Gamblers Ruin Problem.

This problem was challenging for me to understand. Hopefully this work here will make it easier for whoever reads it. I've taken time to try to flush it out with all the details.

You have two gamblers, A and B. Each have a certain amount of dollar bills.

- On each play, the probability that gambler A wins $1 from gambler B is " p ".
- On each play, the probability that gambler B wins $1 from gambler A is " $(1-p)$ ".
- The total dollars to be won are " k ".
- Initially, gambler A has " j " dollars and gambler B has " $(k - j)$ " dollars.
- The gamblers play over and over until one player has all the money.

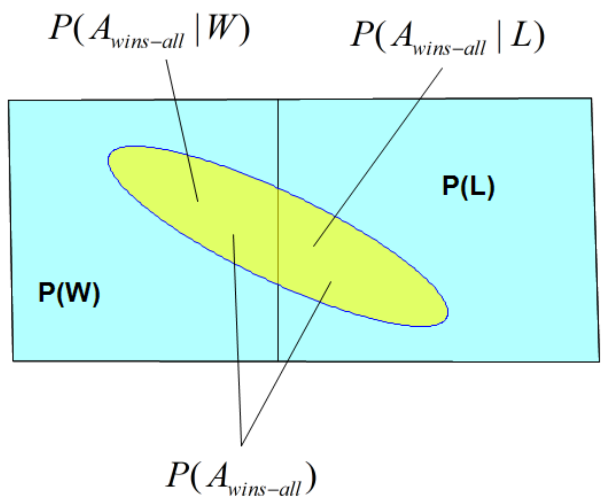
What is the chance that Gambler A will get all the dollars before losing?

Lets break it down:

- Let " W " represent the event that gambler A wins on a try, and " L " the event gambler A loses on a try.
- Gambler A starts with " j " dollars, where "$j = 1, 2,...k\text{-}1$" dollars.
- If Gambler A has 0 dollars (A_0), then the $P(W \mid A_0) = 0$
  - Read: "The probability of a win given Gambler A has zero dollars is zero".
  - Why?....because there is no money and A has already lost.
- If Gambler A has " k " dollars (A_k), then the $P(W \mid A_k) = 1$
  - Read: "The probability of a win given Gambler A has all the dollars is 1".
  - Why?...because A has all the money and has already won, so the probability of winning is 100%.
- **These are called the "End Cases":**
  - $P(W \mid A_0) = 0$
  - $P(W \mid A_k) = 1$

- Look at this diagram (The yellow is the probably of player 'A' winning it all).
  - W = "Win" and $P(W)$ is the probability of winning on a round.
  - L = "Loss" and $P(L)$ is the probability of losing on a round.

- $P\left(A_{wins\ all}\right)$ = the chance that "A" wins it all.
- $P\left(A_{wins\ all} \mid W\right)$ = the probability "A" wins all given a win on a round.
- $P\left(A_{wins\ all} \mid L\right)$ = probability "A" wins all given a loss on a round.



This leads us to our first important equation:

- $P\left(A_{wins\ all}\right) = P\left(A_{wins\ all} \mid W\right)P(W) + P\left(A_{wins\ all} \mid L\right)P(L)$ (1)

This example here uses the **<u>Total Probability Theorem.</u>**

Let X_j be shorthand for $P\left(X_j\right)$, the event that Gambler A wins it all with " j " dollars (0 < j < k). Also, let " p " be the probability of a win or loss on a round.

Then for round 1, we can rewrite (1) as follows:

- $X_1 = p\left(X_1\right) + (1-p)\left(X_1\right)$ (1a)

If we look at round 2, we can rewrite (1) as follows:

- $X_2 = p\left(X_2\right) + (1-p)\left(X_2\right)$ (1b)

These equations can be generalize as follows:

- $X_j = p\left(X_j\right) + (1-p)\left(X_j\right)$ (*)

This is an important equation which will be used below.

I'm going to transform (*) to use some important ideas. Assume player 'A' wins at X_j dollars. Then that allows us to modify (*) as follows:

- $X_j = p(X_{j+1}) + (1-p)(X_{j-1})$ (**)

You see, if A wins, then A has j + 1 dollars and won the first round. Furthermore, A has " j – 1 " dollars in the lose category. **This is also ad critical equation. It lets us devise a plan to solve.**

Look at (*) and (**). Those will get intertwined below.

