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1.4 ANALYSIS OF ALGORITHMS

- *introduction*
- *observations*
- *mathematical models*
- *order-of-growth classifications*
- *theory of algorithms*
- *memory*

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1.4 Analysis of Algorithms

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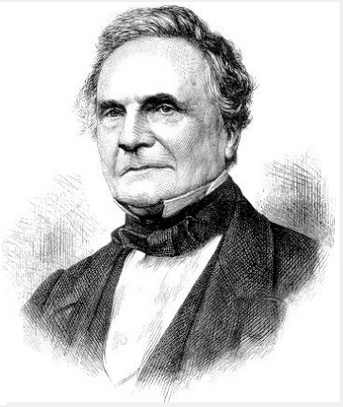
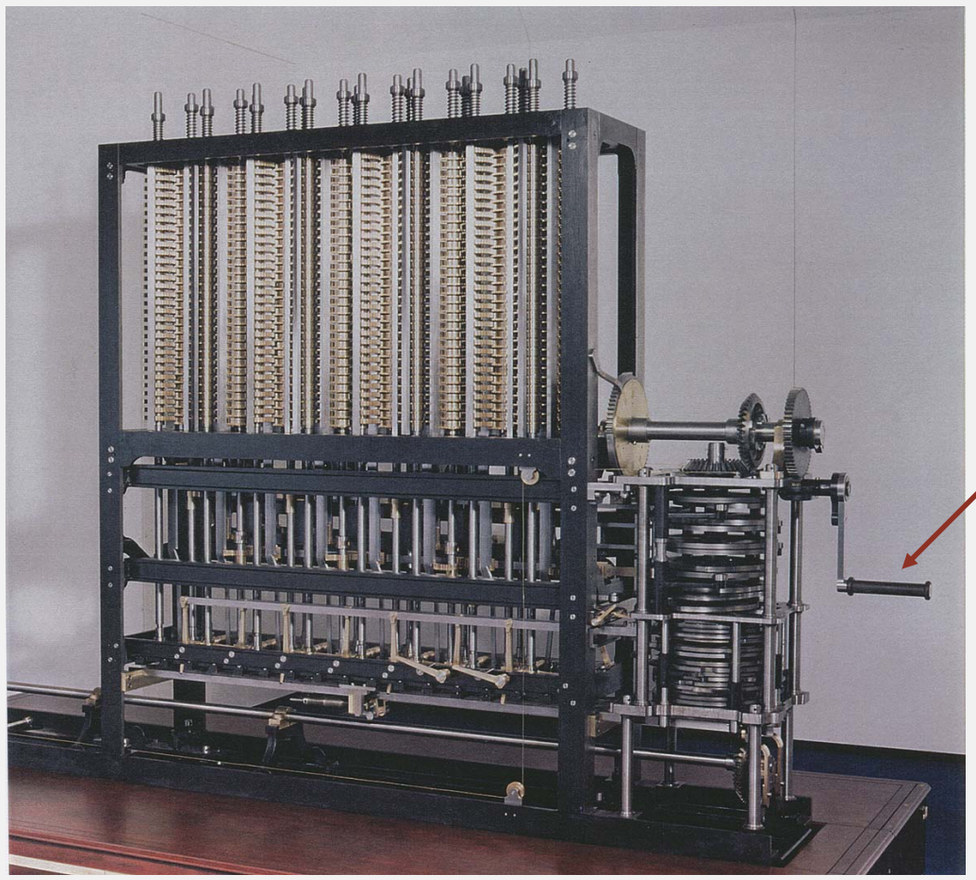
Running time

> *“ As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise—By what course of calculation can these results be arrived at by the machine in the shortest time? ”* — *Charles Babbage (1864)*





how many times do you have to turn the crank?

Analytic Engine

Cast of characters

Programmer needs to develop a working solution.

Client wants to solve problem efficiently.

Student might play any or all of these roles someday.

Theoretician wants to understand.

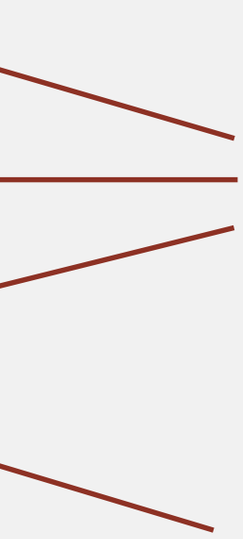
Reasons to analyze algorithms

Predict performance.

Compare algorithms.

Provide guarantees.

Understand theoretical basis.



this course (232)

theory course (338)

Primary practical reason: avoid performance bugs.

client gets poor performance because programmer did not understand performance characteristics

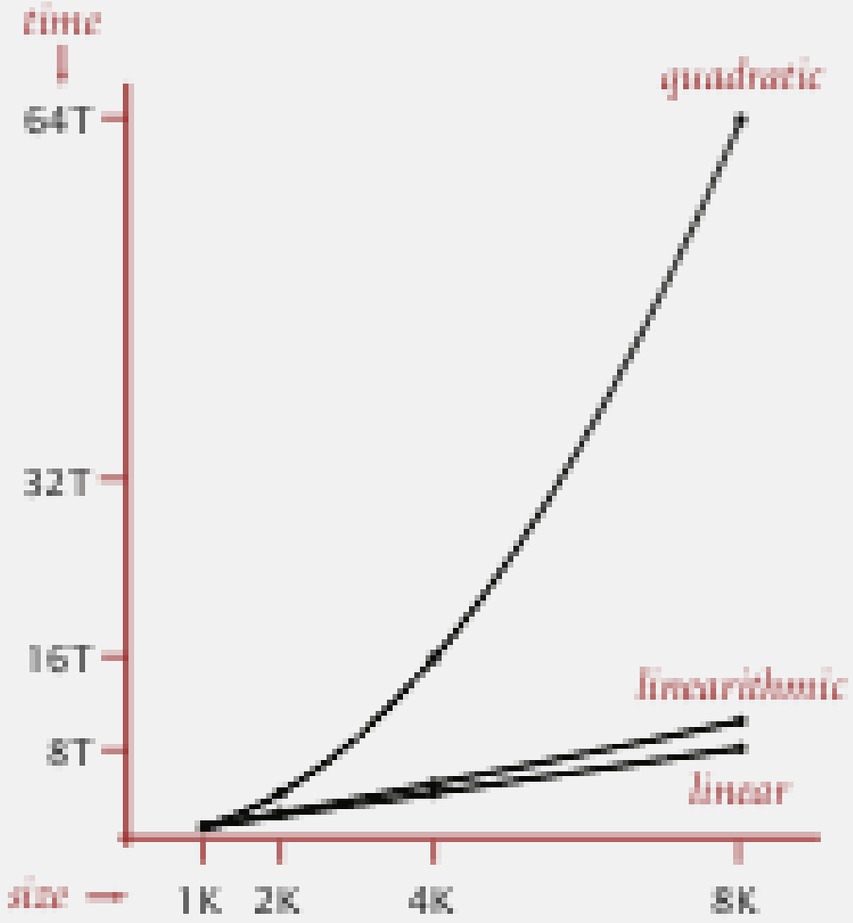
Some algorithmic successes

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: N^2 steps.
- FFT algorithm: $N \log N$ steps, enables new technology.

