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$$\psi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\Omega} \rho(\vec{r}\,')G(\vec{r},\vec{r}\,')d^3x' + \frac{1}{4\pi\epsilon_0}\oint_{\partial\Omega}(G\partial_n\psi - \psi\partial_n G)ds$$

Teorema de Unicidad

→ Asumiendo 2 soluciones, $U = \psi_2 - \psi_1$ con la misma densidad y condiciones de borde (C.B)

Usando la primera desigualdad con $\psi = G = U$

$$\int_{\Omega}(U\vec{\nabla}^2U + |\vec{\nabla}U|^2)d^3x = \oint_{\partial\Omega}(U\partial_n U)ds$$

Como $\vec{\nabla}^2U = \vec{\nabla}^2(\psi_2-\psi_1) = 0$, y como las C.B. pueden ser de Dirchlet $(U=0)$ o Newmann $(\partial_n U = 0)$

Ejemplo: Función de Green con C.B. de Dirchlet (1D)

→ Queremos resolver $\frac{d^2V(x)}{dx^2} = -\frac{\rho(x)}{\epsilon_0}$ Ec. Poisson

con C.B. $V(0) = V(L) = 0$

→ Construimos $\frac{d^2G(x,x')}{dx^2} = -4\pi\delta(x-x')$, $G(0,x') = G(L,x') = 0$

$\therefore \frac{d^2G}{dx^2} = 0$, $x \neq x'$ $\therefore$ $G(x,x') = \begin{cases} A_1x + B_1 & 0 \le x < x' \\ A_2x + B_2, & x' < x \le L \end{cases}$

→ De las cond.
- $G(0,x') = 0 \Rightarrow B_1 = 0$
- $G(L,x') = 0 \Rightarrow A_2L + B_2 = 0$
  $\Rightarrow B_2 = -A_2L$

→ luego

$$G(x,x') = \begin{cases} A_1x & , 0 \le x < x' \\ A_2(x-L) & , x' < x \le L \end{cases}$$

i → Continuidad de $G(x,x')$ en $x=x'$

ii → Discontinuidad de dG/dx en $x=x'$

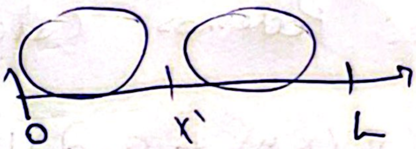
De (i) $$A_1 x' = A_2(x'-L)$$

$$\Rightarrow A_1 = A_2\left(\frac{x'-L}{x'}\right)$$

y de (ii) $$\int_{x'-\varepsilon}^{x'+\varepsilon} \frac{d^2G}{dx^2}\,dx = -4\pi\int_{x'-\varepsilon}^{x'+\varepsilon} \delta(x-x')\,dx = -4\pi$$

