2- Ecuación secular de autovalores tiene raíces múltiples y los correspondientes autovalores son degenerados considere 2 grados de libertad con

22.24 $\det|U_{r\lambda} - \omega^2 m_{r\lambda}| = 0$

$$\overline{U} \equiv \begin{bmatrix} U & U_{12} \\ U_{12} & U \end{bmatrix} \quad \overline{M} \equiv \begin{bmatrix} M & M_{12} \\ M_{12} & m \end{bmatrix}$$

22.34 $\sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \rho_\sigma^{(s)} m_{\sigma\lambda} \rho_\lambda^{(t)} = \delta_{st}$

(a) Encuentre autovalores y autovectores. Muestre que en el límite $(M_{12}, U_{12}) \to 0$ ó $(M, U) \to 0$ los autovalores se vuelven degenerados con $\omega_1^2 = \omega_2^2$

Ecuación generalizada de autovectores (modos normales $\Rightarrow (\overline{U} - \omega^2 \overline{m})\overline{\rho} = 0 \longrightarrow \det(\overline{U} - \omega^2\overline{m}) = 0$

$$\therefore \overline{U} - \omega^2 \overline{m} = \begin{pmatrix} U & U_{12} \\ U_{12} & U \end{pmatrix} - \omega^2 \begin{pmatrix} m & m_{12} \\ m_{12} & m \end{pmatrix} = \begin{pmatrix} U - \omega^2 m & U_{12} - \omega^2 m_{12} \\ U_{12} - \omega^2 m_{12} & U - \omega^2 m \end{pmatrix}$$

$$\Rightarrow \det(\overline{U} - \omega^2\overline{m}) = [(U - \omega^2 m)(U - \omega^2 m)] - [(U_{12} - \omega^2 m_{12})(U_{12} - \omega^2 m_{12})]$$

$$= (U - \omega^2 m)^2 - (U_{12} - \omega^2 m_{12})^2 = 0$$

$A^2 - B^2 = 0$
$(A-B)(A+B) = 0$
dif cuadrados

$$= [\underbrace{(U - \omega^2 m) - (U_{12} - \omega^2 m_{12})}_{I}][\underbrace{(U - \omega^2 m) + (U_{12} - \omega^2 m_{12})}_{II}] = 0$$

I)

$$\therefore (U - U_{12}) - \omega^2 (m - m_{12}) = 0 \Rightarrow \boxed{\omega_1^2 = \frac{U - U_{12}}{m - m_{12}}}$$

II)

$$\text{y} \quad (U + U_{12}) - \omega^2 (m + m_{12}) = 0 \Rightarrow \boxed{\omega_2^2 = \frac{U + U_{12}}{m + m_{12}}}$$

Volviendo a ecuación secular

$(A-B)(A+B) = 0$; $A - B = 0$ (ω_1) , $A + B = 0$ (ω_2)

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = 0 \quad \begin{cases} A\rho_1 + B\rho_2 = 0 \\ B\rho_1 + A\rho_2 = 0 \end{cases} \quad (1)$$

$$\therefore \text{si } A - B = 0 \to A = B \longrightarrow A\rho_1 + B\rho_2 = A(\rho_1 + \rho_2) = 0$$

$$\to \rho_1 = -\rho_2 \longrightarrow \overline{\rho}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Ahora : $A+B=0 \Rightarrow A=-B \rightarrow A\rho_1+B\rho_2 = A(\rho_1-\rho_2)=0$
$\Rightarrow (\rho_1-\rho_2)$

$\therefore \boxed{\bar{\rho}^{(2)} = \begin{pmatrix}1\\1\end{pmatrix}}$

$A+B=0$

Verificación $\begin{pmatrix}A & B\\ B & A\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}A+B\\B+A\end{pmatrix}=0$

límites

$(m_{12}, u_{12}) \rightarrow 0$ o $(m, u)\rightarrow 0$ se vuelven degenerados $\omega_1^2 = \omega_2^2$

$(m_{12}, u_{12}) \rightarrow 0 \Rightarrow \omega_1^2 = \frac{u}{m}$, $\omega_2^2 = \frac{u}{m} \Rightarrow \boxed{\omega_1^2=\omega_2^2}$

$(m, u) \rightarrow 0 \Rightarrow \omega_1^2 = \frac{u_{12}}{m_{12}}$, $\omega_2^2 = \frac{u_{12}}{m_{12}} \Rightarrow \boxed{\omega_1^2=\omega_2^2}$

---

b) Muestre que en estos límites se pierde información concerniente a los correspondientes autovectores y que uno siempre puede encontrar 2 soluciones linearmente independientes de la forma $z_\sigma^{(s)} = e^{i\phi_s}\bar{\rho}_\sigma^{(s)}$ $s=1,2$ y $\bar{\rho}_\sigma^{(s)}$ R

