Lecture 10: Dynamic Programming

Robert Garrard

Sequence Problems

So far we've dealt with optimization problems that look like

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$\text{s.t. some constraints}$$

The solution is a sequence of optimal choices: $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$

Control Variables and State Variables

A **control variable** (or choice variable) is a quantity that we can choose in each period (e.g., consumption).

A **state variable** (or stock variable) is a quantity that's fixed in the current period (e.g., capital stock).

Our constraint gives a flow equation describing how our choice in the current period affects the state in the next

$$k_{t+1} = g(k_t, c_t)$$

In general, we have a vector of states and a vector of controls.

Value Function

A **value function** is a function of the state variables that returns the opitimized value of the objective function. E.g.:

$$V(k_t) = \sum_{i=t}^{\infty} \beta^i U(c_i^*)$$

For a generic objective function, $f(c_t, x_t)$, and flow constraint, $x_{t+1} = g(c_t, x_t)$, the value function is

$$\begin{aligned} V(k_t) = \max_{\{c_t\}} &\quad f(c_t, k_t) \\ \text{s.t.} &\quad k_{t+1} = g(c_t, k_t) \end{aligned}$$

Policy Function

A **policy** is a rule for choosing the control variable based on the state variable.

$$c_t = \sigma(k_t)$$

An **optimal policy** is the policy that maximizes the objective function subject to the constraints.

Note that if our optimization problem has a finite horizon, both the value and policy functions should be indexed by time, since they may differ in different periods.

We'll begin in the infinite horizon setting, so we can look for **stationary** (time invariant) value and policy functions.

The Principle of Optimality

Principle of Optimality (Bellman, 1957)

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

The Bellman Equation

Exploiting the principle of optimality, we can re-write the value function as:

$$\begin{aligned} V(k_t) = \max_{c_t} \quad & U(c_t, k_t) + \beta V(k_{t+1}) \\ \text{s.t.} \quad & k_{t+1} = g(c_t, k_t) \end{aligned}$$

or substituting in the constraint

$$V(k_t) = \max_{c_t} \quad U(c_t, k_t) + \beta V(g(c_t, k_t))$$

This is not a sequence problem but a **functional equation** problem. The solution is a pair of functions: $\sigma(k_t)$ and $V(k_t)$.

The Bellman Equation

If we know the value function, the optimal policy can be found by

$$\sigma(k_t) = \underset{c_t}{\operatorname{argmax}} \left\{ f(c_t, k_t) + \beta V(k_{t+1}) \right\}$$

Exercise 10.1 Show that the Ramsey growth model can be represented with a Bellman equation.

Theorems

A1. The constraint set $\{x \mid x = g(c_t, k_t)\}$ is non-empty for all k_t.

A2. The constraint set is compact and $f(\cdot)$ and $g(\cdot)$ are continuous.

A3. $f(\cdot)$ is strictly concave. The constraint set is convex.

A4. $f(\cdot)$ is strictly increasing.

A5. $f(\cdot)$ is $\mathcal{C}^1$ on the interior of its domain.

Theorems

- A1 and A2 give a unique continuous, bounded, value function that solves the problem. The solution to the sequence problem and functional problem are the same. An optimal policy, σ, exists but may not be unique.

- Under A1-3, the value function is strictly concave. There is a unique optimal policy, σ, which is continuous.
- Under A1, A2, A4, V is strictly increasing.
- Under A1,A2,A3,A5, V is $\mathcal{C}^1$. (Hence we can use envelope conditions)

Ramsay Growth

Recall our Ramsay growth model was

$$\max_{\{c_t, k_{t+1}\}} \quad \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$\text{s.t.} \quad k_{t+1} = F(k_t) + (1-\delta)k_t - c_t$$

With typical assumptions: $U\prime > 0$, $U'' < 0$, $F' > 0$, $F'' < 0$, $F(0) = 0$.

$k_{t+1} = 0$ is always an option. $k_{t+1} \in [0, F(k_t) + (1-\delta)k_t]$ is compact and convex. We specify a $U(c_t)$ that's strictly increasing, strictly concave, and continuous on its domain's interior.

Ramsay Growth

Let's transform it to a recursive problem:

$$\begin{aligned} V(k_t) = \max \quad & U(c_t) + \beta V(k_{t+1}) \\ \text{s.t.} \quad & k_{t+1} = F(k_t) + (1-\delta)k_t - c_t \end{aligned}$$

Using either Lagrange or substitution, the first order condition under which the RHS is maximized is:

$$U'(c_t) - \beta V'(k_{t+1})$$

Ramsay Growth

Substituting in the optimal policy gives

$$V(k_t) = U(\sigma(c_t)) + \beta V(F(k_t) + (1-\delta)k_t - \sigma(c_t))$$

Applying the envelope condition gives:

$$\begin{aligned} V'(k_t) &= [F'(k_t) + 1 - \delta]\beta V'(k_{t+1}) \\ &= U'(c_t)[F'(k_t) + 1 - \delta] \end{aligned}$$

Rolling the envelope condition forward a period and substituting into the first order condition gives the usual Euler equation

$$U'(c_t) = \beta U'(c_{t+1})[F'(k_{t+1}) + 1 - \delta]$$

Ramsay Growth

Exercise 10.2 Verify that the envelope condition holds.

