Solutions due by 10.30am Friday 19th March.

1. Consider the growth problem with full capital depreciation ($\delta = 1$)

$$\max_{\{c_t, k_{t+1}\}} \quad \sum_{t=0}^{\infty} \beta^t \log(c_t)$$
$$\text{s.t.} \quad c_t + k_{t+1} = Ak_t^\alpha$$

a) Write the problem's Bellman equation.

b) Guess and verify that $V(k_t) = a + b\log k_t$.

c) Find the optimal policies for k_{t+1} and c_t.

Solution: a)

$$V(k_t) = \max_{c_t} \ \{\log(c_t) + \beta V(Ak_t^\alpha - c_t)\}$$

b) The first order condition tells us that

$$\frac{1}{c_t} = \beta V'(Ak_t^\alpha - c_t)$$

Plugging in $V(k_t) = a + b\log k_t$ gives:

$$\frac{1}{c_t} = \beta \frac{b}{Ak_t^\alpha - c_t} \implies c_t^* = \frac{1}{1+\beta b} Ak_t^\alpha \implies k_{t+1}^* = \frac{\beta b}{1+\beta b} Ak_t^\alpha$$

Substituting this into the Bellman equation yields

$$a + b\log k_t = \log\left[\frac{1}{1+\beta b} Ak_t^\alpha\right] + \beta\left(a + b\log\left[\frac{\beta b}{1+\beta b} Ak_t^\alpha\right]\right)$$

Expanding and grouping terms gives

$$a + b\log k_t = (1+\beta b)\log A + \beta b\log(\beta b) - (1+\beta b)\log(1+\beta b) + \beta a + \alpha(1+\beta b)\log k_t$$

Equating coefficients on the $\log k_t$ terms gives $b = \frac{\alpha}{1-\alpha\beta}$ (which in turn gives $\beta b = \frac{\alpha\beta}{1-\alpha\beta}$ and $1+\beta b = \frac{1}{1-\alpha\beta}$).

Substituting out and equating the constant terms gives

$$a = \frac{\log A + \alpha\beta \log(\alpha\beta) + (1-\alpha\beta)\log(1-\alpha\beta)}{(1-\beta)(1-\alpha\beta)}$$

c) Plugging those constants into the first order condition gives $c_t^* = (1-\alpha\beta)Ak_t^\alpha)$ and $k_{t+1}^* = \alpha\beta A k_t^\alpha$. That is, save a proportion $\alpha\beta$ of total output, consume the rest.

2. Consider the growth model with labor:

$$\max_{\{c_t,\ell_t,k_{t+1}\}} \quad \sum_{t=0}^{\infty} \beta^t[\log(c_t) + \eta\log(1-\ell_t)]$$
$$\text{s.t.} \quad c_t + k_{t+1} = Ak_t^\alpha \ell^{1-\alpha}$$

Show that the model has the following steady state:

$$\bar{\ell} = \frac{1}{1 + \frac{\eta}{1-\alpha}(1 - \frac{\alpha\beta\delta}{1-\beta(1-\delta)})}$$
$$\bar{k} = \bar{\ell}\left[\frac{\alpha\beta A}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}}$$
$$\bar{c} = A\bar{k}^\alpha\bar{\ell}^{1-\alpha} - \delta\bar{k}$$

Solution: The steady state Euler equation, intra-temporal consumption-leisure tradeoff, and resource constraint are:

$$1 = \beta\left[\alpha A\bar{k}^{\alpha-1}\bar{\ell}^{1-\alpha} + 1 - \delta\right] \tag{1}$$
$$\frac{\eta\bar{c}}{1-\bar{\ell}} = (1-\alpha)A\bar{k}^\alpha\bar{\ell}^{-\alpha} \tag{2}$$
$$\bar{c} = A\bar{k}^\alpha\bar{\ell}^{1-\alpha} - \delta\bar{k} \tag{3}$$

The Euler equation and resource constraint directly give two of the equations above. The tricky one is finding steady state labor. First, define $\bar{Y} = A\bar{k}^\alpha\bar{\ell}^{1-\alpha}$. Substitute consumption out of the labor condition using the resource constraint:

$$\begin{aligned}
\eta[\bar{Y} - \delta\bar{K}] &= (1-\bar{\ell})(1-\alpha)\frac{\bar{Y}}{\bar{\ell}} \\
\therefore \eta[1 - \delta\frac{\bar{K}}{\bar{Y}}]\bar{\ell} &= (1-\bar{\ell})(1-\alpha) \\
\therefore \bar{\ell}\left(1 + \frac{\eta}{1-\alpha}[1-\delta\frac{\bar{K}}{\bar{Y}}]\right) &= 1 \\
\therefore \bar{\ell} &= \frac{1}{1 + \left(\frac{\eta}{1-\alpha}\right)\left[1-\delta\frac{\bar{K}}{\bar{Y}}\right]}
\end{aligned}$$

Noting that $\frac{\bar{k}}{\bar{Y}} = \frac{1}{A\bar{k}^{\alpha-1}\bar{l}^{1-\alpha}} = \frac{1}{A\left(\bar{\ell}\left[\frac{\alpha\beta A}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}}\right)^{\alpha-1}\bar{l}^{1-\alpha}} = \frac{\alpha\beta}{1-\beta(1-\delta)}$ finishes it.