Friedrich Gauss

1805

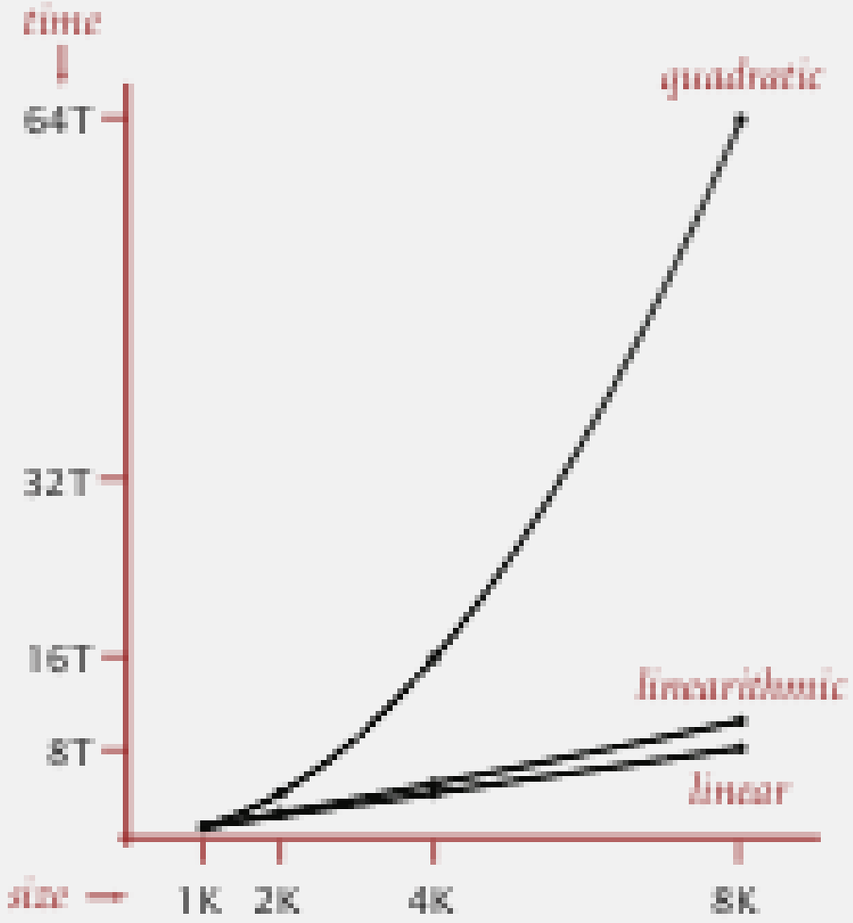


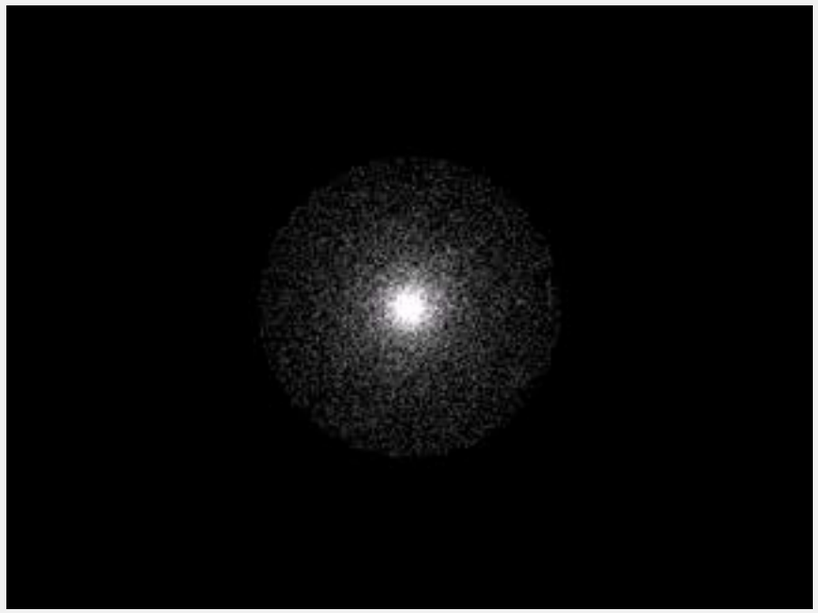
Some algorithmic successes

N-body simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.
- Barnes-Hut algorithm: $N \log N$ steps, enables new research.

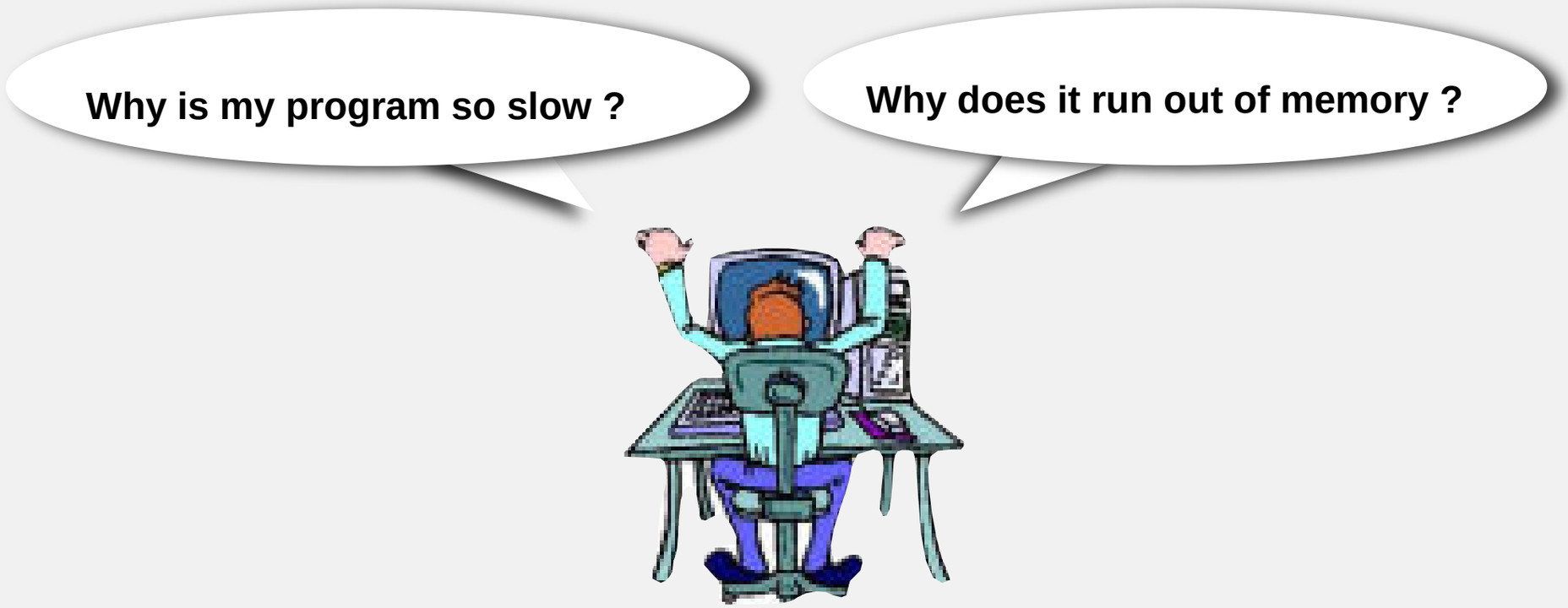
Andrew Appel





The challenge

Q. Will my program be able to solve a large practical input?



Insight. [Knuth 1970s] Use scientific method to understand performance.

Sometimes We Can’t Afford To Do It

- We look at how expensive something is before we buy it
- Same with computers
  - Even if our problem is decidable, the algorithm may take too many resources to decide it
  - How long will it take to compute something
  - Some take so many resources that they become the same as undecidable problems

Taken From Computers Ltd: What They Really Can’t Do by David Harel

What Resources Are We Talking About?

- Running time
- Memory space

- Tied to the input
- Size/amount of the input

Complexity

- Focusing on running time
  - Called time complexity
- Size of the input (not the input itself)
  - Let's call it N to be generic
- If we say an algorithm runs in time 5N
  - We mean that it never performs more than 5 x N (5N) basic actions with N input

Time Complexity

- Big O Notation
- https://www.youtube.com/watch?v=v4cd1O4zkGw

Start Here Wednesday

- Lessons learned with Lab 1 – Starting with Lab 2

My solution to the extra credit

Ways to categorize how long an algorithm will run

- Constant time
  - Not tied to the amount of input it is given
- Linear
  - Proportional to the size of the input
- Quadratic
  - Proportional to n^2
- Polynomial
  - Proportional to n^x
- Exponential
  - Proportional to 2^n

Fastest to Slowest Time Complexities

- Constant
- Linear
- Quadratic
- Polynomial (N^K)
- Exponential (2^N, $N!$, N^N)
- Gets very bad as N gets large

Big O Rules

Given a function that describes the running time of an algorithm, the Big O notation for that function can be determined using the following rules:

1. If f(N) is a sum of several terms, the highest order term (the one with the fastest growth rate) is kept and others are discarded.
2. If f(N) has a term that is a product of several factors, all constants (those that are not in terms of N) are omitted.

Questions on DOCCAM

O(1)	Constant	FindMin(x, y) { if (x < y) { return x } else { return y } }

Taken from Zybook: Data Structures Essentials

O(N)	Linear	``` LinearSearch(numbers, numbersSize, key) { for (i = 0; i < numbersSize; ++i) { if (numbers[i] == key) { return i } } return -1 // not found } ```

Taken from Zybook: Data Structures Essentials

$O(N^2)$	Quadratic	SelectionSort(numbers, numbersSize) { for (i = 0; i < numbersSize; ++i) { indexSmallest = i for (j = i + 1; j < numbersSize; ++j) { if (numbers[j] < numbers[indexSmallest]) { indexSmallest = j } } temp = numbers[i] numbers[i] = numbers[indexSmallest] numbers[indexSmallest] = temp }

The more embedded for loops - polynomial

Taken from Zybook: Data Structures Essentials

$O(c^N)$	Exponential	``` Fibonacci(N) { if ((1 == N) || (2 == N)) { return 1 } return Fibonacci(N-1) + Fibonacci(N-2) } ```

Taken from Zybook: Data Structures Essentials

Measuring Complexity

- Let’s think about searching a long list of names for a particular name
  - Inefficient algorithm
    - Starting at the beginning of the list
    - Search each list item to see if it matches the given name
      - For each list item, determine if we are at the end of the list
- This has 2 actions for every name
- The time complexity is 2N where N stands for the length of the list
- How could we improve this?

