Lecture 4: Linear Algebra II

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Normed vector spaces

Recall that the definition of a vector space allows us to perform the operations of "addition" and "scalar multiplication" on elements of the set. Note that the definition does not require a notion of "distance" or "length".

A **normed vector space** is a pair $(V, ||\cdot||)$ where V is a vector space and $||\cdot|| : V \to \mathbb{R}$ is a function with the following properties

1. $||\mathbf{0}|| = 0$ and $||\mathbf{x}|| > 0$ if $\mathbf{x} \neq \mathbf{0}$
2. $||\alpha\mathbf{x}|| = |\alpha|\,||\mathbf{x}||$ for any scalar α
3. $||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \quad \forall \mathbf{x}, \mathbf{y} \in V$

Exercise 4.1 Show that $||x - y||$ is a metric.

Normed vector spaces

In Euclidean space, we'll most often use one of the ℓ_p norms ($p \geq 1$):

$$||\mathbf{x}||_p = \left(\sum_i |x_i|^p \right)^{\frac{1}{p}}$$

Usually one of the following particular cases:

- The standard Euclidean norm: $||\mathbf{x}||_2 = \sqrt{x_1^2 + \cdots + x_n^2}$
- The ℓ_1 norm: $||\mathbf{x}||_1 = \sum |x_i|$
- The sup-norm, or ℓ_∞ norm: $||\mathbf{x}||_\infty = \max\{|x_1|, \ldots, |x_n|\}$

Inner product spaces

Norms endow a space with a notion of length, which induces a notion of distance between vectors. We can add an extra layer of structure to gain a notion of "angle" between two vectors.

An **inner product space** is a pair $(V, \langle \cdot, \cdot \rangle)$, where V is a vector space and $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{R}$, such that:

1. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ (Positivity)
2. $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ iff $\mathbf{v} = 0$ (Definiteness)
3. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ (Symmetry)
4. $\langle \mathbf{x} + \mathbf{y}, \mathbf{v} \rangle = \langle \mathbf{x}, \mathbf{v} \rangle + \langle \mathbf{y}, \mathbf{v} \rangle$ (Additivity)
5. $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$ (Homogeneity)

Euclidean Inner Product

Let $\mathbf{a} = (a_1, ..., a_n)$ and $\mathbf{b} = (b_1, ..., b_n)$ be vectors in $\mathbb{R}^n$. The **Euclidean inner product** of $\mathbf{a}$ and $\mathbf{b}$ is the scalar

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + ... + a_n b_n$$

Exercise 4.2

1. Show that the dot product is an inner product.
2. What norm is induced by the inner product?

Euclidean Inner Product

Theorem

Let $\mathbf{a}$ *and* $\mathbf{b}$ *be vectors in* $\mathbb{R}^n$*. Let* θ *be the angle between them. Then,*

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \, ||\mathbf{b}|| \cos\theta$$

(This will come in handy when we talk about gradients)

Matrix Algebra

Recall that matrices are $m \times n$ arrays of entries taken from a field (we'll only talk about the reals).

Matrix addition is componentwise:

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & a_{i,j} & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix} + \begin{pmatrix} b_{1,1} & \dots & b_{1,n} \\ \vdots & b_{i,j} & \vdots \\ b_{m,1} & \dots & b_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} a_{1,1} + b_{1,1} & \dots & a_{1,n} + b_{1,n} \\ \vdots & a_{i,j} + b_{i,j} & \vdots \\ a_{m,1} + b_{m,1} & \dots & a_{m,n} + b_{m,n} \end{pmatrix}$$

Matrix Algebra

as is scalar multiplication:

$$r \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & a_{i,j} & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix} = \begin{pmatrix} ra_{1,1} & \dots & ra_{1,n} \\ \vdots & ra_{i,j} & \vdots \\ ra_{m,1} & \dots & ra_{m,n} \end{pmatrix}$$

Matrix Algebra

For two matrices, A and B, we can define a new operation, matrix multiplication, in terms of inner products:

$$(AB)_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j$$

where $\mathbf{a}_i$ and $\mathbf{b}_j$ are the ith row of A and jth column of B respectively.

