# Homework 4

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## Greedy Problems

### Problem 12:

This greedy algorithm can be shown to be correct by an exchange argument.

Let $Alg$ be the process by which the greedy algorithm operates. Assume that there is some input $I$ such that $Alg(I)$ is incorrect. Let $Opt(I)$ be an optimal solution that agrees with the most number of steps with $Alg(I)$.

$Opt(I)$ and $Alg(I)$ must have a first point of disagreement which must occur at some row, since $Alg(I)$ works row by row. Label the row in which the disagreement occurs $r_i$.

A general case of this disagreement can be demonstrated by the following two matrices.

$$Alg(I)_{n,n} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,u} & \cdots & a_{1,v} & \cdots & a_{1.n} \\ a_{2,1} & \cdots & a_{2,u} & \cdots & a_{1,v} & \cdots & a_{1.n} \\ \vdots & \ddots & a_{i,u} & \ddots & a_{i,v} & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,u} & \cdots & a_{n,v} & \cdots & a_{n.n} \end{pmatrix}$$

$$Opt(I)_{n,n} = \begin{pmatrix} o_{1,1} & \cdots & o_{1,u} & \cdots & o_{1,v} & \cdots & o_{1.n} \\ o_{2,1} & \cdots & o_{2,u} & \cdots & o_{1,v} & \cdots & o_{1.n} \\ \vdots & \ddots & o_{i,u} & \ddots & o_{i,v} & \ddots & \vdots \\ o_{n,1} & \cdots & o_{n,u} & \cdots & o_{n,v} & \cdots & o_{n.n} \end{pmatrix}$$

Without loss of generality, allow the disagreement to be between columns $u$ and $v$. In the matrices above, this could mean $a_{i,u} = 1$ and $a_{i,v} = 0$ whereas $o_{i,v} = 0$ and $o_{i,u} = 1$.

Naively defining $Opt'(I)$ as $Opt(I)$ except $o_{i,v} = 1$ and $o_{i,u} = 0$ makes $Opt'(I)$ agree with $Alg(I)$ for an additional step and maintains the necessary sum of row $r_i$, labeled $|r_i|$.

However, this swap invalidates the column sums for $c_u$ and $c_v$, respectively $|c_u|$ and $|c_v|$, since they previously aggreed in $Opt$, but the swap altered the values to $|c_u|' = |c_u| + 1$ and $|c_v|' = |c_v| - 1$, where $|c_u|'$ and $|c_v|'$ are the analogous column values in $Opt'(I)$.

The column sums in $Opt'(I)$ can be rectified, while still maintaining the row sums and the additional step agreement with $Alg(I)$ if there exists a row, $r_j$, such that $i < j$, where $o_{j,v} = 1$ and $o_{j,u} = 1$.

Since $Alg$ operates row-by-row placing 1s in those columns which have the greatest need, $c_i$ required more 1s than $c_j$ at $r_i$. Since $Opt(I)$ deprived $c_i$ a 1 at $r_i$, it must give a 1 to $c_i$ at some $r_k$ such that $i < k$. Additionally, by giving $c_j$ a 1 at $r_i$, $Opt(I)$ is guaranteed to satisfy value $|c_j|$ with fewer 1s than needed to satisfy $|c_i|$. This ensures the existence of a row with the above mentioned property.

Therefore redefine $Opt'(I)$ such that $Opt(I)$ except $o_{i,v} = 1$, $o_{i,u} = 0$ and $o_{j,v} = 0$, $o_{j,u} = 1$ where $j$ is the row where in $Opt(I)$, $o_{j,v} = 1$ and $o_{j,u} = 1$. This gives an optimal solution that agrees with at least 1 more step than $Opt(I)$ does with $Alg(I)$, which contradicts the original assumption that there exists an $I$ such that $Alg$ is incorrect. Therefore $Alg$ is correct for all input $I$.

## Problem 18:

### A:

The following counter example shows that this algorithm does not work.

Let the jobs be

$$J_1 = (1, 5, 5)$$
$$J_2 = (4, 2, 100)$$

This algorithm will do the following scheduling:

| Time | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Job | $J_1$ | $J_1$ | $J_1$ | $J_1$ | $J_2$ | $J_2$ | $J_1$ |

Since $J_1$ was scheduled to finish after its deadline, the algorithm outputs 0. However, there is a feasible schedule for these two jobs.

| Time | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Job | $J_1$ | $J_1$ | $J_1$ | $J_1$ | $J_1$ | $J_2$ | $J_2$ |

### B:

The following counter example shows that this algorithm does not work.

Let the jobs be

$$J_1 = (1, 2, 2)$$
$$J_2 = (1, 1, 100)$$

This algorithm will do the following scheduling:

| Time | 1 | 2 | 3 |
|---|---|---|---|
| Job | $J_2$ | $J_1$ | $J_1$ |

Since $J_1$ finished after its deadline, the algorithm will say that there is no feasible schedule. However, there is a feasible schedule for these two jobs.

| Time | 1 | 2 | 3 |
|---|---|---|---|
| Job | $J_1$ | $J_1$ | $J_2$ |

**C:**

The correctness of this algorithm can be shown by the following exchange argument.