The Bellman Operator

Define the operator $T : X \to X$ (where X is a function space) to be:

$$T(V) = \max_{c} \{f(c,k) + \beta V(g(c,k))\}$$

The value function we're looking for is a fixed point of this operator

$$T(V) = V$$

The Bellman Operator

Exercise 10.3 Recall the infinite horizon consumption-savings problem where an infinitely lived consumer with logarithmic utility can save at a constant rate of return r.

a) Recast the problem as dynamic programming and solve for the intertemporal Euler equation.
b) Consider the function $V(s_t) = 0$. Apply the Bellman operator to this function.
c) Apply the Bellman operator iteratively to $V(s_t) = 0$ three times. Hypothesize a fixed point.
d) Verify your hypothesis.
e) Solve for the optimal policy.

Norms

Suppose we are in X, the space of bounded continuous real valued functions. Define the sup-norm of a function $f \in X$ to be:

$$||f|| = \sup_x |f(x)|$$

Recall that a max is itself a sup. We induce the metric:

$$d(f,g) = ||f-g|| = \sup_x |f(x) - g(x)|$$

Also note this property of sup:

$$|\sup_x f - \sup_x g| \leq \sup_x |f - g|$$

Contraction Mapping

Let V and W be two bounded, continuous, real valued functions. We have

$$
\begin{aligned}
&|T(V) - T(W)| \\
&= |\max_c\{f(c,k) + \beta V(g(c,k))\} - \max_c\{f(c,k) + \beta W(g(c,k))\}| \\
&\leq \max_c |f(c,k) + \beta V(g(c,k)) - f(c,k) + \beta W(g(c,k))| \\
&= \max_c |\beta V(g(c,k)) - \beta W(g(c,k))| \\
&= \beta \max_c |V(g(c,k)) - W(g(c,k))| \\
&= \beta ||V(g(c,k)) - W(g(c,k))||
\end{aligned}
$$

Taking the sup on the left gives

$$||T(V) - T(W)|| \leq \beta ||V(g(c,k)) - W(g(c,k))||$$

When $\beta < 1$, the Bellman operator is a contraction mapping!

Contraction Mapping

Theorem (Blackwell's sufficient conditions)

Let $X \subseteq \mathbb{R}^l$ and let $B(X)$ be the space of bounded functions, $f : X \to \mathbb{R}$, with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying:

a) *(monotonicity) $f, g \in B(X)$ with $f(x) \leq g(x)\ \forall x$ implies $(Tf)(x) \leq (Tg)(x)\ \forall x$.*
b) *(discounting) there exists some $\beta \in (0, 1)$ such that $T(f + a)(x) \leq Tf(x) + \beta a$.*

then T is a contraction mapping.

Contraction Mapping

Exercise 10.4

a) Prove the above theorem.

b) Verify that the Bellman operator satisfies Blackwell's conditions.

Value Function Iteration

VFI

1. Begin with an initial guess for the value function (maybe $V^0 = 0$).
2. Iterate on $V^{j+1} = T(V^j)$.
3. Stop when $\max |V^{j+1} - V^j| < \varepsilon$, for some tolerance ε.
4. Compute the policy function:
   $\sigma(k_t) = \underset{c_t}{\operatorname{argmax}} \{f(c_t, k_t) + \beta V(k_{t+1})\}$.

In practice, we discretize our state space by chopping it up into a bunch of grid points, then we evaluate the function at those grid points.

Remember to make sure that your constraints are satisfied at all times.

VFI

Exercise 10.5 We previously pencil-and-papered the value and policy functions for the simple infinite horizon consumption savings problem.

1. In whatever language you prefer, use a computer to solve the model with VFI for the specification $U(c_t) = \log(c_t)$, $r = 5\%$, and period income $y_t = 0$. Verify that the analytic solution is correct.
2. What happens if we set $y_t = 5$?

Randomness

Suppose there is a random quantity $a_t \sim G$ that we observe at the beginning of each period before we choose c. This becomes another state variable. We can transform the Bellman equation to be:

$$V(k_t, a_t) = f(c_t, k_t, a_t) + \beta \mathbb{E}_t V(k_{t+1}, a_{t+1})$$

We can write out the expectation explicitly as

$$V(k_t, a_t) = f(c_t, k_t, a_t) + \beta \int V(k_{t+1}, a_{t+1}) \, \mathrm{d}G$$

If G is a discrete distribution, the integral becomes a sum. When computing our VFI, we construct a grid for a_{t+1} and numerically evaluate the expectation before applying the Bellman operator.

Randomness

Exercise 10.6 Consider this alteration to the standard consumption/savings model:

$$\max \quad \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$\text{s.t.} \quad c_t + s_{t+1} = (1+r)s_t + y_t$$

$$\mathbb{P}(y_t = 10) = \frac{1}{2} \quad \mathbb{P}(y_t = 0) = \frac{1}{2}$$

1. Solve for the optimality conditions.
2. Use VFI to solve for the optimal policy functions.
3. Simulate the agent's consumption and savings for 100 periods.
4. Suppose the agent begins life with $s_0 = 100$. Simulate the agent's consumption and savings for 1000 periods.

Learning Outcomes

You should be able to:

- Formulate an optimization problem recursively.
- Solve for the first order and envelope conditions of a Bellman equation.
- Verify that an operator satisfies Blackwell's sufficient conditions.
- Pencil-and-paper a few value function iterations.
- Use a computer to numerically apply VFI.