Measuring Complexity

- We could add the given name to the end of the list
- This means we will always find the name
- When we find the name, we check (only once) if we are at the end of the list
- Our new time complexity is now N
  - 50% improvement

Searching a List

- Even though we cut our search time complexity from 2N to N, both are proportional to N
- The time grows linearly with N
  - Runs in linear time
- We want to find order of magnitude improvements not just linear improvements
  - We want to improve the performance greater as N increases
- How can we improve our list search by an order of magnitude?

Searching a List

BinarySearch.java

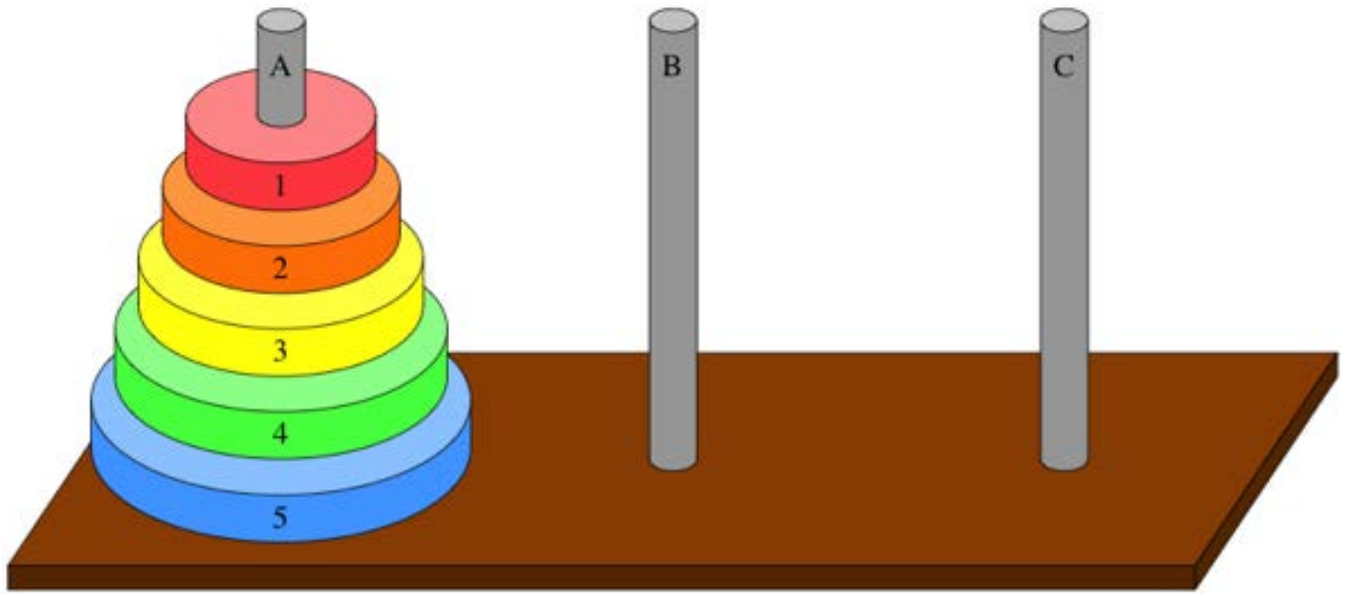
- By using a sorted list
- binary search
  - We can start in the middle and search either the first half
  - We continue to cut down by halves until we get to the string we are looking for
- $O(\log_2 N)$
- logarithmic time much better than linear time

- Using the code in github, show that this is true
  - Use this following list as an example – search for 12
    - 15 2 9 20 3 50 35 72 28 29 16 33 44 1 8 11

Start Here Friday

Towers of Hanoi

- 2^N-1 moves
- Exponential time
- With 64 rings
- 2^{64}-1 moves
- If they could move a ring every 5 seconds, would take 3 trillion years to finish



Question and Review

- Input: a list of 10,000 elements
- Known: each comparison takes 2ms (1/1000th of a sec)
- What is the fastest possible runtime for a linear search?
- What is the longest possible runtime?
- What is Big O for linear search?
- What algorithm would get us to a better time?
- What longest possible runtime of that algorithm with our input and runtime per comparison?

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1.4 Binary Search Demo

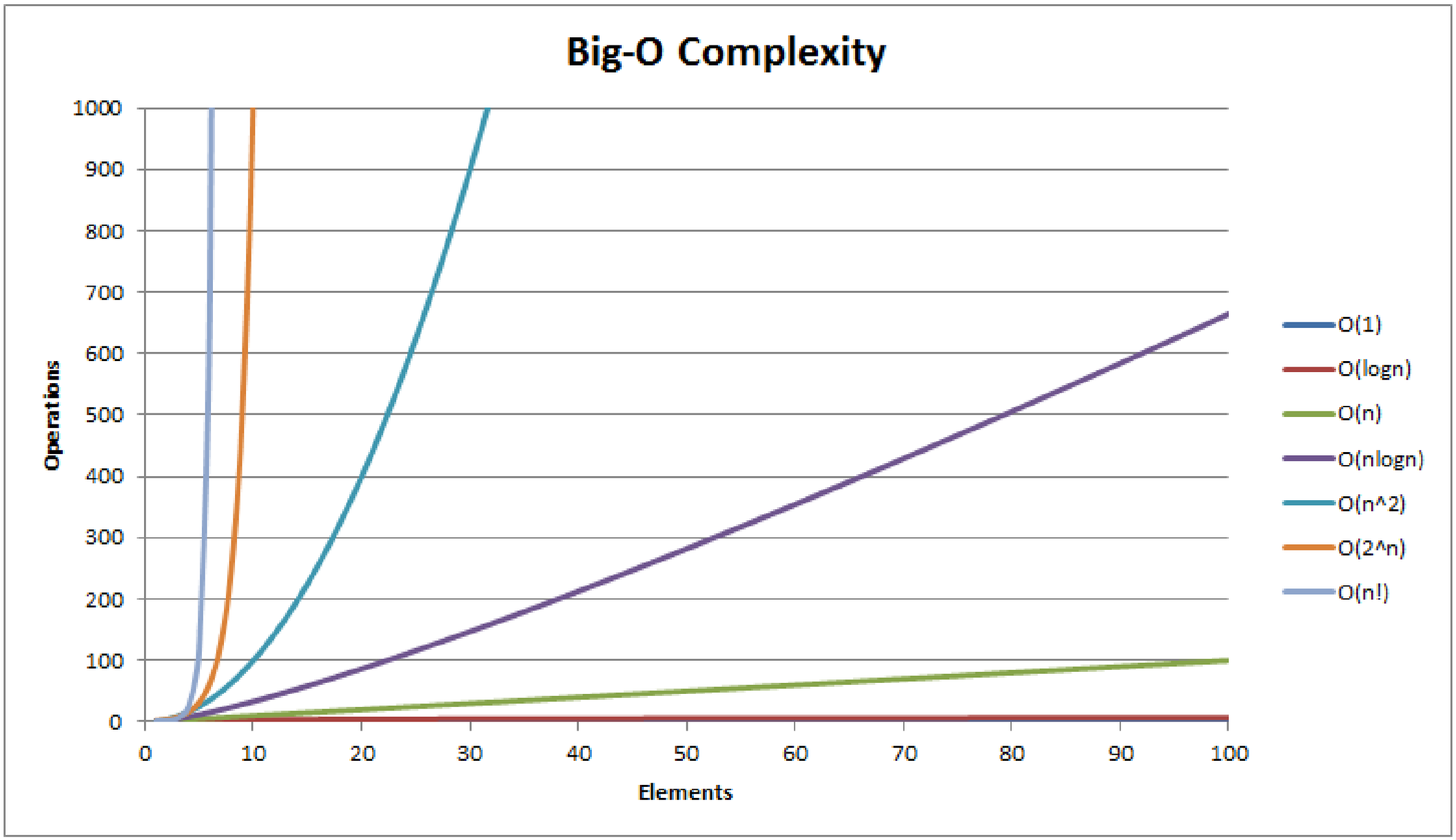
Running Time of 4 Algorithms
Computer capable of a 1M instructions per sec

Input Size

Big O	10	20	50	100	200
N^2	1/10000 sec	1/2500 sec	1/400 sec	1/100 sec	1/25 sec
N^5	1/10 sec	3.2 secs	5.2 mins	2.8 hrs	3.7 days
2^N	1/1000 sec	1 sec	35 years	400 tril cen	45 dig cen
N^N	2.8 hrs	3.3 tril yrs	70 dig cen	185 dig cen	445 dig cen

Taken from H. R. Lewis and C. H. Papdimitrious (1978), The efficiency of algorithms in Computers LTD, what they really can't do by David Harel

Fastest to Slowest Time Complexities



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Example: 3-Sum

What does this algorithm do?

```
% more 8ints.txt
8
30 -40 -20 -10 40 0 10 5

% java ThreeSum 8ints.txt
4
```

ThreeSum.java
*ints.txt

What is the time complexity of this algorithm?