This is only well-defined if the matrices are **conformable** to multiplication: the number of columns in the first matrix must equal the number of rows in the second.

Matrix Algebra

Example

$$\underset{3\times 2}{\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}} \underset{2\times 2}{\begin{pmatrix} W & X \\ Y & Z \end{pmatrix}} = \underset{3\times 2}{\begin{pmatrix} aW + bY & aX + bZ \\ cW + dY & cX + dZ \\ eW + fY & eX + fZ \end{pmatrix}}$$

Observe that the matrices conform to multiplication:
$(3 \times \underline{2}) \ \times \ (\underline{2} \times 2) = 3 \times 2$

Matrix Algebra

We can **transpose** an $m \times n$ matrix A, to form the $n \times m$ matrix A'.

When transposing a product, we reverse the order: $(ABC)' = C'B'A'$.

Matrices that are their own transpose are called **symmetric**: $A' = A$.

If A, B, and C are invertible: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Matrices whose inverse is their transpose are called **orthogonal**: $A^{-1} = A'$.

$(AB)^{-1'} = (B'A')^{-1}$

The **trace** of a square matrix is the sum of its diagonal entries: $trace(A) = \sum_i A_{ii}$.

Matrix Algebra

Exercise 4.3

1. Let A and B be $n \times n$ matrices. Show that $trace(AB) = trace(BA)$.
2. Show that $trace(A + B) = trace(A) + trace(B)$.
3. Show that there are no square matrices with the property $AB - BA = I$.

Matrix Algebra

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, the **kronecker product** of these two matrices is a block matrix of size $mp \times nq$

$$A \otimes B = \begin{pmatrix} a_{1,1}B & \dots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \dots & a_{m,n}B \end{pmatrix}$$

Matrix Algebra

Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 3 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 4 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}$$

Matrix Algebra

The **outer product** is a special case of the kronecker product where we're multiplying a column vector with a row vector.

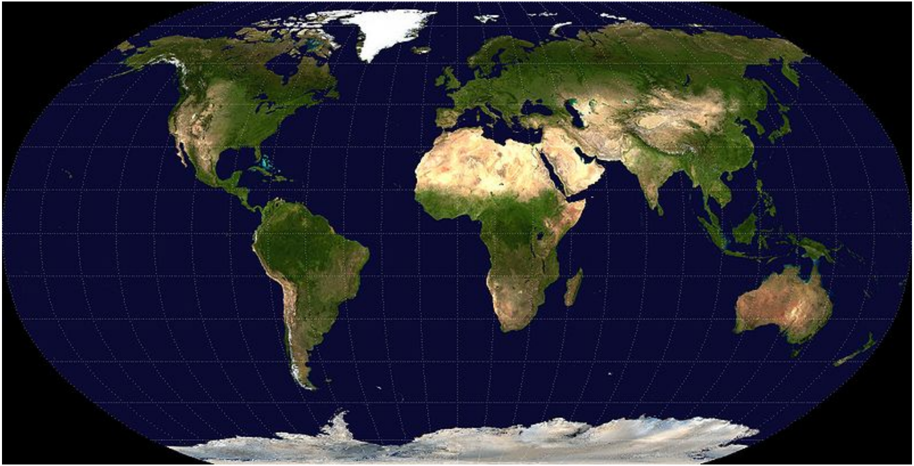
$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{b}^{'} = \begin{bmatrix} b_1, b_2, \ldots, b_n \end{bmatrix}$$

$$\mathbf{ab}' = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \ldots & a_1 b_n \\ \vdots & \ldots & \vdots & \vdots \\ a_n b_1 & a_n b_2 & \ldots & a_n b_n \end{bmatrix}$$

Matrix Algebra

- Conventionally, vectors are assumed to be column vectors.
- When we want to write a row vector, we usually do it as the transpose of a column vector.
- A row vector times a column vector, $\mathbf{a}'\mathbf{b}$, represents an **inner product**
- A column vector times a row vector, $\mathbf{a}\mathbf{b}'$ represents an **outer product**.