En los límites $(m_{12}, u_{12})\rightarrow 0$ $(m,u)\rightarrow 0$

tenemos $\bar{u} \equiv \begin{bmatrix} u & u_{12}\\ u_{12} & u\end{bmatrix} \rightarrow uI$, $\bar{m} \equiv \begin{bmatrix} m & m_{12}\\ m_{12} & m\end{bmatrix} \rightarrow mI$

$\therefore \omega^2 = \frac{u}{m} \Rightarrow (\bar{u}-\omega^2\bar{m})\rho = (uI-\omega^2 mI)\rho = 0$

$= (uI - \frac{u}{m}mI)\rho = 0 \Rightarrow (uI-uI)\rho=0$

$\Rightarrow \rho = 0$ (cualquier vector $\rho\in\mathbb{R}^2$ satisface la ecuación, el espacio propio correspondiente tiene D 2 (todo $\mathbb{R}^2$)).

Se pierde información sobre los autovectores: la ecuación ya no selecciona una única pareja de autovectores, sólo el subespacio, hay infinitas opciones eg

No hay dirección preferida (recta que pasa por el origen, conjunto de vectores proporcionales entre sí), no puedes distinguir entre

por ej: $\rho = (\rho_1, \rho_2)^T$, $(1,0)^T$, $(0,1)^T$, $(1,1)^T$, etc

Cualquier combinación es válida

$\therefore$ Dado que cualquier ρ es solución en el límite degenerado, en particular podemos elegir 2 vectores $\mathbb{R}$ o ind.

$$\bar{\rho}^{(1)}, \bar{\rho}^{(2)} \in \mathbb{R}^2$$

Le agregamos fase multiplicativa $\mathbb{C}$ (origen de fase en la sol. temporal) se define

$$z_r^{(s)} = e^{i\phi_s} \bar{\rho}_r^{(s)}$$; sol. $\mathbb{C}$ de la ec. $(\bar{V} - \omega^2 \bar{m}) z = 0$

Físicamente: 2 modos de misma frecuencia pero con desfases arbitrarios

Se tomó una base $\mathbb{R}$ del subespacio y se multiplicó cada vector por fase $\mathbb{C}$

$$z_j^{(1)} = e^{i\phi_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad z^{(2)} = e^{i\phi_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(c) Muestre que estas soluciones pueden ser ortonormales de acuerdo a la ecuación

$$\sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \rho_{\sigma}^{(t)} M_{\sigma\lambda} \rho_{\lambda}^{(s)} = \delta_{st}$$

escogiendo $\rho_{\sigma}^{(1)} \equiv C_1 \bar{\rho}_{\sigma}^{(1)}$ y $\rho_{\sigma}^{(2)} \equiv C_2\left(\bar{\rho}_{\sigma}^{(2)} - \alpha \bar{\rho}_{\sigma}^{(1)}\right)$

Con $\alpha \equiv \dfrac{\sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(2)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(1)}}{\sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(1)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(1)}}$ ¿Que son C_1 y C_2?

$\longrightarrow$ por lin. ind. del subespacio degenerado)

Queremos vectores $\rho^{(1)}, \rho^{(2)}$ tales que $(\rho^{(t)})^T \bar{M} \rho^{(s)} = \sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \rho_{\sigma}^{(t)} M_{\sigma\lambda} \rho_{\lambda}^{(s)} = \delta_{ts}$

Gram-Schmidt con métrica $\bar{M}$; nos dan: ; y construir:

$$\rho^{(1)} \equiv C_1 \bar{\rho}^{(1)} \quad , \quad \rho^{(2)} \equiv C_2\left(\bar{\rho}^{(2)} - \alpha \bar{\rho}^{(1)}\right); \quad \alpha \equiv \frac{\sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(2)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(1)}}{\sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(1)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(1)}}$$

Comprobar: 1) que con α los vectores $\bar{\rho}^{(2)} - \alpha \bar{\rho}^{(1)}$ y $\bar{\rho}^{(1)}$ son ortogonales en $\bar{M}$

2) Cuáles deben ser C_1 y C_2 para normalizar

1) producto interno en $\bar{M}$

∴ Calcular $\bar{\rho}^{(1)}$ y $\bar{\rho}^{(2)} - \alpha \bar{\rho}^{(1)}$ sin C_1 y C_2)

$$(\bar{\rho}^{(1)})^T \bar{M} (\bar{\rho}^{(2)} - \alpha \bar{\rho}^{(1)}) = (\bar{\rho}^{(1)})^T \bar{M} \bar{\rho}^{(2)} - \alpha (\bar{\rho}^{(1)})^T \bar{M} \bar{\rho}^{(1)}$$

