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The application of Weierstrass elliptic functions to Schwarzschild null geodesics

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Abstract

In this paper, we focus on analytical calculations involving null geodesics in some spherically symmetric spacetimes. We use Weierstrass elliptic functions to fully describe null geodesics in Schwarzschild spacetime and to derive analytical formulae connecting the values of radial distance at different points along the geodesic. We then study the properties of light triangles in Schwarzschild spacetime and give the expansion of the deflection angle to the second order in both M/r_0 and M/b where M is the mass of the black hole, r_0 the distance of the closest approach of the light ray and b the impact parameter. We also use the Weierstrass function formalism to analyze other more exotic cases such as Reissner–Nordström null geodesics and Schwarzschild null geodesics in four and six spatial dimensions. Finally we apply Weierstrass functions to describe the null geodesics in the Ellis wormhole spacetime and give an analytic expansion of the deflection angle in M/b .

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1. Introduction

Geodesics in Schwarzschild spacetime have been studied for a long time and the importance of a good understanding of their behavior is clear. In this paper, we shall focus on analytical calculations involving null geodesics. While these are interesting in their own right, calculations like this are also important for experiments testing general relativity to high levels of accuracy. The examples of two such proposed experiments are the ‘Laser Astrometric Test of Relativity’ (LATOR) and ‘Beyond Einstein Advanced Coherent Optical Network’ (BEACON), both of which use paths of light rays to verify general relativity and are described in detail in [1]. Both are intended to measure second-order effects in light bending. Elliptic functions have been used to describe the geodesics in Schwarzschild spacetime before, mainly in [2] and more recently in [3, 4]. In [2, 4], the focus is mainly on the paths of massive particles and even though they mention the possibility of using Weierstrass functions in the null case, they do not

go into much detail. The discussion in [3] is concerned with null geodesics around a charged neutron star using the Reissner–Nordström metric, a case we also study but with a different emphasis. In this paper, we begin by providing a complete description of Schwarzschild null geodesics in terms of Weierstrass functions and then—this is our principal innovation—using various ‘addition formulae’ for Weierstrass functions [5], we derive some analytical formulae connecting the values of radial distance at different points along the geodesic. The motivation is to develop, as far as possible, optical trigonometry in the presence of a gravitating object such as a star or a black hole. To that end we use these addition formulae to study the properties of light triangles in the Schwarzschild metric and obtain the deflection angle of the scattering geodesics to second order in both M/r_0 and M/b where M is the mass of the black hole, r_0 the distance of the closest approach of the light ray and b the impact parameter.

In the final section, we show how to use the same methods to treat null geodesics in more exotic spacetimes: charged black hole, the Ellis wormhole [6] and Schwarzschild black holes in four and six spatial dimensions. Although not a primary concern of this paper, it is worth remarking that the addition formulae for Weierstrass functions that we make use of are closely related to the existence of an Abelian group multiplication law on any elliptic curve [7] and suggest, in view of the importance of the complex black hole spacetimes at the quantum level, that it might prove fruitful to explore this aspect of the theory further.

The paper is organized as follows. In section 2, we provide the full solution for Schwarzschild null geodesics in terms of Weierstrass elliptic functions and apply it to obtain addition formulae connecting three points on the geodesic. We then calculate the deflection angle of the scattering geodesics to second order in both M/r_0 and M/b , where M is the mass of the black hole, r_0 the distance of the closest approach of the light ray and b the impact parameter. The section is concluded with the discussion of the light triangles and Gauss–Bonnet theorem. In section 3, we apply the Weierstrass function formalism to further examples such as Reissner–Nordström null geodesics and Schwarzschild geodesics in higher spatial dimensions. At the end of the section, we give a detailed description of the Ellis wormhole null geodesics.

2. Schwarzschild null geodesics

The equation obeyed by a null geodesic $r(\phi)$ in the Schwarzschild metric is

$$\left(\frac{dr}{d\phi}\right)^2 = Pr^4 - r^2 + 2Mr, \quad (1)$$

where $P = E^2/L^2 = 1/b^2$. Here E is the energy of the light, L is the angular momentum and b is the impact parameter. Interestingly, the same equation arises for a null geodesic $r(\phi)$ in the Schwarzschild–de Sitter or Kottler metric [8, 9] and many of our results remain valid in that case. Geometrically, one may regard the solutions of (1) as unparameterized geodesics of the optical metric

$$ds_o^2 = \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} + \frac{r^2}{1 - \frac{2M}{r}} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2)$$

with $\theta = \frac{\pi}{2}$. Introducing the isotropic coordinate $\rho = \frac{1}{2}(r - M) + \frac{1}{2}\sqrt{r(r - 2M)}$, we find that

$$ds_o^2 = n^2(\rho) \{d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2)\}, \quad (3)$$

where

$$n(\rho) = \frac{\left(1 + \frac{M}{2\rho}\right)^3}{\left(1 - \frac{M}{2\rho}\right)}. \quad (4)$$

Thus, our results also apply to light rays moving in an isotropic but inhomogeneous optical medium in flat space with the refractive index $n(\rho)$.

