Solutions due by 10.30am Friday 19th February.

1. Show that a closed ball is a closed set.

2. Show that if $A \subset X$ is a closed set, and $a_n \in A$ is a sequence, then $a_n \to a \implies a \in A$.

3. Show that if a sequence is convergent, then it is Cauchy.

4. Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Show that if there is a convergent subsequence, $\{a_{n_k}\}_{k=1}^{\infty}$, such that $a_{n_k} \to c$ then $a_n \to c$.

5. The Bolzano-Weirstrass theorem states that every bounded sequence of real numbers has a convergent subsequence. Use this property to show that the metric space (X, d), where X is a compact subset of reals, is complete.

6. Solve the following systems of linear equations.

a)

$$\begin{aligned} x + 2y + z - w &= 1 \\ 3x + 6y - z - 3w &= 2 \end{aligned}$$

b)

$$\begin{aligned} x + 2z &= 0 \\ x + y + 2z &= 2 \\ 2x + y + 4z &= 3 \\ 5x + 10z &= 0 \end{aligned}$$

c)

$$\begin{aligned} x + y &= 15 \\ 2y &= 20 \\ x + 3y &= 35 \\ 2x + 4y &= 50 \end{aligned}$$

7. Do the vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$, form a basis for $\mathbb{R}^3$?

8. Consider the map $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

   a) Find a basis for the column space.
   b) Find a basis for the nullspace.
   c) Show that the column space and null space are orthogonal.

9. Let $Ax = b$ be an $m \times n$ system of equations and let $\mathcal{S} = \{x \in \mathbb{R}^n \mid Ax = b\}$ be the solution set. Show that if $\mathcal{S}$ is non-empty, such that there is at least one particular solution, x^*, then $\mathcal{S}$ is the affine subspace $\mathcal{S} = \{x^* + v \mid v \in \text{Null}(A)\}$.

   (Hint: Show that if some vector $x' \in \{x \in \mathbb{R}^n \mid Ax = b\}$ then it must also be in $\{x^* + v \mid v \in \text{Null}(A)\}$, and vice versa)

10. Let X be an $n \times p$ matrix with full column rank. Show that $X'X$ is invertible.

    (Hint: Show that the nullspace of $X'X$ only contains 0)