Projection



Projection involves “pushing” a vector from one space into a particular subspace.

Projection

A **linear projection** is a square matrix P that is idempotent. That is:

$$P^2 = P$$

Intuitively, this property means that once the matrix P pushes a vector into a subspace, applying the projection again will just yield the same results. Anything which is already in the subspace stays where it is.

Projection

It's often the case that we wish to break a vector down into the sum of two vectors which are orthogonal.

That is, turn $\mathbf{y}$ into $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{e}$ such that $\mathbf{y}'\mathbf{e} = 0$.

Moreover, we may want $\hat{\mathbf{y}}$ to lie in a particular subspace.

Projection

The **orthogonal projector** of a matrix $\mathbf{X}$ with full column rank is

$$\mathbf{P_X} = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'$$

Exercise 4.4

1. Show that the orthogonal projector is in fact a projection.
2. Show that $\mathbf{P_X}$ is a symmetric matrix.
3. Let the fitted value be $\hat{\mathbf{y}} = \mathbf{P_X y}$ and the residual, $\mathbf{e} = \mathbf{y} - \mathbf{P_X y}$. Show that $\hat{\mathbf{y}}'\mathbf{e} = 0$.
4. Show that $\mathbf{X}'\mathbf{e} = 0$.

Approximate Solutions to Linear Equations

Suppose we had a system of linear equations, $A\mathbf{x} = \mathbf{y}$, for which A is an $n \times k$ matrix with full column rank and $n > k$.

That is, there are more equations than unknowns, but the unknowns are linearly independent.

Suppose further that the augmented system, $[A \mid \mathbf{y}]$ also has full rank.

That is to say, two rows contain contradictory information.

How many solutions are there to the system?

Could we get an approximate solution to this system?

Approximate Solutions to Linear Equations

Let's pick some candidate solution $\mathbf{x}^*$.

The error we make using $\mathbf{x}^*$ as our approximate solution is $\varepsilon = \mathbf{y} - A\mathbf{x}^*$.

A "good" solution should attempt to minimize this error.

Since this error is a vector, a reasonable choice might be to minimize the length of this vector, $||\mathbf{y} - A\mathbf{x}^*||$.

Recall that (under the Euclidean norm) $||\varepsilon|| = \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_n^2}$. Minimising the length of the vector is equivalent to minimising the sum of squared errors.

Approximate Solutions to Linear Equations

So we have a vector $\mathbf{y}$ that is *not* in the column space of A; otherwise there would be a solution.

We're trying to find a vector in the column space of A such that the distance between that vector and $\mathbf{y}$ is minimised.

That is, we're trying to *project* $\mathbf{y}$ into the column space of A.

To be minimum distance, we specifically want to drop a *perpendicular* from the tip of $\mathbf{y}$ into the col space of A.

Recall that the shortest distance between a point and a line forms a right angle between the two.

The same principle is operating here. As you may of guessed, we want the *orthogonal projection* of $\mathbf{y}$ into the column space of A.

Projection

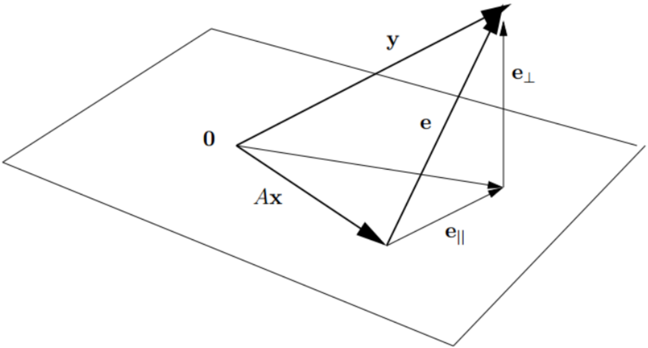


Figure: Orthogonal Projection

Source: http://www.math.ku.edu/ lerner/LAnotes/LAnotes.pdf

Economic Application: Ordinary Least Squares

Let's say we wish to estimate the following multiple linear regression model.

$$y_i = \beta_0 + \beta_1 x_{1,i} + \cdots + \beta_k x_{p,i} + \varepsilon_i \qquad i = 1, \ldots, n$$

Why don't we make this more compact by stacking them in vectors and matrices?