<u>End Cases:</u> If A has 1 dollar, then (*) becomes:

- $X_1 = p(X_2)$ (E1)
  - Because the $(1-p)(X_0) = 0$. A has no money with X_0 .
- $X_{k-1} = p + (1-p)(X_{k-2})$ (E2)
  - Because the $p(X_k) = p(1) = p$ … A has all the money with X_k .

Here is where we start to figure things out for the final equation:

Look at (**).

- $X_j = p(X_{j+1}) + (1-p)(X_{j-1})$ (**)

I'm going to use (**) over and over and write things from $j = 1 \ldots k - 1$while using the end cases:

- $X_1 = p(X_2)$ (E1)
- $X_2 = p(X_3) + (1-p)(X_1)$ (2)
- $X_3 = p(X_4) + (1-p)(X_2)$ (3)
- …
- $X_{k-2} = p(X_{k-1}) + (1-p)(X_{k-3})$ (4)

- $X_{k-1} = p + (1-p)(X_{k-2})$ (E2)

I have equations for X_1 through X_{k-1}.

Look at (*) and (E1):

- $X_1 = p(X_1) + (1-p)(X_1)$ (*)
- $X_1 = p(X_2)$ (E1)

I'm going to substitute the RHS of (*) into the LHS of (E1). That produces the following:

- $p(X_1) + (1-p)(X_1) = p(X_2)$ (E1-a)
- $(1-p)(X_1) = p(X_2) - p(X_1)$ (E1-b)
- $X_2 - X_1 = \frac{(1-p)}{p}(X_1)$ (E1*)

Look at (*) written for $j = 2$ and (2):

- $X_2 = p(X_2) + (1-p)(X_2)$ (* for 2)
- $X_2 = p(X_3) + (1-p)(X_1)$ (2)

Substituting the RHS of (* for 2) into the LHS of (2) gives the following:

- $p(X_2) + (1-p)(X_2) = p(X_3) + (1-p)(X_1)$ (2a)
- $(1-p)(X_2) - (1-p)(X_1) = p(X_3) - p(X_2)$ (2b)
- $X_3 - X_2 = \frac{(1-p)}{p}(X_2 - X_1)$ (2c)
- Now,….substitute (E1*) into the RHS (2c) gives:
- $X_3 - X_2 = \left(\frac{(1-p)}{p}\right)^2 (X_1)$ (2*)

Lets do the same to (3). Look at (*) written for $j=3$ and (3):

- ❖ $X_3=p(X_3)+(1-p)(X_3)$ (* for 3)
- ❖ $X_3=p(X_4)+(1-p)(X_2)$ (3)

Substituting the RHS of (* for 3) into the LHS of (3) gives the following:

- ❖ $p(X_3)+(1-p)(X_3)=p(X_4)+(1-p)(X_2)$ (3a)
- ❖ $(1-p)(X_3)-(1-p)(X_2)=p(X_4)-p(X_3)$ (3b)
- ❖ $X_4-X_3=\frac{(1-p)}{p}(X_3-X_2)$ (3c)

Now,…substitute (2*) into the RHS of (3c) gives:

- ❖ $X_4-X_3=\left(\frac{(1-p)}{p}\right)^3(X_1)$ (3*)

We have a pattern here!!!!

Let's do the same for the last 2 so we know how they line up.

Look at (*) written for $j=k-2$ and (4):

- ❖ $X_{k-2}=p(X_{k-2})+(1-p)(X_{k-2})$ (* for (k-2))
- ❖ $X_{k-2}=p(X_{k-1})+(1-p)(X_{k-3})$ (4)

Substituting the RHS of (* for (k-2)) into the LHS of (4) gives the following:

- ❖ $p(X_{k-2})+(1-p)(X_{k-2})=p(X_{k-1})+(1-p)(X_{k-3})$ (4a)
- ❖ $(1-p)(X_{k-2})-(1-p)(X_{k-3})=p(X_{k-1})-p(X_{k-2})$ (4b)
- ❖ $X_{k-1}-X_{k-2}=\left(\frac{(1-p)}{p}\right)(X_{k-2}-X_{k-3})$ (4c)

Now,…substitute (the prior *) into the RHS of (5c) gives:

- $X_{k-1} - X_{k-2} = \left(\frac{(1-p)}{p}\right)^{k-2} (X_1)$ (4*)

Last one.

