No solutions due in Week 1.

1. Use truth tables to determine whether the following propositions are tautologies, contradictions, or contingencies.

   a) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ b) $(p \Rightarrow q) \Rightarrow (\neg p \Rightarrow \neg q)$ c) $(p \Leftrightarrow q) \Leftrightarrow (\neg p \Leftrightarrow \neg q)$

   Solution

   a) Tautology

p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
T	T	F	F	T
T	F	F	F	T
F	T	F	F	T
F	F	T	T	T

   b) Contingency

p	q	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$	$(p \Rightarrow q) \Rightarrow (\neg p \Rightarrow \neg q)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

   c) Tautology

p	q	$p \Leftrightarrow q$	$\neg p \Leftrightarrow \neg q$	$(p \Leftrightarrow q) \Leftrightarrow (\neg p \Leftrightarrow \neg q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	T	T	T

2. Prove De Morgan's laws for sets.

   a) $(A \cup B)^c = A^c \cap B^c$ b) $(A \cap B)^c = A^c \cup B^c$

   Solution

   a) $x \in (A \cup B)^c \Leftrightarrow x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B \Leftrightarrow x \in A^c$ and $x \in B^c \Leftrightarrow x \in A^c \cap B^c$.
   b) $x \in (A \cap B)^c \Leftrightarrow x \notin A \cap C \Leftrightarrow x \notin A$ or $x \notin B \Leftrightarrow x \in A^c \cup B^c$.

3. Find the power set for each of the following sets.

a) $\{R, G, B\}$ b) $\{\emptyset\}$ c) $\mathcal{P}(\{\emptyset\})$

Solution
a) $\mathcal{P}(\{R, G, B\}) = \{\emptyset, \{R\}, \{G\}, \{B\}, \{R, G\}, \{R, B\}, \{G, B\}, \{R, G, B\}\}$.
b) $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.
c) $\mathcal{P}(\mathcal{P}(\{\emptyset\})) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

4. Prove by contradiction that the sum of a rational number and an irrational number is irrational.

   Solution
   Let $a \in \mathbb{Q}$ be a rational number and $b \in \mathbb{R} \backslash \mathbb{Q}$ be an irrational number. Assume $a + b$ is rational. Then $a + b = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ relatively prime. Since a is rational, a can be represented as $\frac{r}{s}$, with $r, s \in \mathbb{Z}$ relatively prime.

   $b = \frac{p}{q} - \frac{r}{s} = \frac{ps-qr}{qs}$. $ps - qr \in \mathbb{Z}$ and $qs \in \mathbb{Z}$ since integers are closed under addition and multiplication. Therefore b is rational. Contradiction.

   $\blacksquare$

5. Prove by induction that $1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$ for $|x| < 1$.

   Solution
   Base case: $(n = 0)$
   Sum of first 0 powers of x is 1. $\frac{1-x^{0+1}}{1-x} = 1$.

   Induction Step:
   $1 + x + \cdots + x^n + x^{n+1} = \frac{1-x^{n+1}}{1-x} + x^{n+1}$ (by induction hypothesis)
   $= \frac{1-x^{n+1}}{1-x} + \frac{x^{n+1}(1-x)}{1-x} = \frac{1-x^{n+1}}{1-x} + \frac{x^{n+1}-x^{n+2}}{1-x} = \frac{1-x^{n+1}+x^{n+1}-x^{n+2}}{1-x} = \frac{1-x^{(n+1)+1}}{1-x}$ $\blacksquare$

6. Prove by induction that $n! \geq n^2 \quad \forall n \geq 4$.

   Solution
   Base case: $(n = 4)$
   $4! = 4 \times 3 \times 2 \times 1 = 24$. $4^2 = 16$. $24 \geq 16$.

   Induction Step:
   $(n + 1)! = (n + 1)n! \geq (n + 1)n^2$ (by induction hypothesis)
   $= n^3 + n^2 \geq n^2 + 2n + 1$ (since $n^3 \geq 2n + 1 \quad n \geq 4$)

$= (n+1)^2.$

To verify the claim used in the second step, note that $2 + \frac{1}{n} \leq 3 \leq n^2$ if $n = 4$. Since n^2 is increasing, this remains true for $n \geq 4$. Multiply through by n to get the inequality used.

