Lecture 8: Optimization

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The Unconstrained Maximization Problem

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}$.

The **interior** of a set X is the set

$$int\,(X) = \{\mathbf{x} \in X \mid \exists r > 0 \; s.t \; B(\mathbf{x}, r) \subset X\}$$

Consider the following maximisation problem

$$\max_{\mathbf{x} \in int(X)} \quad f(\mathbf{x})$$

The Unconstrained Maximization Problem

The solution to this problem is the set

$$S = \{\mathbf{x} \in int\,(X) \mid f(\mathbf{x}) \geq f(\mathbf{a}) \;\; \forall \mathbf{a} \in int\,(X)\}$$

The fact that we're doing this over the interior of the set makes the problem **unconstrained**.

If we were to allow for maximisers to come from the boundary of the set, then the problem would be **constrained** by the boundary.

First Order Condition

Proposition

Let $f : X \to R$ be a $\mathcal{C}^1$ function defined on a subset of $\mathbb{R}^n$. If $\mathbf{x}_0$ is a local maximum (or minimum) of f and $\mathbf{x}_0 \in int\,(X)$, then

$$\nabla f(\mathbf{x}_0) = 0$$

The proof for this is identical to the one variable case.

Second Order Condition

Proposition

Let $f : X \to \mathbb{R}$ be a $\mathcal{C}^2$ function defined on a subset of $\mathbb{R}^n$. Suppose that $\mathbf{x}_0$ is a critical point of f.

If $D^2 f(\mathbf{x}_0)$ is **negative definite**, then $\mathbf{x}_0$ is a local maximum

If $D^2 f(\mathbf{x}_0)$, is **positive definite**, then $\mathbf{x}_0$ is a local minimum

If $D^2 f(\mathbf{x}_0)$, is **indefinite**, then $\mathbf{x}_0$ is neither a max or a min

The Unconstrained Optimization Recipe

1. Find the partial derivatives with respect to each of your choice variables. Set them equal to zero. Solve the system of equations for the set of critical points.

2. Check the definiteness of the Hessian at each of these critical points. Discard any points which do not meet the necessary conditions for a maximum.

3. Plug each remaining critical point into the objective function a check which gives the highest value.

4. Consider the behaviour of the objective function near the boundary. Remember, there may not even be a solution!

Concavity and Convexity

Theorem

Let $f : X \to \mathbb{R}$ be a $\mathcal{C}^2$ function whose domain is a convex open subset of $\mathbb{R}^n$.

a) *The following conditions are equivalent*
   i) f *is a concave function on* X
   ii) $f(\mathbf{y}) - f(\mathbf{x}) \leq Df(\mathbf{x})(\mathbf{y} - \mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y} \in X$
   iii) $D^2 f(\mathbf{x})$ *is negative semidefinite for all* $\mathbf{x} \in X$

b) *The following conditions are equivalent*
   i) f *is a convex function on* X
   ii) $f(\mathbf{y}) - f(\mathbf{x}) \geq Df(\mathbf{x})(\mathbf{y} - \mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y} \in X$
   iii) $D^2 f(\mathbf{x})$ *is positive semidefinite for all* $\mathbf{x} \in X$

c) *If* f *is a concave function on* X *and* $Df(\mathbf{x}^*) = 0$ *for some* $\mathbf{x}^* \in X$*, then* $\mathbf{x}^*$ *is a global max of* f *on* X

d) *If* f *is a convex function on* X *and* $Df(\mathbf{x}^*) = 0$ *for some* $\mathbf{x}^* \in X$*, then* $\mathbf{x}^*$ *is a global min of* f *on* X

Exercises

Exercise 8.1 For the following functions, find the critical points and classify these as local max, local min, saddle points, or “can’t tell”.

a) $x^4 + x^2 - 6xy + 3y^2$

b) $xy^2 + x^3y - xy$

Exercises

Exercise 8.2

A perfectly competitive firm produces a final consumption good using labour and capital inputs. Labour is paid a wage w and capital is rented at a rate r. The firm faces a constant returns to scale Cobb-Douglas production technology. All capital is returned after production. The price of the final good is normalised to 1. How does the firm choose labour and capital input so as to maximise profit? Solve the firm's problem:

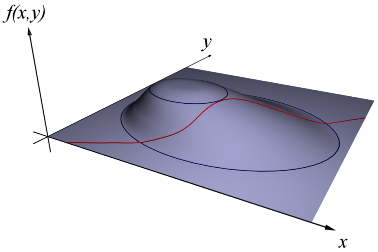
$$\max_{\{L,K\}} \quad K^{\alpha}L^{1-\alpha} - wL - rK$$

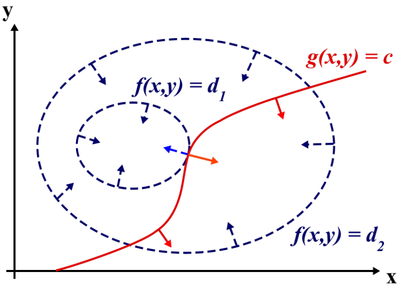
Equality Constrained Optimization

Consider the optimisation problem

$$\max_{\{x_1,x_2\}} \quad f(x_1,x_2)$$
$$\text{s.t} \quad g(x_1,x_2) = h$$

To determine what conditions might be necessary for an optimum, consider these functions and their level sets.





