Lecture 2: Real Analysis II

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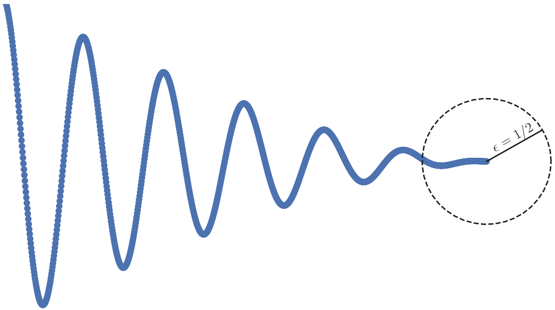

Metric Spaces

A **space** is just a set endowed with some sort of structure.

A **metric space** is a set endowed with a notion of **distance**.

A metric space is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function which satisfies the following conditions

1. $d(x, y) \geq 0 \;\; \forall x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x) \;\; \forall x, y \in X$
3. $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X \;\; (\textit{Triangle inequality})$

Metric Spaces

The metric we encounter most is the familiar **Euclidean metric**, which is one way of defining distance between two points in $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Recall that the Euclidean metric is defined as

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 ... + (x_n - y_n)^2}$$

Exercise 2.1

1. Show that the Euclidean metric in $\mathbb{R}$, $d(x, y) = |x - y|$, is in fact a metric.
2. Show that the **discrete metric** is a metric.

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

Open and Closed Balls

Equipped with a notion of distance, we can construct some useful objects in a metric space, (X, d).

An **open ball** of radius $r > 0$ centered at a is the set:

$$B(a, r) = \{x \in X \mid d(x, a) < r\}$$

The **closed ball** of radius $r > 0$ centered at a is the set

$$\bar{B}(a, r) = \{x \in X \mid d(x, a) \leq r\}$$

Exercise 2.2 What does an open/closed ball of unit radius, centered at the origin, look like in $\mathbb{R}^2$ under the:

1. Euclidean metric?
2. Discrete metric?

Sequences

A **sequence** is an ordered set of elements taken from some set, where the members of the sequence are indexed by the natural numbers.

Specific sequences are often written by enumerating the first few elements if there is an obvious pattern, such as:

“ the sequence $\{2, 4, 6, 8, \dots\}$”

or by explicitly stating what the nth element of the sequence is, such as:

“the sequence $\{x_1, x_2, x_3, \dots\}$ where $x_n = 2n$”

An arbitrary sequence from some set X is often written as $\{x_n\}_{n \in \mathbb{N}}$.

Convergence of Sequences

Consider the sequence $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. As n becomes large, the elements in this sequence become very small. They get closer and closer to zero, although the sequence never actually attains this value at any time. We can use our metric to capture the notion of "closeness" to some limiting point of the sequence.

A sequence, $\{x_n\}_{n\in\mathbb{N}}$, is said to **converge** to a point a if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \;\; s.t \;\; \forall n > N \;\; d(x_n, a) < \varepsilon$$

We often write this proposition as $\lim_{n\to\infty} x_n = a$ or $x_n \to a$.

An sequence which does not converge to any point is said to **diverge**.

Convergence of Sequences

If that definition is tricky to parse, let's try to English it up a little:

For a sequence to converge to some point, $x_n \to a$, it must be that if I place a ball at a of *any* radius, then *all but finitely many* elements of x_n must be inside that ball.

Convergence of Sequences

Example

Show that the sequence $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$ converges to 0 in the metric space $(\mathbb{R}, d)$, where d is the Euclidean metric.

Strategy:

1. Pick an arbitrary $\varepsilon > 0$.
2. Make a good choice for N (as a function of ε).
3. Show that $n > N \Rightarrow |x_n - 0| < \varepsilon$. (eqv. $n > N \Rightarrow x_n \in B(0, \varepsilon)$)

Pick some $\varepsilon > 0$. Choose N to be any integer such that $N > \frac{1}{\varepsilon}$.
Consider some $n \in \mathbb{N}$. We have:
$n > N \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow x_n < \varepsilon \Rightarrow |x_n - 0| < \varepsilon$ (by non-negativity of x_n).
$\therefore x_n \to 0$

$\blacksquare$

Convergence of Sequences

Exercise 2.3

1. Show that the sequence $\{1, 1, 1, 1, \dots\}$ converges to 1 under the discrete metric.
2. Show that the sequence $\{0, 1, 0, 1, 0, 1, \dots\}$ does not converge under the Euclidean metric.