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7. Let R be a binary relation from a set X to itself. Prove the following properties of R:

   a) If R is asymmetric then it is anti-symmetric.

   b) If R is asymmetric then it is irreflexive.

   c) If R is irreflexive and transitive then it is asymmetric.

   Solution
   a) Recall that anti-symmetry is $xRy \wedge yRx \implies x = y$. Asymmetry is $xRy \implies \neg yRx$. If a relation is asymmetric, then $xRy \wedge yRx$ is always false. Implication following from a false proposition is always true. Therefore asymmetry $\implies$ anti-symmetry.

   b) Assume R is not irreflexive. Then $\exists x$ s.t. xRx. By asymmetry, $xRx \implies \neg xRx$. Contradiction. So R must be irreflexive.

   c) Assume R is not asymmetric. Then $\exists x, y \in X$ s.t. xRy and yRx. By transitivity, $xRy \wedge yRx \implies xRx$. This contradicts irreflexivity, so R must be asymmetric.

8. Let $X = \{a, b, c, d\}$, and $(X, \succsim)$ be a rational weak preference relation. Under these preferences we have $a \succ b \sim c \succ d$. What is the graph of the relation?

   Solution

$$G = \{(a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (b,d), (c,b), (c,c), (c,d), (d,d)\}$$

9. Find an example of a function $f : \mathbb{N} \to \mathbb{N}$ that is:

   a) surjective but not injective.

   b) injective but not surjective.

   c) neither injective or surjective.

   d) bijective.

Solution
a) $f(n) = 1$.
b) $f(n) = n + 1$
c) $f(1) = 1$, $f(n) = 2$; $n \geq 2$
d) $f(n) = n$.

10. Let $f : X \to Y$ be a function. Let U_1 and U_2 be subsets of X, and let V_1 and V_2 be subsets of Y. Show that

a) if $U_1 \subseteq U_2$ then $f(U_1) \subseteq f(U_2)$.

b) if $V_1 \subseteq V_2$ then $f^{-1}(V_1) \subseteq f^{-1}(V_2)$.

c) $\forall U$, $U \subseteq f^{-1}(f(U))$.

d) $\forall V$, $f(f^{-1}(V)) \subseteq V$.

For c) and d), produce an example where the left hand side is a *strict subset* of the right hand side.

Solution

a) Recall that $f(U_1) = \{y \in Y \mid \exists x \in U_1 \text{ s.t. } f(x) = y\}$ and $f(U_2) = \{y \in Y \mid \exists x \in U_2 \text{ s.t. } f(x) = y\}$. We're aiming to show that if $y \in f(U_1)$ then $y \in f(U_2)$ also.

Pick a $y \in f(U_1)$. Then $\exists x \in U_1$ such that $f(x) = y$ by definition. Since $U_1 \subset U_2$, $x \in U_2$. Since $f(x) = y$, and $x \in U_2$, then $y \in f(U_2)$ by definition.

b) $f^{-1}(V_1) = \{x \in X \mid f(x) \in V_1\}$. Pick an $x \in f^{-1}(V_1)$. Then $f(x) \in V_1$. Since $V_1 \subset V_2$, $f(x) \in V_2$. So $x \in f^{-1}(V_2)$.

c) Pick an $x \in U$. Then $f(x) \in f(U)$ by definition. $f^{-1}(f(U)) = \{a \in X \mid f(a) \in f(U))$. x satisfies that requirement, so $x \in f^{-1}(f(U))$. Therefore $U \subset f^{-1}(f(U))$.

Consider the mapping $f : \mathbb{N} \to \mathbb{N}$, $f(n) = 1$. Let $U = \{1, 2\}$. We have $f(U) = \{1\}$, $f^{-1}(\{1\}) = \mathbb{N}$, and $U \subset \mathbb{N}$. We get a proper subset because there are other elements in the domain not in our set U that also map to 1.

d) Pick a $y \in f(f^{-1}(V))$. Then there must be an $x \in f^{-1}(V)$ such that $f(x) = y$. Since $x \in f^{-1}(V)$, there must be a $z \in V$ such that $f(x) = z$. From the definition of a function, if $f(x) = y$ and $f(x) = z$, then $z = y$ so $y \in V$. Therefore $f(f^{-1}(V)) \subseteq V$.

