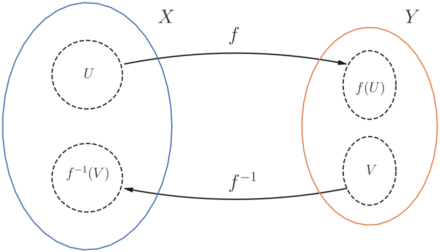
Lecture 1: Real Analysis

Robert Garrard



Housekeeping

Contact:
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Consultation: TBD

Topics Covered:
Real Analysis, Linear Algebra, Probability and Statistics, Calculus, Optimization, Dynamic Programming.

Resources:
Mathematics for Economists. Carl P. Simon & Lawrence Blume.
Econometric Analysis. William H. Greene.
Recursive Methods in Economic Dynamics. Robert E. Lucas, Nancy Stokey, with Edward C. Prescott.

Housekeeping

Assessment:

- Tutorial problems (20%)
  - Either submitted to Blackboard or handed in at beginning of tutorial starting Week 2.
  - Problem set covers the Thursday lecture from previous week and Monday lecture from current week.
  - Can be handwritten or typed at your preference. Please consider using LaTeX (overleaf.com).
- Take-home assessment #1 (30%)
  - Due by end of day 25^{th} February.
  - Covers first 5 lectures.
- Take-home assessment #2 (50%)
  - Due by end of day Wednesday 25^{th} March.
  - Covers entire course.

Vocabulary: Sets

A **set** is an unordered collection of objects.

$$A = \{0, 1\} \qquad B = \{0, \{1\}\} \qquad \emptyset = \{\}$$

Membership is denoted with '$\in$', non-membership with '$\notin$'.

$$0 \in A \qquad \{1\} \in B \qquad 0 \notin C$$

Repeated elements aren't counted seperately: $\{0, 1, 1\} = \{0, 1\}$

Some common sets:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\} \qquad \mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$$
$$\mathbb{Q} = \{\tfrac{a}{b} \mid a, b \in \mathbb{Z}\} \qquad \mathbb{C} = \{a + bi \mid i = \sqrt{-1}; a, b \in \mathbb{R}\}$$

Vocabulary: Logical Statements

A logical **proposition** is a statement that can be either True or False.

P: It rained today.
$Q : \sqrt{2} \in \mathbb{Q}$

We can transform and combine propositions using **logical operators**.

Operator	Notation	Meaning
Negation	$\neg$ p	not p
Conjunction	$p \wedge q$	p and q
Disjunction	$p \vee q$	p or q (or both)
Implication	$p \Rightarrow q$	p implies q/if p then q
Biconditional	$p \iff q$	p if and only if q

Vocabulary: Logical Statements

We can determine under what circumstances a compound proposition is true or false using a **truth table**.

Negation

p	$\neg p$
0	1
1	0

Conjunction

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Disjunction

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Implication

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Biconditional

p	q	$p \iff q$
0	0	1
0	1	0
1	0	0
1	1	1

Vocabulary: Logical Statements

Exercise 1.1

Let $x \in \mathbb{Z}$. What is the truth value of the following propositions?

1. $x \geq 0 \vee x < 0$
2. $x^2 > 0 \Rightarrow x > 0$
3. x^2 is even $\iff$ x is even
4. x is prime $\wedge$ x is even $\Rightarrow x = 2$
5. [Harder] $x^2 = 2 \Rightarrow x = 10$

Vocabulary: Logical Statements

Propositions that are: always true are called **tautologies**; always false are called **contradictions**; sometimes true, sometimes false are called **contingent** statements.

Exercise 1.2

1. Show that $p \wedge \neg p$ is a contradiction.
2. Show that $p \Rightarrow q \iff \neg q \Rightarrow \neg p$ is a tautology.

Vocabulary: Logical Statements

We may want some propositions to refer only to a particular set of quantities. Some common **quantifiers** are:

Quantifier	Notation	Meaning
Universal quantifier	$\forall$	'for all' or 'for every'
Existential quantifier	$\exists$	'there exists'

Example

Write the following statements in logical notation.

1. "Every natural number is greater than zero".
   $\forall n \in \mathbb{N}\ n > 0$
2. "There is a largest integer".
   $\exists N \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}\ \ n \leq N$

Vocabulary: Sets

We can construct new sets out of old with logical operations:

Operator	Notation
Intersection	$A \cap B = \{x \in A \wedge x \in B\}$
Union	$A \cup B = \{x \in A \vee x \in B\}$
Relative Complement	$A \backslash B = \{x \in A \wedge x \notin B\}$
Cardinality	$\lvert A \rvert = \{\text{number of elements in } A\}$

When we have a particular universal set U in mind, we can define an absolute complement: $A^c = U \backslash A$.

