4-10 Una partícula se mueve en el campo de energía Potencial

$$U(x) = a\cos x + \frac{b}{\sin^2 x} \quad , \quad 0 \leq x \leq \pi \quad ; \; a, b \text{ ctes} > 0$$

a) Hallar $F = F(x)$, ¿ donde $F = 0$? $\qquad \pi \to U \; \frac{1}{\sin^2 x}$ diverge $\to \infty$

b) Es un punto de Eq. estable?

c) Determinar periodo de las oscilaciones pequeñas

* Fuerza : $F(x) = -\frac{dU(x)}{dx} = -\underbrace{\left(-a\sin x - \frac{2b\cos x}{\sin^3 x}\right)}_{U'} = a\sin x + \frac{2b\cos x}{\sin^3 x}$

$\Rightarrow F(x) = 0 = a\sin x + \frac{2b\cos x}{\sin^3 x} = 0 \rightarrow a\sin^4 x + 2b\cos x = 0$

$\Rightarrow \boxed{a\sin^4 x_0 = -2b\cos x_0} \longrightarrow -a = \frac{2b\cos x_0}{\sin^4 x_0} \Rightarrow -a\cos x_0 = \frac{2b\cos^2 x_0}{\sin^4 x_0}$ *

$\Rightarrow \boxed{\frac{\sin^4 x}{\cos x} = -\frac{2b}{a}}$

- Ecuación de equilibrio
- Ecuación trascendental
- se resuelve numericamente

* Equilibrio Estable ? $U''(x) = -a\cos x + \frac{6b\cos^2 x}{\sin^4 x} + \frac{2b}{\sin^2 x}$

$\rightarrow U''(x) = -a\cos x + \frac{6b\cos^2 x + 2b\sin^2 x}{\sin^4 x} = -a\cos x + \frac{2b(3\cos^2 x + \sin^2 x)}{\sin^4 x}$

$\Rightarrow U''(x) = -a\cos x + \frac{2b(2\cos^2 x + 1)}{\sin^4 x}$

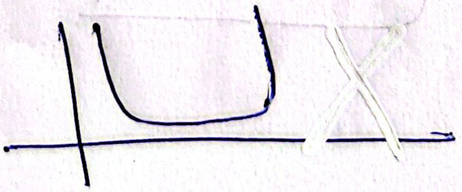
$\overset{*}{=} \frac{2b\cos^2 x_0}{\sin^4 x_0} + \frac{2b(2\cos^2 x + 1)}{\sin^4 x} = \frac{2b\cos^2 x_0 + 2b(2\cos^2 x + 1)}{\sin^4 x_0}$

$= \frac{2b\cos^2 x_0 + 4b\cos^2 x + 2b}{\sin^4 x_0} = \frac{2b(3\cos^2 x + 1)}{\sin^4 x_0}$

$\therefore U''(x) = \frac{2b(3\cos^2 x_0 + 1)}{\sin^4 x_0} > 0 \; , \; b > 0 \quad \sin x_0 \neq 0$

$x_0 \to$ estable

c) $k = U''(x_0) > 0 \Rightarrow T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{m\sin^4 x_0}{2b(3\cos^2 x_0 + 1)}}}$



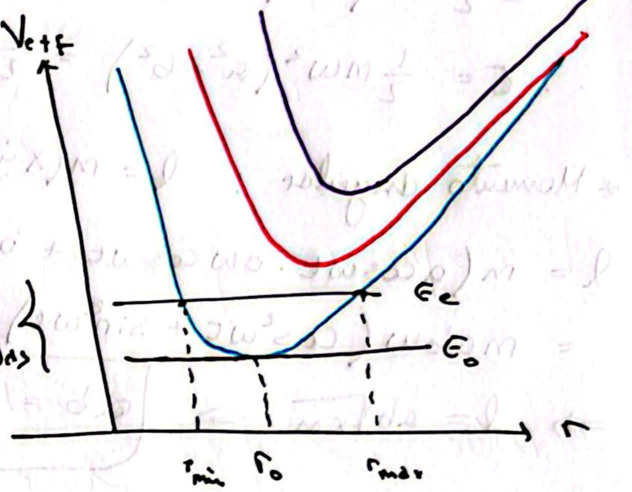
2- $V(r) = \frac{1}{2}k r^2$, $k >$ cte 0