Context. Deeply related to problems in computational geometry.

Example: 3-SUM

Given N distinct integers, how many triples sum to exactly zero?

```
% more 8ints.txt
8
30 -40 -20 -10 40 0 10 5

% java ThreeSum 8ints.txt
4
```

	a[i]	a[j]	a[k]	sum
1	30	-40	10	0
2	30	-20	-10	0
3	-40	40	0	0
4	-10	0	10	0

ThreeSum.java
*ints.txt

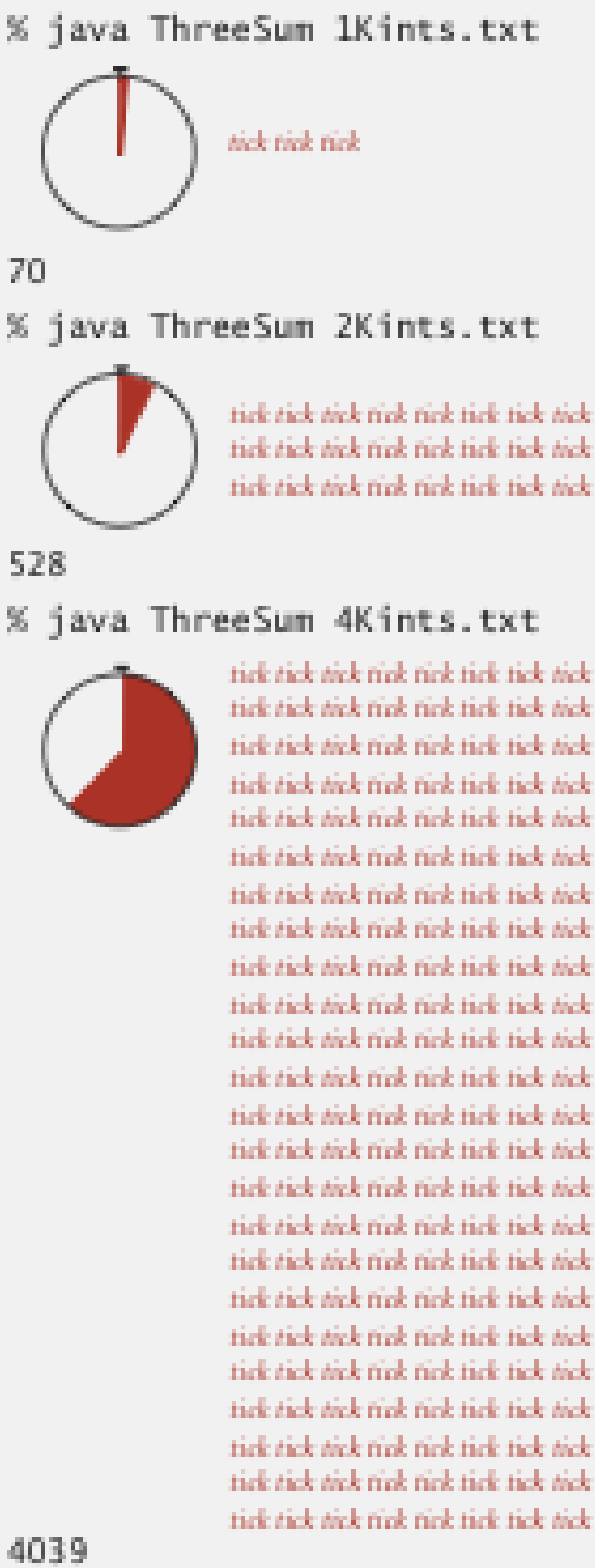
What is the time complexity of this algorithm?

Context. Deeply related to problems in computational geometry.

Measuring the running time

Q. How to time a program?

A. Manual.



Measuring the running time

Q. How to time a program?

A. Automatic.

public class Stopwatch		(part of stdlib.jar)
	Stopwatch()	*create a new stopwatch*
double	elapsedTime()	*time since creation (in seconds)*

ThreeSum.java
StopWatch.java

BinarySearch_time.java
StopWatch.java

StopWatchCPU.java

Empirical analysis

- Run the program for various input sizes and measure running time.

Run with 1Kint.txt – how long did it take?
What do you predict will be the time it takes for 2Kint.txt?
What about 4Kint.txt?

Empirical analysis

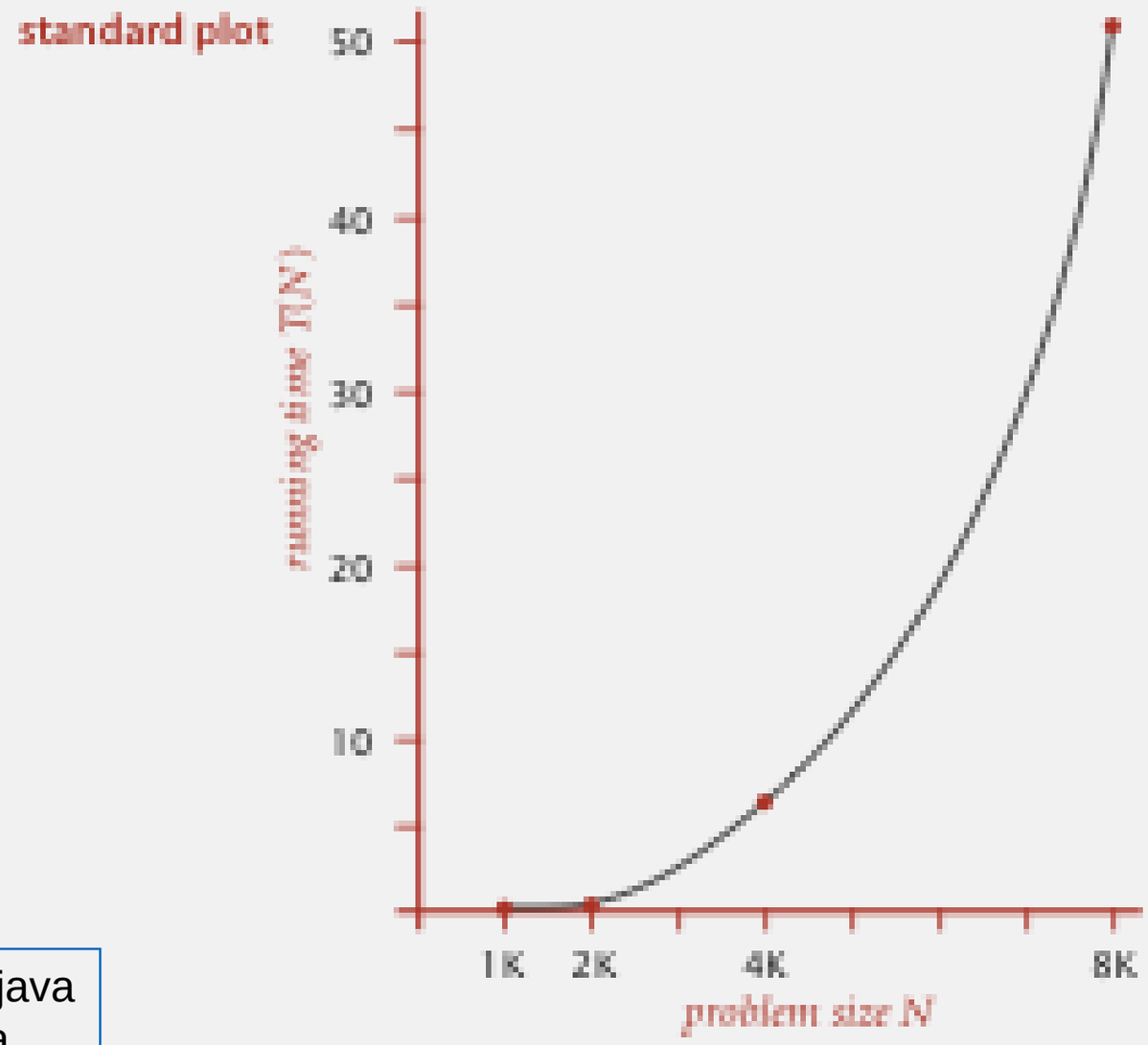
Run the program for various input sizes and measure running time.

N	time (seconds) †
250	0
500	0
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1
16,000	?

ThreeSumDoubling.java
StdRandom.java
StopWatch.java

Data analysis

Standard plot. Plot running time $T(N)$ vs. input size N.



ThreeSumDoubling.java
StdRandom.java
StopWatch.java

Prediction and validation

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

"order of growth" of running time is about N^3 [stay tuned]

Predictions.

- 51.0 seconds for $N = 8{,}000$.
- 408.1 seconds for $N = 16{,}000$.

Observations.

N	time (seconds) †
8,000	51.1
8,000	51
8,000	51.1
16,000	410.8

validates hypothesis!