We can talk about a set being contained in another set with a proposition:
$A \subseteq B$ (eqv. $B \supseteq A$) if $\forall a \in A,\ a \in B$.
$A \subset B$ (eqv. $B \supset A$) if $\forall a \in A,\ a \in B \wedge \exists b \in B$ s.t. $b \notin A$.

And define the **power set** of a set: $\mathcal{P}(A) = \{B \mid B \subset A\}$.

Vocabulary: Sets

Exercise 1.3

$$A = \{-1, 0, 1\} \qquad B = \mathbb{N} \qquad C = \{a, b\}$$

1. $A \cap B$
2. $A \cup B$
3. $B \cap C$
4. $A \subset B$
5. $\mathcal{P}(C)$

Proof

Proof is argumentation we use to establish the truth or falsehood of a proposition.

Common methods of proof:

- **Direct Proof**
  - Deductive reasoning, often by syllogism. E.g., $p \wedge q \Rightarrow r$; p and q are true, therefore r is true.
- **Proof by Contraposition**
  - Transform a proposition $(p \Rightarrow q)$ into its contrapositive $(\neg q \Rightarrow \neg p)$, then use direct proof.
- **Proof by Contradiction**
  - If p is true, $p \wedge q \Rightarrow r$, and r is false; then q must be false.
- **Proof by Induction**
  - We'll explore this in more detail.
- **Proof by Counter-example**
  - To prove that a proposition is false, produce an example where it fails.

Proof

Theorem (De Morgan's Laws)

Let A and B be sets.

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Exercise 1.4

1. Prove directly the first of De Morgan's laws.
2. Prove by contraposition that if x^2 is even, then x is even, $x \in \mathbb{Z}$.

Proof by Induction

Proof by induction is a convenient method availble to us when the proposition can be split into cases indexed by the natural numbers, $P(n)$, $n \in N$.

To establish the truth of the proposition for each n, we must prove:
Base case: $P(1)$ is true.
Induction step: $P(n) \Rightarrow P(n+1)$.

Proof by Induction

Example

Prove that the sum of the first n integers is:
$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Base case:
Sum of first 1 integers = 1.
$\frac{1(1+1)}{2} = \frac{2}{2} = 1$.

Induction step:
Assume true for the n^{th} case, show true for the $(n+1)^{th}$ case.

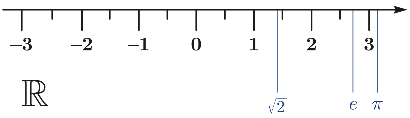
$(n+1)^{th}$ case is:
$1 + 2 + 3 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$
$= \frac{(n+1)(n+1+1)}{2}$

$\blacksquare$

Proof by Induction

Exercise 1.5 Let A be a finite set. Prove that $|\mathcal{P}(A)| = 2^{|A|}$.

The Real Numbers



The set of real numbers, $\mathbb{R}$, is the 'real' in 'real analysis'.

Won't get a rigorous treatment here, since we want to get to the useful results from real analysis in the next lecture. We may briefly return to touch on the definition of the real numbers once we've covered Cauchy sequences and completeness, time permitting.

We'll need an important feature of the reals:
Recall that a subset of numbers, X, is **bounded above** if $\exists B$ s.t. $\forall x \in X \quad x \leq B$.

E.g., the interval $(0, 1)$ is bounded above by 2.

The Real Numbers

Least Upper Bound property: Any non-empty subset of the reals that is bounded above has a *least* upper bound.

The interval $(0, 1)$ has many upper bounds. $2, 3, \pi$, etc. However, 1 is the unique LUB.

Analogously, subsets of real numbers bounded *below* have a **greatest lower bound**.

We call the least upper bound of a set the **supremum**, denoted sup. We call the greatest lower bound of a set the **infimum**, denoted inf.

This is not true for the rationals, $\mathbb{Q}$.

The Real Numbers

Exercise 1.6

1. Show that the set $\{x \in \mathbb{Q} \mid x^2 < 2\}$ has no supremum.
2. What are the sup and inf of $\{x \in \mathbb{R} \mid x^2 < 2\}$.
3. What are the sup and inf of bounded intervals in $\mathbb{R}$?

Cartesian Product

The **Cartesian Product** of two sets, A and B, is the set of all ordered pairs formed by taking the first element of the pair from A and the second element of the pair from B:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Example

$$A = \{1, 2\} \qquad B = \{1, 2, 3\}$$
$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

The Cartesian product of n sets is the set of all n-tuples formed in the same fashion as above.