$\langle \bar{\rho}^{(1)} | \hat{M} | \bar{\rho}^{(2)} \rangle$

$$= \sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(1)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(2)} - \alpha \sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(1)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(1)}$$

$$\therefore \sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(1)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(2)} - \frac{\sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(2)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(1)}}{\cancel{\sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(1)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(1)}}} \cancel{\sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(1)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(1)}} =$$

$$= \sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(1)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(2)} - \sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(2)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(1)} \xrightarrow{(M_{\sigma\lambda} = M_{\lambda\sigma}) \text{ simétrica}} \sum_{\sigma}^{\mu}\sum_{\lambda}^{\mu} \bar{\rho}_{\lambda}^{(2)} M_{\lambda\sigma} \bar{\rho}_{\sigma}^{(1)}$$

$$= \sum_{\lambda}^{\mu}\sum_{\sigma}^{\mu} \bar{\rho}_{\sigma}^{(1)} M_{\sigma\lambda} \bar{\rho}_{\lambda}^{(2)} = 0$$

$$\therefore (\bar{\rho}^{(1)})^T \bar{M} (\bar{\rho}^{(2)} - \alpha \bar{\rho}^{(1)}) = 0 \quad \text{(ortogonales)}$$

2) Normalizamos con C_1 y C_2 para que las normas $(\rho^{(s)})^T \overline{m} \rho^{(s)}$ ... 1

~~notación de dirac~~

$$\alpha = \frac{(\bar{\rho}^{(1)})^T \overline{m} \bar{\rho}^{(2)}}{(\bar{\rho}^{(1)})^T \overline{m} \bar{\rho}^{(1)}} = \frac{\sum_\lambda^\ell \sum_\sigma^\ell \bar{\rho}_\sigma^{(2)} m_{\sigma\lambda} \bar{\rho}_\lambda^{(1)}}{\sum_\lambda^\ell \sum_\sigma^\ell \bar{\rho}_\sigma^{(1)} m_{\sigma\lambda} \bar{\rho}_\lambda^{(1)}}$$

---

2) * C_1 y C_2 ctes de Normalización

* $\bar{\rho}^{(1)}$ y $\bar{\rho}^{(2)}$ autovectores $\mathbb{R}$, linealmente ind. del Subespacio del caso degenerado :

defi $\tilde{\rho}^{(2)} = \bar{\rho}^{(2)} - \alpha \bar{\rho}^{(1)}$

* Cuánto debería valer C_1 y C_2 tal que los vectores $\rho^{(1)} = C_1 \bar{\rho}^{(1)}$, $\rho^{(2)} = \tilde{\rho}^{(2)}$ satisfagan $(\rho^{(t)})^T \overline{m} \rho^{(s)} = \delta_{ts}$

ortonormalidad $\sum_\lambda^\ell \sum_\sigma^\ell \rho_\sigma^{(t)} m_{\sigma\lambda} \rho_\lambda^{(s)} = \delta_{st}$

$\therefore$ C_1 | norma al cuadrado con la métrica $\overline{m}$

$$\|\bar{\rho}^{(1)}\|_m^2 = (\bar{\rho}^{(1)})^T \overline{m} \bar{\rho}^{(1)} = \sum_\sigma^\ell \sum_\lambda^\ell \bar{\rho}_\sigma^{(1)} m_{\sigma\lambda} \bar{\rho}_\lambda^{(1)} = 1$$

$\therefore$ para $\rho^{(1)} = C_1 \bar{\rho}^{(1)} \longrightarrow (\rho^{(1)})^T \overline{m} \rho^{(1)} = C_1^2 \|\bar{\rho}^{(1)}\|^2 = 1$

$$\rightarrow C_1^2 = \frac{1}{\sqrt{\|\bar{\rho}^{(1)}\|_m^2}} = \left(\sum_\sigma^\ell \sum_\lambda^\ell \bar{\rho}_\sigma^{(1)} m_{\sigma\lambda} \bar{\rho}_\lambda^{(1)}\right)^{-1/2}$$

C_2 | expandir norma al cuadrado de $\tilde{\rho}^{(2)} = \bar{\rho}^{(2)} - \alpha \bar{\rho}^{(1)}$ con $\overline{m}$

$$\|\tilde{\rho}^{(2)}\|_m^2 = (\bar{\rho}^{(2)} - \alpha \bar{\rho}^{(1)})^T \overline{m} (\bar{\rho}^{(2)} - \alpha \bar{\rho}^{(1)})$$

$$= (\bar{\rho}^{(2)})^T \overline{m} \bar{\rho}^{(2)} - \alpha (\bar{\rho}^{(1)})^T \overline{m} \bar{\rho}^{(2)} - \alpha (\bar{\rho}^{(2)})^T \overline{m} \bar{\rho}^{(1)} + \alpha^2 (\bar{\rho}^{(1)})^T \overline{m} \bar{\rho}^{(1)}$$