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon$$

Where $\mathbf{y}$ and ε are $n \times 1$ vectors, $\mathbf{X}$ is an $n \times (p+1)$ matrix with $n > p+1$ and full column rank, and β is a $(p+1) \times 1$ vector.

Economic Application: Ordinary Least Squares

What is it we're asking for when we run an OLS regression?

We're trying to find the causal effect of each x variable on y; typically interpreted as "if x_1 goes up by one unit, y goes up β_1 units on average".

Another way to say this is that we're looking for a solution to the deterministic model $\mathbf{y} = \mathbf{X}\beta$.

To solve this we need a β in the column space of $\mathbf{X}$.

The system typically has no solution (why?), so we need to approximate a solution by projecting $\mathbf{y}$ into the column space of $\mathbf{X}$.

OLS picks β to minimize the sum of square residuals. I.e., the projection of $\mathbf{y}$ into $\mathrm{Col}(\mathbf{X})$ must be an *orthogonal* projection.

Economic Application: Ordinary Least Squares

Exercise 4.5

1. Show that $\hat{\beta}_{OLS} = (\mathbf{X}'\mathbf{X})^{-1}\,\mathbf{X}'\mathbf{y}$ solves the least squares problem.
2. Show that $\hat{\beta}_{OLS}$ is unique.
3. Suppose $\mathbf{X}$ is rank deficient. What is the solution set? How does it affect fitted values $\hat{\mathbf{y}}$?

Eigenvalues and Eigenvectors

First let's recall that a matrix A may be thought of as a linear map between two vector spaces. That is, it takes as input a vector $\mathbf{x} \in \mathbb{R}^m$ and transforms it into another vector, $A\mathbf{x} \in \mathbb{R}^n$.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. An **eigenvector** $\mathbf{v}$ of A and its corresponding **eigenvalue**, λ, are a vector and scalar satisfying

$$A\mathbf{v} = \lambda\mathbf{v}$$

Exercise: Why are eigenvectors only defined for square matrices?

Eigenvalues and Eigenvectors

$$
\begin{aligned}
& A\mathbf{v} = \lambda\mathbf{v} \\
\Rightarrow \quad & \lambda\mathbf{v} - A\mathbf{v} = 0 \\
\Rightarrow \quad & (\lambda\mathbf{I} - A)\mathbf{v} = 0
\end{aligned}
$$

But if $(\lambda\mathbf{I} - A)\mathbf{v} = 0$ for some non-zero vector $\mathbf{v}$, then $(\lambda\mathbf{I} - A)$ can't be full rank (why?).

Since it's a square matrix, if it's not full rank then it must be singular! That is, it must be the case that

$$\det(\lambda\mathbf{I} - A) = 0$$

This determinant expands to an nth order polynomial in λ called the **characteristic equation**.

Eigenvalues and Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

Find the characteristic equation for A and determine A's eigenvalues.

$$\begin{aligned} \det(\lambda \mathbf{I} - A) &= \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{pmatrix} \right| \\ &= \lambda(\lambda + 3) - (-1)(2) \\ &= \lambda^2 + 3\lambda + 2 \end{aligned}$$

Solving $\lambda^2 + 3\lambda + 2 = 0$ yields $\lambda_1 = -1$, $\lambda_2 = -2$.

Once we've obtained the set of eigenvalues, we can find their corresponding eigenvectors by solving the homogeneous system $(\lambda \mathbf{I} - A)\mathbf{v} = 0$ for $\mathbf{v}$.