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n} = \{(x_1, x_2, ..., x_n) \mid x_i \in \mathbb{R}\}$$

Binary Relations

A **binary relation**, (X, Y, G) is an ordered triple where the set X is called the domain, the set Y is called the codomain, and the set G, called the graph, is a subset of the cartesian product of these sets, $G \subseteq X \times Y$.

The graph is a set of ordered pairs, $G = \{(x, y) \mid x \in X, y \in Y\}$. If the pair $(x, y) \in G$, we often write xRy and say "x relates to y".

Example

1. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. Define the relation (X, Y, G) such that
$$G = \{(1, a), (1, b), (1, c), (2, a), (3, c)\}$$
2. Define the unit circle to be the relation $(\mathbb{R}, \mathbb{R}, G)$ such that
$$G = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

Economic Application: Preference relations

Consider a set of bundles $X = \{\text{Apples}, \text{Bananas}, \text{Carrots}\}$.

Our economic agent likes carrots more than apples, and likes apples and bananas equally.

We can represent these **preferences** with a binary relation from X to itself, $(X, X, \succsim)$, where:

$$\begin{aligned}\succsim = \{&(\text{Apples, Apples}), (\text{Apples, Bananas}), (\text{Bananas, Bananas}),\\ &(\text{Bananas, Apples}), (\text{Carrots, Carrots}), (\text{Carrots, Apples}),\\ &(\text{Carrots, Bananas})\}\end{aligned}$$

We call $\succsim$ the "weak preference" relation.

Representing preferences as a binary relation is very general. We'll need to place some additional structure on these relations to conform with how we expect real-world preferences to behave.

Properties of Relations

When the domain and codomain of a relation are the same set, the relation *may* exhibit some interesting properties.

Properties of Relations

Reflexive if $\forall x \in X \quad xRx$

Irreflexive if $\forall x \in X \quad \neg xRx$

Symmetric if $xRy \Rightarrow yRx \quad \forall x, y \in X$

Anti-symmetric if $xRy \wedge yRx \Rightarrow x = y \quad \forall x, y \in X$

Asymmetric if $xRy \Rightarrow \neg yRx \quad \forall x, y \in X$

Transitive if $xRy \wedge yRz \Rightarrow xRz \quad \forall x, y, z \in X$

Weakly Connected if $xRy \vee yRx \quad \forall x \neq y \in X$

Complete if reflexive and weakly connected.

Properties of Relations

Exercise 1.7

1. Recall the unit circle relation above. Is this relation reflexive, irreflexive, symmetric or transitive?
2. Show that if a relation is asymmetric, then it is irreflexive.
3. Consider the game Rock-Paper-Scissors. Let R be the relation "beats". Which properties does RPS satisfy?
4. Which of the properties would the weak preference relation, $\succsim$, satisfy? Verify that they are satisfied for the preferences on slide 22.

Economic Application: Preference relations

A weak preference relation, $\succsim$, is defined to be **rational** if it is **complete** and **transitive**.

A rational weak preference relation induces two other useful relations:

Name	Symbol	Meaning
Strict Preference	$x \succ y$	$x \succsim y \ \wedge \neg \ y \succsim x$
Indifference	$x \sim y$	$x \succsim y \ \wedge y \succsim x$

Functions

A **function**, f, is a binary relation, (X, Y, G), such that:

1. $\forall x \in X \ \exists y \in Y \ s.t \ (x, y) \in f$
2. $(x, y) \in G \wedge (x, z) \in G \Rightarrow y = z$

That is, each element in the domain *must* be mapped to *exactly* one object in the codomain.

Functions (or mappings) are usually written as $f : X \rightarrow Y$, where f is the name of the function, X is its domain and Y is its codomain; along with a "rule" for assigning each element in X to an element in Y.

Functions

Example

The following are functions:

- $f : \mathbb{R} \to \mathbb{R} \quad f(x) = x^2$
- $\pi : \mathbb{N} \to \mathbb{N} \cup \{0\} \quad \pi(x) = \#\text{ primes} \leq x.$
- $g : \{1, 2, 3\} \to \{a, b\} \quad G = \{(1, a), (2, a), (3, a)\}.$

Exercise 1.8

Which of the following are functions?