Equality Constrained Optimization

If the level set of the objective function and the constraint set aren't tangential at some point, then they cross.

This means we could move along the constraint set while at the same time moving up to a higher level curve (and so a higher value for the objective function).

It must be the case then that at an optimum, the level set of the function and the constraint set are tangential.

$$\nabla f(\mathbf{x}^*) = \lambda \nabla g(\mathbf{x}^*)$$

This is true whether we are maximising or minimising the objective.

However, for λ to be uniquely defined, we need to require that $\nabla g(\mathbf{x}^*) \neq 0$.

Equality Constrained Optimization

Now consider the optimisation problem with several binding constraints

$$\begin{aligned}
\max_{\{x_1,x_2\}} \quad & f(x_1, x_2, \dots, x_n) \\
\text{s.t} \quad & g_1(x_1, x_2, \dots, x_n) = h_1 \\
\text{s.t} \quad & g_2(x_1, x_2, \dots, x_n) = h_2 \\
& \vdots \\
\text{s.t} \quad & g_m(x_1, x_2, \dots, x_n) = h_m
\end{aligned}$$

At the optimum, $\mathbf{x}^*$, it's still necessary that each of the constraint sets are tangential to the level curve of the objective function.

We still need $\mathbf{x}^*$ not to be a critical point of *any* of the constraint functions.

Non-degenerate Constraint Qualification

Consider the Jacobian matrix of the constraint functions at x^*

$$D\mathbf{h}(\mathbf{x}^*) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{pmatrix}$$

The largest the rank of this Jacobian can be is m (since we assume to have $m < n$). Not having x^* be a critical point of any constraint is equivalent to saying that this Jacobian has full rank. This is known as the **Non-degenerate Constraint Qualification** (NDCQ).

Linear constraints always satisfy the NDCQ.

The Theorem of Lagrange

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Consider the problem of maximising (or minimising) f on the constraint set

$$C_h = \{\mathbf{x} \in \mathbb{R}^n \mid h_i(\mathbf{x}) = a_i \ \forall i = 1, 2, \ldots, m\}$$

Suppose that $\mathbf{x}^* \in C_h$ is a local max or min of f on C_h and x^* satisfies the NDCQ.
Then there exist $\lambda^* = (\lambda_1^*, \ldots, \lambda_n^*)$ such that $(\mathbf{x}^*, \lambda^*)$ is a critical point of the Lagrangian

$$\mathcal{L} = f(\mathbf{x}) + \lambda_1 \left[a_1 - h_1(\mathbf{x})\right] + \cdots + \lambda_n \left[a_n - h_n(\mathbf{x})\right]$$

The Theorem of Lagrange

Example

A consumer has a budget of w dollars which they can spend on a mixture of consumption goods, c_1 and c_2; costing p_1 and p_2 dollars respectively. Their preferences are represented by $U(c_1, c_2) = c_1^\alpha c_2^\beta$, $\alpha + \beta \leq 1$. Set up and solve the consumer's utility maximization problem.

$$\max_{\{c_1, c_2\}} \; c_1^\alpha c_2^\beta$$
$$\text{s.t.} \quad p_1 c_1 + p_2 c_2 = w$$

which yields the lagrangian

$$\mathcal{L} = c_1^\alpha c_2^\beta + \lambda \left[w - p_1 c_1 - p_2 c_2 \right]$$

The Theorem of Lagrange

Example

Setting $\nabla \mathcal{L} = 0$ yields the first order conditions:

$$\alpha c_1^{\alpha-1} c_2^{\beta} - \lambda p_1 = 0 \tag{1}$$

$$\beta c_1^{\alpha} c_2^{\beta-1} - \lambda p_2 = 0 \tag{2}$$

$$p_1 c_1 + p_2 c_2 = w \tag{3}$$

Dividing (1) and (2) gives

$$\frac{\alpha c_2}{\beta c_1} = \frac{p_1}{p_2} \Rightarrow c_2 = \frac{\beta p_1}{\alpha p_2} c_1$$

Subbing into (3) gives

$$p_1 c_1 = \frac{\alpha}{\alpha + \beta} w \qquad p_2 c_2 = \frac{\beta}{\alpha + \beta} w$$

Exercises

Exercise 8.3

$$\max_{\{x,y\}} \quad x$$
$$\text{s.t.} \quad x^3 + y^2 = 0$$

Second Order Conditions

Second order conditions are substantially more difficult to check in the constrained case.

