Solutions due by 10.30am Friday 19th February.

1. Show that a closed ball is a closed set.

   Solution:
   Show that the complement of $\bar{B}(x,\varepsilon)$ is open. For any $y \in \bar{B}(x,\varepsilon)^c$, pick $B(y, d(x,y)-\varepsilon)$ to be the open ball around y. To show that $B(y, d(x,y)-\varepsilon) \subset \bar{B}(x,\varepsilon)^c$, consider any $z \in B(y, d(x,y)-\varepsilon)$. By the triangle inequality, $d(x,y) \leq d(x,z) + d(z,y)$. Since $z \in B(y, d(x,y)-\varepsilon)$, then $d(z,y) < d(x,y)-\varepsilon$. Plugging into the triangle inequality gives $d(x,y) < d(x,z) + d(x,y) - \varepsilon$. Rearranging gives $d(x,z) > \varepsilon$, hence it is not in $\bar{B}(x,\varepsilon)$, and is instead in the complement. ■

2. Show that if $A \subset X$ is a closed set, and $a_n \in A$ is a sequence, then $a_n \to a \implies a \in A$.

   Solution:
   We will prove by contradiction. Suppose that $a \in A^c$. Then since the complement of a closed set is open, $\exists \varepsilon > 0$ such that $B(a,\varepsilon) \subset A^c$. By the definition of convergence, this ball must contain all but finitely many elements of a_n; however, since it's fully contained in A^c, it in fact has no elements of a_n. Contradiction. Therefore $a \in A$. ■

3. Show that if a sequence is convergent, then it is Cauchy.

   Solution:
   For any $\varepsilon > 0$, recall that the defintion of convergence requires that we can find an N such that $\forall m,n > N$, $d(x_m,c) < \frac{\varepsilon}{2}$ and $d(x_n,c) < \frac{\varepsilon}{2}$. The triangle inequality gives us that $d(x_m,x_n) \leq d(x_m,c) + d(c,x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus $\forall \varepsilon > 0\ \exists N$ s.t. $\forall m,n > N$ $d(x_m,x_n) < \varepsilon$, which is the definition of a Cauchy sequence.

4. Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Show that if there is a convergent subsequence, $\{a_{n_k}\}_{k=1}^{\infty}$, such that $a_{n_k} \to c$ then $a_n \to c$.

   Solution:
   For $\varepsilon > 0$, since $a_{n_k} \to c$, there exists an N large enough such that $d(a_{n_k},c) < \frac{\varepsilon}{2}$ for all $n_k > N$. Since a_n is Cauchy, there's also an N large enough that $d(a_n,a_m) < \frac{\varepsilon}{2}$ for $m,n > N$. Pick N large enough such that both of these properties hold at the same time for $n_k > n > N$. By the triangle inequality, $d(a_n,c) \leq d(a_n,a_{n_k}) + d(a_{n_k},c) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

5. The Bolzano-Weirstrass theorem states that every bounded sequence of real numbers has a convergent subsequence. Use this property to show that the metric space (X,d), where X is a compact subset of reals, is complete.

Solution:
To show completeness we need to show that every Cauchy sequence converges to a point in the set. Let $\{x_n\}$ be a Cauchy sequence in X. Since X is a compact subset of the reals, it is closed and bounded. Since X is bounded, by the Bolzano-Weirstrass theorem, the sequence must have a convergent subsequence. From the solution above, if a Cauchy sequence has a convergent subsequence, then the whole sequence also converges to that point. Since X is closed, the subsequence converges to a point in X. Therefore every Cauchy sequence converges to a point in X, and the space is complete.

6. Solve the following systems of linear equations.

a)
$$\begin{aligned} x + 2y + z - w &= 1 \\ 3x + 6y - z - 3w &= 2 \end{aligned}$$

b)
$$\begin{aligned} x + 2z &= 0 \\ x + y + 2z &= 2 \\ 2x + y + 4z &= 3 \\ 5x + 10z &= 0 \end{aligned}$$

c)
$$\begin{aligned} x + y &= 15 \\ 2y &= 20 \\ x + 3y &= 35 \\ 2x + 4y &= 50 \end{aligned}$$

Solution:
a)
$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 1 \\ 3 & 6 & -1 & -3 & 2 \end{array}\right) \overset{\text{rref}}{\rightarrow} \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & \frac{3}{4} \\ 0 & 0 & 1 & 0 & \frac{1}{4} \end{array}\right)$$