Example (continued)

First solve for λ_1. In order to solve $(-1\mathbf{I} - A)\mathbf{v} = 0$, simplify the term in brackets and construct an augmented system.

$$\left(\begin{array}{cc|c} -1 & -1 & 0 \\ 2 & 2 & 0 \end{array}\right) \overset{rref}{\rightarrow} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Recall that to solve we set x_2 to be a free variable.

$$x_1 = -x_2 \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} x_2$$
$$x_2 = x_2$$

Eigenvalues and Eigenvectors

Example (continued)

Doing the same for λ_2 gives

$$\left(\begin{array}{cc|c} -2 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right) \overset{rref}{\longrightarrow} \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Set x_2 to be a free variable.

$$\begin{aligned} x_1 &= -\frac{1}{2}x_2 \\ x_2 &= x_2 \end{aligned} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} x_2$$

Thus the eigenvector corresponding to $\lambda_1 = -1$ is $[-1, 1]'$ and corresponding to $\lambda_2 = -2$ is $[-\frac{1}{2}, 1]'$.

Eigenvalues and Eigenvectors

Exercise 4.6

1. Verify manually that the above are actually eigenvectors.
2. $[\frac{1}{2}, 1]'$ is not very nice to look at. Is $[-1, 2]$ an eigenvector corresponding to λ_2?
3. Suppose $\mathbf{v}$ is an eigenvector of A. Is $k\mathbf{v}$ also an eigenvector for any scalar k?

Diagonalization

Theorem

Let A be an $n \times n$ matrix. Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Construct the matrix

$$P = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$$

whose columns are A's eigenvectors. If P in invertible, then

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

Exercise 4.7 Verify that the matrix A in the previous example is diagonalizable.

Diagonalization

But how do we know that the invertibility condition will be satisfied? We have a sufficient (but not necessary) condition!

Theorem

Let $\lambda_1, \ldots, \lambda_n$ be n distinct eigenvalues of an $n \times n$ matrix. Then the corresponding eigenvectors are linearly independent.

If a matrix is not diagonalizable (called a **defective** matrix), other methods may be available for solving the problem at hand (see Schur decomposotion).

Diagonalization

Theorem (The Spectral Theorem)

Let A be an $n \times n$ matrix of real entries that is symmetric (i.e., $A' = A$). Then A can be diagonalized

$$A = QDQ'$$

where Q is an orthogonal matrix ($Q^{-1} = Q'$).

Systems of Difference Equations

$$x_{t+1} = ax_t$$

Perhaps x_t is a bank balance and $a = 1 + r$ is the gross rate of return, for example. The solution to this is fairly straight forward and can be achieved through back substitution:

$$x_t = a^t x_0$$

An alternative way to say this is that

$$x_t = ca^t$$

where c is a constant determined by the boundary condition (initial value).

Systems of Difference Equations

But what about an arbitrary *system* of first order linear difference equations.

$$\begin{array}{ccccccccc}
x_{1,t+1} & = & a_{11}x_{1,t} & + & a_{12}x_{2,t} & + & \cdots & + & a_{1n}x_{n,t} \\
x_{2,t+1} & = & a_{21}x_{1,t} & + & a_{22}x_{2,t} & + & \cdots & + & a_{2n}x_{n,t} \\
x_{3,t+1} & = & a_{31}x_{1,t} & + & a_{32}x_{2,t} & + & \cdots & + & a_{3n}x_{n,t} \\
\vdots & = & \vdots & + & \vdots & + & \cdots & + & \vdots \\
x_{n,t+1} & = & a_{n1}x_{1,t} & + & a_{m2}x_{2,t} & + & \cdots & + & a_{nn}x_{n,t}
\end{array}$$

Written more compactly as

$$\mathbf{x}_{t+1} = A\mathbf{x}_t$$

Systems of Difference Equations

Suppose that A is diagonalizable. Such that we can right

$$A = PDP^{-1}$$

We can consider the transformed system

$$\begin{aligned} \mathbf{x}_{t+1} &= A\mathbf{x}_t \\ P^{-1}\mathbf{x}_{t+1} &= D\left(P^{-1}\mathbf{x}_t\right) \\ \mathbf{z}_{t+1} &= D\mathbf{z}_t \end{aligned}$$

Such that $\mathbf{x}_t = P\mathbf{z}_t$ and D is a diagonal matrix of the eigenvalues of A.

