Fun with Lambda Calculus

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1. Beta-reduce the following expressions to their normal form:

A. $(\lambda a\ \lambda y\ .\ y\ a)\ (z\ z)$

$[(z\ z)/a]$ in $\lambda y\ .\ y\ a$
$\underset{\beta}{\rightarrow} \lambda y\ .\ y\ (z\ z)$

B. $(\lambda x\ \lambda y\ .\ (x\ y))\ (\lambda z.y)$

$(\lambda x\ \lambda y\ .\ (x\ y))\ (\lambda z\ y)$
$\underset{\alpha}{=} (\lambda x\ \lambda a\ .\ (x\ a))\ (\lambda z\ y)$
$[(\lambda z\ y)/x]$ in $\lambda a\ .\ (x\ a)$
$\underset{\beta}{\rightarrow} \lambda a\ .\ ((\lambda z\ y)\ a)$
$[a/z]$ in y
$\underset{\beta}{\rightarrow} \lambda a\ .\ y$

C. $(\lambda x\ .\ (x\ x))\ (\lambda y\ .\ (y\ y))$

$[(\lambda y\ .\ (y\ y))/x]$ in $(x\ x)$
$\underset{\beta}{\rightarrow} (\lambda y\ .\ (y\ y))\ (\lambda y\ .\ (y\ y))$
$\underset{\alpha}{=} (\lambda y\ .\ (y\ y))\ (\lambda a\ .\ (a\ a))$
$[(\lambda a\ .\ (a\ a))/y]$ in $(y\ y)$
$\underset{\beta}{\rightarrow} (\lambda a\ .\ (a\ a))\ (\lambda a\ .\ (a\ a))$ //Never ending!

D. $K\ x\ y$

$(\lambda xy\ .\ x)\ x\ y$
$\underset{\alpha}{=} (\lambda az\ .\ a)$
$[x/a]$ in $(\lambda z\ .\ a)$
$\underset{\beta}{\rightarrow} (\lambda z\ .\ x)\ y$
$[y/z]$ in x
$\underset{\beta}{\rightarrow} x$

E. $S\ K$

$(\lambda epz\ .\ ez(pz))$ K
$[K/e]$ in $\lambda pz\ .\ ez(pz)$
$\underset{\beta}{\rightarrow} \lambda pz\ .\ Kz(pz)$
$\lambda pz\ .\ (\lambda xy\ .\ x)\ z\ (pz)$
$[z/x]$ in $\lambda pz\ .\ (\lambda y\ .\ x)$
$\underset{\beta}{\rightarrow} \lambda pz\ .\ (\lambda y\ .\ z)\ (pz)$
$[(pz)/y]$ in $\lambda pz\ .\ z$
$\underset{\beta}{\rightarrow} \lambda pz\ .\ z$

F. $(S\ K)\ y\ y\ z$

$(\lambda pz\ .\ z)\ y\ y\ z$
$\underset{\alpha}{=} (\lambda px\ .\ x)\ y\ y\ z$
$[y/p]$ in $(\lambda x\ .\ x)\ yz$
$\underset{\beta}{\rightarrow} (\lambda x\ .\ x)\ yz$
$[y/x]$ in $(x)\ z$
$\underset{\beta}{\rightarrow} (y)\ z$

G. $K'\ y\ y\ z$

$(\lambda xy\ .\ y)\ y\ y\ z$
$\underset{\alpha}{=} (\lambda xa\ .\ a)\ y\ y\ z$
$[y/x]$ in $(\lambda a\ .\ a)\ yz$
$\underset{\beta}{\rightarrow} (\lambda a\ .\ a)\ y\ z$
$[y/a]$ in $(a)\ z$
$\underset{\beta}{\rightarrow} (y)\ z$