Another interpretation of (1), recently exploited in [10], is provided by substituting $r = \frac{1}{u}$ and differentiating to obtain

$$\frac{d^2 u}{d\phi^2} + u = \frac{1}{h^2 u^2} F(u), \quad (5)$$

with

$$F(u) = 3Mh^2 u^4. \quad (6)$$

Now (5) is the equation governing the motion of a non-relativistic particle of angular momentum per unit mass h moving under the influence of a central force $F(u)$. In our case, the effective force $F(u)$ is attractive and varies inversely as the fourth power of the distance. A search of the voluminous 19th century literature on such problems reveals that it was comparatively well known that although this problem admits some simple exact solutions, which we shall detail below, the general solution requires elliptic functions.

If we had adopted isotropic coordinates and substituted $\rho = \frac{1}{u}$, we would have obtained a very different formula for $F(u)$. In fact in that case we would have

$$F(u) = 2Mu^2 \frac{\left(1 + \frac{Mu}{2}\right)^5 \left(1 - \frac{Mu}{4}\right)}{\left(1 - \frac{Mu}{2}\right)^3}. \quad (7)$$

As we shall see in detail in a later section, the null geodesics of neutral Tangherlini black holes in D spacetime dimensions correspond, in Schwarzschild coordinates, to the motion of a non-relativistic particle with a force $F(u) \propto r^{-D}$. The cases $D = 4, 5, 7$ are the only cases known to be integrable in terms of elliptic functions. In fact the cases $D = 4$ and $D = 7$ may be related by a conformal mapping introduced in this context by Bohlin [14] and elaborated upon by Arnold [15]. The Bohlin–Arnold mapping is a type of duality, i.e. it is involutive, and the case $D = 5$ is self-dual.

2.1. Weierstrass function solution

Substituting $y = M/2r - 1/12$ into (1) gives

$$(y')^2 = 4y^3 - \frac{1}{12}y - g_3, \quad (8)$$

where

$$g_3 = \frac{1}{216} - \left(\frac{M}{2}\right)^2 P. \quad (9)$$

In the case $g_3 \neq \pm 1/216$, the general solution to this equation is $y(\phi) = \wp(\phi + C)$ where $\wp(z)$ is the Weierstrass elliptic function and $C = \text{const}$. A detailed description of these functions together with the proof of the above statement can be found in [5] (pp 429–44 and p 484). For the critical values of g_3 , the equation for r can be integrated to give

$$r(\phi) = M(1 + \cos(\phi)) \quad (10)$$

in the case $g_3 = 1/216$, and

$$\frac{M}{r(\phi)} = \frac{1}{3} - \frac{1}{1 \pm \cosh(\phi)} \quad (11)$$

in the case $g_3 = -1/216$. The former, geometrically a *cardioid* in the (r, ϕ) coordinates, starts at the singularity, reaches the horizon from below and then returns back. The latter describes two types of trajectories, one starting at infinity, the other at the singularity and both approaching the photon sphere, never reaching it.

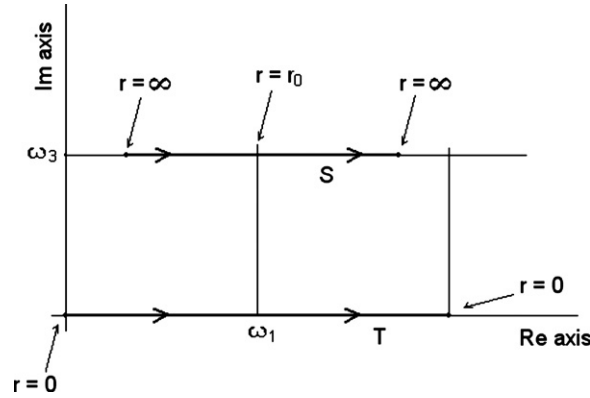


Figure 1. The argument of \wp in the complex plane, S corresponds to scattering trajectories and T to the trapped ones.