3. Consider a tree whose growth is determined by a function h. This is, if k_t is the size of the tree in period t, then $k_{t+1} = h(k_t)$, $t = 0, 1, \ldots$. Suppose h is strictly increasing, strictly conncave, and $h(0) > 0$. Assume that the price of wood and the interest rate are constant over time, with $p = 1$ and $\beta = \frac{1}{1+r}$. Assume further that it is costless to cut down the tree. If the tree cannot be replanted, present value maximization leads to the functional equation $V(k_t) = \max\{k_t, \beta V(h(k_{t+1})\}$.

   a) Show that the above operator satisfies Blackwell's conditions for a contraction mapping.

   b) Let k_0 be the height of the tree that solves $\beta h(k_0) = k_0$. Show that the rule "cut down the tree if $k \geq k_0$, leave it standing otherwise" is optimal.

   Solution:
   a) Let $f(x) \leq g(x)\ \forall x$. Then $\max\{k_t, \beta f(h(k_t))\} \leq \max\{k_t, \beta g(h(k_t))\}$ (B1). $T(f(x) + a) = \max\{k_t, \beta(f(h(k_t)+a)\} = \max\{k_t, \beta f(h(k_t))+\beta a\} \leq \max\{k_t+\beta a, \beta f(h(k_t))+\beta a\} = \max\{k_t, \beta f(h(k_t))\} + \beta a = T(f) + \beta a$ (B2). Therefore the operator is a contraction mapping.

   b)Note that we have $\beta h(k_t) > k_t$ if $k_t < k_0$, and $\beta h(k_t) < k_t$ if $k_t > k_0$.

Suppose $k_t < k_0$. Then

$$\begin{aligned} V(k_t) &= \max\{k_t, \beta V(h(k_t))\} \\ &> \max\{k_t, \beta V(k_t/\beta)\} \\ &= \beta \max\{k_t/\beta, V(k_t/\beta)\} \\ &= \beta V(k_t/\beta) \\ &> \beta k_t/\beta \\ &= k_t \end{aligned}$$

So if $k_t < k_0$, then $V(k_t) > k_t$. So we should leave the tree up. By the same logic in the other direction, we should take the tree down if $k_t \geq k_0$.

4. [Challenge Problem] You don't have to submit this one if you don't want to.

   Check out the setup of an odd game here:

   https://www.youtube.com/watch?v=6_yU9eJ0NxA&ab_channel=Numberphile.

   We can define the expected payoff of this game recursively with value functions. Let $V(R_t)$ be the expected payoff of the game.

   a) Show that

   $$V(R_t) = 1 + \mathbb{P}(d_t \leq R_t) \int_0^{R_t} \mathbb{P}(d_t = x \mid d_t \leq R_t) V\left(\sqrt{R_t^2 - x^2}\right) \mathrm{d}x$$

   where R_t is the current radius of the board, and d_t is the distance the dart lands from the origin.

   b) Find expressions for $\mathbb{P}(d_t \leq R_t)$ and $\mathbb{P}(d_t = x \mid d_t \leq R_t)$.

   c) Verify that the above operator satisfies the Blackwell conditions.

   d) Guess $V^0 = 0$ and manually perform 3 value function iterations.

   e) Make a guess for the value function and verify it.

   f) What's the expected score for the game with dart board of initial radius 1?

   Solution:
   a) For a given radius R_t, we get to throw one dart; so the period payoff is 1. If we fail to hit the board, the game ends and our future payoff is zero. If we do hit the board,

which happens with $\mathbb{P}(d_t \leq R_t)$ (probability that the distance the dart lands is within R_t of the origin), we continue the game with a board of new radius $R_{t+1} = \sqrt{R_t^2 - d_t^2}$. Since d_t is random, we must take expectation over future values $\mathbb{E}[V(R_{t+1})]$. The expectation is over the condtional density, $\mathbb{P}(d_t = x \mid d_t \leq R_t)$, since we must condition on the dart hitting the board.

b) $\mathbb{P}(d_t \leq R_t)$ is the area of the board divided by the area the dart could land in. The dart could land in a 2×2 square centered at the origin, so $\mathbb{P}(d_t \leq R_t) = \frac{\pi R_t^2}{4}$.

