Homework 2

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Question 4 Part A

Let S be the set of all gas stations. Let $G(A)$ be the output of the greedy algorithm on input A. Assume $G_A(A)$ is not optimal for proof by contradiction.

Let *Opt* be an optimal solution.

There are two possible scenarios: Either $G(A)$ and *Opt* select the same gas stations to stop at or they do not. First, we address the former before tackling the latter.

Assume that $G(A)$ and *Opt* select all of the same gas stations. Then, for an arbitrary gas station p, let the amount of time $G(A)$ spends filling up be x. Now, since *Opt* must also stop at that gas station, and $Opt \neq G(A)$, *Opt* must spend more time filling up than $G(A)$, since $G(A)$ fills up the minimum at every stop. Thus, *Opt* stops for some $y : x \leq y$. Since *Opt* and $G(A)$ stop at all the same gas stations, total time t is

$$t_{Opt} = \sum_{p \in S} y_p$$

for *Opt*, and

$$t_{G(A)} = \sum_{p \in S} x_p$$

for $G(A)$. By the properties of addition, $t_{G(A)} \leq t_{Opt}$, and thus is an optimal solution. $\perp$. Since we have resolved a contradiction, our assumption must be false. Thus, $G(A)$ is optimal and a solution, in the case where $G(A)$ and Opt stop at all of the same gas stations.

Now, we address the case where Opt and $G(A)$ do not stop at the same gas stations. Since $\forall p \in S.G(A)$ stops at $p.$, then $\exists p \in S.Opt$ does not stop at p.

Let S be the gas station before p.

Let $Opt' = Opt$ such that Opt stops at S and fills the distance necessary to reach p, just as $G(A)$ does, and subtract that amount from the amount filled up at p.

Opt' is at least as optimal as Opt since the total amount of time spent filling is the same. It is obvious that Opt' is more like $G(A)$ than Opt, which violates our assumption. $\perp$.

Thus, $G(A)$ is an optimal solution.

Question 4 Part B

This algorithm does not work. Consider the following counterexample:

Let $F = 1$liter/95kilo. Let $C = 1$liter. Let $r = 1$liter/95minutes. Let the distance between A and B be 100kilos. Let there be a single gas station 95 miles from A and 5 miles from B (that is, at the 95th mile marker).

The greedy algorithm specified will stop at the gas station and fill the tank 1 liter. Filling the tank 1 liter takes 95 minutes. An optimal solution is to stop at the gas station and fill the tank 1/19 liters, which will only take 5 minutes.

Question 5 Part A

We believe we have a counterexample.

Consider the set $A = \{0.25, 1, 1.25, 1.75, 2, 2.75\}$

The greedy algorithm can cover four points by selecting the interval $[1, 2]$. Then, the greedy algorithm must cover the remaining two points individually, selecting an interval that covers 0.25 and another interval that covers 2.25, resulting in a total of 3 intervals used.

However, an optimal solution would cover A in two intervals, selecting $[0.25, 1.25]$ and $[1.75, 2.25]$. Thus, greedy is not a solution.

Question 5 Part B

Let $G(A)$ be the output of the Greedy algorithm on input A. For proof by contradiction, assume G is not optimal. Let $Opt(A)$ be an optimal output on input A that is closest in selection order to $G(A)$.

Let i be the index of the first disagreement between $G(A)$ and $Opt(A)$, let $G(A)_i$ be the interval $[x_i, x_i + 1]$ selected by G, and $Opt(A)_i$ be the interval $[x_i', y_i']$ selected by Opt.

Since the intervals are not identical, either $x_i < x_i'$, or $x_i > x_i'$.

Consider the case where $x_i < x_i'$:

Since x_i is the leftmost point, and x_i', Opt has yet to cover x_i. Thus, there is an index $j : j > i$ where $Opt(A)_j$ covers x_i'.

Let $Opt(A)' = Opt(A) - Opt(A)_j + G(A)_i$. Now, it is obvious that $Opt(A)'$ has the same cardinality as $Opt(A)$. $Opt(A)'$ is still an optimal solution, because it has the optimal cardinality, and because $Opt(A)'$ doesn't cover any points left uncovered by $G(A)_i$; there are no points less than x_i because if there were, they would have been selected by greedy algorithm, and there are no points greater than x_i covered by $Opt(A)_j$ not covered by $G(A)_i$ because

$G(A)_i$ covers all points within the range of 1 full interval, since it covers $[x_i, x_i+1]$ by definition. Thus, $Opt(A)'$ is an optimal solution, which violates the premise that $Opt(A)$ is the optimal solution closest to G. $\perp$.

Now, consider the case where $x_i > x_i'$:

Let $Opt(A)' = Opt(A) - Opt(A)_i + G(A)_i$. It is obvious that $Opt(A)'$ has the same cardinality as $Opt(A)$. $Opt(A)'$ does not leave any points less than x_i uncovered because by the definition of the greedy algorithm x_i is the leftmost point. $Opt(A)'$ does not leave any points greater than x_i uncovered because any point covered by the interval $[x_i, y_i']$ will be covered by $[x_i, x_i+1]$ since y_i' is necessarily less than $x_i + 1$. Thus, $Opt(A)'$ is an optimal solution, which violates the premise that $Opt(A)$ is the optimal solution closest to G. $\perp$.

Since both cases result in contradiction, by disjunctive elimination the assumption introduces contradiction, and thus the assumption is proven false and $G(A)$ is necessarily a solution.