Now suppose that $g_3 \neq \pm 1/216$ and $M^2 P < 1/27$. Then the polynomial $4y^3 - y/12 - g_3$ has three real roots $e_1 > e_2 > e_3$ and the half-periods of the corresponding Weierstrass function \wp are

$$\omega_1 = \int_{e_1}^{\infty} \frac{dt}{\sqrt{4t^3 - t/12 - g_3}}, \quad (12)$$

$$\omega_3 = -i \int_{-\infty}^{e_3} \frac{dt}{\sqrt{g_3 + t/12 - 4t^3}}, \quad (13)$$

where $\omega_1 \in \mathbb{R}$ and $i\omega_3 \in \mathbb{R}$. In this case, $\wp(z)$ is real on a rectangular grid with vertices $0, \omega_1, \omega_3, \omega_1 + \omega_3$ and since $y(\phi)$ is real, the only physical solutions to (2) are $y(\phi) = \wp(\phi + \phi_0)$ or $y(\phi) = \wp(\phi + \phi_0 + \omega_3)$ where $\phi_0 \in \mathbb{R}$. We have the following two cases.

(i) Scattering paths: the point $r = \infty$ corresponds to $y = -1/12$ and $\wp(z)$ takes the value $-1/12$ at z such that $\text{Im}(z) = \omega_3$. Therefore, choosing the line $\phi = 0$ to be the axis of symmetry, we have $y(\phi) = \wp(\phi + \omega_1 + \omega_3)$ and so

$$\frac{M}{r(\phi)} = \frac{1}{6} + 2\wp(\phi + \omega_1 + \omega_3), \quad (14)$$

where of course the function \wp depends on P . Here the range of ϕ is $[-\beta, \beta]$ where

$$\beta = \omega_1 - \int_{e_3}^{-1/12} \frac{dt}{\sqrt{4t^3 - t/12 - g_3}}. \quad (15)$$

In this notation, the angle of deflection $\delta\phi$ is $\delta\phi = 2\beta - \pi$.

(ii) Trapped paths: these begin and end at the singularity and $r = 0$ corresponds to $y = \infty$. So choosing the line $\phi = 0$ to be the axis of symmetry once again gives $y(\phi) = \wp(\phi + \omega_1)$ where $\phi \in [-\omega_1, \omega_1]$. So in this case

$$\frac{M}{r(\phi)} = \frac{1}{6} + 2\wp(\phi + \omega_1). \quad (16)$$

Figure 1 shows the argument of \wp in the scattering and trapped cases.

Now suppose that $M^2 P > 1/27$. Then we have

(iii) Absorbed paths: these go from infinity to $r = 0$ or from $r = 0$ to infinity and therefore the solution is uniquely determined by P . There is only one real root of the rhs of the Weierstrass equation, $e_1 < -1/12$ and ω_1 defined as before is again a half-period. For each P ,

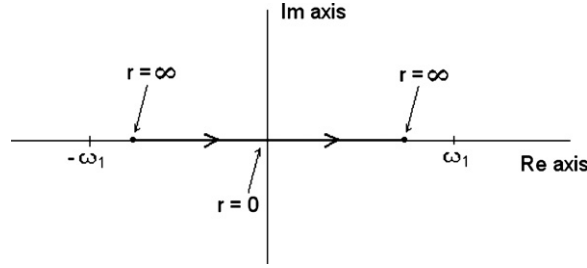


Figure 2. The argument of \wp in the complex plane corresponding to absorbed trajectories.

there is a solution of the form $y(\phi) = \wp(\phi + \phi_0)$, $\phi_0 \in \mathbb{R}$, and so by uniqueness, all physical solutions are of this form.

We can take $\phi_0 = 0$ which means defining the line $\phi = 0$ by the direction in which the path leaves/hits $r = 0$. Then the range of ϕ is $[-\alpha, \alpha]$ where

$$\alpha = \int_{-1/12}^{\infty} \frac{dt}{\sqrt{4t^3 - t/12 - g_3}}, \quad (17)$$

and the solution is

$$\frac{M}{r(\phi)} = \frac{1}{6} + 2\wp(\phi). \quad (18)$$

A diagram of the complex plane corresponding to absorbed trajectories may be found in figure 2.

2.2. Addition formulae

As shown in [5], p 440, Weierstrass functions satisfy an addition formula of the form

$$\wp(x+y) = \frac{1}{4} \left[\frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)} \right]^2 - \wp(x) - \wp(y) \equiv F(\wp(x), \wp(y)), \quad (19)$$

where

$$F(x, y) = \frac{1}{4} \left[\frac{\sqrt{4x^3 - x/12 - g_3} - \sqrt{4y^3 - y/12 - g_3}}{x - y} \right]^2 - x - y. \quad (20)$$

