Solutions due by 10.30am Friday 5$^{\text{th}}$ March.

1. Show that $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]]$.

2. Suppose $\sqrt{n}(\mathbf{x} - \mu) \xrightarrow{d} N(0, \Sigma)$, where $\mathbf{x}$ is a p-vector and Σ a $p \times p$ covariance matrix. Show that $n(\mathbf{x} - \mu)'\Sigma^{-1}(\mathbf{x} - \mu) \xrightarrow{d} \chi^2(p)$.

3. Suppose $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2)$. Let $g(x)$ be a differentiable function.

   a) Show that if $g'(\theta) \neq 0$, then $\sqrt{n}(g(\hat{\theta} - g(\theta)) \xrightarrow{d} N(0, g'(\theta)^2\sigma^2)$.

   b) What's the asymptotic distribution of $\bar{X}^2$?

4. Let $X \sim N(\mu, \sigma^2)$. Show that $\mathbb{P}(|X - \mu| > k\sigma) \leq 2e^{-k^2/2}$.

   [Hint: Markov's inequality + MGF]

5. The clinical trial research for the Pfizer vaccine was published in the New England Journal of Medicine: 10.1056/NEJMoa2034577. The trial consists of two populations: the treated (who received the vaccine) and the placebo (who did not). The study measures the number of COVID infections in each population one week after receiving the second dose.

   Let p_T and p_P be the true proportion of people who would became infected with COVID if treated and not treated respectively. Let n_T and n_P be the number of people in each of the treated and placebo populations respectively, with $\hat{p}_T$ and $\hat{p}_P$ the sample proportions observed in the study. The incident risk ratio is defined to be $IRR = p_T/p_P$. The effectiveness of the vaccine is defined to be $1 - IRR$.

   The Pfizer study reports: 8 COVID infections in the treated group of size $n_T = 18,198$; 162 infections in the placebo group of size $n_P = 18,325$.

   a) What is the point estimate for vaccine effectiveness: $1 - \widehat{IRR}$?

   b) What asymptotic distribution does $\hat{p} - p$ have for each population?

   c) What asymptotic distribution does $\ln \hat{p}_T - \ln \hat{p}_P$ have?

   d) Find a 95% confidence interval for the vaccine effectiveness.

6. Suppose an agent needs to know a parameter θ, but cannot observe it directly. Suppose they are given two noisy signals:

$$x = \theta + \varepsilon$$
$$y = \theta + \nu$$

where $\theta \sim N(0, \sigma_\theta^2)$, $\varepsilon \sim N(0, \sigma_\varepsilon^2)$, $\nu \sim N(0, \sigma_\nu^2)$, all pairwise independent. The agent estimates θ with a linear combination of the signals. What weights should they choose such that their estimator is minimum mean square error?

7. Consider the Solow growth model described by

$$k_{t+1} = sk_t^\alpha + (1-\delta)k_t \tag{1}$$
$$c_t = (1-s)k_t^\alpha \tag{2}$$

where $0 \leq \alpha < 1$, $s \in [0,1]$ is a constant savings rate, and $\delta \in [0,1]$ is the capital depreciation rate.

a) Solve for the unique steady state levels of capital and consumption.
b) Find the savings rate that maximizes consumption in the steady state.
c) Log-linearize the two equations.
d) Describe the approximate solution as a system of first order difference equations.

8. Log-linearize the following equations assuming a steady state exists:

a) $c_t + k_{t+1} = F(k_t) + (1-\delta)k_t$
b) $U'(c_t) = \beta U'(c_{t+1})[F'(k_{t+1}) + 1 - \delta]$