$$\overset{(\overline{m}\ \text{simétrica})}{=} (\bar{\rho}^{(2)})^T \overline{m} \bar{\rho}^{(2)} - 2\alpha (\bar{\rho}^{(1)})^T \overline{m} \bar{\rho}^{(2)} + \alpha^2 (\bar{\rho}^{(1)})^T \overline{m} \bar{\rho}^{(1)}$$

def: $A \equiv (\bar{\rho}^{(1)})^T \bar{m} \bar{\rho}^{(1)}$, $B \equiv (\bar{\rho}^{(1)})^T \bar{m} \bar{\rho}^{(2)}$

$D \equiv (\bar{\rho}^{(2)})^T \bar{m} \bar{\rho}^{(2)}$; $\alpha = \frac{B}{A}$

$\therefore \|\tilde{\rho}^{(2)}\|_M^2 = D - 2\alpha B + \alpha^2 A$,

$\overset{\alpha}{=} D - 2\frac{B^2}{A} + \frac{B^2}{A} = D - \frac{B^2}{A}$

$\therefore$ normalizando para $\rho^{(2)} = C_2 \tilde{\rho}^{(2)}$

$$(\rho^{(2)})^T \bar{m} \rho^{(2)} = C_2^2 \|\tilde{\rho}^{(2)}\|_M^2 = 1$$

$$\Rightarrow C_2 = \frac{1}{\sqrt{\|\tilde{\rho}^{(2)}\|_M^2}} = \frac{1}{\sqrt{D - B^2/A}}$$

ó

$$C_2 = \left[\sum_\sigma^n \sum_\lambda^n (\bar{\rho}_\sigma^{(2)} - \alpha \bar{\rho}_\sigma^{(1)}) m_{\sigma\lambda} (\bar{\rho}_\lambda^{(2)} - \alpha \bar{\rho}_\lambda^{(1)})\right]^{-1/2}$$

Notación de dirac

$|\rho^{(1)}\rangle = C_1 |\bar{\rho}^{(1)}\rangle \longrightarrow C_1^2 \langle \bar{\rho}^{(1)} | \hat{m} | \bar{\rho}^{(1)}\rangle = 1$

$$\boxed{C_1 = \frac{1}{\sqrt{\langle \bar{\rho}^{(1)} | \hat{m} | \bar{\rho}^{(1)}\rangle}}}$$

$|\tilde{\rho}^{(2)}\rangle = |\bar{\rho}^{(2)}\rangle - \alpha |\bar{\rho}^{(1)}\rangle$

$\alpha = \frac{\langle \bar{\rho}^{(1)} | \hat{m} | \bar{\rho}^{(2)}\rangle}{\langle \bar{\rho}^{(1)} | \hat{m} | \bar{\rho}^{(1)}\rangle}$; $\langle \bar{\rho}^{(1)} | \hat{m} | \tilde{\rho}^{(2)}\rangle = \langle \bar{\rho}^{(1)} | \hat{m} | \bar{\rho}^{(2)}\rangle - \alpha \langle \bar{\rho}^{(1)} | \hat{m} | \bar{\rho}^{(1)}\rangle = 0$

$|\rho^{(2)}\rangle = C_2 |\tilde{\rho}^{(2)}\rangle$

$\rightarrow C_2^2 \langle \tilde{\rho}^{(2)} | \hat{m} | \tilde{\rho}^{(2)}\rangle = 1 \Rightarrow \boxed{C_2 = \frac{1}{\sqrt{\langle \tilde{\rho}^{(2)} | \hat{m} | \tilde{\rho}^{(2)}\rangle}}}$

$\langle \tilde{\rho}^{(2)} | \hat{m} | \tilde{\rho}^{(2)}\rangle = \langle \bar{\rho}^{(2)} | \hat{m} | \bar{\rho}^{(2)}\rangle - \alpha \langle \bar{\rho}^{(2)} | \hat{m} | \bar{\rho}^{(1)}\rangle$

$- \alpha^* \langle \bar{\rho}^{(1)} | \hat{m} | \bar{\rho}^{(2)}\rangle + |\alpha|^2 \langle \bar{\rho}^{(1)} | \hat{m} | \bar{\rho}^{(1)}\rangle$