We can apply this result to null geodesics to obtain an expression for $r(\phi_1 + \phi_2)$ as a function of $r(\phi_1)$ and $r(\phi_2)$. Of course, if any such formula is useful in some experimental setup, we need to be able to easily find the line $\phi = 0$. Also, because of the additive constant in the argument of the Weierstrass function, we cannot apply the addition formula directly because the sum of the two arguments will not correspond to the sum of the two angles. Fortunately, in the case of the scattering and trapped orbits, choosing the line $\phi = 0$ to be the axis of symmetry takes care of both problems. Take the scattering orbit for example. The axis of symmetry is easy to find, and we can apply the addition formula for \wp to three points on the orbit $y_1 = y(\phi_1)$, $y_2 = y(\phi_2)$ and $y_3 = y(\phi_3)$ as

$$y\left(\sum_{i=1}^3 \phi_i\right) = \wp\left(\sum_{i=1}^3 \phi_i + 3\omega_1 + 3\omega_3\right) = F(F(y_1, y_2), y_3) \quad (21)$$

which works because $2\omega_1$ and $2\omega_3$ are periods of \wp .

Now, letting $\phi_3 = 0$ gives $y_3 = e_2$ with e_2 directly related to the distance of the closest approach d_{\min} as $e_2 = M/2d_{\min} - 1/12$. Then we obtain an addition formula for three points on the orbit in the form

$$\frac{M}{2r(\phi_1 + \phi_2)} = \frac{1}{12} + F\left(F\left(\frac{M}{2r(\phi_1)} - \frac{1}{12}, \frac{M}{2r(\phi_2)} - \frac{1}{12}\right), e_2\right). \quad (22)$$

The same procedure for trapped orbits gives the same formula only with e_1 instead of e_2 where e_1 is related to the maximal attained distance d_{\max} by $e_1 = M/2d_{\max} - 1/12$.

In the absorbed case, the lack of additive constant in the argument of the Weierstrass function means that we can apply the addition formula directly to obtain algebraically the simpler result

$$\frac{M}{2r(\phi_1 + \phi_2)} = \frac{1}{12} + F\left(\frac{M}{2r(\phi_1)} - \frac{1}{12}, \frac{M}{2r(\phi_2)} - \frac{1}{12}\right). \quad (23)$$

In this case, we can use the Euclidean angle between the direction of the ray and the ϕ -direction ψ which satisfies

$$\tan \psi = \frac{1}{r} \frac{dr}{d\phi}. \quad (24)$$

Then the addition formula can be written as

$$\begin{aligned} & \left(\frac{1}{r(\phi_1)} - \frac{1}{r(\phi_2)}\right)^2 \left(2M\left(\frac{1}{r(\phi_1)} + \frac{1}{r(\phi_2)} + \frac{1}{r(\phi_1 + \phi_2)}\right) - 1\right) \\ &= \left(\frac{\tan \psi(\phi_1)}{r(\phi_1)} - \frac{\tan \psi(\phi_2)}{r(\phi_2)}\right)^2. \end{aligned} \quad (25)$$

However, in this case this is not very useful since it is practically impossible to identify the line $\phi = 0$ for such a choice. We could of course make a different choice, like $r(\phi = 0) = R$ for some chosen R , but then the obtained addition formula would not be analytic anymore because we would need to find the corresponding additive constant ϕ_0 given by the integral

$$\phi_0 = \int_{\frac{M}{2R} - \frac{1}{12}}^{\infty} \frac{dt}{\sqrt{4t^3 - t/12 - g_3}}. \quad (26)$$

2.3. The deflection angle

We start from the equation for $u = 2M/r$ which is

$$\left(\frac{du}{d\phi}\right)^2 = u^3 - u^2 + 4M^2P = u^3 - u^2 - \mu^3 + \mu^2, \quad (27)$$

where $\mu = 2M/r_0$, r_0 is the distance of the closest approach, for scattering orbits. Then the deflection angle $\delta\phi$ is given by

$$\delta\phi = 2 \int_0^\mu \frac{du}{\sqrt{u^3 - u^2 - \mu^3 + \mu^2}} - \pi = 2I - \pi. \quad (28)$$

The integral can be rewritten in terms of $x = u/\mu$ which gives

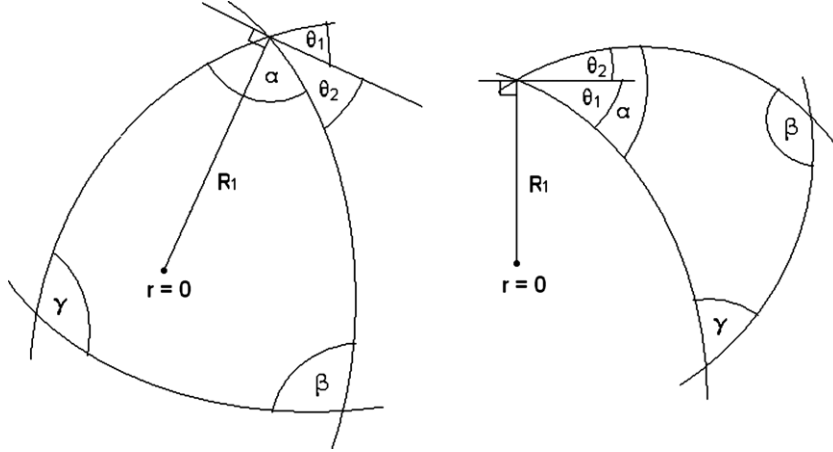
$$I(\mu) = \int_0^1 \frac{dx}{\sqrt{(1-x^2) - (1-x^3)\mu}}. \quad (29)$$

