Ordinals

Roger Bishop Jones

Abstract

This document provides support for a kind of ordinals in ProofPower.

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Contents

1 INTRODUCTION 3

2 PRELIMINARIES 4
- 2.1 Cardinality 4

3 GENERIC ORDINALS 5
- 3.1 The Ordering 5
- 3.2 Zero 6

4 CONDITONAL RESULTS 7

A The Theory ordinals 8
- A.1 Parents 8
- A.2 Constants 8
- A.3 Fixity 8
- A.4 Definitions 8
- A.5 Theorems 9

B INDEX 10

1 INTRODUCTION

We begin here from the theory ordered_sets in which the theorem that over any set there exists an initial strict well-ordering is proven. This enables us to define a polymorphic constant which denotes such an ordering over any type to which it is instantiated. Each type is thereby made isomorphic to a initial segment of the ordinals, permitting the theory of ordinals to be developed without introducing any new types. To get a rich theory of ordinals we would need a strong axiom of infinity, but the theory can be developed in the first instance using claims about the cardinality of the type as conditions or assumptions.

In a subsequent document a new type constructor will be defined with an axiom which ensures that the resulting type is strictly larger (in cardinality) than the parameter type, and is at least inaccessible. This is placed in a separate theory and document so that any results here which may prove useful in a strictly conservative development need not feel tainted by an unnecessary axiomatic extension.

In this document the development takes place in the following rough stages.

- first, some preliminary matters before the 'ordinals' are introduced
- second, the introduction of the polymorphic initial strict well-ordering in terms of which the development of a theory of ordinals is conducted, and the specialisation to the ordering of the results about well-orderings, well-foundedness, induction, recursion, and any other matters which we later find convenient in the development, and which are true of all non-empty initial strict well-orderings.
- A progression of developments which depend upon assumptions about cardinality of the type to which our ordering is instantiated.

The development of the theory is focussed on those features which support two special applications. The first of these is the definition of recursive datatypes. In this area of application, in each particular application, a certain repertoir of methods for constructing data objects is to be supported, and one or more 'datatypes' result from the indefinite iteration of these constructions. Indefinite iteration is expected ultimately to exhaust all possible result, and the resulting types together constitute a fixed point or closure of a composite constructor functor which augments any starting point by those objects which can be constructed in one step from it. This application area is addressed en-passant and to whatever extent it contributes to the second application.

Similar methods may also be applied to the estabishment of foundational ontologies and of logical foundation systems. In this application the constructors may be guaranteed to raise cardinality, and will therefor have no fixed point. The resulting abstract onology will have the same cardinality as the ordinal type over which the inductive definition is performed, and the ontologoy will not be unconditionally closed under the constructions. The simplest example is the construction of the ontology of well-founded set by addin at each stage all the elements of the powerset of the preceding ontology. In this case the failure of closure in any resulting ontology is shown by the limitation of abstraction to separation, and the secure a rich enough ontology, such as would be obtained in an axiomatic set theory via the axion of replacement (or large cardinal axioms), an order type of large cardinality is required for our ordinals. Tough these application provide my primary motivation, any material particular to them which depends upon principles like replacement, will be the subject matter of a subsequent document.

In both of these applications, the ordinals enumerate the entities created, which are then represented by their place in the enumeration, the combined constructor (a single function with a disjoin union domain encapsulating all the individual constructors) is the inverse of this enumerating function defined by induction over the relevant type of ordinals.

2 PRELIMINARIES

SML

```
open_theory "rbjmisc";
force_new_theory "ordinals";
new_parent "U_orders";
new_parent "trees";
new_parent "wf_relp";
new_parent "wf_recp";
new_parent "fun_rel_thms";
force_new_pc "'ordinals";
merge_pcs ["'savedthm_cs_∃_proof"] "'ordinals";
set_merge_pcs ["rbjmisc", "'ordinals"];
```

2.1 Cardinality

This is a treatment of cardinality as a property of sets, not a theory of cardinal numbers.

The original motivation is in fact not far removed from the present motivation, which is nice ways of expressing strong axioms of infinity. Of course, the niceness which is most desirable is in the application of such axioms rather than in the aesthetics of their statement, and at the time when I starting the material in this section I didn't have much clue about the application.

The document as a whole reflects my present feeling that the applications (at least those of particular interest to me, but possible more generally) are best mediated by types of infinitary sequences and infinitary trees, and that other aspects of the set theories in which strong axioms are usually placed are less important in this context. In particular, whereas I had at times felt that the development of the treatment of functions was important, I now feel that it is not, and that the notion of function already available in HOL is sufficient. So the whole business of coding up functions as graphs of ordered pairs in set theory now seems unnecessary (in this context).