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power-law relationship.

Q. How to estimate a (assuming we know b) ?

A. Run the program (for a sufficient large value of N) and solve for a.

N	time (seconds) †
8,000	51.1
8,000	51
8,000	51.1

$$51.1 = a \times 8000^3$$

$$\Rightarrow \quad a = 0.998 \times 10^{-10}$$

Hypothesis. Running time is about $0.998 \times 10^{-10} \times N^3$ seconds.

almost identical hypothesis

to one obtained via linear regression

Experimental algorithmics

System independent effects.

- Algorithm.
- Input data.

} determines exponent in power law

System dependent effects.

- Hardware: CPU, memory, cache, …
- Software: compiler, interpreter, garbage collector, …
- System: operating system, network, other apps, …

} determines constant in power law

Bad news. Difficult to get precise measurements.

Good news. Much easier and cheaper than other sciences.

e.g., can run huge number of experiments

ThreeSumFast

- Let’s look at a method of speeding up this algorithm

ThreeSumFast.java

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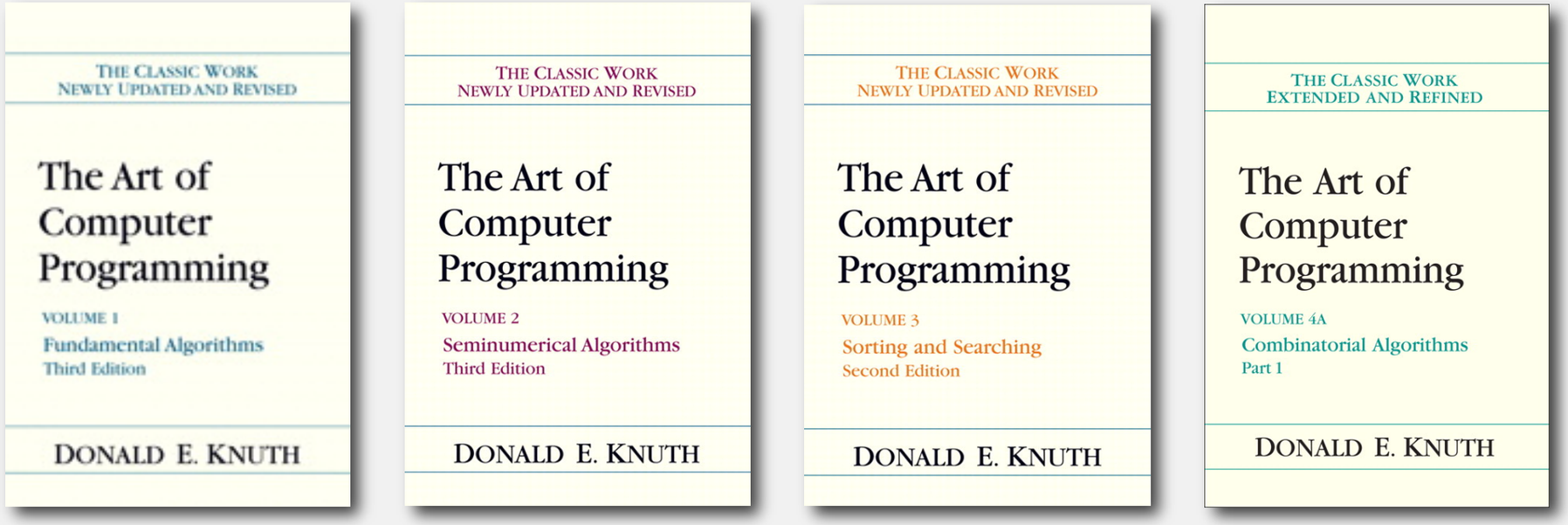
Mathematical models for running time

Total running time: sum of cost $\times$ frequency for all operations.

- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.



Donald Knuth
1974 Turing Award

In principle, accurate mathematical models are available.

Cost of basic operations

Challenge. How to estimate constants.

operation	example	nanoseconds †
integer add	a + b	2.1
integer multiply	a * b	2.4
integer divide	a / b	5.4
floating-point add	a + b	4.6
floating-point multiply	a * b	4.2
floating-point divide	a / b	13.5
sine	Math.sin(theta)	91.3
arctangent	Math.atan2(y, x)	129
...	...	...

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM

Cost of basic operations

Observation. Most primitive operations take constant time.

operation	example	nanoseconds †
variable declaration	int a	c_1
assignment statement	a = b	c_2
integer compare	a < b	c_3
array element access	a[i]	c_4
array length	a.length	c_5
1D array allocation	new int[N]	$c_6 \ N$
2D array allocation	new int[N][N]	$c_7 \ N^2$

Caveat. Non-primitive operations often take more than constant time.