2. What is the normal form of S (K S) (K I)?

$(\lambda epz\ .\ ez\ (pz))\ (KS)\ (KI)$

$[(KS)/e]$ in $(\lambda pz\ .\ ez\ (pz))\ (KI)$

$\underset{\beta}{\rightarrow} (\lambda pz\ .\ (KS)z\ (pz))\ (KI)$

$[(KI)/p]$ in $\lambda z\ .\ (KS)z(pz)$

$\underset{\beta}{\rightarrow} \lambda z\ .\ (KS)z((KI)z)$

$\lambda z\ .\ ((\lambda xy\ .\ x)S)z((KI)z)$

$[S/x]$ in $\lambda z\ .\ (\lambda y\ .\ x)z((KI)z)$

$\underset{\beta}{\rightarrow} \lambda z\ .\ (\lambda y\ .\ S)z((KI)z)$

$[z/y]$ in $\lambda z\ .\ (S)((KI)z)$

$\underset{\beta}{\rightarrow} \lambda z\ .\ (S)((KI)z)$

$\lambda z\ .\ (\lambda epz\ .\ ez(pz))((KI)z)$

$[((KI)z)/e]$ in $\lambda z\ .\ (\lambda pz\ .\ ez(pz))$

$\underset{\beta}{\rightarrow} \lambda z\ .\ (\lambda pz\ .\ ((KI)z)z(pz))$

$[z/p]$ in $\lambda z\ .\ (\lambda z\ .\ ((KI)z)z(pz))$

$\underset{\beta}{\rightarrow} \lambda z\ .\ (\lambda z\ .\ ((KI)z)(pz))$

$[(pz)/z]$ in $\lambda z\ .\ ((KI)z)$

$\underset{\beta}{\rightarrow} \lambda z\ .\ ((KI)(pz))$

$\lambda z\ .\ (((\lambda xy\ .\ x)I)(pz))$

$[I/x]$ in $\lambda z\ .\ ((\lambda y\ .\ x)(pz))$

$\underset{\beta}{\rightarrow} \lambda z\ .\ ((\lambda y\ .\ I)(pz))$

$[(pz)/y]$ in $\lambda z\ .\ (I)$

$\underset{\beta}{\rightarrow} \lambda z\ .\ I$

$\lambda z\ .\ (\lambda x\ .\ x)$

3. Prove the following equivalencies by reducing each side to its normal form.

A. $I = S\ K\ K$

$\lambda x\ .\ x = (\lambda epz\ .\ ez(pz))KK$
$\lambda x\ .\ x = [K/e]$ in $(\lambda pz\ .\ ez(pz))K$
$\lambda x\ .\ x = \underset{\beta}{\rightarrow} (\lambda pz\ .\ Kz(pz))K$
$\lambda x\ .\ x = [K/p]$ in $(\lambda z\ .\ Kz(pz))$
$\lambda x\ .\ x = \underset{\beta}{\rightarrow} \lambda z\ .\ Kz(Kz)$
$\lambda x\ .\ x = \lambda z\ .\ (\lambda xy\ .\ x)z(Kz)$
$\lambda x\ .\ x = [z/x]$ in $\lambda z\ .\ (\lambda y\ .\ x)(Kz)$
$\lambda x\ .\ x = \underset{\beta}{\rightarrow} \lambda z\ .\ (\lambda y\ .\ z)(Kz)$
$\lambda x\ .\ x = [(Kz)/y]$ in $\lambda z\ .\ (z)$
$\lambda x\ .\ x = \underset{\beta}{\rightarrow} \lambda z\ .\ z \underset{\alpha}{=} \lambda x\ .\ x$
$\lambda x\ .\ x = \lambda x\ .\ x$

B. $S\ K\ K = K\ I\ I$

$\lambda x\ .\ x = (\lambda xy\ .\ x)I\ I$
$\lambda x\ .\ x = [I/x]\ (\lambda y\ .\ x)I$
$\lambda x\ .\ x = \underset{\beta}{\rightarrow} (\lambda y\ .\ I)I$
$\lambda x\ .\ x = [I/y]\ I$
$\lambda x\ .\ x = \underset{\beta}{\rightarrow} I$
$\lambda x\ .\ x = \lambda x\ .\ x$

4. Given the definition of Church numerals below, what does $(m\ n)$ do when m and n are Church numerals? For example $(\bar{2}\ \bar{3})$. It may be easier to work out as $\lambda m\ \lambda n\ .\ (m\ n)$. Show your work (or at least an example).

$\lambda m\ n\ .\ (m\ n)$
$\lambda m\ n\ .\ (\bar{2}\ \bar{3})$
$\lambda m\ n\ .\ (\lambda f\ x\ .\ (f(fx))\ \bar{3})$
$\lambda m\ n\ .\ (\lambda f\ x\ .\ (f(fx))\ \bar{3})$
$[\bar{3}/f]$ in $\lambda m\ n\ .\ (\lambda f\ x\ .\ (f(fx)))$
$\underset{\beta}{\rightarrow} \lambda m\ n\ .\ (\lambda x\ .\ (\bar{3}(\bar{3}x)))$
$\lambda m\ n\ .\ (\lambda x\ .\ (\lambda fy\ .\ (f(f(fy)))\ (\bar{3}x)))$

$[\bar{3}x/f]$ in $\lambda m\ n\ .\ (\ \lambda x\ .\ (\lambda y\ .\ (f(f(fy)))))$
$\underset{\beta}{\rightarrow} \lambda m\ n\ .\ (\ \lambda x\ .\ (\lambda y\ .\ (\bar{3}x(\bar{3}x((\bar{3}x)y)))))$

Church numerals have not been my strongest area in this assignment. I still do not fully understand it, but I hope I was able to get somewhere with the reductions. I did not reduce farther because I saw the pattern arising that it was going to go one for a while and I could not for-see the end of the reductions. It looked like it would just keep growing the more I went along with the beta reductions.