From here on in we have the original commentary (at least, pro-tem), which may not be entirely appropriate here.

The relations defined here with subscript s on their names are cardinality comparisons on sets.

SML

```
declare_infix(300, "≤s");
declare_infix(300, "<s");
declare_infix(300, "∼s");
```

HOL Constant

```
$≤s : 'a SET → 'b SET → BOOL
────────────────────────────────────────
∀ A B• A ≤s B ⇔ ∃f•
    ∀x y• x ∈ A ∧ y ∈ A ⇒ f x ∈ B ∧ f y ∈ B ∧ (f x = f y ⇒ x = y)
```

$\leqslant_{s\_refl}$ =

$\vdash \forall$ A• A $\leqslant_s$ A

$\subseteq\_\leqslant_{s\_thm}$ =

$\vdash \forall$ A B• A $\subseteq$ B $\Rightarrow$ A $\leqslant_s$ B

$\leqslant_{s\_trans}$ =

$\vdash \forall$ A B C• A $\leqslant_s$ B $\wedge$ B $\leqslant_s$ C $\Rightarrow$ A $\leqslant_s$ C

HOL Constant

$\$<_s$: 'a SET → 'b SET → BOOL

$\forall$ A B• A $<_s$ B $\Leftrightarrow$ A $\leqslant_s$ B $\wedge$ $\neg$ B $\leqslant_s$ A

lt_{s_irrefl} =

$\vdash \forall$ A• $\neg$ A $<_s$ A

lt_{s_trans} =

$\vdash \forall$ A B C• A $<_s$ B $\wedge$ B $<_s$ C $\Rightarrow$ A $<_s$ C

$lt_{s\_}\leqslant_{s\_trans}$ =

$\vdash \forall$ A B C• A $<_s$ B $\wedge$ B $\leqslant_s$ C $\Rightarrow$ A $<_s$ C

$\leqslant_{s\_lts\_trans}$ =

$\vdash \forall$ A B C• A $\leqslant_s$ B $\wedge$ B $<_s$ C $\Rightarrow$ A $<_s$ C

HOL Constant

$\$\sim_s$: 'a SET → 'b SET → BOOL

$\forall$ A B•
 A $\sim_s$ B $\Leftrightarrow$ $\exists$f g•
 ($\forall$x• x $\in$ A $\Rightarrow$ f x $\in$ B $\wedge$ g (f x) = x)
 $\wedge$ ($\forall$y• y $\in$ B $\Rightarrow$ g y $\in$ A $\wedge$ f (g y) = y)

$card\_equiv\_lemma$ =

$\vdash \forall$ x y z• x $\sim_c$ x $\wedge$ (x $\sim_c$ y $\Leftrightarrow$ y $\sim_c$ x) $\wedge$ (x $\sim_c$ y $\wedge$ y $\sim_c$ z $\Rightarrow$ x $\sim_c$ z)

3 GENERIC ORDINALS

3.1 The Ordering

The existence of initial strict well-orderings has been established in [1], which allows us to define the following constant:

HOL Constant

$\$<_o$: 'a → 'a → BOOL

UInitialStrictWellOrdering $\$<_o$

This is a polymorphic constant each instance of which is an initial strict well-ordering over the instance type, which may have any cardinality greater than zero. The cardinality uniquely determines the order-type of the defined ordering over that type, which are in one to one correspondence with initial ordinals or cardinals.

HOL Constant

$\$\leqslant_o$: 'a → 'a → BOOL

$\forall$ x y• x $\leqslant_o$ y ⇔ x $<_o$ y ∨ x = y

In axiomatic set theory the least ordinal of a set of ordinals is the distributed intersection over that set, for which a large cap symbol is often used. Though these ordinals are not sets, a similar notation seems reasonable. The function Minr, defined in [1].

HOL Constant

$\bigcap_o$: 'a SET → 'a

∀s• $\bigcap_o$ s = Minr(Universe, $\$<_o$) s

$lt_{o\_clauses}$ = ⊢
(∀ x• ¬ x $<_o$ x)
∧ (((∀ x y• ¬ x = y ⇒ ¬ (x $<_o$ y ∧ y $<_o$ x))
∧ (∀ x y z• x $<_o$ y ∧ y $<_o$ z ⇒ x $<_o$ z))
∧ (∀ x y• ¬ x = y ⇒ x $<_o$ y ∨ y $<_o$ x))
∧ (∀ A• ¬ A = {} ⇒ $\bigcap_o$ A ∈ A ∧ (∀ y• y ∈ A ⇒ y = $\bigcap_o$ A ∨ $\bigcap_o$ A $<_o$ y))

