Theoretical Assignment
DeepBayes Summer School 2018 (deepbayes.ru)

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0.1 Problem 1

Goal: show that η follows a Poisson distribution with parameter $p\lambda$

0.1.1 Generative model

Our problem is like a generative model with the following distributions. Note that I am using different variable naming.

1. I use N and n to denote the random variable ζ and its realization
2. I use X and x to denote the random variable η and its realization

$$N \sim Poisson(n; \lambda) \tag{1}$$
$$X \sim Binomial(x; p, n) \tag{2}$$

The joint distribution over our generative model is

$$p(x, n|\lambda, p) = p(x|p, n)p(n|\lambda)$$

0.1.2 Marginalize over N

To get the distribution over x, we marginalize over N:

$$\begin{aligned} p(x|\lambda, p) &= \int p(x, n|\lambda, p)dn && (3) \\ &= \int p(x|p, n)p(n|\lambda)dn && (4) \\ &= \int \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}\ e^{-\lambda}\lambda^n\frac{1}{n!}dn && (5) \\ &= \frac{1}{x!}p^x e^{-\lambda}\int \frac{1}{(n-x)!}(1-p)^{n-x}\lambda^n dn && (6) \\ &= \frac{1}{x!}(\lambda p)^x e^{-\lambda p}\int \frac{1}{(n-x)!}e^{-\lambda(1-p)}(\lambda(1-p))^{n-x}dn && (7) \end{aligned}$$

In the last two lines, we re-arranged the terms. Now we actually recognize the integral as an integral over a shifted Poisson distribution, with parameter $\lambda(1-p)$. Therefore, the integral evaluates to 1. We are left with:

$$p(x|\lambda, p) = \frac{1}{x!}(\lambda p)^x e^{-\lambda p} \tag{8}$$

We recognize this distribution as a Poisson distribution over x with parameter λp, which proves the goal.

0.2 Problem 2

Goal: find the probability of the datum $t = 10$ given that the reviewer is kind

0.2.1 Model

Our problem consist of a model over two reviewers.
Let's denote with R the random variable of the reviewers. The sample space for R is $\{r1, r2\}$ where $r1$ and $r2$ are reviewer 1 and 2, respectively.
Now our model is

$$p(t|R = r_1) = \mathcal{N}(t; \mu_1, \sigma_1^2) \tag{9}$$

$$p(t|R = r_2) = \mathcal{N}(t; \mu_2, \sigma_2^2) \tag{10}$$

We seek for $p(R = r_2|t = 10)$.
We use Bayes rule to find this probability:

$$p(R = r_2|t = 10) = \frac{p(t = 10|R = r_2)p(R = r2)}{\sum_{r \in \{r_1, r_2\}} p(t = 10|R = r)p(R = r)}$$

We fill in all the numbers and get

$$p(R = r_2|t = 10) = \frac{\mathcal{N}(10; 20, 5^2)\frac{1}{2}}{\mathcal{N}(10; 30, 10^2)\frac{1}{2} + \mathcal{N}(10; 20, 5^2)\frac{1}{2}} = \frac{2}{3}$$