novice mistake: abusive string concatenation

Example: 1-SUM

Q. How many instructions as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   if (a[i] == 0)
      count++;
```

N array accesses

operation	frequency
variable declaration	2
assignment statement	2
less than compare	$N + 1$
equal to compare	N
array access	N
increment	N to $2\,N$

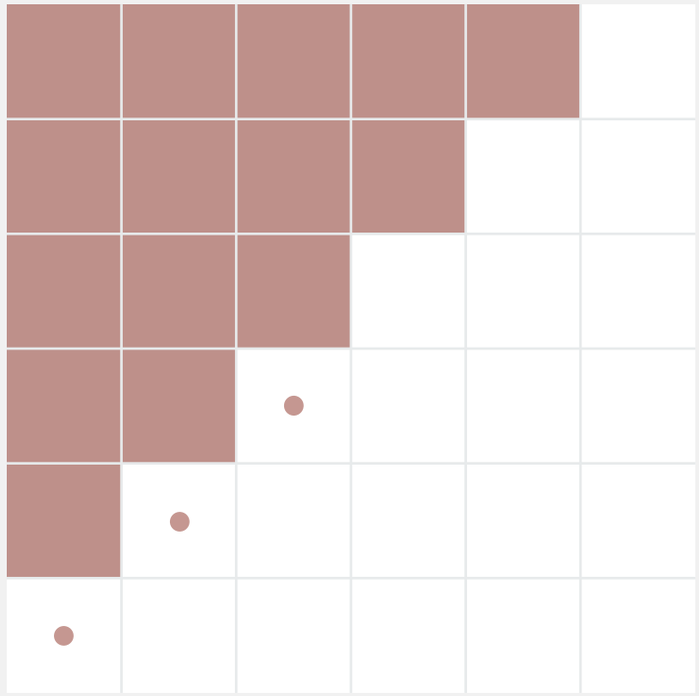
Example: 2-SUM

Q. How many instructions as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1)$$
$$= \binom{N}{2}$$

Pf. [n even]



$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2}N^2 - \frac{1}{2}N$$

half of square — half of diagonal

Example: 2-SUM

Q. How many instructions as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) \;=\; \tfrac{1}{2}\, N\,(N-1)$$

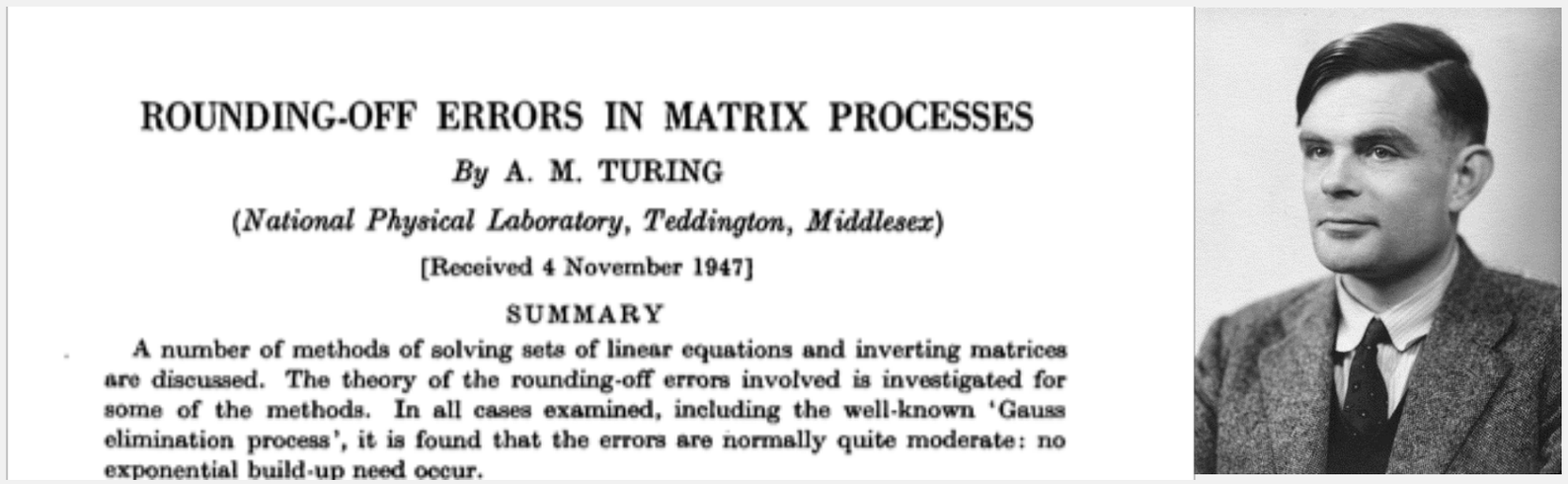
$$=\; \binom{N}{2}$$

operation	frequency
variable declaration	$N + 2$
assignment statement	$N + 2$
less than compare	$\tfrac{1}{2}\,(N+1)\,(N+2)$
equal to compare	$\tfrac{1}{2}\, N\,(N-1)$
array access	$N\,(N-1)$
increment	$\tfrac{1}{2}\, N\,(N-1)$ to $N\,(N-1)$

tedious to count exactly

Simplifying the calculations

> *“ It is convenient to have a measure of the amount of work involved in a computing process, even though it be a very crude one. We may count up the number of times that various elementary operations are applied in the whole process and then given them various weights. We might, for instance, count the number of additions, subtractions, multiplications, divisions, recording of numbers, and extractions of figures from tables. In the case of computing with matrices most of the work consists of multiplications and writing down numbers, and we shall therefore only attempt to count the number of multiplications and recordings. ”* — *Alan Turing*



ROUNDING-OFF ERRORS IN MATRIX PROCESSES

By A. M. TURING

(*National Physical Laboratory, Teddington, Middlesex*)

[Received 4 November 1947]

SUMMARY

A number of methods of solving sets of linear equations and inverting matrices are discussed. The theory of the rounding-off errors involved is investigated for some of the methods. In all cases examined, including the well-known ‘Gauss elimination process’, it is found that the errors are normally quite moderate: no exponential build-up need occur.

Simplification 1: cost model

Cost model. Use some basic operation as a proxy for running time.

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) \;=\; \frac{1}{2} N (N-1)$$

$$= \binom{N}{2}$$

operation	frequency
variable declaration	$N + 2$
assignment statement	$N + 2$
less than compare	$\frac{1}{2} (N + 1) (N + 2)$
equal to compare	$\frac{1}{2} N (N - 1)$
array access	$N (N - 1)$
increment	$\frac{1}{2} N (N - 1)$ to $N (N - 1)$

cost model = array accesses

(we assume compiler/JVM do not optimize any array accesses away!)

Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care

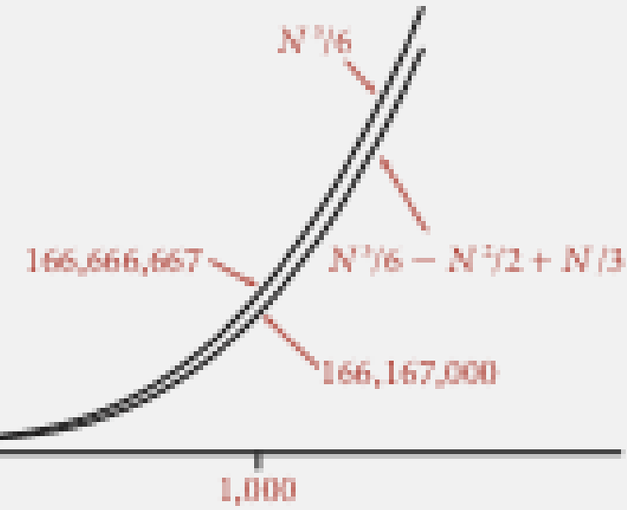
Ex 1. $\frac{1}{6} N^3 + 20 N + 16 \sim \frac{1}{6} N^3$

Ex 2. $\frac{1}{6} N^3 + 100 N^{4/3} + 56 \sim \frac{1}{6} N^3$

Ex 3. $\frac{1}{6} N^3 - \frac{1}{2} N^2 + \frac{1}{3} N \sim \frac{1}{6} N^3$

discard lower-order terms

(e.g., N = 1000: 166.67 million vs. 166.17 million)



Leading-term approximation

Technical definition. $f(N) \sim g(N)$ means $\lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$

Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care

operation	frequency	tilde notation
variable declaration	$N + 2$	$\sim N$
assignment statement	$N + 2$	$\sim N$
less than compare	$\frac{1}{2}(N + 1)(N + 2)$	$\sim \frac{1}{2} N^2$
equal to compare	$\frac{1}{2} N (N - 1)$	$\sim \frac{1}{2} N^2$
array access	$N (N - 1)$	$\sim N^2$
increment	$\frac{1}{2} N (N - 1)$ to $N (N - 1)$	$\sim \frac{1}{2} N^2$ to $\sim N^2$

Example: 2-SUM

Q. Approximately how many array accesses as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

"inner loop"

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1)$$

$$= \binom{N}{2}$$

A. $\sim N^2$ array accesses.

Bottom line. Use cost model and tilde notation to simplify counts.

Example: 3-SUM

Q. Approximately how many array accesses as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

"inner loop"

$$\binom{N}{3} = \frac{N(N-1)(N-2)}{3!} \sim \frac{1}{6}N^3$$

A. $\sim \frac{1}{2} N^3$ array accesses.

Bottom line. Use cost model and tilde notation to simplify counts.

Diversion: estimating a discrete sum

Q. How to estimate a discrete sum?

A1. Take a discrete mathematics course.

A2. Replace the sum with an integral, and use calculus!

Ex 1. $1 + 2 + \ldots + N$.

$$\sum_{i=1}^{N} i \sim \int_{x=1}^{N} x\,dx \sim \frac{1}{2} N^2$$

Ex 2. $1^k + 2^k + \ldots + N^k$.

$$\sum_{i=1}^{N} i^k \sim \int_{x=1}^{N} x^k dx \sim \frac{1}{k+1} N^{k+1}$$

Ex 3. $1 + 1/2 + 1/3 + \ldots + 1/N$.

$$\sum_{i=1}^{N} \frac{1}{i} \sim \int_{x=1}^{N} \frac{1}{x} dx = \ln N$$

Ex 4. 3-sum triple loop.

$$\sum_{i=1}^{N} \sum_{j=i}^{N} \sum_{k=j}^{N} 1 \sim \int_{x=1}^{N} \int_{y=x}^{N} \int_{z=y}^{N} dz\,dy\,dx \sim \frac{1}{6} N^3$$

Estimating a discrete sum

Q. How to estimate a discrete sum?

A1. Take a discrete mathematics course.

A2. Replace the sum with an integral, and use calculus!

Ex 4. $1 + ½ + ¼ + ⅛ + \ldots$

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 2$$

$$\int_{x=0}^{\infty} \left(\frac{1}{2}\right)^x dx = \frac{1}{\ln 2} \approx 1.4427$$

Caveat. Integral trick doesn't always work!

Mathematical models for running time

In principle, accurate mathematical models are available.

In practice,

- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.

costs (depend on machine, compiler)

$$T_N = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E$$

A = array access
B = integer add
C = integer compare
D = increment
E = variable assignment

frequencies

(depend on algorithm, input)

Bottom line. We use approximate models in this course: $T(N) \sim c\,N^3$.

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1.4 Analysis of Algorithms

- *introduction*
- *observations*
- *mathematical models*
- *order-of-growth classifications*
- *theory of algorithms*
- *memory*

Common order-of-growth classifications

Definition. If $f(N) \sim c\, g(N)$ for some constant $c > 0$, then the order of growth of $f(N)$ is $g(N)$.

- Ignores leading coefficient.
- Ignores lower-order terms.

Ex. The order of growth of the running time of this code is N^3.

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

Typical usage. With running times.

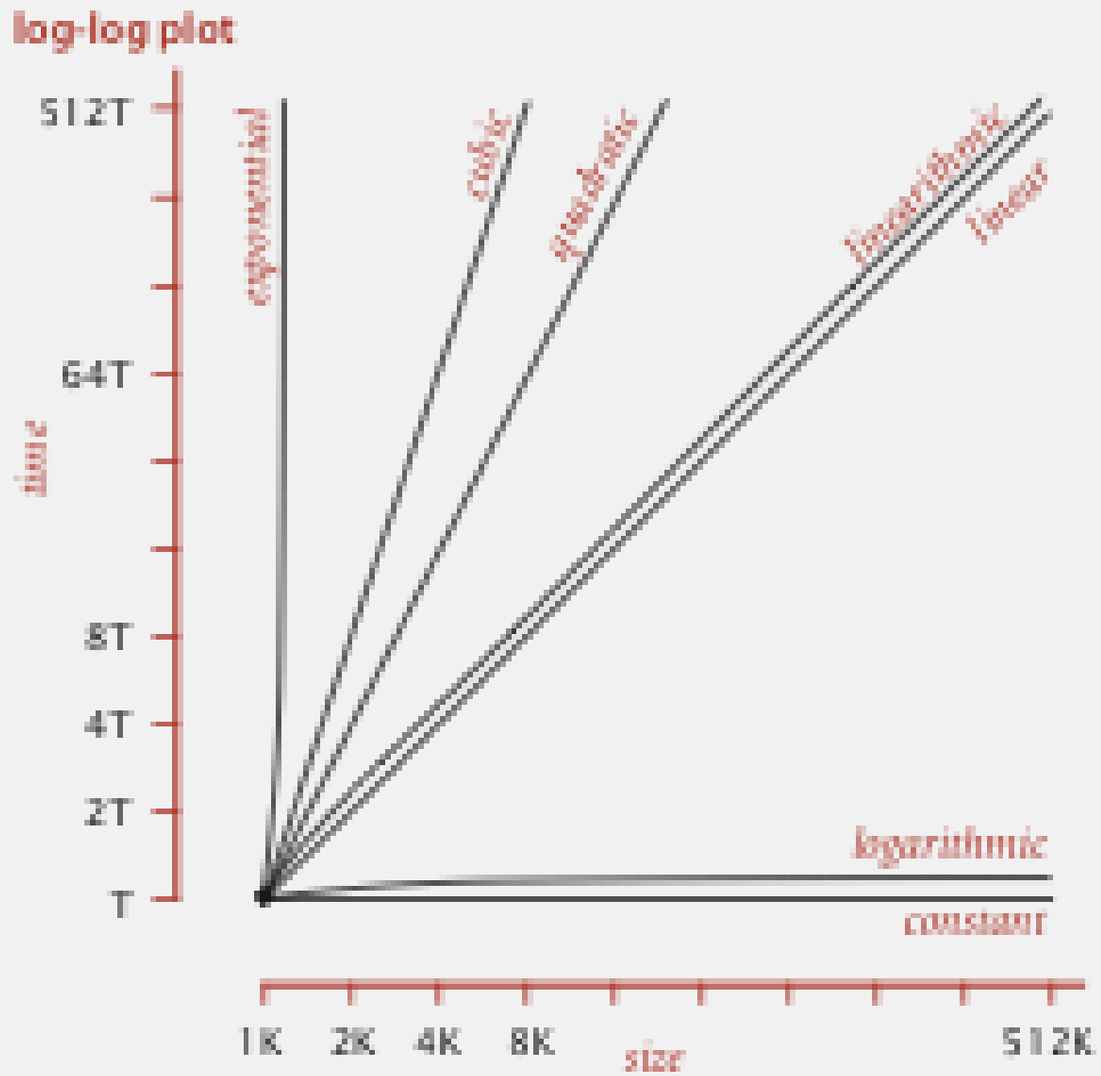
where leading coefficient depends on machine, compiler, JVM, ...

Common order-of-growth classifications

Good news. The set of functions

$$1,\ \log N,\ \ N,\ \ N \log N,\ \ N^2,\ \ N^3, \text{ and } 2^N$$

suffices to describe the order of growth of most common algorithms.



Typical orders of growth

Common order-of-growth classifications

order of growth	name	typical code framework	description	example	$T(2N)$ / $T(N)$
1	**constant**	a = b + c;	statement	add two numbers	1
$\log N$	**logarithmic**	while (N > 1) { N = N / 2; ... }	divide in half	binary search	~ 1
N	**linear**	for (int i = 0; i < N; i++) { ... }	loop	find the maximum	2
$N \log N$	**linearithmic**	[see mergesort lecture]	divide and conquer	mergesort	~ 2
N^2	**quadratic**	for (int i = 0; i < N; i++) for (int j = 0; j < N; j++) { ... }	double loop	check all pairs	4
N^3	**cubic**	for (int i = 0; i < N; i++) for (int j = 0; j < N; j++) for (int k = 0; k < N; k++) { ... }	triple loop	check all triples	8
2^N	**exponential**	[see combinatorial search lecture]	exhaustive search	check all subsets	$T(N)$

Practical implications of order-of-growth

growth rate	problem size solvable in minutes			
	1970s	1980s	1990s	2000s
1	any	any	any	any
log N	any	any	any	any
N	millions	millions	millions	billions
N log N	hundreds of thousands	millions	millions	hundreds of millions
N^2	hundreds	thousand	thousands	tens of thousands
N^3	hundred	hundreds	thousand	thousands
2^N	20	20s	20s	30

Bottom line. Need linear or linearithmic alg to keep pace with Moore's law.

Practical implications of order-of-growth

growth rate	problem size solvable in minutes				time to process millions of inputs			
	1970s	1980s	1990s	2000s	1970s	1980s	1990s	2000s
1	any	any	any	any	instant	instant	instant	instant
log N	any	any	any	any	instant	instant	instant	instant
N	millions	tens of millions	hundreds of millions	billions	minutes	seconds	second	instant
N log N	hundreds of thousands	millions	millions	hundreds of millions	hour	minutes	tens of seconds	seconds
N^2	hundreds	thousand	thousands	tens of thousands	decades	years	months	weeks
N^3	hundred	hundreds	thousand	thousands	never	never	never	millennia

Practical implications of order-of-growth

growth rate	name	description	effect on a program that runs for a few seconds	
			time for 100x more data	size for 100x faster computer
1	constant	independent of input size	–	–
log N	logarithmic	nearly independent of input size	–	–
N	linear	optimal for N inputs	a few minutes	100x
N log N	linearithmic	nearly optimal for N inputs	a few minutes	100x
N^2	quadratic	not practical for large problems	several hours	10x
N^3	cubic	not practical for medium problems	several weeks	4–5x
2^N	exponential	useful only for tiny problems	forever	1x

Binary search: Java implementation

Trivial to implement?

- First binary search published in 1946.
- First bug-free one in 1962.
- Bug in Java's Arrays.binarySearch() discovered in 2006.