Consider the mapping $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n + 1$. Let $V = \{1, 2\}$. Then $f^{-1}(V) = \{1\}, f(\{1\}) = \{2\} \subset V$. We get a proper subset because there's an element in our set V, namely 1, that nothing in the domain maps to.

11. Show that a preference relation $(X, X, \succsim)$ can be represented by a utility function *only if* preferences are complete and transitive.

    Solution
    For any $x, y \in X$, either $U(x) \geq U(y)$ or $U(y) \geq U(X)$. Since U represents the preferences, either $x \succsim y$ or $y \succsim x$. Further, $\forall x \in X$, $U(x) \leq U(x)$, so $x \succsim x$. Therefore the preferences must be complete.

    For any $x, y, z \in X$, $U(x) \geq U(y) \wedge U(y) \geq U(Z) \implies U(x) \geq U(z)$, since $\geq$ is transitive. Therefore $x \succsim y \wedge y \succsim z \implies x \succsim z$, so preferences are also transitive.

12. Let (X, d_1) and (X, d_2) be metric spaces. Show that $d_3(x, y) = \max\{d_1(x, y), d_2(x, y)\}$ is a valid metric.

    Solution
    Assumption 1: if $x = y$, then $d_1(x, y) = d_2(x, y) = 0$. $d_3(x, y) = \max\{0, 0\} = 0$. If $x \neq y$, then d_1 and d_2 are both non-zero. The max of two non-zero numbers is non-zero.

    Assumption 2: $d_3(x, y) = \max\{d_1(x, y), d_2(x, y)\} = \max\{d_1(y, x), d_2(y, x)\} = d_3(y, x)$.

    Assumption 3: Note that
    $d_1(x, y) \leq d_1(x, z) + d_1(z, y) \leq \max\{d_1(x, z), d_2(x, z)\} + \max\{d_1(z, y), d_2(z, y)\}$.
    $d_2(x, y) \leq d_2(x, z) + d_2(z, y) \leq \max\{d_1(x, z), d_2(x, z)\} + \max\{d_1(z, y), d_2(z, y)\}$.

    If the inequality holds for *both* d_1 and d_2 then it must hold for the *maximum* of the two (since the max will be either one or the other).
    $\max\{d_1(x, y), d_2(x, y)\} \leq \max\{d_1(x, z), d_2(x, z)\} + \max\{d_1(z, y), d_2(z, y)\}$.

13. Prove that the union of any number of open sets is open.

    Solution
    Consider the union of two open sets $A \cup B$ (this generalises to any number of open sets). Pick any point $x \in A \cup B$. x must be in A or B or both.

    If $x \in A$, then since A is an open set,

    $$\exists \varepsilon_1 > 0 \ s.t \ B(x, \varepsilon_1) \subset A \subset A \cup B$$

If $x \in B$, then since B is an open set,

$$\exists \varepsilon_2 > 0 \ s.t \ B(x, \varepsilon_2) \subset B \subset A \cup B$$

If x is in both, pick either of $\varepsilon_1, \varepsilon_2$.

So for any point in $A \cup B$, we can fit an open ball around it completely contained in $A \cup B$. So $A \cup B$ is an open set.

14. Show that the intersection of finitely many open sets is open.

Solution
Consider the intersection of two open sets, $A \cap B$. Pick any $x \in A \cap B$.

$x \in A$, which is open, so

$$\exists \varepsilon_1 > 0 \ s.t \ B(x, \varepsilon_1) \subset A$$

$x \in B$, which is open, so

$$\exists \varepsilon_2 > 0 \ s.t \ B(x, \varepsilon_2) \subset B$$

Pick $\min\{\varepsilon_1, \varepsilon_2\}$.

$$B(x, \min\{\varepsilon_1, \varepsilon_2\}) \subset B(x, \varepsilon_1) \subset A$$
$$B(x, \min\{\varepsilon_1, \varepsilon_2\}) \subset B(x, \varepsilon_2) \subset B$$
$$B(x, \min\{\varepsilon_1, \varepsilon_2\}) \subset A \cap B$$

We can't generalise this to the intersection of any number of sets because, in general, $\min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, ...\}$ may not exist. It does, however, exist for finitely many intersections. So the intersection of finitely many open sets is open.