* Ayudantía Mecánica

* Cuántica

* Electro

$$\left.\begin{aligned}\alpha &= \langle \bar{\rho}^{(1)} | \hat{M} | \bar{\rho}^{(2)} \rangle \\ \alpha^* &= \langle \bar{\rho}^{(2)} | \hat{M} | \bar{\rho}^{(1)} \rangle\end{aligned}\right\} \quad \hat{M} = \hat{M}^\dagger \text{ hermitiana, simetrica}$$

$$\therefore \langle \tilde{\rho}^{(2)} | \hat{M} | \tilde{\rho}^{(2)} \rangle = \langle \bar{\rho}^{(2)} | \hat{M} | \bar{\rho}^{(2)} \rangle - \frac{\langle \bar{\rho}^{(1)} | \hat{M} | \bar{\rho}^{(2)} \rangle \langle \bar{\rho}^{(2)} | \hat{M} | \bar{\rho}^{(1)} \rangle}{\langle \bar{\rho}^{(1)} | \hat{M} | \bar{\rho}^{(1)} \rangle}$$

$$= \langle \bar{\rho}^{(2)} | \hat{M} | \bar{\rho}^{(2)} \rangle - \frac{|\langle \bar{\rho}^{(1)} | \hat{M} | \bar{\rho}^{(2)} \rangle|^2}{\langle \bar{\rho}^{(1)} | \hat{M} | \bar{\rho}^{(1)} \rangle}$$

$$\therefore \boxed{C_\alpha = \frac{1}{\sqrt{\langle \bar{\rho}^{(2)} | \hat{M} | \bar{\rho}^{(2)} \rangle - \dfrac{|\langle \bar{\rho}^{(1)} | \hat{M} | \bar{\rho}^{(2)} \rangle|^2}{\langle \bar{\rho}^{(1)} | \hat{M} | \bar{\rho}^{(1)} \rangle}}}}$$

$\therefore$ Gram-Schmidt

$$|\rho^{(1)}\rangle = \frac{|\bar{\rho}^{(1)}\rangle}{\sqrt{\langle \bar{\rho}^{(1)} | \hat{M} | \bar{\rho}^{(1)} \rangle}}$$

$$|\rho^{(2)}\rangle = \frac{|\bar{\rho}^{(2)}\rangle - \dfrac{\langle \rho^{(1)} | \hat{M} | \bar{\rho}^{(2)} \rangle}{\langle \rho^{(1)} | \hat{M} | \bar{\rho}^{(1)} \rangle} |\rho^{(1)}\rangle}{\sqrt{\langle \bar{\rho}^{(2)} | \hat{M} | \bar{\rho}^{(2)} \rangle}}$$

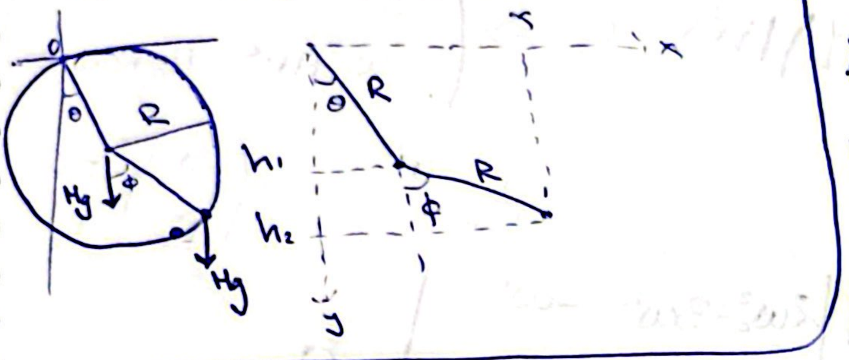
Small Oscillations

1.- Un aro delgado de R y M oscila en su propio plano con un punto del aro fijo. Unido al aro hay una masa puntual M obligada a moverse sin fricción a lo largo del aro. El sistema está en un campo gravitacional g. oscilaciones pequeñas

$*\cos\theta \approx 1-\theta^2/2$

$*\theta,\phi \ll 1 \rightarrow \cos(\theta-\phi)\approx 1$

(a) Muestre que las frecuencias de modos normales $\omega_1 = \frac{1}{2}\sqrt{\frac{2g}{R}}$ $\omega_2 = \sqrt{\frac{2g}{R}}$



1) Energía Cinética y potencial

$h_1 = R - R\cos\theta = R(1-\cos\theta)$
$\overset{(os)}{=} R(1-1+\theta^2/2) = R\theta^2/2$

$h_2 = R - R\cos\theta - R\cos\phi$
$= R(1-\cos\theta - \cos\phi)$
$\overset{(os)}{=} R(-1+\theta^2/2+\phi^2/2)$

Masa puntual: $x = R(\sin\theta + \sin\phi)$
$y = -R(\cos\theta + \cos\phi)$

$\dot{x} = R(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi) \longrightarrow \dot{x}^2 = R^2(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi)^2 = R^2(\dot{\theta}^2\cos^2\theta + 2\dot{\theta}\dot{\phi}\cos\theta\cos\phi + \dot{\phi}^2\cos^2\phi)$

