Solutions due by 5pm Friday 26$^{\text{th}}$ February.

1. (2 points) Prove by induction that the sum of the first n odd numbers is n^2.

2. (2 points) Show that the intersection of any number of closed sets is closed.

3. (2 points) Let (X, d) be a metric space and $\{x_n\}_{n=1}^{\infty}$ be a sequence. Suppose $x_n \to a$ and $x_n \to b$. Show that $a = b$. That is, a sequence can converge to at most one point.

4. (3 points) For each of the following matrices, state the dimensions of the column and null spaces.

   a) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$

5. (3 points) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   a) Find the eigenvalues and eigenvectors of A. [2 points]

   b) Can A be diagonalized? If yes, verify that $A = P^{-1}DP$; if no, state why. [1 point]

6. (2 points) Let $X \sim N(0, 1)$ and $Y \sim N(0, 1)$, X and Y independent. Suppose I want to generate another standard normal variable, $Z \sim N(0, 1)$, such that $\text{Cov}(X, Z) = \rho$ for some arbitrary $\rho \in [-1, 1]$. Find constants a and b such that the linear combination $Z = aX + bY$ yields such a variable.

7. (2 points) Let (X, d) be a complete metric space. Let $f : X \to X$ be a contraction mapping, such that $\forall x, y \in X \; d(f(x), f(y)) \leq Md(x, y)$, $0 \leq M < 1$. Show that the sequnce formed by iterating the function starting from an arbitrary point $x_0 \in X$, $x_n = f(x_{n-1})$ for $n \geq 1$, is a Cauchy sequence.