Binet

Determinar fuerza, potencial, aceleración radial y tangencial

$$r = a(1+\cos\phi)$$

(a) F ; $u = \dfrac{1}{a(1+\cos\phi)} \Rightarrow u' = \dfrac{\sin\phi}{a(1+\cos\phi)^2} = a\sin\phi\, u^2$

$$u'' = a\cos\phi\, u^2 + 2a\sin\phi\, u u' = a\cos\phi\, u^2 + 2a^2\sin^2\phi\, u^3$$

$\star$ $u = \dfrac{1}{a(1+\cos\phi)} \Rightarrow 1+\cos\phi = \dfrac{1}{ua} \Rightarrow \cos\phi = \dfrac{1}{ua} - 1$

$\sin^2\phi = 1 - \cos^2\phi$

$$\therefore u'' = au^2\left(\frac{1}{ua} - 1\right) + 2a^2u^3\left(1 - \frac{1}{a^2u^2} + \frac{2}{au} - 1\right)$$

$$= u - au^2 + \cancel{2a^2u^3} - 2u + 4au^2 - \cancel{2a^2u^3} = 3au^2 - u$$

$$\therefore u^2u'' = 3au^4 - u^3 \Rightarrow u^2u'' + u^3 = 3au^4$$

$$\therefore F(1/u) = -\frac{l^2}{m}(3au^4) \Rightarrow F(r) = -3a\frac{l^2}{m}\frac{1}{r^4}$$

b) $\dfrac{dV}{dr} = 3a\dfrac{l^2}{m}\dfrac{1}{r^4} \Rightarrow V(r) = 3a\dfrac{l^2}{m}\displaystyle\int_{r_0}^{r} r^{-4}\,dr = 3a\dfrac{l^2}{m}\left(-\dfrac{1}{3r^3}\right)_{r_0}^{r}$

$\Rightarrow V(r) = -a\dfrac{l^2}{m}\dfrac{1}{r^3} + C_1$; C_1 cte integración

c) $a_c = \dfrac{F}{m} = -\dfrac{3al^2}{m^2}\dfrac{1}{r^4}$; $a_t = 0$

Potencial Gravitacional

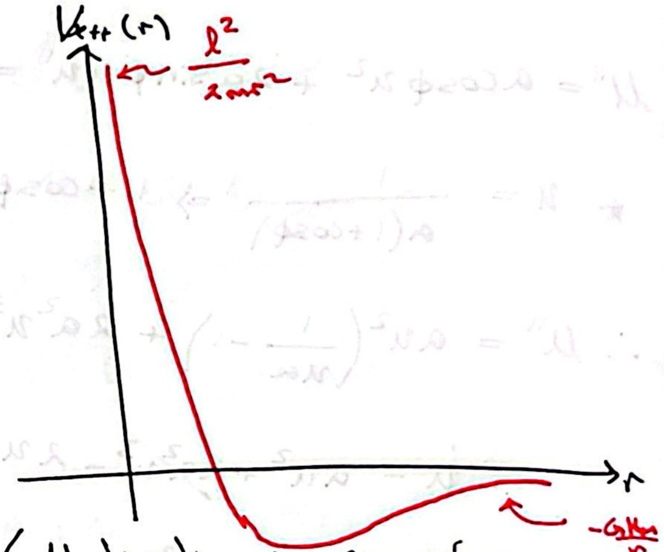
Graficar $V_{eff}(r)$ para el potencial gravitacional