1. $f(x) = \frac{1}{x}$
2. $f : \mathbb{R} \to \mathbb{R} \quad f(x) = \frac{1}{x}$
3. $g : \mathbb{R} \to \mathbb{R} \quad G = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
4. The weak preference relation on slide 22
5. $\mathbb{P} : \mathcal{P}(\{H, T\}) \to [0, 1]$ where
   $\mathbb{P}(\{H\}) = \frac{1}{2},\ \mathbb{P}(\{T\}) = \frac{1}{2},\ \mathbb{P}(\emptyset) = 0,\ \mathbb{P}(\{H, T\}) = 1.$

Properties of Functions

The **image** of a set A under a function f is the set

$$f(A) = \{y \in Y \mid \exists x \in A \text{ s.t. } f(x) = y\}$$

The **pre-image** of a set B under a function f is the set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

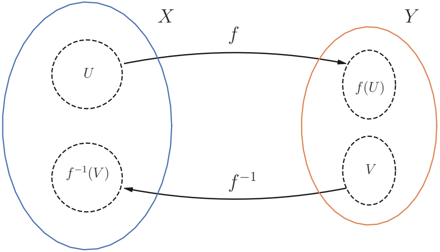
The **range** of a function, $im(f)$, is the image of its domain.

Example

Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2$, and the sets $A = \{1, 2, 3\}$, $B = \{5, 6\}$, $C = \{2\}$

$$f(A) = \{2\} \qquad f^{-1}(B) = \emptyset \qquad f^{-1}(C) = \mathbb{R}$$

Properties of Functions



Exercise 1.9 Let $f : X \to Y$ be a function. Let $U \subseteq X$ and $V \subseteq Y$. Show that

1. $U \subseteq f^{-1}(f(U))$.
2. $f(f^{-1}(V)) \subseteq V$.

Properties of Functions

A function $f : X \to Y$ is said to be an

Injection (one-to-one) if $f(a) = f(b) \Rightarrow a = b \;\; \forall a, b \in X$

Surjection (onto) if $\forall y \in Y \; \exists x \in X \; s.t \; f(x) = y$

Bijection if it is both injective and surjective

Example

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is neither an injection ($-x$ and x both map to the same thing) or surjection (nothing maps to -1).

$f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(x) = x^2$, is a bijection.

Properties of Functions

Functions that are bijective possess an **inverse function**, denoted f^{-1}.

A continuous function $f : \mathbb{R} \to \mathbb{R}$, is invertible iff it is strictly increasing everywhere or strictly decreasing everywhere. If the function is differentiable, it is sufficient to show that $\forall x$, $f^{'}(x) > 0$ or $f^{'}(x) < 0$.

Economic Application: A Utility Representation Theorem

A weak preference relation can be **represented** by a **utility function** if there exists a function U such that

$$x \succsim y \iff U(x) \geq U(y)$$

Theorem

Let X be a finite set and $(X, X, \succsim)$ be a weak preference relation. The preferences can be represented by a utility function if and only if they are complete and transitive.

Cardinality, Revisited

How do we assign cardinality to sets with infinitely many elements, like $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$?

$|A| = |B|$ iff there exists a bijection, $f : A \to B$.

Defining cardinality in this way, we see that not all infinities are equal. In particular, $|\mathbb{N}| \neq |\mathbb{R}|$. (see Cantor's Diagonal Argument)

We refer to sets whose cardinality is either finite or equal to $|\mathbb{N}|$ as having **countably** many elements. Set's that do not satisfy this criterion, such as $\mathbb{R}$, have **uncountably** many elements.

Cardinality, Revisited

Exercise 1.10

1. Show that the set of non-negative even numbers $E = \{0, 2, 4, 6, \dots\}$ has the same cardinality as the naturals.
2. Show that the interval $(0, 1)$ has the same cardinality as the interval $(0, a)$, $a > 0$.

Metric Spaces

A **space** is just a set endowed with some sort of structure.

A **metric space** is a set endowed with a notion of **distance**.

A metric space is a pair (X, d) where X is a set and $d : X \times X \rightarrow \mathbb{R}$ is a function which satisfies the following conditions

1. $d(x, y) \geq 0 \;\; \forall x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x) \;\; \forall x, y \in X$
3. $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X \;\;$ (*Triangle inequality*)

Metric Spaces

The metric we encounter most is the familiar **Euclidean metric**, which is one way of defining distance between two points in $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Recall that the Euclidean metric is defined as

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 ... + (x_n - y_n)^2}$$

Exercise 1.11

1. Show that the Euclidean metric in $\mathbb{R}$, $d(x, y) = |x - y|$, is in fact a metric.
2. Show that the **discrete metric** is a metric.

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

Learning Outcomes

You should be able to:

- Use a truth table to show that a proposition is a tautology, contradiction, or contingent statement.
- Write propositions using logical operators and quantifiers.
- Prove propositions using an appropriate method.
- Manipulate sets with set operations.
- Explain what binary relations and functions are.
- Prove that a binary relation/function has a given property and use that property to prove further propositions.
- Show that a function is a valid metric.