Rather than checking the definiteness of the Hessian, we need to construct a **bordered** Hessian, and check its definiteness.

We won't go into detail here, if interested, see Simon & Bloom (pp. 386 and 457).

These only guarantee *local* maxima/minima anyway.

For now we will manually check the values at the critical points on the interior and check the boundary.

We'll try to set up problems such that there is exactly one critical point, and next lecture we'll find some assumptions that guarantee an interior solution.

Exercises

Exercise 8.4 A Two Period Consumption/Savings Model

An agent lives for two periods. The agent receives utility in each period from consumption with the period utility function $U(c) = \log c$. Second period utility is discounted by a factor $\beta \in (0, 1)$.

Each period, the agent receives income, y. In the first period, the agent is allowed to save some of their income at a gross rate of return, r. Consumption is financed from period income and savings. The agent chooses consumption in each period and savings to take into the second period to maximise lifetime utility.

a) Set up and solve the agents problem.
b) When will savings be strictly positive?
c) Solve for c_1, c_2, and s_2.
d) What is $\mathrm{d}c_1/\mathrm{d}r$?

Finite Horizon Consumption-Savings Problem

We can push our little two-period problem out to T periods. Let's also allow the agent to borrow (so s_t can be negative), as long as they don't die in debt.

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{T} \beta^t U(c_t) \\
\text{s.t.} \quad & c_t + s_{t+1} = y_t + (1+r)s_t \quad t = 0, \dots, T \\
& s_{T+1} = 0 \\
& s_0
\end{aligned}
$$

Giving a Lagrangian

$$
\mathcal{L} = \sum_{t=0}^{T} \left\{ \beta^t U(c_t) + \lambda_t [y_t + (1+r)s_t - c_t - s_{t+1}] \right\} + \lambda_{T+1} s_{T+1}
$$

Finite Horizon Consumption-Savings Problem

Our optimality conditions are:

$$
\begin{aligned}
c_t: &\quad \beta^t U'(c_t) - \lambda_t = 0 &\quad t = 0, \ldots, T \\
s_{t+1}: &\quad -\lambda_t + (1+r)\lambda_{t+1} = 0 &\quad t = 0, \ldots, T-1 \\
s_{T+1}: &\quad -\lambda_T + \lambda_{T+1} = 0 & \\
\lambda_t: &\quad c_t + s_{t+1} = y_t + (1+r)s_t &\quad t = 0, \ldots, T \\
\lambda_{T+1}: &\quad s_{T+1} = 0 &
\end{aligned}
$$

Finite Horizon Consumption-Savings Problem

Suppose we have $U(c_t) = \log c_t$. Suppose further $y_t = 0$, such that the agent just consumes out of their initial savings. The Euler equation and resource constraint become:

$$c_{t+1} = \beta(1+r)c_t$$
$$s_{t+1} = (1+r)s_t - c_t$$

Which is matrix form is

$$\begin{bmatrix} c_{t+1} \\ s_{t+1} \end{bmatrix} = \begin{pmatrix} \beta(1+r) & 0 \\ -1 & (1+r) \end{pmatrix} \begin{bmatrix} c_t \\ s_t \end{bmatrix}$$

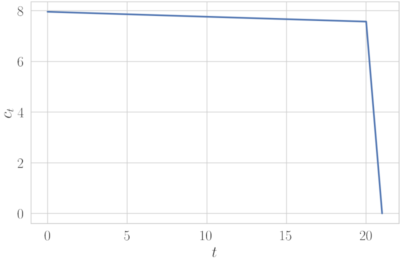
which has solution

$$\begin{bmatrix} c_t \\ s_t \end{bmatrix} = a_1[\beta(1+r)]^t \begin{bmatrix} (1-\beta)(1+r) \\ 1 \end{bmatrix} + a_2(1+r)^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

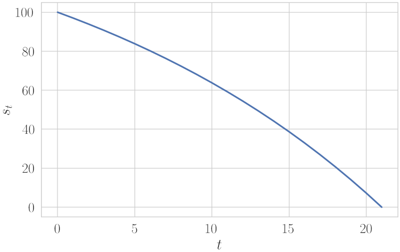
Finite Horizon Consumption-Savings Problem

Plugging in boundary conditions, s_0 and $s_{T+1} = 0$ gives

$$a_1 = \frac{s_0}{1 - \beta^{T+1}} \qquad a_2 = \frac{s_0}{1 - \beta^{-(T+1)}}$$



(a) Consumption



(b) Savings

$T = 20$, $\beta = 0.95$, $r = 0.05$, $s_0 = 100$.