Systems of Difference Equations

Can we solve this diagonal system? Actually, it's very easy. The diagonal matrix makes this system **decoupled**. That is, each difference equation only depends on its own lag, not the lag of the other variables. That is

$$
\begin{aligned}
z_{1,t+1} &= \lambda_1 z_{1,t} \\
z_{2,t+1} &= \lambda_2 z_{2,t} \\
&\vdots \\
z_{n,t+1} &= \lambda_n z_{n,t}
\end{aligned}
$$

But we've seen how to easily solve these separate difference equations!

Systems of Difference Equations

$$\begin{pmatrix} z_{1,t} \\ \vdots \\ z_{n,t} \end{pmatrix} = \begin{pmatrix} c_1\lambda_1^t \\ \vdots \\ c_n\lambda_n^t \end{pmatrix}$$

But we don't wish to know the solution in terms of $\mathbf{z}_t$, we want it in terms of $\mathbf{x}_t$. But recall that $\mathbf{x}_t = P\mathbf{z}_t$! Yielding

$$\begin{aligned} \mathbf{x}_t &= P\mathbf{z}_t \\ &= [\mathbf{v}_1, \ldots, \mathbf{v}_n]\mathbf{z}_t \\ &= z_{1,t}\mathbf{v}_1 + \ldots z_{n,t}\mathbf{v}_n \\ &= c_1\lambda_1^t\mathbf{v}_1 + \cdots + c_n\lambda_n^t\mathbf{v}_n \end{aligned}$$

The solution to a system of difference equations can be written as a linear combination of its eigenvectors!

Quadratic Forms

After linear functions, the next simplest are the **quadratic forms**. A quadratic form on $\mathbb{R}^n$ is a real-valued function of the form

$$\mathcal{Q}(x_1, x_2, \ldots, x_n) = \sum_{i \leq j} a_{ij} x_i x_j$$

Any quadratic form can be written more compactly as

$$\mathcal{Q}(x_1, \ldots, x_n) = \mathbf{x}^{'} A \mathbf{x}$$

where A is a unique real-valued symmetric matrix.