By similar logic, $\mathbb{P}(d_t \leq x \mid d_t \leq R_t)$ is the probability that the dart lands in a circle of radius x given than it lands in the board of radius R_t. So $\mathbb{P}(d_t \leq x \mid d_t \leq R_t) = \frac{\pi x^2}{\pi R_t^2} = \frac{x^2}{R_t^2}$. Differentiating gives $\mathbb{P}(d_t = x \mid d_t \leq R_t) = \frac{2x}{R_t^2}$.

c) Let $f(x) \leq g(x)\ \forall x$. By monotonicity of integrals: $1 + \mathbb{P}(d_t \leq R_t) \int_0^{R_t} \mathbb{P}(d_t = x \mid d_t \leq R_t) f\left(\sqrt{R_t^2 - x^2}\right) \mathrm{d}x \leq 1 + \mathbb{P}(d_t \leq R_t) \int_0^{R_t} \mathbb{P}(d_t = x \mid d_t \leq R_t) g\left(\sqrt{R_t^2 - x^2}\right) \mathrm{d}x$ (B1).

$T(f(x)+a) = 1+\mathbb{P}(d_t \leq R_t) \int_0^{R_t} \mathbb{P}(d_t = x \mid d_t \leq R_t) \left[f\left(\sqrt{R_t^2 - x^2} + a\right]\right) \mathrm{d}x = 1+\mathbb{P}(d_t \leq R_t) \int_0^{R_t} \mathbb{P}(d_t = x \mid d_t \leq R_t) f\left(\sqrt{R_t^2 - x^2}\right) \mathrm{d}x + \mathbb{P}(d_t \leq R_t) \int_0^{R_t} \mathbb{P}(d_t = x \mid d_t \leq R_t) a \mathrm{d}x = 1 + \mathbb{P}(d_t \leq R_t) \int_0^{R_t} \mathbb{P}(d_t = x \mid d_t \leq R_t) f\left(\sqrt{R_t^2 - x^2}\right) \mathrm{d}x + \mathbb{P}(d_t \leq R_t) a = T(f) + \mathbb{P}(d_t \leq R_t) a$. Since $\mathbb{P}(d_t \leq R_t) < 1$, the discounting property is satisfied (B2).

d) Plugging in the probabilities we found above, the value function is:

$$V(R_t) = 1 + \frac{\pi}{2} \int_0^{R_t} x V(\sqrt{R_t^2 - x^2}) \mathrm{d}x$$

If we begin with $V^0(R_t) = 0$, then $V^1(R_t) = 1$.

$V^2(R_t) = 1 + \frac{\pi}{2} \int_0^{R_t} x \mathrm{d}x = 1 + \frac{\pi R_t^2}{4}$

$V^3(R_t) = 1+\frac{\pi}{2} \int_0^{R_t} x(1+\frac{\pi(\sqrt{R_t^2 - x^2})^2}{4}) \mathrm{d}x = 1+\frac{\pi}{2} \int_0^{R_t} x+\frac{x\pi R_t^2}{4} - \frac{x^3}{4} \mathrm{d}x = 1+\frac{\pi}{2} \left[\frac{1}{2}x^2 + \frac{x^2 \pi R_t^2}{8} - \frac{x^4}{16}\right]_0^{R_t} = 1 + \frac{\pi R_t^2}{4} + \frac{\pi^2 R_t^4}{32} = 1 + \left(\frac{\pi R_t^2}{4}\right) + \frac{1}{2}\left(\frac{\pi R_t^2}{4}\right)^2$.

e) This is starting to look like the Taylor series for $\exp\{\frac{\pi R_t^2}{4}\}$. So guess $V(R_t) = \exp\{\frac{\pi R_t^2}{4}\}$.

$$
\begin{aligned}
1 + \frac{\pi}{2}\int_0^{R_t} x\exp\{\frac{\pi(\sqrt{R_t^2 - x^2})^2}{4}\}\mathrm{d}x &= 1 + \frac{\pi}{2}\int_0^{R_t} x\exp\{\frac{\pi R_t^2}{4}\}\exp\{-\frac{\pi x^2}{4}\}\mathrm{d}x \\
&= 1 + \exp\{\frac{\pi R_t^2}{4}\}\int_0^{R_t} \frac{\pi x}{2}\exp\{-\frac{\pi x^2}{4}\}\mathrm{d}x \\
&= 1 + \exp\{\frac{\pi R_t^2}{4}\}\left[-\exp\{-\frac{\pi x^2}{4}\}\right]_0^{R_t} \\
&= 1 + \exp\{\frac{\pi R_t^2}{4}\}\left[-\exp\{-\frac{\pi R_t^2}{4}\} + 1\right] \\
&= 1 - 1 + \exp\{\frac{\pi R_t^2}{4}\} \\
&= \exp\{\frac{\pi R_t^2}{4}\}
\end{aligned}
$$

So we've confirmed that our guess is a fixed point. Therefore we've found the value function.

f) $V(1) = e^{\pi/4} \approx 2.2$.