Let *Alg* be the process by which this algorithm operates. Assume that there exists some input $I$ such that $Alg(I)$ is incorect. Let $Opt(I)$ be the optimal solution for input $I$ that agrees with the most number of steps with $Alg(I)$.

As a result, there must be some first point of disagreement between $Alg(I)$ and $Opt(I)$. Label this step $u$ and let $Alg_u(I)$ be $J_i$ and $Opt_u(I)$ be $J_j$.

The process by which *Alg* operates is to schedule the job with the least laxity, defined as

$$l = d_i - t - x_i(t)$$

This quantity represents the urgency with which a job must be scheduled, with *Alg* scheduling the most urgent job at every step. Therefore, since at step $u$ $Alg(I)$ chose $J_i$ whereas $Opt(I)$ has $J_j$ at step $u$, $J_i$ must have a lower laxity than $J_j$.

Alternatively, laxity can be viewed as the number of available slots a job has between time $t$ and its deadline where it does not need to be schedule while still completing before its deadline. Since $J_j$ has a higher laxity than $J_i$, it must have free slots in which another job could be scheduled.

Therefore, define $Opt'(I)$ as $Opt(I)$ except at step $u$ schedule $J_i$ and schedule $J_j$ down by one unit, thereby using one of its lax spaces.

To show that $Opt'(I)$ is at least as optimal as $Opt(I)$ it needs to be recognized that $J_i$ must have fewer lax slots than $J_j$, otherwise *Alg* would have picked $J_j$ at step $u$. This includes the case in which $J_j$ can't be shifted because it would violate its deadline $d_j$. Additionally, if $J_i$ and $J_j$ must overlap due, both $Opt'(I)$ and $Opt(I)$ would produce 0 for no viable schedule since only a single unit is changed in $Opt(I)$ and deadlines remain constant.

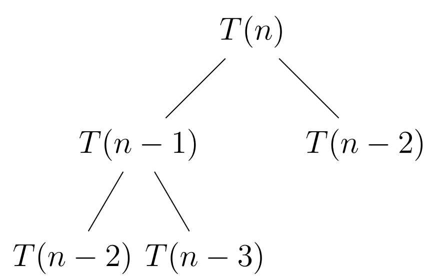
As a result, $Opt'(I)$ agrees with $Alg(I)$ for more steps than $Opt(I)$, contradicting the initial assumptions. Therefore there is not an input $I$ such that $Alg(I)$ is incorrect.

# Dynamic Programming

## Problem 1:

### A:

A direct implementation of this recurrence relation leads to an exponential runtime because every $T(i)$ requires $2^i$ recurvsive calls. This can seen using a tree diagram.




This tree will have a depth of $i$ and at every point branches by into two at every node. Hence the $\mathrm{O}\left(2^i\right)$ calculations.

### B:

To show that only $\mathrm{O}\left(n^2\right)$ operations are needed if every duplicate $T(i)$ is calculated only once, begin by expanding the sum in the recurrence.

$$T(n) = \sum_{i=1}^{n-1} T(i)T(i-1)$$

$$T(n) = T(1)T(0)+T(2)T(1)+T(3)T(2)\cdots+T(n-2)T(n-3)+T(n-1)T(n-2)$$

Since every $T(i)$ will only be calculated once, following sequence can be observed by counting the number of operations needed to determine each $T(i)$.

| T (i) | T (2) | T (3) | T (4) | T (5) | T (6) |
|---|---|---|---|---|---|
| Ops | 1 | 3 | 5 | 7 | 9 |

It can be shown that the $T(i+1)$ element of the sum requires two additional operations to calculate: *a multiplication and an addition.* Hence, this sequence will continue. It can be proven inductively that a closed form expression for the sum of operations required is $n^2$. Therefore, in this case $\mathrm{O}\big(n^2\big)$ operations are required.

**C:**

A $\mathrm{O}\big(n\big)$ algorithm can be derived from the original recurrence relationship by first eliminating the summation by calculating $T(n+1)$ in the following manner.

$$T(n+1) = \sum_{i=1}^{n} T(i)T(i-1)$$

$$T(n) = \sum_{i=1}^{n-1} T(i)T(i-1)$$

$$T(n+1) - T(n) = \sum_{i=1}^{n} T(i)T(i-1) - \sum_{i=1}^{n-1} T(i)T(i-1)$$

$T(n+1)$ and $T(n)$ overlap for all values $i : 1 \leq i \leq n-1$, therefore subtracting the two sums leaves only the the final in the sum for $T(n+1)$.

$$T(n+1) - T(n) = T(n)T(n-1)$$

The values for n can be shifted by setting $n = m - 1$.

$$T(m) - T(m-1) = T(m-1)T(m-2)$$

However, the label $m$ is without meaning, so label $m = n$.

$$T(n) - T(n-1) = T(n-1)T(n-2)$$

Equivalently,

$$T(n) = T(n-1)[1 + T(n-2)]$$

This expression is easily expressed as a single $\mathrm{O}(n)$ loop.

$Array: \; T$
$T[0] = 2$
$T[1] = 2$
**for** $i \leftarrow 2$ *to* $n$ **do**
$\mid \quad T[i] = T[i-1] * (1 + T[i-2])$
**end**
$Output: \; T[n]$