Definiteness

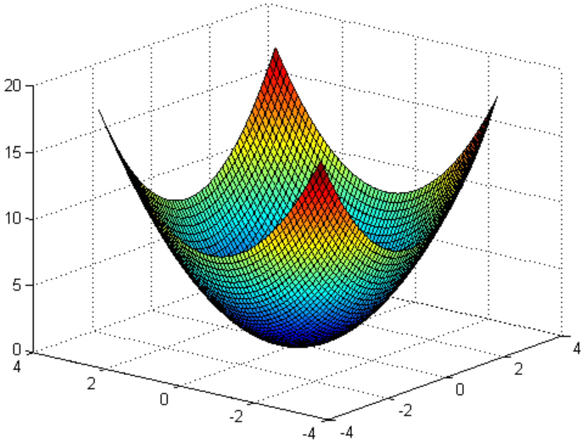


Figure: Positive Definite: $\mathbf{x}'A\mathbf{x} > 0$

Definiteness

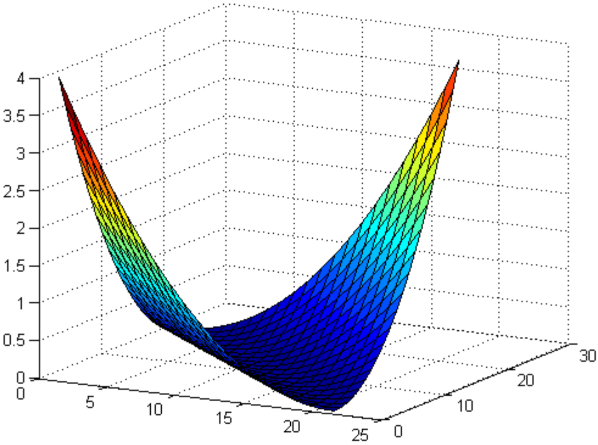


Figure: Positive Semi-definite: $\mathbf{x}' A \mathbf{x} \geq 0$

Definiteness

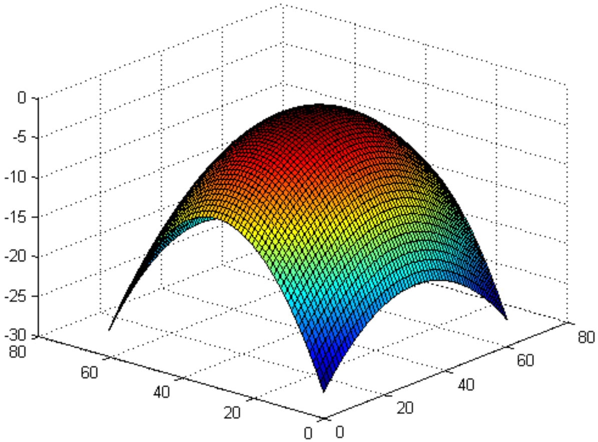


Figure: Negative Definite: $\mathbf{x}'A\mathbf{x} < 0$

Definiteness

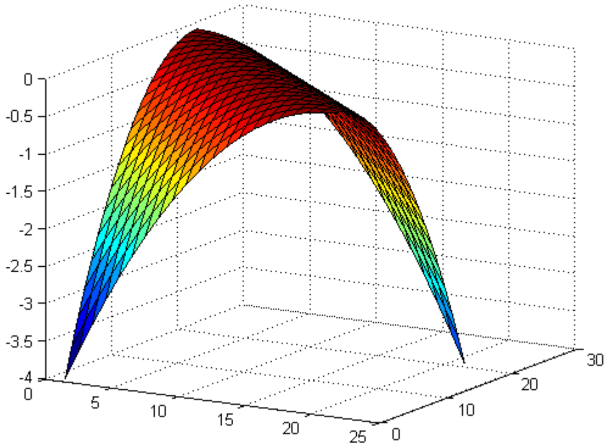


Figure: Negative Semi-definite: $\mathbf{x}'A\mathbf{x} \leq 0$

Definiteness

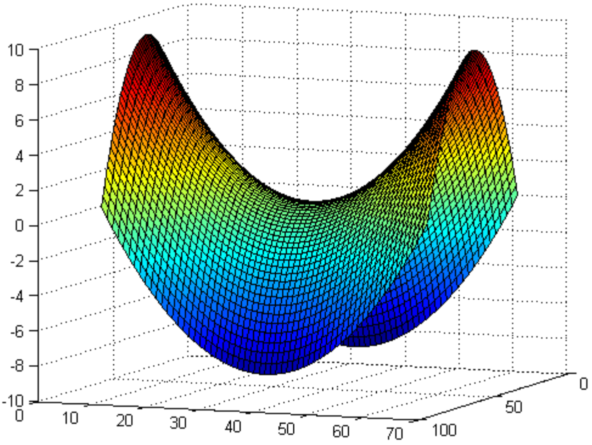


Figure: Indefinite: $\mathbf{x}'A\mathbf{x} > 0$ for some $\mathbf{x}$, $\mathbf{x}'A\mathbf{x} < 0$ for others.

Definiteness

Theorem

Let A be a symmetric matrix. Then

a) *A is positive definite iff all eigenvalues are > 0.*
b) *A is negative definite iff all eigenvalues are < 0.*
c) *A is positive semidefinite iff all eigenvalues are ≥ 0.*
d) *A is negative semidefinite iff all eigenvalues are ≤ 0*
e) *A is indefinite iff it has at least one positive and one negative eigenvalue.*

Cholesky Decomposition

Theorem (Cholesky Decomposition)

If A is symmetric and positive definite, then A can be decomposed as

$$A = CC'$$

where C is a lower-triangular matrix.

Learning Outcomes

You should be able to:

- Show that a function is a valid norm or inner product.
- Perform common matrix operations.
- Construct the orthogonal projector for a matrix.
- Find the eigenvalues and eigenvectors of a matrix.
- Diagonalize a matrix.
- Solve very simple linear dynamical systems.
- Determine the definiteness of a square matrix.