This integral can be expanded in the powers of μ , for μ sufficiently small, as

$$I = \sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} \left(\int_0^1 \left(\frac{1-x^3}{1-x^2} \right)^n \frac{1}{\sqrt{1-x^2}} dx \right) \mu^n. \quad (30)$$

We are only interested in small values of μ and this expansion clearly converges at least for $\mu < 2/3$ since $(1-x^3)/(1-x^2) < 3/2$ for $x \in (0, 1)$. Calculating the first three terms in this expansion results in an expansion for the deflection angle (substituting for μ),

$$\delta\phi = \frac{4M}{r_0} + 3 \left(\frac{5\pi}{4} - \frac{4}{3} \right) \frac{M^2}{r_0^2} + O(r_0^{-3}). \quad (31)$$

**Figure 3.** Light triangles.

Now, we have $P = 1/b^2$ and so we define $v = 2M/b$. Then

$$v^2 = \mu^2 - \mu^3, \quad (32)$$

and working to the second order gives

$$\mu = v + \frac{1}{2}\mu^2. \quad (33)$$

Substituting into the expansion for $\delta\phi$ then gives

$$\delta\phi = 2v + \frac{15\pi}{16}v^2 + O(v^3) = \frac{4M}{b} + \frac{15\pi}{4} \frac{M^2}{b^2} + O(b^{-3}), \quad (34)$$

which is the expansion of the deflection angle to second order in $1/b$.

2.4. Angular sum in light triangles

Because the Gauss curvature of the optical metric restricted to the equatorial plane is negative, the angular sum of a triangle made up of geodesics must be less than π unless the triangle encloses the horizon [16]. One might hope to get a more precise statement using the addition formulae. To this end, let Θ be the physical angle between the direction of the light and the ϕ -direction. Then

$$\tan \Theta = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \frac{1}{r} \frac{dr}{d\phi} = \sqrt{\frac{Pr^3 - r + 2M}{r - 2M}}. \quad (35)$$

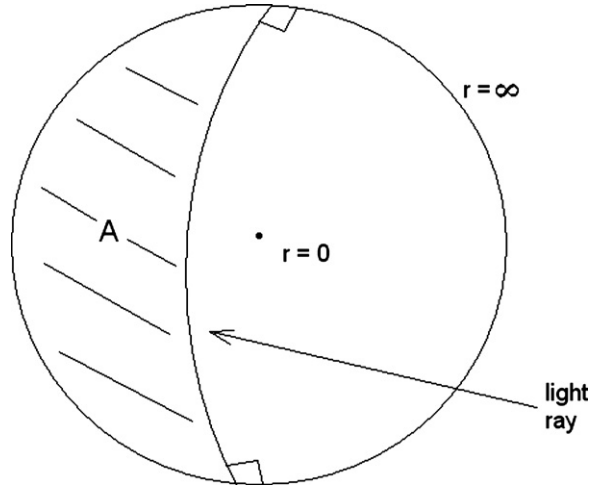
Note that this formula is valid for the Schwarzschild solution but not the Kottler solution with a non-vanishing cosmological constant [9].

Now consider three light rays, forming a triangle around the origin with P_1, P_2 and P_3 and vertices at the radial coordinates R_1, R_2 and R_3 . To simplify the notation, we define

$$W_{ij} = \sqrt{\frac{P_i R_j^3 - R_j + 2M}{R_j - 2M}}. \quad (36)$$

Then from figure 3, it is clear that

$$\alpha = \pi - \tan^{-1} W_{11} - \tan^{-1} W_{21}, \quad (37)$$

**Figure 4.** Scattering light ray.

$$\beta = \pi - \tan^{-1} W_{22} - \tan^{-1} W_{32}, \quad (38)$$

$$\gamma = \pi - \tan^{-1} W_{13} - \tan^{-1} W_{33}. \quad (39)$$

Alternatively, if the origin is not inside of the triangle, then from figure 3 it follows that