Look at (*) written for $j = k-1$ and (E2):

- $X_{k-1} = p(X_{k-1}) + (1-p)(X_{k-1})$ (* for (k-1))

- $X_{k-1} = p + (1-p)(X_{k-2})$ (E2)

Substituting the RHS of (* for (k-1)) into the LHS of (E2) gives the following:

- $p(X_{k-1}) + (1-p)(X_{k-1}) = p + (1-p)(X_{k-2})$ (E2-a)

- $(1-p)(X_{k-1}) - (1-p)(X_{k-2}) = p - p(X_{k-1})$ (E2-b)

- $1 - X_{k-1} = \left(\frac{(1-p)}{p}\right)(X_{k-1} - X_{k-2})$ (E2-c)

Now,…substitute (4*) into the RHS of (E2-c) gives:

- $1 - X_{k-1} = \left(\frac{1-p}{p}\right)^{k-1} (X_1)$ (E2*)

I'm going to line up all of the (*) ones and we are going to **add** them all together. There will be a cancelation that I will mark in RED.

- $X_2 - X_1 = \frac{(1-p)}{p} (X_1)$ (E1*)

- $X_3 - X_2 = \left(\frac{(1-p)}{p}\right)^2 (X_1)$ (2*)

- $X_4 - X_3 = \left(\frac{(1-p)}{p}\right)^3 (X_1)$ (3*)

- …

- $X_{k-1} - X_{k-2} = \left(\frac{(1-p)}{p}\right)^{k-2} (X_1)$ (4*)

❖ $1 - X_{k-1} = \left(\frac{1-p}{p}\right)^{k-1} (X_1)$ (E2*)

This leaves us with the following:

❖ $1 - X_1 = X_1 \sum_{j=1}^{k-1} \left(\frac{(1-p)}{p}\right)^j$ (***)

That was a lot of work, but it does give us a starting point to get the final equations. I am going to check for two cases:

1. The game is "fair". That means we have our probability of winning a round given by $p = 1/2$.
2. The game is not "fair". That means we have $p \neq (1-p)$.

First, looking at #1. If $p = 1/2$, we can substitute that into (***) as follows:

❖ $1 - X_1 = X_1 \sum_{i=1}^{k-1} \left(\frac{\left(1 - \frac{1}{2}\right)}{\frac{1}{2}}\right)^i$ (5a)

❖ $1 - X_1 = X_1 \sum_{i=1}^{k-1} \left(\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}\right)^i$ (5b)

❖ $1 - X_1 = X_1 \sum_{i=1}^{k-1} (1)^i$ (5c)

❖ $1 - X_1 = X_1 (k-1)$ (5d)

This gives our solution to X_1 as:

❖ $X_1 = \frac{1}{k}$ (5e)

What happens if we start with $j = 2$ dollars instead of 1? To solve that, we need to "generalize" (5e).

Going back to (E1*), we can see this:

- $X_2 - X_1 = \frac{(1-p)}{p}(X_1)$ (E1*)

From (5e) I know that $X_1 = 1/k$ so substituting this into (E1*) gives the following:

- $X_2 - \frac{1}{k} = \left(\frac{1/2}{1/2}\right)\left(\frac{1}{k}\right)$ (5f)

- $X_2 = \frac{(1+1)}{k} = \frac{2}{k}$ (5g)

If I keep going, I come up with this formula, which is our final for the "fair" game.

- $X_j = \frac{j}{k}$ (Fair*)

We need to now figure out what happens when the game is not fair….when $p \neq (1-p)$.

To start, we have this formula from above:

- $1 - X_1 = X_1 \sum_{j=1}^{k-1}\left(\frac{(1-p)}{p}\right)^j$ (***)

Using the "Sum of Powers formula", this (***) can be rewritten as follows:

- $1 - X_1 = X_1\left(\frac{\left(\frac{(1-p)}{p}\right)^k - \frac{(1-p)}{p}}{\frac{(1-p)}{p} - 1}\right)$ (6)

- $X_1\left(\frac{\left(\frac{(1-p)}{p}\right)^k - \frac{(1-p)}{p}}{\frac{(1-p)}{p} - 1}\right) + X_1 = 1$ (6a)