$irrefl_{o\_thm}$ = ⊢ ∀ x•
¬ x $<_o$ x

$antisym_{o\_thm}$ = ⊢ ∀ x y•
¬ x = y ⇒ ¬ (x $<_o$ y ∧ y $<_o$ x)

$trans_{o\_thm}$ = ⊢ ∀ x y z•
x $<_o$ y ∧ y $<_o$ z ⇒ x $<_o$ z

$linear_{o\_thm}$ = ⊢ ∀ x y•
¬ x = y ⇒ x $<_o$ y ∨ y $<_o$ x

$\bigcap_{o\_thm}$ = ⊢ ∀ A•
¬ A = {} ⇒ $\bigcap_o$ A ∈ A ∧ (∀ y• y ∈ A ⇒ y = $\bigcap_o$ A ∨ $\bigcap_o$ A $<_o$ y)

$lt_{o\_well\_founded\_thm}$ = ⊢ WellFounded $\$<_o$

$lt_{o\_induction\_thm}$ = ⊢ ∀ p• (∀ x• (∀ y• y $<_o$ x ⇒ p y) ⇒ p x) ⇒ (∀ x• p x)

3.2 Zero

Zero (0_o) is the smallest ordinal. Every type has one.

HOL Constant

$\mathbf{0}_o$: 'a

0_o = $\bigcap_o$ {x|T}

$|\boldsymbol{zero_{o\_thm}} = \vdash \forall\ \text{ord}\bullet\ 0_o \leqslant_o \text{ord}$
$|\boldsymbol{zero_{o\_thm2}} = \vdash \forall\ \text{ord}\bullet\ 0_o <_o \text{ord} \vee 0_o = \text{ord}$

4 CONDITONAL RESULTS

References

[1] R.D. Arthan and R.B. Jones. Well-orderings and Well-foundedness. RBJones.com, 2010. http://www.rbjones.com/rbjpub/pp/doc/t009.pdf.

A The Theory ordinals

A.1 Parents

fun_rel_thmswf_relp U_orders
wf_recp trees rbjmisc

A.2 Constants

$\$\leqslant_s$	$'a\ \mathbb{P} \to {'b}\ \mathbb{P} \to BOOL$
$\$<_s$	$'a\ \mathbb{P} \to {'b}\ \mathbb{P} \to BOOL$
$\$\sim_s$	$'a\ \mathbb{P} \to {'b}\ \mathbb{P} \to BOOL$
$\$<_o$	$'a \to {'a} \to BOOL$
$\$\leqslant_o$	$'a \to {'a} \to BOOL$
$\bigcap_o$	$'a\ \mathbb{P} \to {'a}$
$\mathbf{0}_o$	$'a$

A.3 Fixity

Right Infix 300:

$<_o \quad <_s \quad \sim_s \quad \leqslant_o \quad \leqslant_s$

A.4 Definitions

$\leqslant_s$
$$\vdash \forall\ A\ B \bullet A \leqslant_s B \Leftrightarrow (\exists\ f \bullet \forall\ x\ y \bullet x \in A \land y \in A \Rightarrow f\ x \in B \land f\ y \in B \land (f\ x = f\ y \Rightarrow x = y))$$

$<_s$
$$\vdash \forall\ A\ B \bullet A <_s B \Leftrightarrow A \leqslant_s B \land \neg B \leqslant_s A$$

$\sim_s$
$$\vdash \forall\ A\ B \bullet A \sim_s B \Leftrightarrow (\exists\ f\ g \bullet (\forall\ x \bullet x \in A \Rightarrow f\ x \in B \land g\ (f\ x) = x) \land (\forall\ y \bullet y \in B \Rightarrow g\ y \in A \land f\ (g\ y) = y))$$

$<_o$
$$\vdash \mathrm{UInitialStrictWellOrdering}\ \$<_o$$

$\leqslant_o$
$$\vdash \forall\ x\ y \bullet x \leqslant_o y \Leftrightarrow x <_o y \lor x = y$$

$\bigcap_o$
$$\vdash \forall\ s \bullet \bigcap_o s = \mathrm{Minr}\ (\mathrm{Universe},\ \$<_o)\ s$$

$\mathbf{0}_o$
$$\vdash 0_o = \bigcap_o \{x|T\}$$

A.5 Theorems

$\leqslant_{s\_refl}$ $\vdash \forall\ A\bullet\ A \leqslant_s A$

$\subseteq\_\leqslant_{s\_thm}$ $\vdash \forall\ A\ B\bullet\ A \subseteq B \Rightarrow A \leqslant_s B$

$\leqslant_{s\_trans}$ $\vdash \forall\ A\ B\ C\bullet\ A \leqslant_s B \land B \leqslant_s C \Rightarrow A \leqslant_s C$

lt_{s_irrefl} $\vdash \forall\ A\bullet\ \neg\ A <_s A$

lt_{s_trans} $\vdash \forall\ A\ B\ C\bullet\ A <_s B \land B <_s C \Rightarrow A <_s C$

