Homework 12

Robbie McKinstry, Jack McQuown, Cyrus Ramavarapu

30 September 2016

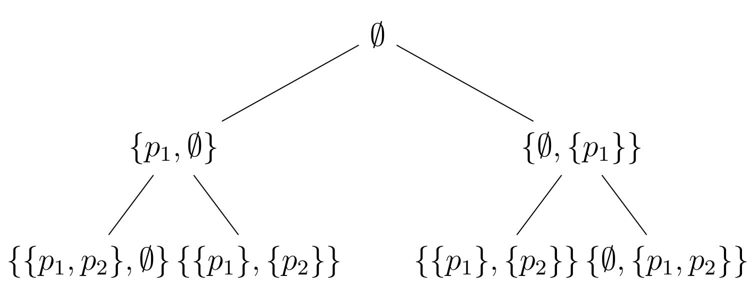
Please provide writing or oral feedback on this and future assignments.

Dynamic Programming

Problem 20:

Given a series of n points in the Euclidean plane labeled $p_1, \ldots, p_n$, dynammic programming can be used to find the minimum routing distance to service these points in order with two tracers, t_1 and t_2. In order tracing of these points implies that if p_j is serviced aftered p_i by either tracer, it implies that $i < j$.

A dynamic programming algorithm for this problem can be developed for this problem by initally considering all possible routes the two tracers can take after each point is considered. This produces the following tree.



At every level in the tree a new point is considered and can be taken by either t_1 or t_2. However, if both t_1 and t_2 are considered in the tree, significant duplication is observed at the leaves because the assignment of either t_1 or t_2 to a set of points is arbitrary. For example, if the information in a leaf node is represented by the set pt_1, pt_2 where pt_1 and pt_2 are the sets of points visited respectively by t_1 and t_2 then the node containing p_1, p_2, p_3 is equivalent to the node containing p_2, p_3, p_1. This is because the labels t_1 and t_2 do not really have meaning and can be interchanged. As a consequence of this result, half the tree can be immediately eliminated.

Although removing half of the initial tree significantly reduces the search space, the number of possible tours remaining is still $2^{|P|-1}$, where $|P|$ is the number of points in the entire trace. To further reduce the search space, two nodes at the same level have both t_1 and t_2 respectively at the same point, then the node with greater total distance traced can be pruned. This is a consequence that adding the next point to either node will result in the same additional distance because the tracers t_1 and t_2 are respectively equividistant from the next point. As a result of this pruning rule, only the minimum of each combination of end points such as $(p1, p2)$, needs to be kept at every level. Applying this rule to this problem produces the following algorithm, SlowPokeTaxi or SPT.

Function: *SlowPokeTaxi*
Input: $P = p_1, \ldots, p_2$
Globals: $A[\,][\,][\,]$
for $i = 0$ *to* $|P|$ **do**
  for $j = 0$ *to* $|P|$ **do**
    for $k = 0$ *to* $|P|$ **do**
      if $A[i][j][k]$ *is defined* **then**
        $A[i+1][j][k] = \min(A[i+1][j][k], +A[i][j][k] + dist(j, i+1)$
        $A[i+1][j+1][k] =$
        $\min(A[i+1][j+1][k] + A[i][j+1][k] + dist(j, i+1)$
Return: $\min(A[|P|][*][*])$

This algorithm runs in $\mathcal{O}(n^3)$ because it has to consider all possible available end points at a given level. In this case recovery of the actual sequence of traces requires determining which point from the previous matches the difference in the distance between the current point and the previous. This value be subtracted from the currently minimum traversed values of t_1 and t_2. Locating these values in the previous rows will determine which tracer covered this point.

This question now is if a faster algorithm can be developed. By the will of Cthulhu, we found an algorithm can be found to run in $\mathcal{O}(n^2)$. It is believed that this algorithm can be developed by recognizing that if two nodes contain information for t_1 and t_2 that are end on respectively opposite points, the node with the longer path can be pruned. This results from the next point being equal distance from both tracers in each case; however, one tracer may have taken a longer route to the node. Applying this pruning rule gives the following algorithm.

