TheoryExerice

Tom Demont, Louise Rieupouilh, Emmanuelle Denove

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1 Iterative formula derivation

We want to prove that $\mathbf{c}_b = \sum_{i=0}^{+\infty}(1-\alpha_i)(\prod_{k=0}^{i-1}\alpha_k)\mathbf{c}_i$ by derivating the recursive formula $\mathbf{c}_b = (1-\alpha_0)\mathbf{c}_0+\alpha_0\mathbf{c}^1$ where $\mathbf{c}^i = (1-\alpha_i)\mathbf{c}_i+\alpha_i\mathbf{c}^{i+1}$. The $\mathbf{c}^i$s correspond to the the color obtained after reflections i to N. We will conventionally call $\mathbf{c}^0$ the value $\mathbf{c}_b$, the color of the pixel after considering all our reflections.

1.1 Considering N steps

For the first part of this proof, we'll consider that at most N reflections occur. This means that $\mathbf{c}_i = 0$ for $i > N$ and consequently, $\mathbf{c}^i = 0$ for $i > N$. This leads to easily compute our color $\mathbf{c}^N = (1-\alpha_N)\mathbf{c}_N$ as $\mathbf{c}^{N+1} = 0$.

We will now prove by recurrence that:

$$\mathbf{c}^m = \sum_{i=N}^{m}(1-\alpha_i)(\prod_{k=i-1}^{m}\alpha_k)\mathbf{c}_i$$

for all $m = N, N-1, ..., 0$. We will consider that in the product, for all the values of k that are strictly less than m, the product yields 1 (as it seems to be done in the given original formula).

1.1.1 Initial step

Using both recursive and iterative formulas, we observe that $\mathbf{c}^N = (1-\alpha_N)\mathbf{c}_N$. Our initial steps goes toward our hypothesis.

1.1.2 Iterative step

We will now consider that $\mathbf{c}^{m-1} = \sum_{i=N}^{m-1}(1-\alpha_i)(\prod_{k=i-1}^{m-1}\alpha_k)\mathbf{c}_i$. We can now recall that the recursive formulas gives $\mathbf{c}^{m-1} = (1-\alpha_{m-1})\mathbf{c}_{m-1} + \alpha_{m-1}\mathbf{c}^m$. Taking both equations we should have:

$$\sum_{i=N}^{m-1}(1-\alpha_i)(\prod_{k=i-1}^{m-1}\alpha_k)\mathbf{c}_i = (1-\alpha_{m-1})\mathbf{c}_{m-1} + \alpha_{m-1}\mathbf{c}^m$$

By isolating $\mathbf{c}^m$ we have:

$$\mathbf{c}^m = \frac{1}{\alpha_{m-1}}((\sum_{i=N}^{m-1}(1-\alpha_i)(\prod_{k=i-1}^{m-1}\alpha_k)\mathbf{c}_i) - (1-\alpha_{m-1})\mathbf{c}_{m-1})$$

By putting the last term out of the sum, we see it cancels the rightmost part:

$$\mathbf{c}^m = \frac{1}{\alpha_{m-1}}(\sum_{i=N}^{m}(1-\alpha_i)(\prod_{k=i-1}^{m-1}\alpha_k)\mathbf{c}_i)$$

Finally, by dividing each term of the sum by $\frac{1}{\alpha_{m-1}}$ we can retrieve the last element of the product in every summed element leading to:

$$\mathbf{c}^m = \sum_{i=N}^{m}(1-\alpha_i)(\prod_{k=i-1}^{m}\alpha_k)\mathbf{c}_i$$

Which completes our induction. We then have that the relation is true for every m from N to 0 (we do not consider going below 0 for the physical sense of our demonstration).

1.1.3 Conclusion

The previous parts show us that the relation is true especially for $m = 0$, which, considering the introduction of the demonstration, yields:

$$\mathbf{c}_b = \mathbf{c}^0 = \sum_{i=0}^{N}(1-\alpha_i)(\prod_{k=0}^{i-1}\alpha_k)\mathbf{c}_i$$

By inverting the order of our product and sum (which does not change the result by commutativity of real addition and multiplication).

1.2 For ∞ steps

From the previous formula, the recurrence being true makes us able to make a similar proof for any $N \geq 0$. Considering $N \to \infty$, we have that:

$$\mathbf{c}_b = \sum_{i=0}^{+\infty}(1-\alpha_i)(\prod_{k=0}^{i-1}\alpha_k)\mathbf{c}_i$$

2 Iterative formula for our use

As we only consider at most N reflections, we can restrain ourselves to the previously demonstrated formula:

$$\mathbf{c}_b = \sum_{i=0}^{N}(1-\alpha_i)(\prod_{k=0}^{i-1}\alpha_k)\mathbf{c}_i$$

Our `lighting` function returns the $\mathbf{c}_i$ color on the point intersected by the ith reflected ray. We can easily compute the $\mathbf{c}_b$ light by iterating N times and computing the returned color of the new point and iteratively compute the weighting of reflection $(1 - \alpha_i)(\prod_{k=0}^{i-1} \alpha_k)$.