Sobre x región + ; dibujo de x

$$\left.\frac{dG}{dx}\right|_{x'+} - \left.\frac{dG}{dx}\right|_{x'-} = -4\pi$$

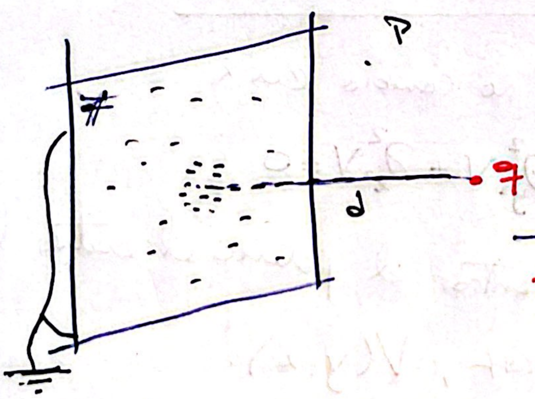


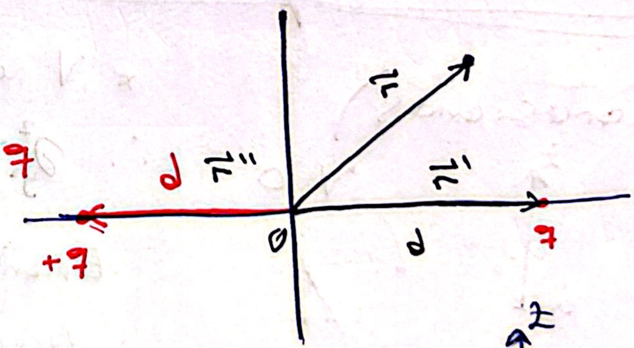
$$A_2 - A_1 = -4\pi$$

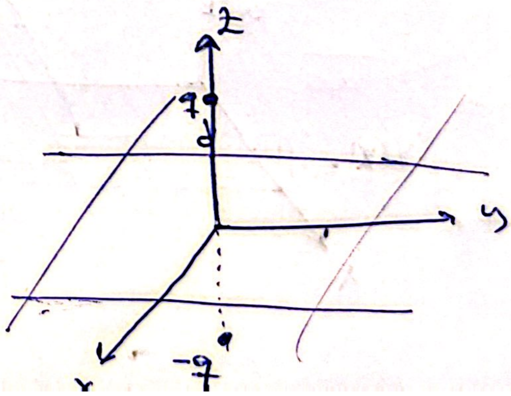
y resolvemos $\longrightarrow$ $A_1 = -4\pi\left(\frac{x'-L}{L}\right)$; $A_2 = -4\pi\frac{x'}{L}$

escribiendo $$G(x,x') = \begin{cases} 4\pi\, \dfrac{x(L-x')}{L}, & 0 \le x < x' \\ 4\pi\, \dfrac{x'(L-x)}{L}, & x' < x \le L \end{cases}$$

Método de Imágenes eléctricas







$$V(\vec r) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec r-\vec r'|} - \frac{q}{|\vec r-\vec r'|}\right]$$

$\vec r = x\hat i + y\hat j + z\hat k$

$\vec r' = d\hat k$, $\vec r'' = d(-\hat k)$

$$|\vec{r}-\vec{r}'| = \sqrt{x^2+y^2+(z-d)^2} \; ; \; |\vec{r}-\vec{r}''| = \sqrt{x^2+y^2+(z+d)^2}$$

$$\therefore V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

→ En el plano $z=0 \Rightarrow V=0$

→ claramente
- $V=0$ en $z=0$
- $V \to 0$ cuando $r^2 \gg d^2$

obs: Podemos calcular la carga inducida $\sigma = -\epsilon_0 \partial_n V\big|_s$

$$\to \sigma(x,y) = \epsilon_0 \partial_z V\big|_{z=0} \Rightarrow \sigma(x,y) = \frac{-qd}{2\pi (x^2+y^2+d^2)^{3/2}}$$

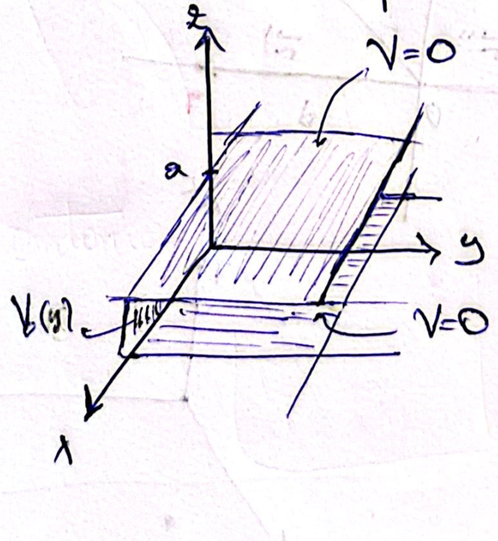
obs 2: $\int \sigma \, dx\,dy = -q \Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(x,y)\,dx\,dy$

obs 3: $\vec{E} = -\vec{\nabla} V$

obs 4: Calcular la fuerza que el plano conductor, ejerce sobre la carga q. (★★) Griffiths

Método de Separación de Variables

(Coor. Cartesianas)



- V no cambia en x

$\therefore \partial^2_{y^2} V + \partial^2_{z^2} V = 0$

Encontrar el potencial entre las placas, $V(y,z)$!

- Definir C.B.
  - (i) $V=0$ en $z=0$
  - (ii) $V=0$; $z=a$
  - (iii) $V\to 0$, $y \to \infty$
  - (iv) $V = V_0(z)$, en $y=0$

$V(y,z) = Y(y)Z(z)$, se obtienen 2 ecuaciones

$$Z'' + k^2 Z = 0 \quad , \quad Y'' - k^2 Y = 0$$

→ luego, $V(y,z) = (Ae^{ky} + Be^{-ky})(C \sin kz + D\cos kz)$

→ aplicando C.B.