$\dot{y} = R(\dot{\theta}\sin\theta + \dot{\phi}\sin\phi) \longrightarrow \dot{y}^2 = R^2(\dot{\theta}\sin\theta + \dot{\phi}\sin\phi)^2 = R^2(\dot{\theta}^2\sin^2\theta + 2\dot{\theta}\dot{\phi}\sin\theta\sin\phi + \dot{\phi}^2\sin^2\phi)$

$\therefore \dot{x}^2 + \dot{y}^2 = R^2(\dot{\theta}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi) + \dot{\phi}^2) \overset{(*)}{=} R^2(\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2)$

$\cos\theta + \cos\phi$
$1-\theta$

Así, para el CM:

$I = I_d + I_{CM} = MR^2 + MR^2 = 2MR^2$

$T_1 = \frac{1}{2} I\dot{\theta}^2 = MR^2\dot{\theta}^2$

$V_1 = \frac{1}{2} MgR\theta^2$

$T_2 = \frac{1}{2}MR^2(\dot{\theta}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi) + \dot{\phi}^2)$

$V_2 = \frac{1}{2}MgR(\theta^2+\phi^2)$

$= \frac{1}{2}MR^2(\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2)$

2) Formas matriciales $q = \begin{pmatrix}\theta \\ \phi\end{pmatrix}$ $q^T = (\theta\ \phi)$ $T = \frac{1}{2}\dot{q}^T \mathbb{M}\dot{q}$; $\mathbb{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix}$

$\mathbb{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{pmatrix}$

$V = \frac{1}{2} q^T \mathbb{K} q$

$T = T_1 + T_2 = MR^2\dot{\theta}^2 + \frac{1}{2}MR^2(\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2) = \frac{1}{2}MR^2(3\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2)$

$V = V_1 + V_2 = \frac{1}{2}MgR\theta^2 + \frac{1}{2}MgR(\theta^2 + \phi^2) = \frac{MgR}{2}(2\theta^2 + \phi^2)$

$$\therefore T = \frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2}(\dot{\theta}\,\dot{\phi})\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix} \qquad T = \frac{1}{2}MR^2(3\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2)$$

$$\rightarrow \frac{1}{2}(M_{11}\dot{\theta}^2 + M_{12}\dot{\phi}\dot{\theta} + M_{21}\dot{\phi}\dot{\theta} + D\dot{\phi}^2)$$

$$V = \frac{1}{2}q^T M q = \frac{1}{2}(\theta\,\phi)\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{pmatrix}\theta\\ \phi\end{pmatrix} \qquad V = \frac{MgR}{2}(2\theta^2 + \phi^2)$$

$$= \frac{1}{2}(K_{11}\theta^2 + K_{12}\phi\theta + K_{21}\phi\theta + D\phi^2)$$

$$\therefore M = MR^2\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \quad K = MgR\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

3) Autovalores $(K - \omega^2 M)|\vec{u}\rangle = 0 \Rightarrow \begin{pmatrix} 2MgR - 3\omega^2 MR & -\omega^2 MR^2 \\ -\omega^2 MR^2 & MgR - \omega^2 MR^2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$

$\frac{1}{MR^2}$

$$\rightarrow \det(K - \omega^2 M) = 0$$

$$\therefore \begin{pmatrix} 2\frac{g}{R} - 3\omega^2 & -\omega^2 \\ -\omega^2 & \frac{g}{R} - \omega^2 \end{pmatrix}_{\omega_0^2 = \frac{g}{R}} \Longrightarrow \begin{vmatrix} 2\omega_0^2 - 3\omega^2 & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (2\omega_0^2 - 3\omega^2)(\omega_0^2 - \omega^2) - \omega^4 = 0, \quad x = \omega^2$$

$$\therefore (2\omega_0^2 - 3x)(\omega_0^2 - x) - x^2 = 0$$

$$2\omega_0^4 - 2\omega_0^2 x - 3\omega_0^2 x + 3x^2 - x^2 = 0 \longrightarrow 2x^2 - 5\omega_0^2 x + 2\omega_0^4$$

$$x = \frac{5\omega_0^2 \pm \sqrt{25\omega_0^4 - 16\omega_0^4}}{4} = \frac{5\omega_0^2 \pm 3\omega_0^2}{4} \left\{ \begin{array}{l} x_1 = 2\omega_0^2 \\ x_2 = \frac{1}{2}\omega_0^2 \end{array} \right.$$

$$\therefore \omega_1^2 = 2\frac{g}{R} \Rightarrow \boxed{\omega_1 = \sqrt{2g/R}}$$

$$\boxed{\omega_2 = \sqrt{g/2R}}$$

b) Encuentre los autovectores de modos normales. Dibuje mov.