```
public static int binarySearch(int[] a, int key)
{
   int lo = 0, hi = a.length-1;
   while (lo <= hi)
   {
      int mid = lo + (hi - lo) / 2;
      if      (key < a[mid]) hi = mid - 1;
      else if (key > a[mid]) lo = mid + 1;
      else return mid;
   }
   return -1;
}
```

one "3-way compare"

Invariant. If key appears in the array a[], then a[lo] ≤ key ≤ a[hi].

Comparing programs

Hypothesis. The sorting-based $N^2 \log N$ algorithm for 3-SUM is significantly faster in practice than the brute-force N^3 algorithm.

N	time (seconds)
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1

ThreeSum.java

N	time (seconds)
1,000	0.14
2,000	0.18
4,000	0.34
8,000	0.96
16,000	3.67
32,000	14.88
64,000	59.16

ThreeSumDeluxe.java

Guiding principle. Typically, better order of growth $\Rightarrow$ faster in practice.

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1.4 Analysis of Algorithms

- *introduction*
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- *theory of algorithms*
- *memory*

Types of analyses

Best case. Lower bound on cost.

- Determined by "easiest" input.
- Provides a goal for all inputs.

Worst case. Upper bound on cost.

- Determined by "most difficult" input.
- Provides a guarantee for all inputs.

Average case. Expected cost for random input.

- Need a model for "random" input.
- Provides a way to predict performance.

this course

Ex 1. Array accesses for brute-force 3-SUM.

Best: $\sim \frac{1}{2} N^3$

Average: $\sim \frac{1}{2} N^3$

Worst: $\sim \frac{1}{2} N^3$

Ex 2. Compares for binary search.

Best: ~ 1

Average: $\sim \lg N$

Worst: $\sim \lg N$

Types of analyses

Best case. Lower bound on cost.

Worst case. Upper bound on cost.

Average case. "Expected" cost.

Actual data might not match input model?

- Need to understand input to effectively process it.
- Approach 1: design for the worst case.
- Approach 2: randomize, depend on probabilistic guarantee.

Theory of algorithms

Goals.

- Establish "difficulty" of a problem.
- Develop "optimal" algorithms.

Approach.

- Suppress details in analysis: analyze "to within a constant factor."
- Eliminate variability in input model: focus on the worst case.

Upper bound. Performance guarantee of algorithm for any input.

Lower bound. Proof that no algorithm can do better.

Optimal algorithm. Lower bound = upper bound (to within a constant factor).

Commonly-used notations in the theory of algorithms

notation	provides	example	shorthand for	used to
Big Theta	asymptotic order of growth	$\Theta(N^2)$	$\frac{1}{2} N^2$ $10 N^2$ $5 N^2 + 22 N \log N + 3N$ ⋮	classify algorithms
Big Oh	$\Theta(N^2)$ and smaller	$O(N^2)$	$10 N^2$ $100 N$ $22 N \log N + 3 N$ ⋮	develop upper bounds
Big Omega	$\Theta(N^2)$ and larger	$\Omega(N^2)$	$\frac{1}{2} N^2$ N^5 $N^3 + 22 N \log N + 3 N$ ⋮	develop lower bounds

Theory of algorithms: example 1

Goals.

- Establish "difficulty" of a problem and develop "optimal" algorithms.
- Ex. 1-SUM = "*Is there a* 0 *in the array?* "

Upper bound. A specific algorithm.

- Ex. Brute-force algorithm for 1-SUM: Look at every array entry.
- Running time of the optimal algorithm for 1-SUM is $\mathrm{O}(N)$.

Lower bound. Proof that no algorithm can do better.

- Ex. Have to examine all *N* entries (any unexamined one might be 0).
- Running time of the optimal algorithm for 1-SUM is $\Omega(N)$.

Optimal algorithm.

- Lower bound equals upper bound (to within a constant factor).
- Ex. Brute-force algorithm for 1-SUM is optimal: its running time is $\Theta(N)$.

Theory of algorithms: example 2

Goals.

- Establish "difficulty" of a problem and develop "optimal" algorithms.
- Ex. 3-SUM.

Upper bound. A specific algorithm.

- Ex. Brute-force algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is $O(N^3)$.

Theory of algorithms: example 2

Goals.

- Establish "difficulty" of a problem and develop "optimal" algorithms.
- Ex. 3-SUM.

Upper bound. A specific algorithm.

- Ex. Improved algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is $O(N^2 \log N)$.

Lower bound. Proof that no algorithm can do better.

- Ex. Have to examine all N entries to solve 3-SUM.
- Running time of the optimal algorithm for solving 3-SUM is $\Omega(N)$.

Open problems.

- Optimal algorithm for 3-SUM?
- Subquadratic algorithm for 3-SUM?
- Quadratic lower bound for 3-SUM?

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1.4 Analysis of Algorithms

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Basics

Bit. 0 or 1.

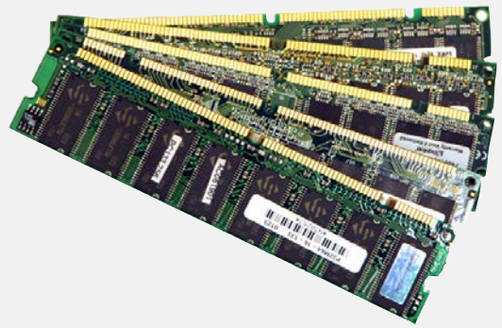
Byte. 8 bits.

Megabyte (MB). 1 million or 2^{20} bytes.

NIST

most computer scientists

Gigabyte (GB). 1 billion or 2^{30} bytes.



64-bit machine. We assume a 64-bit machine with 8-byte pointers.

- Can address more memory.
- Pointers use more space.

some JVMs "compress" ordinary object pointers to 4 bytes to avoid this cost

Typical memory usage for primitive types and arrays

type	bytes
boolean	1
byte	1
char	2
int	4
float	4
long	8
double	8

primitive types

type	bytes
char[]	$2N + 24$
int[]	$4N + 24$
double[]	$8N + 24$

one-dimensional arrays

type	bytes
char[][]	$\sim 2MN$
int[][]	$\sim 4MN$
double[][]	$\sim 8MN$

two-dimensional arrays

Typical memory usage for objects in Java

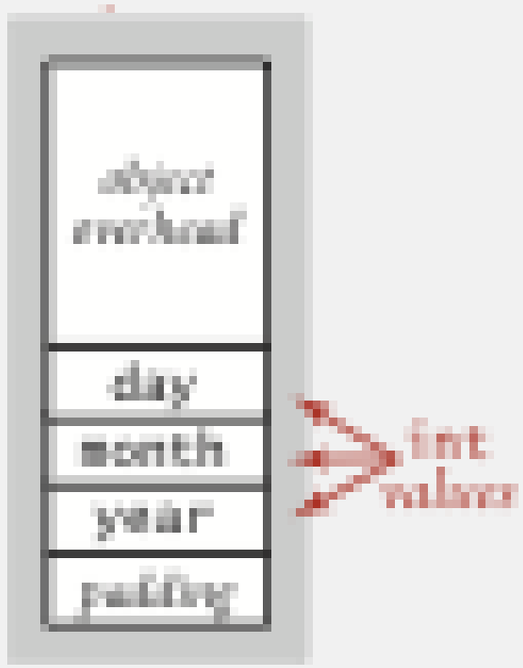
Object overhead. 16 bytes.

Reference. 8 bytes.

Padding. Each object uses a multiple of 8 bytes.

Ex 1. A Date object uses 32 bytes of memory.