- (iii) $\Longrightarrow$ $A = 0$ //
- (i) $\Longrightarrow$ $D = 0$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} V(y,z) = B' e^{-ky} \sin(kz)$$

- (ii) $\Longrightarrow$ $\sin(ka) = 0 \Rightarrow k = \frac{n\pi}{a}$; $(n = 1, 2, \dots)$

$$\therefore V(y,z) = \sum_n B_n e^{-\frac{n\pi y}{a}} \sin\left(\frac{n\pi z}{a}\right)$$

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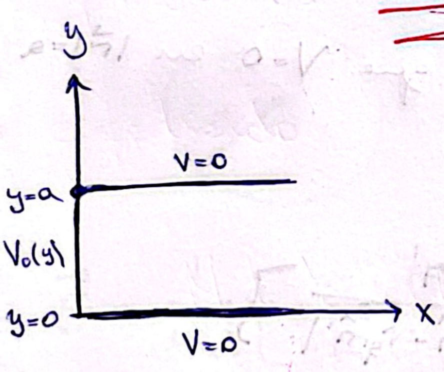


$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

→ imponiendo condiciones de Borde (iii)

$$(*) \quad V_0(y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) \quad \text{(Fourier!!)}$$

→ Sabemos: $\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right) dy$

→ multiplicamos $(*)$ por $\sin\left(\frac{m\pi y}{a}\right)$ e integramos obteniendo

$$C_n = \frac{2}{a}\int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

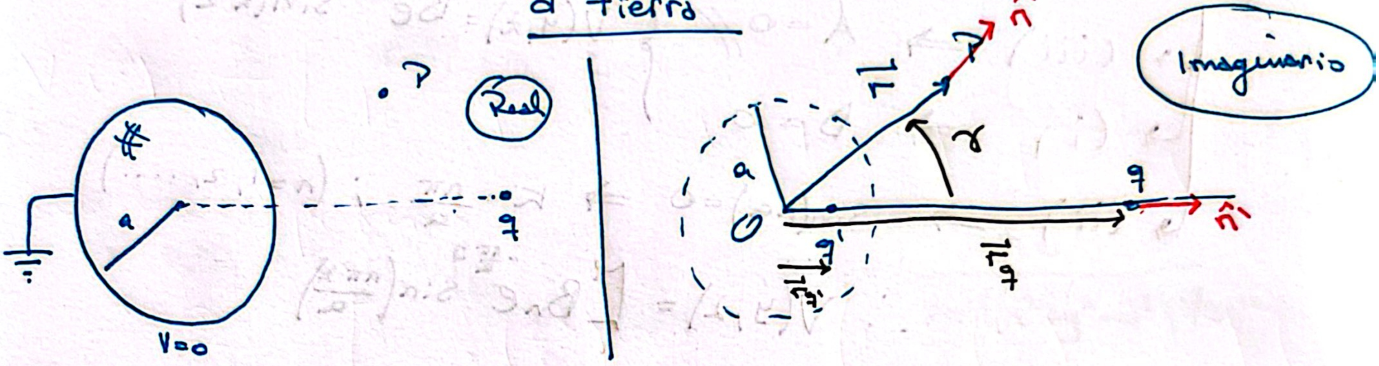
Ejemplo: $V_0(y) = V_0$ $\Longrightarrow$ $C_n = \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right) dy = \frac{2V_0}{n\pi}(1 - \cos n\pi)$

$$= \begin{cases} 0 & , n \text{ par} \\ \frac{4V_0}{n\pi} & , n \text{ impar} \end{cases}$$

$$\Rightarrow V(x,y) = \frac{4V_0}{\pi} \sum_{n\,\text{impar}} \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$V_0(y) = V_0 \sin\left(\frac{5\pi y}{a}\right)$?