Columns 1 and 3 are the basic variables. Columns 2 and 4 the free variables. We have $x_1 = \frac{3}{4} - 2x_2 + x_4$. $x_2 = x_2$. $x_3 = \frac{1}{4}$. $x_4 = x_4$. Or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ 0 \\ \frac{1}{4} \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_4$$

b)

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 2 & 2 & 2 \\ 2 & 1 & 4 & 3 \\ 5 & 0 & 10 & 0 \end{array}\right) \overset{\text{rref}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Last column is a basic column. No solutions.

c)

$$\left(\begin{array}{cc|c} 1 & 1 & 15 \\ 0 & 2 & 20 \\ 1 & 3 & 35 \\ 2 & 4 & 50 \end{array}\right) \overset{\text{rref}}{\to} \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Unique solution. $x_1 = 5$, $x_2 = 10$.

7. Do the vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$, form a basis for $\mathbb{R}^3$?

   Solution:
   Since $\mathbb{R}^3$ is 3 dimensional, a basis must have exactly 3 vectors, which we have. To determine whether they are linearly independent, we can use the determinant:

$$\begin{aligned}
\det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix} &= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 3 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\
&= 1 \cdot (2-6) - 2 \cdot (4-3) + 3 \cdot (4-1) \\
&= -4 - 2 + 9 \\
&= 3
\end{aligned}$$

   Determinant is non-zero, so vectors are linearly independent.

8. Consider the map $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

   a) Find a basis for the column space.
   b) Find a basis for the nullspace.
   c) Show that the column space and null space are orthogonal.

Solution:
a) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \stackrel{\text{rref}}{\to} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. The basic column form a basis for the column space, $col(A) = \text{span}\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$.

b) The nullspace is the set of vectors, x that solve $Ax = 0$. Set up the augmented matrix and put in rref.
$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) \stackrel{\text{rref}}{\to} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$
Which gives solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$. So $\text{Null}(A) = \text{span}\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$.

c) Each element of $col(A)$ has the form $\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Each element of the nullspace has the form $\beta \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. To show orthogonality, the inner product of these must be zero.

$\langle \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \beta \begin{bmatrix} -2 \\ 1 \end{bmatrix} \rangle = \alpha\beta(1 \cdot -2 + 2 \cdot 1) = 0.$

9. Let $Ax = b$ be an $m \times n$ system of equations and let $\mathcal{S} = \{x \in \mathbb{R}^n \mid Ax = b\}$ be the solution set. Show that if $\mathcal{S}$ is non-empty, such that there is at least one particular solution, x^*, then $\mathcal{S}$ is the affine subspace $\mathcal{S} = \{x^* + v \mid v \in \text{Null}(A)\}$.

   (Hint: Show that if some vector $x' \in \{x \in \mathbb{R}^n \mid Ax = b\}$ then it must also be in $\{x^* + v \mid v \in \text{Null}(A)\}$, and vice versa)

   Solution:
   Let $x' \in \{x \in \mathbb{R}^n \mid Ax = b\}$. Consider the vector $v = x' - x^*$. $Av = A(x' - x^*) = Ax' - Ax^* = b - b = 0$. Therefore $v \in \text{Null}(A)$. So $x' = x^* + v$, $v \in \text{Null}(A)$, and $x' \in \{x^* + v \mid v \in \text{Null}(A)\}$.

   Let $x' \in \{x^* + v \mid v \in \text{Null}(A)\}$. Then $x' = x^* + v$, for some $v \in \text{Null}(A)$. $Ax' = A(x^* + v) = Ax^* + Av = b + 0 = b$. Therefore $x' \in \{x \in \mathbb{R}^n \mid Ax = b\}$. ■

10. Let X be an $n \times p$ matrix with full column rank. Show that $X'X$ is invertible.

(Hint: Show that the nullspace of $X'X$ only contains 0)

Solution:
Suppose there is a vector v in the nullspace of $X'X$, such that $X'Xv = 0$. It must be that $v'X'Xv = 0$, since $v'(X'Xv) = v'0 = 0$. But $v'X'Xv = (Xv)'(Xv) = \langle Xv, Xv\rangle = ||Xv||^2 = 0$.

$||Xv||^2 = 0 \implies v = 0$, since X is full column rank and so Null(X) = $\{0\}$. Therefore the only $v \in \text{Null}(X'X)$ is $v = 0$. Therefore $X'X$ is full column rank. Since $X'X$ is square, it must be invertible.