$$V(r) = -\frac{GMm}{r}$$

* $V_{eff}(r) = V(r) + \frac{l^2}{2mr^2} = -\frac{GMm}{r} + \frac{l^2}{2mr^2}$

* $E - V_{eff}(r) = \frac{1}{2}m\dot{r}^2 \geq 0$

$\therefore V_{eff}(r) = \frac{1}{r}\left(\frac{l^2}{2mr} - GMm\right)$



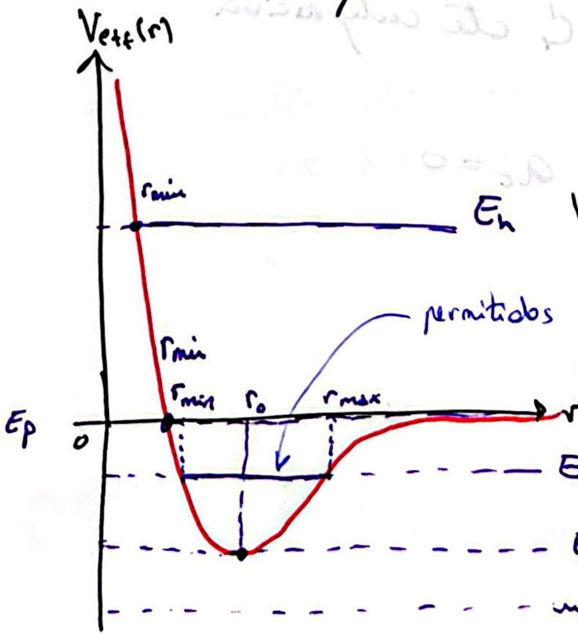
Para graficar, se toman los casos límites donde $r \to 0$ y $r \to \infty$

$\therefore V_{eff\,r\to 0} \longrightarrow V_{eff} \sim \frac{l^2}{2mr^2} \to \infty$

(Al dividir por un número pequeño el valor se hace grande, por lo que va a dominar sobre la cte GMm)

$V_{eff\,r\to\infty} \sim -\frac{GMm}{r} \to 0$

(la cte GMm será mayor que algo que se acerque a 0, además se acerca a 0 por el lado negativo)



E_h Hiperbólica $r_{max} \to \infty$

E_p : $r_{max} \to \infty$

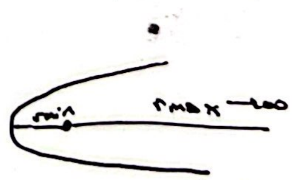
$E_e \longrightarrow E_e$: $r_{min} \leq r \leq r_{max}$ elíptica

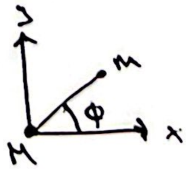
$E_0 = V_{eff}(r_0)$

imposible

$\therefore E - V_{eff}(r) \geq 0$

Posiciones radiales permitidas son aquellas donde la línea E está por encima o tocando la curva $V_{eff}(r)$





Condición órbita circular, r_0 punto de equilibrio si la fuerza radial neta es cero;

$$\left.\frac{dV_{ef}}{dr}\right|_{r=r_0} = 0 \quad ; \; V_{ef} = -\frac{GMm}{r} + \frac{l^2}{2mr^2}$$

$$\Rightarrow \frac{dV_{eff}}{dr} = \frac{GMm}{r^2} - \frac{l^2}{mr^3} \xrightarrow{\text{cond. eq.}} \frac{GMm}{\cancel{r_0^2}} - \frac{l^2}{mr_0^3} = 0$$

$$\Rightarrow GMm = \frac{l^2}{mr_0} \Rightarrow \boxed{r_0 = \frac{l^2}{GMm^2}} \Rightarrow \boxed{l^2 = GMm^2 r_0}$$

* Fuerza: $F_{eff}(r) = -\frac{dV_{eff}}{dr} \Rightarrow F_{eff}(r) = -\frac{GMm}{r^2} + \frac{l^2}{mr^3}$

* Periodo: $V''_{eff}(r_0) > 0 \;;\; V''_{eff}(r_0) = k \;,\; T = 2\pi\sqrt{\frac{m}{k}}$

$$\text{Si } V'_{eff}(r) = \frac{GMm}{r^2} - \frac{l^2}{mr^3} \Rightarrow V''_{eff}(r) = -2\frac{GMm}{r^3} + \frac{3l^2}{mr^4}$$

$$\Rightarrow V''_{eff}(r_0) = -2\frac{GMm}{r_0^3} + \frac{3}{\cancel{m}r_0^{\cancel{4}}} GMm^{\cancel{2}}\cancel{r_0} = -2\frac{GMm}{r_0^3} + \frac{3GMm}{r_0^3}$$

$$V''_{eff}(r_0) = \frac{GMm}{r_0^3} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{r_0^3}{GM}}}$$