$$\alpha = \tan^{-1} W_{11} + \tan^{-1} W_{21}, \quad (40)$$

$$\beta = \pi - \tan^{-1} W_{22} - \tan^{-1} W_{32}, \quad (41)$$

$$\gamma = \tan^{-1} W_{13} + \tan^{-1} W_{33}. \quad (42)$$

Further analytical work in this general case does not seem to lead anywhere, because the distances R_1 , R_2 and R_3 are not independent, but finding a formula for the relation between them is impossible. We can however consider a symmetric case with all R s and P s equal. Then its angles are given by

$$\alpha = \pi - 2 \tan^{-1} \sqrt{\frac{PR^3 - R + 2M}{R - 2M}}. \quad (43)$$

2.5. Gauss–Bonnet theorem

An alternative approach to finding the angular deflection is using the Gauss–Bonnet theorem [16]. Consider the setup in figure 4. Then by the Gauss–Bonnet theorem, we have

$$\alpha + \pi + \int_A K \, dA = 2\pi. \quad (44)$$

One of the ways to calculate this is to transform the optical metric into the form

$$ds^2 = d\rho^2 + C(\rho)^2 d\phi^2. \quad (45)$$

Then

$$K \, dA = -\frac{d^2 C}{d\rho^2} d\rho \, d\phi, \quad (46)$$

and so

$$\int_A K \, dA = \int_{-\alpha/2}^{\alpha/2} \left[-\frac{dC}{d\rho} \Big|_{r=\infty} + \frac{dC}{d\rho} \Big|_{r=r(\phi)} \right] d\phi. \quad (47)$$

Now,

$$\frac{dC}{d\rho} = \frac{dC}{dr} \frac{dr}{d\rho} = \frac{r-M}{\sqrt{r}\sqrt{r-2M}}. \quad (48)$$

Therefore, we obtain

$$\int_{-\alpha/2}^{\alpha/2} \frac{r(\phi)-M}{\sqrt{r(\phi)}\sqrt{r(\phi)-2M}} d\phi = \pi, \quad (49)$$

which holds for any scattering path. It does not seem to be very useful when it comes to evaluating α but it is an interesting expression. Rewriting this in terms of r gives another interesting identity

$$\int_{r_0}^{\infty} \frac{r-M}{\sqrt{r}\sqrt{r-2M}\sqrt{Pr^4-r^2+2Mr}} dr = \frac{\pi}{2}, \quad (50)$$

where r_0 is the distance of the closest approach.

3. Further applications of Weierstrass functions

3.1. Reissner–Nordström null geodesics

As mentioned in the introduction, these geodesics have been studied previously using Weierstrass functions in [3, 11, 12]. In this case, the relevant equation for $u = 1/r$ is

$$\left(\frac{du}{d\phi} \right)^2 = P - u^2 + 2Mu^3 - Q^2u^4. \quad (51)$$

One may verify using the formulae in [9] or directly that as in the case of the Schwarzschild–de Sitter metrics, with the Reissner–Nordström metrics the cosmological constant does occur in (51). Thus, some of the results in [13], which appeared on the archive subsequently to the first version of this paper, follow directly from the work of the present section.

The rhs of (51) always has a real root so let x_0 be 1. Then define $s = u - x_0$. This gives

$$(s')^2 = As + Bs^2 + Cs^3 + Ds^4, \quad (52)$$

where

$$A = 6x_0^2M - 2x_0 - 4x_0^3Q^2, \quad (53)$$

$$B = 6x_0M - 1 - 6x_0^2Q^2, \quad (54)$$

$$C = 2M - 4x_0Q^2, \quad (55)$$

$$D = -Q^2. \quad (56)$$

Now the substitution $\psi = 1/s$ takes it into the form

$$(\psi')^2 = A\psi^3 + B\psi^2 + C\psi + D, \quad (57)$$

and finally setting $\psi = 4y/A - B/3A$ gives

$$(y')^2 = 4y^3 - g_2y - g_3, \quad (58)$$

where

$$g_2 = \frac{B^2}{12} - \frac{AC}{4}, \quad (59)$$

$$g_3 = \frac{ABC}{48} - \frac{A^2D}{16} - \frac{B^3}{216}. \quad (60)$$

Therefore, this time the solution will be given by

$$\frac{1}{r(\phi)} = x_0 + \frac{3A}{12\wp(\phi + \xi_0) - B}, \quad (61)$$

where ξ_0 is a complex constant. However, the more complicated relation between r , \wp and also many different constants make it algebraically very challenging to analyze the situation any further and find a suitable ξ_0 or addition formula similar to the Schwarzschild case. For that purpose, consider the equation for $r(\lambda)$ where λ is an affine parameter of the path. This equation is

$$\frac{1}{L^2} \left(\frac{dr}{d\lambda} \right)^2 = P - \left(\frac{1}{r^2} - \frac{2M}{r^3} + \frac{Q^2}{r^4} \right) \equiv f(r). \quad (62)$$

Now the motion is only possible in regions where $f(r) > 0$. If these split into two disconnected ones, then it must be the case in which the rhs of the corresponding Weierstrass equation has three real roots and we have scattering and trapped paths. If there is only one such region, we know we have the case where the above-mentioned rhs has only one real root and we have absorbed paths.

