Solutions due by 10.30am Friday 19$^{\text{th}}$ March.

1. Consider the growth problem with full capital depreciation ($\delta = 1$)

$$\max_{\{c_t, k_{t+1}\}} \quad \sum_{t=0}^{\infty} \beta^t \log(c_t)$$
$$\text{s.t.} \quad c_t + k_{t+1} = A k_t^{\alpha}$$

   a) Write the problem's Bellman equation.
   b) Guess and verify that $V(k_t) = a + b \log k_t$.
   c) Find the optimal policies for k_{t+1} and c_t.

2. Consider the growth model with labor:

$$\max_{\{c_t, \ell_t, k_{t+1}\}} \quad \sum_{t=0}^{\infty} \beta^t [\log(c_t) + \eta \log(1 - \ell_t)]$$
$$\text{s.t.} \quad c_t + k_{t+1} = A k_t^{\alpha} \ell^{1-\alpha}$$

   Show that the model has the following steady state:

$$\bar{\ell} = \frac{1}{1 + \frac{\eta}{1-\alpha}\left(1 - \frac{\alpha\beta\delta}{1-\beta(1-\delta)}\right)}$$
$$\bar{k} = \bar{\ell} \left[\frac{\alpha\beta A}{1 - \beta(1-\delta)} \right]^{\frac{1}{1-\alpha}}$$
$$\bar{c} = A \bar{k}^{\alpha} \bar{\ell}^{1-\alpha} - \delta \bar{k}$$

3. Consider a tree whose growth is determined by a function h. This is, if k_t is the size of the tree in period t, then $k_{t+1} = h(k_t)$, $t = 0, 1, \ldots$. Suppose h is strictly increasing, strictly conncave, and $h(0) > 0$. Assume that the price of wood and the interest rate are constant over time, with $p = 1$ and $\beta = \frac{1}{1+r}$. Assume further that it is costless to cut down the tree. If the tree cannot be replanted, present value maximization leads to the functional equation $V(k_t) = \max\{k_t, \beta V(h(k_{t+1}))\}$.

a) Show that the above operator satisfies Blackwell's conditions for a contraction mapping.

b) Let k_0 be the height of the tree that solves $\beta h(k_0) = k_0$. Show that the rule "cut down the tree if $k \geq k_0$, leave it standing otherwise" is optimal.

4. [Challenge Problem] You don't have to submit this one if you don't want to.

Check out the setup of an odd game here:

https://www.youtube.com/watch?v=6_yU9eJ0NxA&ab_channel=Numberphile.

We can define the expected payoff of this game recursively with value functions. Let $V(R_t)$ be the expected payoff of the game.

a) Show that

$$V(R_t) = 1 + \mathbb{P}(d_t \leq R_t) \int_0^{R_t} \mathbb{P}(d_t = x \mid d_t \leq R_t) V\left(\sqrt{R_t^2 - x^2}\right)$$

where R_t is the current radius of the board, and d_t is the distance the dart lands from the origin.

b) Find expressions for $\mathbb{P}(d_t \leq R_t)$ and $\mathbb{P}(d_t = x \mid d_t \leq R_t)$.

c) Verify that the above operator satisfies the Blackwell conditions.

d) Guess $V^0 = 0$ and manually perform 3 value function iterations.

e) Make a guess for the value function and verify it.

f) What's the expected score for the game with dart board of initial radius 1?