Solutions due by 10.30am Friday 12th March.

1. Last week we showed that if $X_i \sim N(\mu, \sigma^2)$, then: if $\mu \neq 0$, $\sqrt{n}(\bar{X}^2 - \mu^2) \xrightarrow{d} N(0, 4\mu^2\sigma^2)$; if $\mu = 0$, $\frac{n\bar{X}^2}{\sigma^2} \xrightarrow{d} \chi^2(1)$.

   In whatever language you prefer, verify with a Monte Carlo simulation that this is correct. Consider $n \in \{50, 500\}$, $\mu \in \{-2, -1, 0, 1, 2\}$, and $\sigma^2 = 1$. For each combination of parameters, draw n observations from a $N(\mu, \sigma^2)$ distribution and compute the sample mean $ITER = 1000$ times. Plot the empirical distribution of $\sqrt{n}(\bar{X}^2 - \mu^2)$ or $\frac{n\bar{X}^2}{\sigma^2}$ as appropriate, and overlay the density that it should converge to.

2. Solve the following problem:

$$\begin{aligned} &\max \ xyz \\ \text{s.t. } &y + 2x = 15 \\ &2z + y = 7 \\ &y \geq 5 \end{aligned}$$

3. Suppose there is a worker who chooses consumption c and labor ℓ to maximize the utility function $u(c, \ell) = \log(c) + \eta \log(1 - \ell)$, where $1 - \ell$ is their leisure time, $\ell \in [0, 1]$, and $\eta \in \mathbb{R}_+$ is the elasticity of leisure. When labor ℓ is provided, the worker can produce $A\ell$ units of the consumption good. The worker's output is taxed at a rate $\tau \in [0, 1]$.

   a) Is the utility function concave, convex, or neither?

   b) Solve the worker's optimization problem for how much labor they choose to supply and how much consumption they obtain.

   c) How does labor supply change with the tax rate in this model? How much revenue does the government receive?

4. Consider the problem of a student who is working to solve a problem. The student receives payoff $\bar{z}$ if they correctly solve the problem, and payoff $\underline{z}$ if they do not, $\bar{z} > \underline{z}$. Unfortunately, thinking about the problem requires effort. Let the amount of effort exerted by the student be $e \geq 0$. For a given level of effort, the probability that they solve the problem is given by the function $p(e) \in [0, 1]$, with $p' > 0$, $p'' < 0$, $\lim_{e \to \infty} p' = 0$. For any level of effort, the student experiences disutility, $v(e)$, where $v' > 0$, $v'' > 0$, $v(0) = 0$.

   Write down the student's optimization problem. Solve for the optimality conditions.

5. Consider adding labor to the Ramsay growth model. The agent has one unit of time each period that they can split between labor ℓ, and leisure, $1-\ell$. Their utility function, $U(c_t, 1-\ell_t)$ is increasing and concave in both consumption and leisure. Let the production technology be $F(k_t, \ell_t)$.

   a) Solve for the agent's optimality condition governing the intra-temporal consumption-leisure tradeoff.

   b) Assume a steady state exists and log-linearize this equation around it.