First, the roots of $f'(r)$ are

$$r_{\pm} = \frac{3M \pm \sqrt{9M^2 - 8Q^2}}{2}. \quad (63)$$

Physically we want $M^2 > Q^2$ and so $r_{\pm} \in \mathbb{R}^+$. Clearly the regions where $f(r) > 0$ will be disconnected (and there will be 2) if $P > 0$ and $f(r_+) < 0$. Suppose that this is the case and consider scattering paths. Let a be the distance of the closest approach. Then $a = e_i$ and we can write $y(\phi) = \wp(\phi + \omega_i)$ for some $i \in \{1, 2, 3\}$ because the path is symmetric. As before, this corresponds to choosing the line $\phi = 0$ to be the axis of symmetry. We can compute a as the largest root of $f(r) = 0$ and so we obtain an addition formula

$$\frac{1}{12} \left(B + \frac{3A}{\frac{1}{r(\phi_1 + \phi_2)} - x_0} \right) = F \left(F \left(\frac{1}{12} \left(B + \frac{3A}{\frac{1}{r(\phi_1)} - x_0} \right), \frac{1}{12} \left(B + \frac{3A}{\frac{1}{r(\phi_2)} - x_0} \right) \right), a \right).$$

For trapped orbits, the same addition formula applies, only in that case a is the largest attained distance and is given by the second largest root of $f(r) = 0$.

If $f(r_+) > 0$, then we have orbits that go in, miss the singularity and continue to another asymptotically flat region of spacetime. These satisfy the same addition formula like the scattering ones. In the case $P = 0$, the equation can be integrated and the solution is

$$\frac{r}{\sqrt{2Mr - Q^2 - r^2}} = \arctan(\phi - \phi_0), \quad (64)$$

where ϕ_0 is the constant of integration. Another special solution can be found when

$$P = \frac{r_+^2 - Q^2}{3r_+^4}, \quad (65)$$

which is equivalent to $f(r_+) = 0$ and corresponds to the situation when the two periods of the corresponding Weierstrass function become linearly dependent. This leads to a pair of solutions

$$r(\phi) = \frac{4c e^{\sqrt{c}\phi}}{-2b e^{\sqrt{c}\phi} \pm (1 + (b^2 - 4ac) e^{2\sqrt{c}\phi})} - r_+, \quad (66)$$

where

$$a = \frac{r_+^2 - Q^2}{3r_+^4}, \quad (67)$$

$$b = 4 \frac{r_+^2 - Q^2}{3r_+^3}, \quad (68)$$

$$c = 2 \frac{r_+^2 - Q^2}{r_+^2} - 1. \quad (69)$$

3.2. 5D Schwarzschild null geodesics

Here by 5D it is meant four spatial dimensions. The relevant equation for $u = 1/r$ in this case is

$$(u')^2 = 2Mu^4 - u^2 + P, \quad (70)$$

where M is proportional to the five-dimensional mass. There is an interesting self-duality here, which is in fact a special case of Bohlin–Arnold duality, in which when we write the equation in terms of r , we obtain

$$(r')^2 = Pr^4 - r^2 + 2M, \quad (71)$$

which is exactly the same as (70) with the constants interchanged. First consider the equation for u . The substitution $u^2 = \frac{1}{2M}(y + \frac{1}{3})$ will take it into a form

$$(y')^2 = 4y^3 - g_2y - g_3, \quad (72)$$

where

$$g_2 = \frac{4}{3} - 8MP, \quad (73)$$

$$g_3 = \frac{8}{3}(\frac{1}{9} - MP). \quad (74)$$

As usual, the rhs of (72) has three real roots if $g_2 > 0$ and $g_3^2 < (g_2/3)^3$. The first condition is $MP < 1/6$, while the second is $8(MP)^3 - (MP)^2 < 0$. Therefore, we have four real roots if $MP < 1/8$. Also note that the rhs of equation (32) always has root $-1/3$ and expanding it into a power series around this point quickly shows that in fact $e_3 = -1/3$. Finally, the point $y = -1/3$ corresponds to $r = \infty$. Hence in this case, with ω_1, ω_3 defined as before, we have two classes of solutions, depending on the initial conditions, scattering or trapped. The case of trapped paths is exactly the same as before, with the same addition formula for y and $y(\phi) = \wp(\phi + \omega_1)$. However, the scattering case is more interesting in 5D. This is because now the point ω_3 in the \mathbb{C} -plane corresponds to $r = \infty$ and so we can write the solution as $y(\phi) = \wp(\phi + \omega_3)$ where the line $\phi = 0$ is in the direction of the ray incoming from ∞ and $\phi \in [0, 2\omega_1]$.