Carga puntual frente a una esfera conductora conectada a tierra



→ El potencial en $\vec{r}$ es: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{r}-\vec{r}_q|} + \frac{q'}{|\vec{r}-\vec{r}_{q'}|}\right]$

→ Tenemos que elegir q' y $\vec{r}_{q'}$ tal que $V=0$ en $|\vec{r}|=a$

→ Si $\hat{n} = \frac{\vec{r}}{|\vec{r}|}$, $\hat{n}' = \frac{\vec{r}_q}{|\vec{r}_q|}$

$$\therefore V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|r\hat{n} - r_q\hat{n}'|} + \frac{q'}{|r\hat{n} - r_{q'}\hat{n}'|}\right]$$

→ Evaluando en la superficie en $|\vec{r}|=a$

$$0 = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{a\left|\hat{n} - \frac{r_q}{a}\hat{n}'\right|} + \frac{q'}{r_{q'}\left|\hat{n}' - \frac{a}{r_{q'}}\hat{n}\right|}\right]$$

→ entonces : $\frac{q}{a} = -\frac{q'}{r_{q'}} \wedge \frac{r_q}{a} = \frac{a}{r_{q'}}$

→ o sea $\boxed{r_{q'} = \frac{a^2}{r_q}}$ $\boxed{q' = -\frac{a q}{r_q}}$

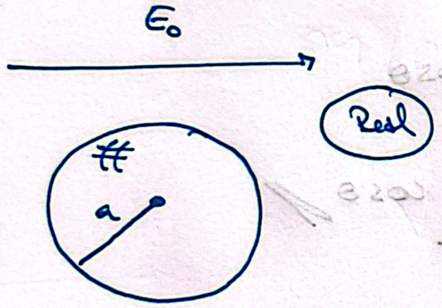
→ finalmente : $V(\vec r) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|r\hat n - r_q\hat n'|} - \frac{a q/r_q}{|r\hat n - \frac{a^2}{r_q}\hat n'|}\right]$

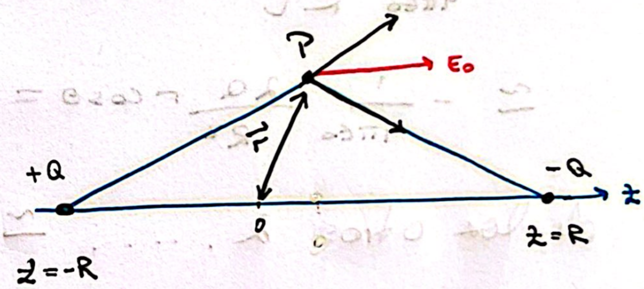
→ Distribución superficial de carga

$$\sigma = -\epsilon_0\, \partial_r V\Big|_{r=a} = -\frac{q}{4\pi a^2}\left(\frac{a}{r_q}\right)\frac{1-a^2/r_q^2}{\left(1+\frac{a^2}{r_q^2}-\frac{2a}{r_q}\cos\gamma\right)^{3/2}}$$

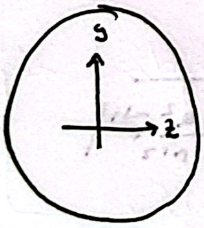
→ Fuerza entre q y carga superficial inducida!
(Coulomb y dF integrando σ)

Esfera conductora en presencia de un campo eléctrico homogéneo





$$\vec E = \frac{1}{4\pi\epsilon_0}\left[\frac{Q\,(\vec r-\vec r_Q)}{|\vec r-\vec r_Q|^3} - \frac{Q\,(\vec r-\vec r_{-Q})}{|\vec r-\vec r_{-Q}|^3}\right];$$



