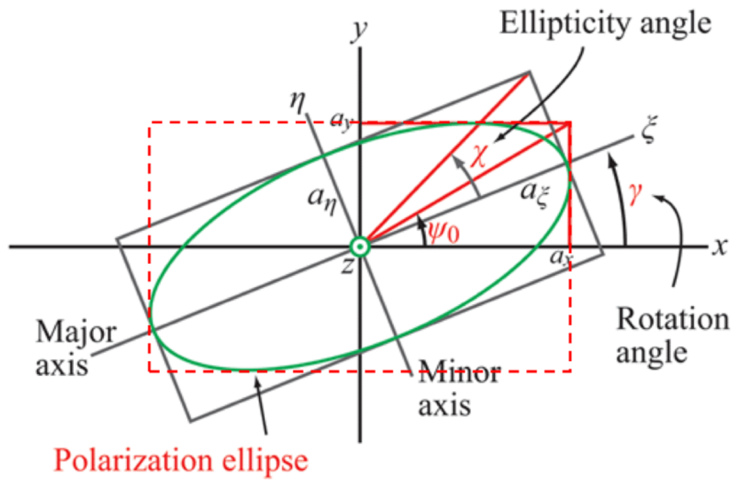

Figure 17: Polarization ellipse variable definitions.

Relationship between ax and ay:

We can use a normalized amplitude parameterization with the auxiliary angle ψ_0:

$$a_x = \cos\psi_0\,,\ a_y = \sin\psi_0\,, \tan\psi_0 = \frac{a_y}{a_x} \qquad (1.9)$$

We can observe that ψ_0 directly sets the amplitude ratio**.** When $\psi_0 = 45°$, $a_x = a_y = \frac{\sqrt{2}}{2}$ (circular limit). When ψ_0 is small, $a_x > a_y$ (the ellipse stretches along the x-axis). When ψ_0 is near 90°, $a_y > a_x$ (the ellipse stretches along the y-axis). The auxiliary angle is linked to the set angles (γ, χ) and the phase δ through relations stated in *Equations 1.5 and 1.6*.

Handedness and sign-tracking:

Handedness is determined by the sign of sin (δ) (stated below):

- $\sin(\delta) > 0$ → Left-hand (LHP)
- $\sin(\delta) < 0$ → Right-hand (RHP)
- $\sin(\delta) \approx 0$ → Linear

To keep signs and quadrants consistent we did the following in the code:

1. Compute both $\sin(\delta)$ and $\cos(\delta)$ from the two equations (to avoid relying on arcsin):

- $\sin(\delta) = \frac{\sin(2\chi)}{\sin(2\psi_0)}$ (when $\sin(2\psi_0)$ is not small)
- $\cos(\delta) = \frac{\tan(2\gamma)}{\tan(2\psi_0)}$ (when $\tan(2\psi_0)$ is not small)

2. Use $atan2(\sin(\delta)\cos(\delta))$ to recover δ in the correct quadrant**.** This prevents ambiguities with π that occur if you use only $arcsin$ or $arccos$.

3. Numerical safety: clip intermediate ratios to $[-1,1]$ before inverse trig, and treat near-zeros with a small value (e.g. 10^{-10}). Special cases are:

- $\chi = 0°$ (linear)**:** force $\delta = 0$.
- $|\chi| = 45°$ (circular): set $\delta = \pm 90°$ with the sign from χ.
- $\boldsymbol{\gamma = \pm 90°}$**:** $\tan(2\gamma) \to 0$, utilize thesin (2χ) relation to compute ψ_0, then set δ by sign convention.

4. Normalize ψ_0 to $[0, \frac{\pi}{2}]$ after solving, so $a_x = \cos\,\psi_0 \geq 0$ and $a_y = \sin\,\psi_0 \geq 0$. This keeps the amplitude signs positive to avoid "absorbing" what should be represented by δ.

5. The computation for the handedness is found last and uses the computed sign of $\sin(\delta)$ (or equivalently the sign of χ in the circular limit). This avoids the confusion of when the value of γ changes sign.

Challenges Encountered:

Many problems stemmed from the use of many different trigonometric functions at once causing multiple sign issues and principal angles to be misrepresented. In addition, I believed that *Equation 1.7* was simply a ratio which allowed me to set one of the amplitudes equal to 1 which led to many complications. This was resolved with parametrization which fixed my amplitude errors.

References

Schriemer, H. (2025). *Module 3: Plane wave propagation (ELG3106: Electromagnetic Engineering)* [Lecture slides] https://uottawa.brightspace.com/d2l/le/content/524143/viewContent/7037208/View.

Schriemer, H. (2025). *Assignment 1: Polarization of a uniform plane wave (ELG3106: Electromagnetic Engineering)* [Course assignment handout]. https://uottawa.brightspace.com/d2l/le/dropbox/524143/369112/DownloadAttachment?fid=22093107.

Appendix:

Derivation of ψ_0:

Equation 1:

$$\tan(2\gamma) = \tan(2\psi_0)\cos(\delta)$$

Equation 2:

$$\sin(2\chi) = \sin(2\psi_0)\sin(\delta)$$

Equation 3 (Re-arranged Eq. 1):

$$\delta = \cos^{-1}(\frac{\tan(2\gamma)}{\tan(2\psi_0)})$$

Equation 3 → 2:

$$\sin(2\chi) = \sin(2\psi_0)\sin(\cos^{-1}(\frac{\tan(2\gamma)}{\tan(2\psi_0)}))$$

Note:

$$\sin(\cos^{-1} x) = \sqrt{1 - x^2}$$

Therefore:

$$\sin(2\chi) = \sin(2\psi_0)\sqrt{1 - (\frac{\tan(2\gamma)}{\tan(2\psi_0)})^2}$$

$$\sin^2(2\chi) = \sin^2(2\psi_0)(1 - (\frac{\tan(2\gamma)}{\tan(2\psi_0)})^2)$$

Note:

$$\sin^2(x) = 1 - \cos^2(x)$$

$$\frac{\sin^2(x)}{\tan^2(x)} = \cos^2(x)$$

Therefore:

$$\sin^2(2\chi) = 1 - \cos^2(2\psi_0) - \cos^2(2\psi_0)\tan^2(2\gamma)$$

$$\frac{\sin^2(2\chi) - 1}{1 + \tan^2(2\gamma)} = -\cos^2(2\psi_0)$$

$$\psi_0 = \frac{1}{2}\arccos\left(\sqrt{\frac{1-\sin^2(2\chi)}{1+\tan^2(2\gamma)}}\right) = \frac{1}{2}\arcsin\,(\sqrt{\sin^2(2\chi)\cos^2(2\gamma)}$$