$\omega_1=\sqrt{2g/R}$ $\quad \omega_2=\sqrt{g/2R}$; $M = MR^2\begin{pmatrix}3&1\\1&1\end{pmatrix}$ $\quad K = MgR\begin{pmatrix}2&0\\0&1\end{pmatrix}$

$$\therefore (K-\omega_1^2 M)|\vec{u}_1\rangle = 0 \rightarrow \begin{pmatrix}\frac{2g}{R}-3\frac{2g}{R} & -2g/R\\ -2g/R & g/R - 2g/R\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=0 \qquad |u_1\rangle = \begin{pmatrix}1\\-2\end{pmatrix}$$

$$\rightarrow \begin{pmatrix}-4&-2\\-2&-1\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=0 \quad \left.\begin{matrix}4A+2B=0\\2A+B=0\end{matrix}\right\} 2A=-B \rightarrow \boxed{|\vec u_1\rangle = (1,-2)^T}$$

$$(K-\omega_2^2M)|\vec u_2\rangle = 0 \Rightarrow \begin{pmatrix}2-\frac32 & -\frac12\\ -\frac12 & 1-\frac12\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=0 \rightarrow \begin{pmatrix}1/2&-1/2\\-1/2&1/2\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=0 \Big\} A=B$$

$$|u_2\rangle = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\rightarrow \boxed{|\vec u_2\rangle = (1,1)^T}$$

c) Construya la matriz modal $U = (u_1' \;\; u_2')$ ortonormales

$$\langle u,u\rangle_M = u^T M u \quad , \quad \langle u_1,u_1\rangle = \langle u_1\rangle_M^2$$

$$\Rightarrow |u_1'\rangle = \frac{u_1}{\langle u_1\rangle_M} \quad , \quad |u_2'\rangle = \frac{u_2}{\langle u_2\rangle_M}$$

$$\therefore \langle u_1\rangle_M^2 = \langle u_1,u_1\rangle_M = u_1^T M u_1 = (1\ -2)\, MR^2 \begin{pmatrix}3&1\\1&1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}$$

$$= MR^2(1\ -2)\begin{pmatrix}1\\-1\end{pmatrix} = 3MR^2 \Rightarrow \langle u_1\rangle_M = \sqrt{3MR^2}$$

$$\therefore \boxed{|u_1'\rangle = \frac{1}{\sqrt{3MR^2}}\begin{pmatrix}1\\-2\end{pmatrix}}$$

$$\langle u_2\rangle_M^2 = \langle u_2,u_2\rangle_M = u_2^T M u_2 = (1\ 1)\, MR^2\begin{pmatrix}3&1\\1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = MR^2(1\ 1)\begin{pmatrix}4\\2\end{pmatrix}$$

$$= 6MR^2 \Rightarrow \langle u_2\rangle_M = \sqrt{6MR^2} \Rightarrow \boxed{|u_2'\rangle = \frac{1}{\sqrt{6MR^2}}\begin{pmatrix}1\\1\end{pmatrix}}$$

$$\therefore \boxed{U = \frac{1}{\sqrt{MR^2}}\begin{pmatrix}1/\sqrt3 & 1/\sqrt6\\ -2/\sqrt3 & 1/\sqrt6\end{pmatrix}} \rightarrow \boxed{U^T = \frac{1}{\sqrt{MR^2}}\begin{pmatrix}1/\sqrt3 & -2/\sqrt3\\ 1/\sqrt6 & 1/\sqrt6\end{pmatrix}}$$

(d) Encuentre Coor. Normales y Muestre que ellas diagonalizan el Lagrangiano $\mathcal{A} = \frac{1}{\sqrt{MR^2}}\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ -2/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \longrightarrow \mathcal{A}^T = \frac{1}{\sqrt{MR^2}}\begin{pmatrix} 1/\sqrt{3} & -2/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$

Fetter.

Normal Coordinates

→ new set of generalized coor. $\zeta_1, \dots, \zeta_n$, linearly related to the original generalized coor. $\eta_1, \dots, \eta_n$ by the modal matrix

$$\underline{\eta}(t) \equiv \underline{\mathcal{A}}\,\underline{\zeta}(t) \quad / \underline{\mathcal{A}}^T \underline{m} \rightarrow \underline{\zeta}(t) = \mathcal{A}^T \underline{m}\,\underline{\eta}(t)$$

$$\underline{x}^T \underline{a}\, \underline{x} \equiv \sum_i^e \sum_j^e x_i a_{ij} x_j \longrightarrow$$

$$\mathcal{L} = T - V = \frac{1}{2}\sum_\sigma^e \sum_\lambda^e \left(m_{\lambda\sigma}\dot{\eta}_\lambda \dot{\eta}_\sigma - v_{\lambda\sigma}\eta_\lambda \eta_\sigma \right) - V_0 \Bigg\} \; 2\mathcal{L} = \dot{\underline{\eta}}^T \underline{m}\, \dot{\underline{\eta}} - \underline{\eta}^T \underline{v}\, \underline{\eta}$$