Monotone Sequences

A sequence $\{x_n\}$ in $\mathbb{R}$ is **bounded** if

$$\exists M \in R \;\; s.t \;\; |x_n| \leq M \;\forall n \in \mathbb{N}$$

Monotonicity

A sequence $\{x_n\}$ in $\mathbb{R}$ is called

- **Monotone Increasing** if $x_{n+1} \geq x_n \;\; \forall n \in \mathbb{N}$
- **Monotone Strictly Increasing** if $x_{n+1} > x_n \;\; \forall n$
- **Monotone Decreasing** if $x_{n+1} \leq x_n \;\; \forall n$
- **Monotone Strictly Decreasing** if $x_{n+1} < x_n \;\; \forall n$

Monotone Sequences

Theorem (Monotone Convergence Theorem)

A monotone sequence of real numbers converges iff it is bounded. Further, if it is increasing (resp. decreasing), it converges to its supremum (infimum).

Proof.

(Bounded $\Rightarrow$ converges, increasing case)

Let $\{a_n\}$ be an increasing bounded sequence of real numbers. By the LUB property, $c = \sup\{a_n\}$ exists.

$\forall \varepsilon > 0 \; \exists N$ sufficiently large that $a_N > c - \varepsilon$, otherwise $c - \varepsilon$ would be a smaller lower bound.

Since a_n is increasing and c is an upper bound: $\forall n \geq N$
$|a_n - c| \leq |a_N - c| < \varepsilon$.

$\therefore a_n \rightarrow c$.

∎

Cauchy Sequences

A sequence $\{x_n\}$ is called a **Cauchy sequence** if:

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \forall m, n > N \ d(x_m, x_n) < \varepsilon$$

In English: terms of the sequence must get *closer together* as the sequence progresses and *stay* close.

Example

Show that $x_n = \frac{1}{n}$ is Cauchy.

For any ε, we need to find an $N \in \mathbb{N}$ such that $|\frac{1}{m} - \frac{1}{n}| < \varepsilon$ when $m, n > N$. Note that $\frac{1}{m} < \frac{1}{N}$, and similarly for n.

$|\frac{1}{m} - \frac{1}{n}| \leq |\frac{1}{m}| + |\frac{1}{n}| < \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \varepsilon$

Setting $N > \frac{2}{\varepsilon}$ achieves the inequality.

Cauchy Sequences

Theorem (Cauchy Convergence Criterion)

A sequence of real numbers is convergent iff it is Cauchy

In general, convergent $\Rightarrow$ Cauchy. (Tutorial exercise)

But in general Cauchy need not imply convergent.

Usually we won't be trying to prove that a specific sequence converges to a specific value. Instead we'll be considering some sequence with a particular property, and be interested in whether it converges and, if so, what the properties of its limit point are.

So we need some generic conditions under which convergence is guaranteed.

Open and Closed Sets

A set X is **open** if

$$\forall x \in X \quad \exists \varepsilon > 0 \;\; s.t \;\; B(x, \varepsilon) \subseteq X$$

That is, for any point in the set, there is some open ball centered at that point whose radius is sufficiently small for the ball to be entirely contained within the set.

A set X is **closed** if its complement is open.

Sets are not like doors! A set can be both open and closed.

Open and Closed Sets

Some useful properties:

- The union of any collection of open sets is open.
- The intersection of any collection of closed sets is closed.
- The intersection of a finite collection of open sets is open.
- The union of a finite collection of closed sets is closed.

Open and Closed Sets

Exercise 2.4

1. For a metric space (X, d), show that the whole space X is both open and closed.
2. Show that a single point, $\{x\}$, is a closed set.
3. Show that an open ball is an open set.

Open and Closed Sets

Proposition: Closed sets contain their limit points

Let (X, d) be a metric space. If $A \subseteq X$ is a closed subset and $a_n \in A$ is a sequence, then

$$a_n \to a \Rightarrow a \in A$$

Proof (tutorial exercise)

Completeness

Exercise 2.5

Let $(\mathbb{Q}, d)$ be a metric space over the rationals where d is the Euclidean metric, $d(x, y) = |x - y|$. Let $A \subseteq \mathbb{Q}$, $A = \mathbb{Q} \cap [0, 2]$.

1. Show that A is closed.
2. Does the sequence $x_1 = 1$, $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ converge?

Completeness

A metric space, (X, d), is **complete** if *every* Cauchy sequence converges.

Every Euclidean space with the Euclidean metric, $(\mathbb{R}^n, d)$, is complete.

Proposition: Closedness preserves completeness

If (X, d) is a complete metric space and $A \subseteq X$ is a closed subset, then every Cauchy sequence in A converges in A.