(a) todas órbitas ligadas y que E debe exceder $E_{min} = l\sqrt{\frac{k}{m}}$

* $V_{eff} = V(r) + \frac{l^2}{2mr^2} = \frac{1}{2}kr^2 + \frac{l^2}{2mr^2}$; órbita ligada $\rightarrow$ mov. confinado

Si $r \to 0$: $V_{eff} \sim \frac{l^2}{2mr^2} \longrightarrow \infty$

si $r \to \infty$: $V_{eff} \sim \frac{1}{2}kr^2 \longrightarrow \infty$



$\therefore$ V_{eff} tiene un min. global a algún r_0 y todas las órbitas están ligadas (órbitas ligadas)

Es decir, la partícula no llega al centro ni se escapa al ∞

* Valor mínimo E: $E_{min} = V_{eff}(r)$; Condición de mínimo $F = \frac{dV_{eff}}{dr} = 0$

$$\frac{dV_{eff}}{dr} = kr_0 - \frac{l^2}{mr_0^3} = 0 \Rightarrow k - \frac{l^2}{mr_0^4} = 0 \Rightarrow k = \frac{l^2}{mr_0^4} \Rightarrow r_0^4 = \frac{l^2}{mk}$$

$$\Rightarrow \boxed{r_0^2 = \frac{l}{\sqrt{mk}}} \; ; \; E_{min} = V_{eff}(r_0) = \frac{1}{2}kr_0^2 + \frac{l^2}{2mr_0^2} = \frac{1}{2}k\frac{l}{\sqrt{mk}} + \frac{l^2\sqrt{mk}}{2ml}$$

$$= \frac{1}{2}l\sqrt{\frac{k}{m}} + \frac{1}{2}l\sqrt{\frac{k}{m}} = l\sqrt{\frac{k}{m}} \Rightarrow \boxed{E_{min} = l\sqrt{\frac{k}{m}}}$$

(b) Verificar órbita elipse cerrada con origen en el centro. Si la relación $\frac{E}{E_{min}} = \cosh\xi$ define ξ, verifique parámetros $a^2 = \frac{e^{\xi} l}{\sqrt{mk}}$ y $b^2 = \frac{e^{-\xi} l}{\sqrt{mk}}$
$\epsilon^2 = 1 - e^{-2\xi}$. Discutir lím $E \to E_{min}$, $E \gg E_{min}$

* Potencial isotrópico (igual en todas direcciones) $V(r) \rightarrow V = \frac{1}{2}kr^2$

$$\vec{F} = -\vec{\nabla}V(r) = -k\vec{r} \Rightarrow m\ddot{\vec{r}} = -k\vec{r} \rightarrow \ddot{\vec{r}} + \frac{k}{m}\vec{r} = 0 \; , \; \omega^2 = \sqrt{\frac{k}{m}}$$

* Soluciones generales: $x(t) = a\cos\omega t$, $y(t) = b\sin\omega t$; a y b semiejes de trayectoria elíptica.

Centrada en el origen: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\cancel{a^2}\cos^2\omega t}{\cancel{a^2}} + \frac{\cancel{b^2}\sin^2\omega t}{\cancel{b^2}} = 1$

$E = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}k(x^2+y^2)$; * Energía

$\dot{x} = -a\omega\sin\omega t$, $\dot{y} = b\omega\cos\omega t$

$k = m\omega^2 \Rightarrow \omega = \sqrt{\frac{k}{m}}$

$m = \frac{k}{\omega^2}$

$\therefore E = \frac{1}{2}m(a^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t) + \frac{1}{2}(a^2\cos^2\omega t + b^2\sin^2\omega t)$

$= \frac{1}{2}m\omega^2(a^2(\sin^2\omega t+\cos^2\omega t) + b^2(\cos^2\omega t+\sin^2\omega t))$

$\therefore E = \frac{1}{2}m\omega^2(a^2+b^2) = \frac{1}{2}k(a^2+b^2) \Rightarrow \boxed{a^2+b^2 = \frac{2E}{k}}$

* Momento angular : $l = m(x\dot{y}-y\dot{x})$

$l = m(a\cos\omega t \cdot b\omega\cos\omega t + b\sin\omega t \cdot a\omega\sin\omega t)$

$= mab\omega(\cos^2\omega t+\sin^2\omega t) \Longrightarrow l = abm\omega = abm\sqrt{\frac{k}{m}} = ab\sqrt{km}$