- $X_1\left(\frac{\left(\frac{(1-p)}{p}\right)^k - \frac{(1-p)}{p}}{\frac{(1-p)}{p} - 1} + 1\right) = 1$ (6b)

- $$X_1\left(\frac{\left(\frac{(1-p)}{p}\right)^k-\frac{(1-p)}{p}}{\frac{(1-p)}{p}-1}+\frac{\frac{(1-p)}{p}-1}{\frac{(1-p)}{p}-1}\right)=1 \qquad (6c)$$

- $$X_1\left(\frac{\left(\frac{(1-p)}{p}\right)^k-\frac{(1-p)}{p}+\frac{(1-p)}{p}-1}{\frac{(1-p)}{p}-1}\right)=1 \qquad (6e)$$

- $$X_1\left(\frac{\left(\frac{(1-p)}{p}\right)^k-1}{\frac{(1-p)}{p}-1}\right)=1 \qquad (6f)$$

- $$X_1=\frac{\frac{(1-p)}{p}-1}{\left(\frac{(1-p)}{p}\right)^k-1} \qquad (6^*)$$

We now have the formula for an unfair game when $j=1$. Let's look at our (E1*) from above, and flush this one out like we did on the "fair" game:

- $$X_2-X_1=\frac{(1-p)}{p}(X_1) \qquad (E1^*)$$

- $$X_2=(X1)\left(\frac{(1-p)}{p}+1\right)$$ (in terms X_2)

Substituting (6*) into (E1*) gives the following:

- $$X_2=\frac{\left(\frac{(1-p)}{p}-1\right)}{\left(\left(\frac{(1-p)}{p}\right)^k-1\right)}\left(\frac{\frac{(1-p)}{p}+1}{1}\right) \qquad (7)$$

- $$X_2=\frac{\left(\frac{(1-p)}{p}-1\right)\left(\frac{(1-p)}{p}+1\right)}{\left(\frac{(1-p)}{p}\right)^k-1} \qquad (7a)$$

❖ $$X_2 = \frac{\left(\frac{(1-p)}{p}\right)^2 - 1}{\left(\frac{(1-p)}{p}\right)^k - 1} \qquad (7^*)$$

Let's go for one more. There appears to be a pattern on this as well. We have formula (2*) from above as follows:

❖ $$X_3 - X_2 = \left(\frac{(1-p)}{p}\right)^2 (X_1) \qquad (2^*)$$

❖ $$X_3 = X_2 + \left(\frac{(1-p)}{p}\right)^2 (X_1)$$ (in terms of X_3)

Now, let's substitute (7*) in for X_2 and (6*) in for X_1.

❖ $$X_3 = \frac{\left(\frac{(1-p)}{p}\right)^2 - 1}{\left(\frac{(1-p)}{p}\right)^k - 1} + \left(\frac{(1-p)}{p}\right)^2 \left(\frac{\left(\frac{1-p}{p}\right) - 1}{\left(\frac{(1-p)}{p}\right)^k - 1}\right) \qquad (8a)$$

❖ $$X_3 = \frac{\left(\frac{(1-p)}{p}\right)^2 - 1}{\left(\frac{(1-p)}{p}\right)^k - 1} + \left(\frac{\left(\frac{(1-p)}{p}\right)^3 - \left(\frac{(1-p)}{p}\right)^2}{\left(\frac{(1-p)}{p}\right)^k - 1}\right) \qquad (8b)$$

❖ $$X_3 = \frac{\left(\frac{(1-p)}{p}\right)^3 - 1}{\left(\frac{(1-p)}{p}\right)^k - 1} \qquad (8^*)$$

Yes….we have a pattern here as well. If we keep going, we end up with the generalized "not fair" formula as follows:

- $$X_j = \frac{\left(\frac{(1-p)}{p}\right)^j - 1}{\left(\frac{(1-p)}{p}\right)^k - 1} \qquad (NF^*)$$

This means our final solution for Gambler's Ruin is given by the following:

The chance of player 'A' winning the game with " j " dollars to start is:

$$X_j = \begin{cases} X_j = \dfrac{j}{k} & \textit{when } \; p = (1-p) \\[2ex] X_j = \dfrac{\left(\dfrac{(1-p)}{p}\right)^j - 1}{\left(\dfrac{(1-p)}{p}\right)^k - 1} & \textit{when } \; p \neq (1-p) \end{cases} \qquad \blacksquare$$