Compactness

An **open cover** for a set X is a (possibly infinite) collection of open sets, $\{A_i\}$, whose union contains X: $\bigcup_i A_i \subseteq X$.

A set is called **compact** if *every* open cover has a finite sub-cover.

Theorem (Heine-Borel)

A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Exercise 2.6

Which of the following subsets of R are compact?

- $(-\infty, 0]$
- $(0, 1)$
- $[0, \pi] \cap [0, e]$

Compactness

Some properties of compactness

- A compact metric space is complete.
- A closed subset of a compact set is compact.
- A compact subset of R has a maximum and minimum value.

Continuity

A function, $f : X \to Y$, is **continuous** at a point $c \in X$ if

$\varepsilon - \delta$ Continuity
$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; s.t \;\; d(x, c) < \delta \Rightarrow d\left(f(x), f(c)\right) < \varepsilon$

Sequential Continuity
$\forall \{x_n\} \quad x_n \to c \Rightarrow f\left(x_n\right) \to f(c)$

Topological continuity
Every open image, V, has an open preimage $f^{-1}(V)$

These definitions (and many others) are equivalent in a metric space.

A function is called continuous if it is continuous at every point in its domain.

Continuity

Some Properties

- Function composition preserves continuity.
- Continuous functions preserve compactness.

Theorem (Weirstrass Extreme Value Theorem)

Let $f : X \to Y$ be a continuous function and X be a compact set. f attains a maximum and a minimum.

Theorem (Intermediate Value Theorem)

Let $I = [a, b]$ be a closed interval and $f : I \to \mathbb{R}$ a continuous function. If there is a u such that $\min\{f(a), f(b)\} < u < \max\{f(a), f(b)\}$ then there exists a $c \in (a, b)$ st $f(c) = u$.

Economic Application: Utility Representation

Let $(X, \succsim)$ be a weak preference relation. Define this relation to be **continuous** if the following sets are closed:

$$\{y \in X \mid y \succsim x\} \text{ (upper contour set)}$$
$$\{y \in X \mid x \succsim y\} \text{ (lower contour set)}$$

Theorem (MWG 3.C.1)

Suppose that the rational preference relation $\succsim$ on X is continuous. Then there is a continuous utility function $U(x)$ that represents $\succsim$.

Economic Application: Utility Maximization

Let $\mathcal{B}(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{x} \cdot \mathbf{p} \leq w\}$ be an agent's budget set, and let $U : \mathbb{R}^n \to \mathbb{R}$ be a continuous utility function.

Show that the utility maximization problem

$$\max_{\mathbf{x}} U(\mathbf{x})$$
$$\text{s.t. } \mathbf{x} \in \mathcal{B}$$

has a solution.

Proof:
$\mathcal{B}$ is a closed and bounded subset of $\mathbb{R}^n$, therefore it is compact. The image of $\mathcal{B}$, $U(\mathcal{B})$ is also compact, since continuity preserves compactness. Compact subset's of $\mathbb{R}$ contain a maximum and minimum, so a solution exists.

Contraction Mappings

Let (X, d) be a metric space. A function $f : X \to X$ is called a **contraction mapping** if there exists an $0 \leq M < 1$ s.t. $\forall x, y \in X$,

$$d\left(f(x), f(y)\right) \leq Md(x, y)$$

M is called the **modulus** of contraction.

Example

Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}x$ is a contraction mapping.
$|\frac{1}{2}x - \frac{1}{2}y| = |\frac{1}{2}(x - y)| = \frac{1}{2}|x - y| \leq \frac{1}{2}|x - y|$
$\therefore f$ is a contraction mapping with $M = \frac{1}{2}$. ■

Contraction Mappings

A function $f : X \to X$ has a **fixed-point** if $\exists x \in X$ s.t. $f(x) = x$.

Theorem (Banach Fixed-point Theorem)

Let (X, d) be a non-empty complete metric space with a contraction mapping $f : X \to X$. Then f admits a unique fixed-point x^*, $f(x^*) = x^*$.

Furthermore, x^* can be found by iterating f on an arbitrary initial value.

Exercise 2.7

What's the unique fixed-point of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}x$?

Learning Outcomes

You should be able to:

- Use properties of metric spaces, and open/closed balls in proofs.
- Determine whether a sequence is convergent, divergent, and prove that a sequence converges to a given point.
- Apply the monotone convergence theorem and Cauchy convergence criterion.
- Prove that a set is open, closed, both, or neither.
- Define a complete metric space.
- Use the Heine-Borel theorem to identify compact sets.
- Apply the Weirstrass and contraction mapping theorems.

You need not be able to:

- Prove that a particular function is continuous.
- Prove that a particular function is a contraction mapping.