```
public class Date
{
   private int day;
   private int month;
   private int year;
...
}
```



16 bytes (object overhead)

4 bytes (int)

4 bytes (int)

4 bytes (int)

4 bytes (padding)

32 bytes

Typical memory usage for objects in Java

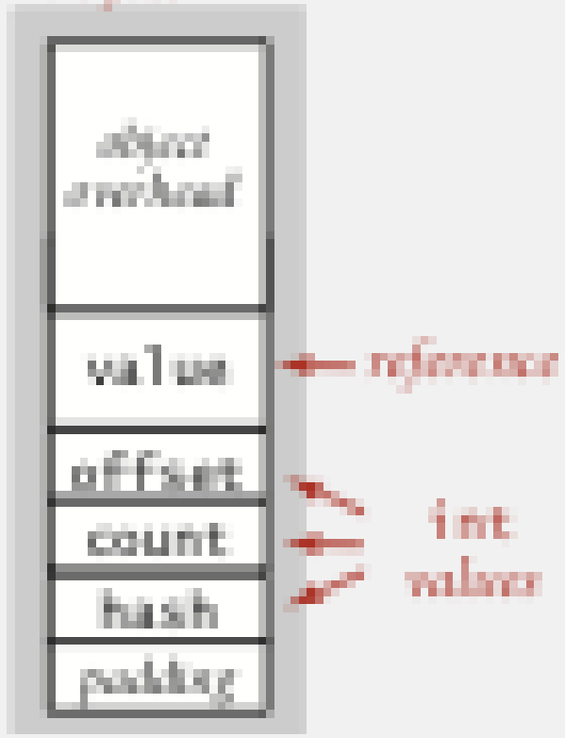
Object overhead. 16 bytes.

Reference. 8 bytes.

Padding. Each object uses a multiple of 8 bytes.

Ex 2. A virgin String of length N uses $\sim 2N$ bytes of memory.

```
public class String
{
   private char[] value;
   private int offset;
   private int count;
   private int hash;
...
}
```



16 bytes (object overhead)

8 bytes (reference to array)
2N + 24 bytes (char[] array)

4 bytes (int)

4 bytes (int)

4 bytes (int)

4 bytes (padding)

2N + 64 bytes

Typical memory usage summary

Total memory usage for a data type value:

- Primitive type: 4 bytes for int, 8 bytes for double, …
- Object reference: 8 bytes.
- Array: 24 bytes + memory for each array entry.
- Object: 16 bytes + memory for each instance variable.
- Padding: round up to multiple of 8 bytes.

+ 8 extra bytes per inner class object

(for reference to enclosing class)

Shallow memory usage: Don't count referenced objects.

Deep memory usage: If array entry or instance variable is a reference, count memory (recursively) for referenced object.

Example

Q. How much memory does WeightedQuickUnionUF use as a function of N?
Use tilde notation to simplify your answer.

```
public class WeightedQuickUnionUF
{
   private int[] id;
   private int[] sz;
   private int count;

   public WeightedQuickUnionUF(int N)
   {
      id = new int[N];
      sz = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
      for (int i = 0; i < N; i++) sz[i] = 1;
   }
   ...
}
```

16 bytes (object overhead)

8 + (4N + 24) bytes each (reference + int[] array)

4 bytes (int)

4 bytes (padding)

8N + 88 bytes

A. $8N + 88 \sim 8N$ bytes.

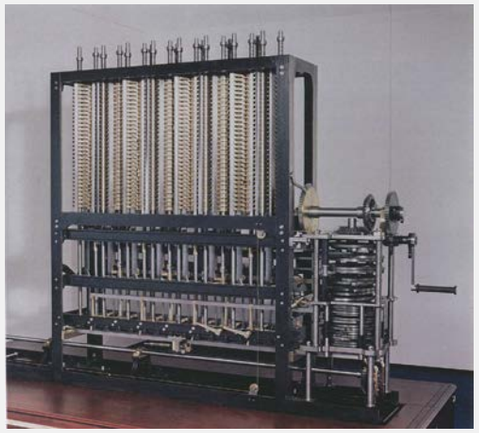
Turning the crank: summary

Empirical analysis.

- Execute program to perform experiments.
- Assume power law and formulate a hypothesis for running time.
- Model enables us to make predictions.

Mathematical analysis.

- Analyze algorithm to count frequency of operations.
- Use tilde notation to simplify analysis.
- Model enables us to explain behavior.



Scientific method.

- Mathematical model is independent of a particular system; applies to machines not yet built.
- Empirical analysis is necessary to validate mathematical models and to make predictions.