Function: *FastPokeTaxi*
Input: $P = p_1, \ldots, p_2$
Globals: $A[\ ][\ ][\ ]$

```
for i = 0 to |P| do
    for j = 0 to |P| do
        /* A[i][j][0] holds distance for t_1 */
        /* A[i][j][1] holds distance for t_2 */
        if A[i][j][*] is defined then
            /* Add to t_1 */
            A[i+1][j][0] = min(A[i][j][0], A[i][j][0] + dist(j, i+1))
            A[i+1][j][1] = A[i+1][j][1]
            /* Add to t_2 */
            A[i+1][j+1][1] = min(A[i][j+1][1], A[i][j][1] + dist(j, i+1))
            A[i+1][j][0] = A[i+1][j][0]
```

Return: $\min(A[|P|][*][*])$

Since this algorithm only considers unique pairs and applies the pruning rule regarding opposite endpoints on the same node, it runs in $\mathcal{O}(n^2)$.

To recover the exact sequence each tracer must travere, begin at the index where the minimum value was found. From this index find the difference between points n and the $n-1$. This value can be subtracted from where the minimum depending on whether the point was added to t_1 or t_2 to recover the tracer of the previous point. This processed can then be repeated for all each previous n up to the first point.

Problem 21:

Since all points are colinear, any visit to a point P_i if $P_i < 0 \implies \forall P_j : P_i < P_j < 0$, an optimal good path must visit P_j before visiting P_i. This is obvious because any trip that skips over a point in between Pi and the origin would have to spend time traveling back to P_j, and this would increase the response time. An immediate corollary is that this is also true for all points farther along the real line in the positive direction as well.

Thus, an algorithm to find the optimal good path has one one choice: include in the path either the next point less than the current point, or the next point greater than the current point (colloquially, go left or go right). The search tree for an optimal good path can be represented as a tree of nodes storing a triple: the set of points unvisited on the left, the current position, and the set of points unvisited on the right. Each child node is populated by selecting either the greatest number of the left set (the first point on the left) or the least number in the second set (the first number on the right) and visiting that point. As a result, each parent has two children in the general case. When either set is empty, the remaining choices are obvious, and thus the node becomes a leaf. Without loss of generality, this occurs when the good path moves from the origin left l times before moving right r times, where l is the total number of node left of the origin and r is the total number of nodes right of the origin.

There are l ways to move left before moving right (take the first point, take the first point and the second point, take the first point and ... and the nth point). Then, once moving right, there are r ways to move right. Thus, there are lr ways to move when changing direction no more than twice. The number of times you can change direction is limited the minimum of l and r, since the number of times you can interleave right and left can not exceed the total number of times moving right and it cannot exceed the total number of times moving left. Assume l is larger than r, without loss of generality. Then the total number of unique good paths is l^2r.

This leads to the following iterative algorithm:

Let k be the minimum of r and l. $\forall i < k$ consider changing direction i times:

while $k > 0$: $\forall j < 1 + l - k$, go left j times, before changing direction. $\forall m < 1 + r - k$ go right m times before changing direction. Decrement k by 1. Jump to the while condition.

Reduction

Problem 1:

In order to show that there is a $O(n^2)$ time algorithm for matrix multiplication, it is first necessary to reduce matrix multiplication (MM) to lower triangle multiplication (LTM). Because we reduce matrix multiplication to LTM, we have an algorithm for MM and are stating that LTM is at least as hard as MM.

The reduction is as follows:

Transform matrices A and B into matrices C and D to use as input for LTM, which will take $O(n^2)$ time.

$$C = \begin{bmatrix} 0 & \frac{A}{3} \\ \frac{A}{3} & \frac{A}{3} \end{bmatrix} \; D = \begin{bmatrix} 0 & \frac{B}{3} \\ \frac{B}{3} & \frac{B}{3} \end{bmatrix}$$

Now from this transformed input, we can now use C and D in our LTM algorithm, which will also take $O(n^2)$ time.

$$\text{LTM(C,D)} = \begin{bmatrix} \frac{AB}{9} & \frac{AB}{9} \\ \frac{AB}{9} & \frac{2AB}{9} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} AB & AB \\ AB & 2AB \end{bmatrix}$$

Get AB (the solution to MM(A,B)) from the above matrix in $O(n^2)$ time

return AB

Therefore, because there is a $O(n^2)$ time algorithm for lower triangle multiplication that is used in the matrix multiplication algorithm and every other step is also no more than $O(n^2)$, there is a $O(n^2)$ time algorithm for matrix multiplication.