$\Rightarrow l = ab\sqrt{km} \Rightarrow \boxed{ab = \frac{l}{\sqrt{mk}}}$

* a^2 y b^2 como soluciones de polinomio cuadrático $(\lambda-a^2)(\lambda-b^2)=0$

$\Rightarrow \lambda^2-(a^2+b^2)\lambda + a^2b^2 = 0$

$\therefore a^2 = \frac{(a^2+b^2)+\sqrt{(a^2+b^2)^2-4(ab)^2}}{2}$; $b^2 = \frac{(a^2+b^2)-\sqrt{(a^2+b^2)^2-4(ab)^2}}{2}$

$\Delta = (a^2+b^2)^2-4(ab)^2 = \left(\frac{2E}{k}\right)^2 - 4\frac{l^2}{mk} = \frac{4}{k}\left(E^2-\frac{kl^2}{m}\right)$

Energía mínima : $E_{min}^2 - \frac{kl^2}{m} = 0 \Rightarrow E_{min} = l\sqrt{\frac{k}{m}} = \frac{lk}{\sqrt{km}} \Rightarrow ab = E_{min}k$

$\Rightarrow \frac{E_{min}}{k} = ab$

* Sea $\frac{E}{E_{min}} = \cosh\xi \Longrightarrow \sinh\xi = \frac{\sqrt{E^2-E_{min}^2}}{E_{min}}$

$a^2 = \frac{2E/k + \sqrt{4E^2/k^2-4E_{min}^2/k^2}}{2} = \frac{E+\sqrt{E^2-E_{min}^2}}{k}$; $E = E_{min}\cosh\xi$

$\sqrt{E^2-E_{min}^2} = E_{min}\sinh\xi$

$b^2 = \frac{E-\sqrt{E^2-E_{min}^2}}{k}$; $a^2 = \frac{E_{min}\cosh\xi + E_{min}\sinh\xi}{k} = \frac{E_{min}}{k}(\cosh\xi+\sinh\xi)$

$\Rightarrow \boxed{a^2 = \frac{E_{min}}{k}e^{\xi}}$; $b^2 = \frac{E_{min}\cosh\xi - E_{min}\sinh\xi}{k} = \frac{E_{min}}{k}(\cosh\xi-\sinh\xi)$

$\Rightarrow \boxed{b^2 = \frac{E_{min}}{k}e^{-\xi}}$; * Excentricidad $\epsilon^2 = 1-\frac{b^2}{a^2} = 1-\frac{\cancel{l}e^{-\xi}/\cancel{\sqrt{mk}}}{\cancel{l}e^{\xi}/\cancel{\sqrt{mk}}} = 1-e^{-2\xi}$

$$\Rightarrow \boxed{e^2 = 1 - e^{-2\xi}}$$

* Casos límites. $E \to E_{min} \Rightarrow E_{min} = E_{min} \cosh\xi \Rightarrow \cosh\xi = 1$

$$\Rightarrow \xi \to 0$$

$$a^2 = \frac{E_{min}}{K} e^{\xi} \xrightarrow{\xi \to 0} \frac{E_{min}}{K} \quad ; \quad b^2 = \frac{E_{min}}{K} e^{-\xi} \xrightarrow{\xi \to 0} \frac{E_{min}}{K}$$

$$e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{E_{min}/K}{E_{min}/K} = 0 \quad \text{órbita circular}$$

$$E \gg E_{min} \Rightarrow \frac{E}{E_{min}} \gg 1 \Rightarrow \cosh\xi \gg 1 \Rightarrow \xi \text{ grande}$$

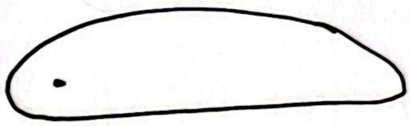
$\cosh 0 = 1$

$\cosh 1 \approx 1.54$

$\cosh 2 \approx 3.76$

$\cosh 5 \approx 74$

$$\therefore \left.\begin{array}{l} e^{\xi} \gg 1 \Rightarrow a^2 \gg b^2 \\ e^{-\xi} \ll 1 \Rightarrow b^2 \ll a^2 \end{array}\right\} \text{Elipse muy alargada}$$



c) $T = 2\pi\sqrt{\dfrac{m}{K}}$

* Periodo ind. de E y l
* iso-frecuencias: todas oscilan a la misma ω
* orbita ligada.