Infinite Horizon Consumption-Savings Problem

What happens if we allow the agent to be infinitely lived and push $T \to \infty$.

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \beta^t U(c_t) \\
\text{s.t.} \quad & c_t + s_{t+1} = y_t + (1+r)s_t \quad t = 0, \ldots, T \\
& \lim_{t \to \infty} \beta^t U'(c_t) s_{T+1} = 0 \\
& s_0
\end{aligned}
$$

Notice that our previous boundary condition now looks like $\lim_{t \to \infty} \beta^t U'(c_t) s_{T+1} = 0$. This is called the **transversality condition**.

Infinite Horizon Consumption-Savings Problem

Our Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{T} \left\{ \beta^t U(c_t) + \lambda_t [y_t + (1+r)s_t - c_t - s_{t+1}] \right\}$$

With optimality conditions:

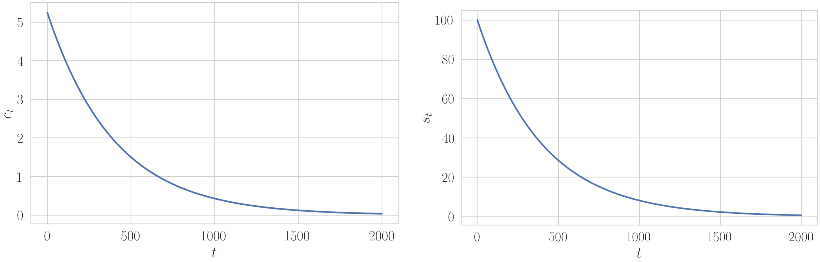
$$\begin{aligned} c_t : \quad && \beta^t U'(c_t) - \lambda_t = 0 \quad && \forall t \\ s_{t+1} : \quad && -\lambda_t + (1+r)\lambda_{t+1} = 0 \quad && \forall t \\ \lambda_t : \quad && c_t + s_{t+1} = y_t + (1+r)s_t \quad && \forall t \end{aligned}$$

That's a little bit nicer!

Infinite Horizon Consumption-Savings Problem

The solution looks the same as before, but now the constants come out a little cleaner. After some algebra we get:

$$a_1 = s_0 \qquad a_2 = 0$$



(a) Consumption (b) Savings

$\beta = 0.95$, $r = 0.05$, $s_0 = 100$

Ramsey Growth Model

There is an infinitely lived agent with logarithmic utility in consumption and discount factor $\beta \in (0,1)$. This agent begins their life with a stock of capital, k_0. Each period, the agent uses their capital stock to produce output according to the production technology $F(k_t) = Ak_t^\alpha$, $0 < \alpha < 1$.

The production process causes capital to depreciate at rate δ. After production, the agent chooses how to allocate output and the remaining capital between consumption in the present period and capital to take into the next period.

Exercise 8.5

a) Set up the problem and solve for optimality conditions.
b) Determine if the system admits a steady state.
c) Log-linearize the system around the steady state.
d) Characterize the system of first order difference equations.

Ramsey Growth Model

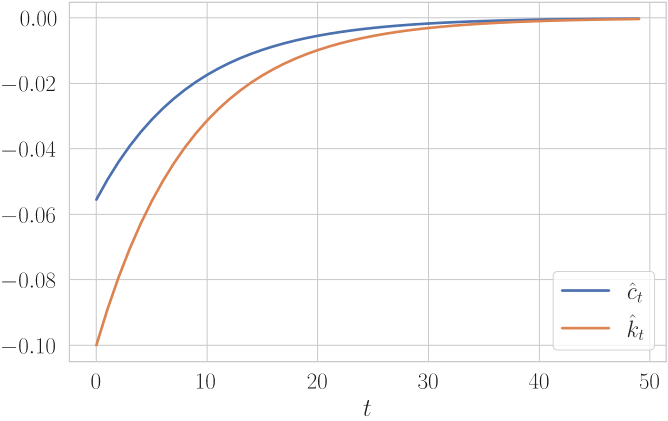


Figure: Ramsey model with $\alpha = 0.3$, $\beta = 0.95$, $\delta = 0.05$, $A = 1$, $k_0 = -0.1$

Learning Outcomes

You should be able to:

1. Solve unconstrained optimization problems and verify second order conditions.
2. Set up the Lagrangian and solve.
3. Log-linearize the optimality conditions around a steady state.