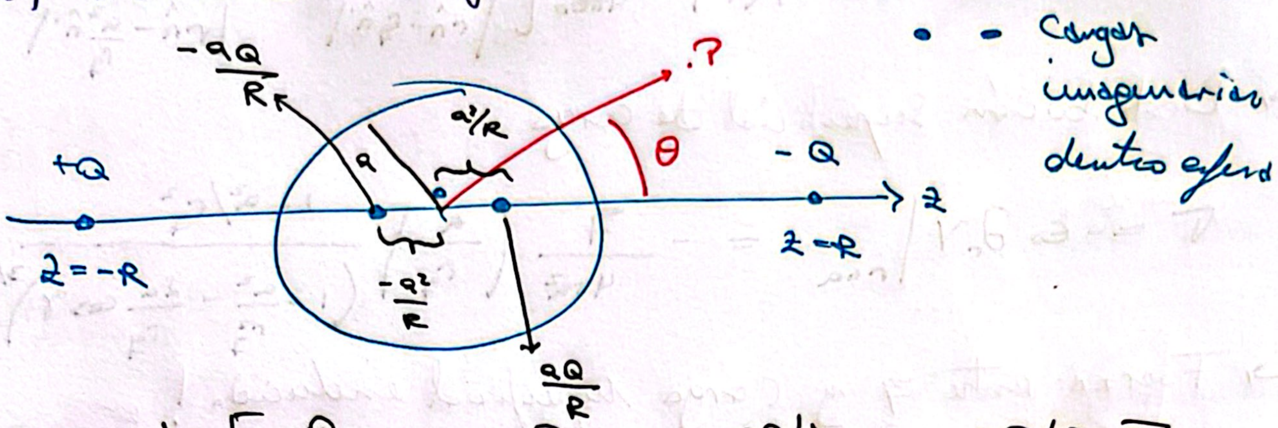
$\vec r = y\hat j$; $\vec r_Q = -R\hat k$; $\vec r_{-Q} = R\hat k$; $|\vec r-\vec r_Q| = \sqrt{y^2+R^2} = |\vec r-\vec r_{-Q}|$

$$\therefore \vec E = \frac{Q}{4\pi\epsilon_0}\left[\frac{\cancel{y\hat j}+R\hat k}{(\sqrt{y^2+R^2})^3} - \frac{\cancel{y\hat j}-R\hat k}{(\sqrt{y^2+R^2})^3}\right]$$

$\Rightarrow \vec E = \frac{Q}{4\pi\epsilon_0}\frac{2R\hat k}{(y^2+R^2)^{3/2}}$ lím $R\to\infty$, $Q\to\infty$ Manteniendo Q/R^2 cte

$$E_0 \simeq \frac{1}{4\pi\epsilon_0}\frac{2Q}{R^2}$$

(ii) Insertamos la esfera conductora



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|\vec{r}+R\hat{k}|} - \frac{Q}{|\vec{r}-R\hat{k}|} - \frac{aQ/R}{|\vec{r}+\frac{a^2}{R}\hat{k}|} + \frac{aQ/R}{|\vec{r}-\frac{a^2}{R}\hat{k}|}\right]$$

con los 2 primeros

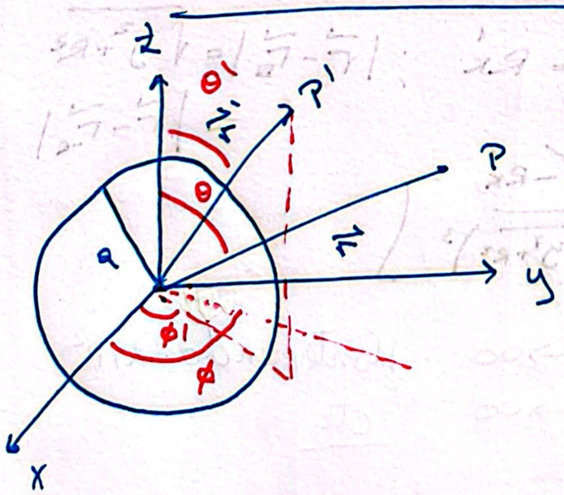
$$\simeq \frac{Q}{4\pi\epsilon_0}\frac{1}{R}\left[1-\frac{1}{2}\left(\frac{2r}{R}\right)\cos\theta+\ldots-\left(1+\frac{1}{2}\left(\frac{2r}{R}\right)\cos\theta+\ldots\right)\right]$$

$$\simeq -\frac{1}{4\pi\epsilon_0}\frac{2Q}{R^2}r\cos\theta = -E_0 r\cos\theta$$

de los otros 2 ..... $\simeq -E_0\frac{a^3}{r^2}\cos\theta$ //

luego $V(r) \simeq -E_0\left(r-\frac{a^3}{r^2}\right)\cos\theta$ //

Función de Green



$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{a}{r'|\vec{r}-\frac{a^2}{r'^2}\vec{r}'|}$$

$$G(\vec{r},\vec{r}') = \frac{1}{(r^2+r'^2-2rr'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{r^2r'^2}{a^2}+a^2-2rr'\cos\gamma\right)^{1/2}}$$

* $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$