Lag. para oscilaciones pequeñas

cte suppressed

$(\underline{a}\underline{b})^T \equiv \underline{b}^T \underline{a}^T$

$$\therefore\; 2\mathcal{L} = \dot{\underline{\zeta}}^T \underline{\mathcal{A}}^T \underline{m}\, \underline{\mathcal{A}}\, \dot{\underline{\zeta}} - \underline{\zeta}^T \underline{\mathcal{A}}^T \underline{v}\, \underline{\mathcal{A}}\, \underline{\zeta}$$

; $\mathcal{A}$ diagonalizes both $\underline{m}$ and $\underline{v}$

$$\mathcal{L}' = \frac{1}{2}\left(\dot{\underline{\zeta}}^T \dot{\underline{\zeta}} - \underline{\zeta}^T \underline{\omega_0^2}\, \underline{\zeta} \right) \longrightarrow \mathcal{L}' = \frac{1}{2}\sum_{\sigma=1}^{n} \left(\dot{\zeta}_\sigma^2 - \omega_\sigma^2 \zeta_\sigma^2 \right)$$

modos normales

Coor. Normal $\quad Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \mathcal{A}^T M q \;;\; M = MR^2\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix},\; MgR\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = K$

Lagrangiano $\quad \mathcal{L} = \frac{1}{2}\dot{q}^T M \dot{q} - \frac{1}{2} q^T K q \xrightarrow[\text{coor. Normales}]{\text{diagonalizado}} \mathcal{L}' = \frac{1}{2}\dot{Q}^T M' \dot{Q} - \frac{1}{2} Q^T K' Q$

$\mathcal{A}$ diagonaliza M y K $\quad M' = \mathcal{A}^T M \mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{ij}$,

$\quad K = \mathcal{A}^T K \mathcal{A} = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$

M' y K': $\quad M' = \mathcal{A}^T M \mathcal{A} = \frac{1}{\cancel{MR^2}}\begin{pmatrix} 1/\sqrt{3} & -2/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \cancel{MR^2}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ -2/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$

$$= \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} \\ 4/\sqrt{6} & 2/\sqrt{6} \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ -2/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{ij}$$

$$K' = \mathcal{A}^T \cdot \mathcal{A} = \frac{1}{\cancel{MR^2}}\begin{pmatrix} 1/\sqrt{3} & -2/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \cancel{M}gR\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ -2/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$$

$\frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{4}{3} = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$

$$= \frac{g}{R}\begin{pmatrix} 2/\sqrt{3} & -2/\sqrt{3} \\ 2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ -2/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} = \omega_0^2\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

Coor. Normales: $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = A^T M q$

$$= \frac{1}{\sqrt{MR^2}} \begin{pmatrix} 1/\sqrt{3} & -2/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} MR^2 \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix}$$

$$= \sqrt{MR^2} \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} \\ 4/\sqrt{6} & 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix}$$

$$\therefore \boxed{\begin{aligned} Q_1 &= \sqrt{MR^2}\left(\frac{\theta}{\sqrt{3}} - \frac{\phi}{\sqrt{3}}\right) = \sqrt{\frac{MR^2}{3}}(\theta - \phi) \\ Q_2 &= \sqrt{MR^2}\left(\frac{4}{\sqrt{6}}\theta + \frac{2}{\sqrt{6}}\phi\right) = \sqrt{\frac{MR^2}{6}}(4\theta + 2\phi) \end{aligned}}$$

$$\therefore \mathcal{L}' = \frac{1}{2}\dot{Q}^T M' \dot{Q} - \frac{1}{2} Q^T K' Q \quad ; M' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, K' = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

$$= \frac{1}{2}\left\{ (\dot{Q}_1 \; \dot{Q}_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \end{pmatrix} - (Q_1 \; Q_2) \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \right.$$

$$= \frac{1}{2}\left\{ (\dot{Q}_1 \; \dot{Q}_2) \begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \end{pmatrix} - (\omega_1^2 Q_1 \;\; \omega_2^2 Q_2) \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \right\}$$

$$= \frac{1}{2}\left\{ \dot{Q}_1^2 + \dot{Q}_2^2 - \omega_1^2 Q_1^2 - \omega_2^2 Q_2^2 \right\}$$

$$= \frac{1}{2}\sum_{n=1}^{2} \dot{Q}_n^2 - \omega_n^2 Q_n^2 \longrightarrow \boxed{\mathcal{L}' = \frac{1}{2}\sum_{n=1}^{2} \dot{Q}_n^2 - \omega_n^2 Q_n^2}$$

Scanned with CamScanner