$lt_{s\_\leqslant_{s\_trans}}$

$\vdash \forall\ A\ B\ C\bullet\ A <_s B \land B \leqslant_s C \Rightarrow A <_s C$

$\leqslant_{s\_lts\_trans}$

$\vdash \forall\ A\ B\ C\bullet\ A \leqslant_s B \land B <_s C \Rightarrow A <_s C$

$card\_equiv\_lemma$

$\vdash \forall\ x\ y\ z$
$\quad \bullet\ x \sim_s x$
$\qquad \land\ (x \sim_s y \Leftrightarrow y \sim_s x)$
$\qquad \land\ (x \sim_s y \land y \sim_s z \Rightarrow x \sim_s z)$

$lt_{o\_clauses}$ $\vdash (\forall\ x\bullet\ \neg\ x <_o x)$
$\quad \land\ (((\forall\ x\ y\bullet\ \neg\ x = y \Rightarrow \neg\ (x <_o y \land y <_o x))$
$\qquad \land\ (\forall\ x\ y\ z\bullet\ x <_o y \land y <_o z \Rightarrow x <_o z))$
$\quad \land\ (\forall\ x\ y\bullet\ \neg\ x = y \Rightarrow x <_o y \lor y <_o x))$
$\quad \land\ (\forall\ A$
$\quad \bullet\ \neg\ A = \{\}$
$\qquad \Rightarrow \bigcap_o A \in A$
$\qquad\quad \land\ (\forall\ y\bullet\ y \in A \Rightarrow y = \bigcap_o A \lor \bigcap_o A <_o y))$

$irrefl_{o\_thm}$ $\vdash \forall\ x\bullet\ \neg\ x <_o x$

$antisym_{o\_thm}$

$\vdash \forall\ x\ y\bullet\ \neg\ x = y \Rightarrow \neg\ (x <_o y \land y <_o x)$

$trans_{o\_thm}$ $\vdash \forall\ x\ y\ z\bullet\ x <_o y \land y <_o z \Rightarrow x <_o z$

$linear_{o\_thm}$ $\vdash \forall\ x\ y\bullet\ \neg\ x = y \Rightarrow x <_o y \lor y <_o x$

$\bigcap_{o\_thm}$ $\vdash \forall\ A$
$\quad \bullet\ \neg\ A = \{\}$
$\qquad \Rightarrow \bigcap_o A \in A$
$\qquad\quad \land\ (\forall\ y\bullet\ y \in A \Rightarrow y = \bigcap_o A \lor \bigcap_o A <_o y)$

$lt_{o\_well\_founded\_thm}$

$\vdash$ UWellFounded $\$<_o$

$lt_{o\_induction\_thm}$

$\vdash \forall\ p\bullet\ (\forall\ x\bullet\ (\forall\ y\bullet\ y <_o x \Rightarrow p\ y) \Rightarrow p\ x) \Rightarrow (\forall\ x\bullet\ p\ x)$

$zero_{o\_thm}$ $\vdash \forall\ ord\bullet\ 0_o \leqslant_o ord$

$zero_{o\_thm2}$ $\vdash \forall\ ord\bullet\ 0_o <_o ord \lor 0_o = ord$

B INDEX

$'ordinals$ 4
$<_o$ 8
$<_s$ 5, 8
$\$ <_o$ 5
$\$ \leqslant_o$ 6
$\bigcap_{o\_thm}$ 6, 9
$\bigcap_o$ 6, 8
$\leqslant_o$ 8
$\leqslant_{s\_lts\_trans}$ 5, 9
$\leqslant_{s\_refl}$ 5, 9
$\leqslant_{s\_trans}$ 5, 9
$\leqslant_s$ 4, 8
$\sim_s$ 5, 8
$\subseteq_{\_} \leqslant_{s\_thm}$ 5, 9
0_o 6, 8

$antisym_{o\_thm}$ 6, 9

$card\_equiv\_lemma$ 5, 9

$irrefl_{o\_thm}$ 6, 9

$linear_{o\_thm}$ 6, 9
$lt_{o\_clauses}$ 6, 9
$lt_{o\_induction\_thm}$ 6, 9
$lt_{o\_well\_founded\_thm}$ 6, 9
lt_{s_irrefl} 5, 9
lt_{s_trans} 5, 9
$lt_{s\_} \leqslant_{s\_trans}$ 5, 9

$ordinals$ 4

$trans_{o\_thm}$ 6, 9

$zero_{o\_thm2}$ 7, 9
$zero_{o\_thm}$ 7, 9