Things are even simpler when we solve the equation for r directly. The substitution $r^2 = \frac{1}{P}(y + \frac{1}{3})$ takes it into equation (32) but now the difference is that point 0 corresponds to $r = \infty$ and so we can write the (scattering) solution simply as

$$r(\phi) = \frac{1}{\sqrt{P}} \sqrt{\wp(\phi) + \frac{1}{3}}, \quad (75)$$

and the addition formula in this case is simply

$$r(\phi_1 + \phi_2) = \frac{1}{\sqrt{P}} \sqrt{F\left(P(r(\phi_1))^2 - \frac{1}{3}, P(r(\phi_2))^2 - \frac{1}{3}\right) + \frac{1}{3}}. \quad (76)$$

Finally, if $MP > 1/8$, then we have only one root of the rhs of the Weierstrass equation and thus absorbing paths for which the solution is

$$\frac{1}{(r(\phi))^2} = \frac{1}{2} \left(\wp(\phi + \omega_1) + \frac{1}{3} \right), \quad (77)$$

where again the line $\phi = 0$ is given by the direction of the ray incoming from ∞ .

As before, we can obtain several special solutions by imposing $g_2^3 = 27g_3^2$ which in this case gives $MP = 0$ or $MP = 1/8$. In the case $MP = 0$, we obtain a special *circular* solution,

$$r(\phi) = \sqrt{2M} \cos \phi, \quad (78)$$

while in the case $MP = 1/8$, we obtain

$$r(\phi) = 2\sqrt{M}(\tanh(\phi/\sqrt{2}))^{\pm 1}. \quad (79)$$

3.3. Duality and 7D Schwarzschild null geodesics

By the Bohlin–Arnold duality [15], if we have a particle moving in the Newtonian potential $V \propto r^{2p-2}$ and following the trajectory $r(\phi) = f(\phi)$, then there will be a particle with accordingly modified energy moving in the potential $\hat{V} \propto r^{\frac{2-2p}{p}}$ following the trajectory $r(\phi) = f(\phi)^p$. In this case, if $V = -kr^{2p-2}$ and the particle has the energy E , then $\hat{V} = -Er^{\frac{2-2p}{p}}$ and $\hat{E} = k$.

Null geodesics in the $(n+1)$ D Schwarzschild geometry correspond to Newtonian motion in an r^{-n} potential and so the duality applies to these geodesics as well. As we already saw, 5D corresponds to the case $p = -1$ and is self-dual. A quick check reveals that the case $p = -1/2$ gives a duality between null geodesics in 7D and 4D.

Given the potential $V = -kr^{-n}$, Newton's equation of motion is

$$(r')^2 = \frac{2E}{L^2}r^4 - r^2 + \frac{2k}{L^2}r^{4-n}. \quad (80)$$

The equation for null geodesics in 4D is

$$(r')^2 = Pr^4 - r^2 + 2Mr, \quad (81)$$

and in 7D it is

$$(r')^2 = Pr^4 - r^2 + 2Mr^{-2}. \quad (82)$$

So under the duality with $p = -1/2$, we have $E \leftrightarrow k$ and thus $P \leftrightarrow 2M$. Therefore, if we have a 4D black hole with mass M and light with $(E/L)^2 = P$ following the trajectory $r(\phi)$ and a 7D black hole with mass $P/2G$ and light with $(E/L)^2 = 2M$ following the trajectory $r = f(\phi)$, then

$$r(\phi) = \left(\frac{1}{f(\phi)} \right)^2. \quad (83)$$

Making the substitution $r^2 = y/P + 1/(3P)$ in the equation for r in the 7D case takes it into the Weierstrass form with $g_2 = 4/(3MP^2)$ and

$g_3 = 8/27 - 8M_6P^2$. Therefore, the orbits in 7D satisfy

$$r(\phi) = \frac{1}{\sqrt{P}} \sqrt{\wp(\phi + C) + \frac{1}{3}}. \quad (84)$$

In this case, the formula for scattering paths looks especially simple, which is

$$r(\phi) = \frac{L}{E} \sqrt{\wp(\phi) + \frac{1}{3}}. \quad (85)$$

By the Bohlin–Arnold duality, the special solutions in 7D corresponding to the special solutions in 4D given by $P = 1/27$ have

$$P = \frac{2}{\sqrt{54M}}, \quad (86)$$

where M is proportional to the mass of the 7D black hole. The corresponding special solutions thus are

$$r(\phi) = \sqrt[4]{54M} \sqrt{\frac{1}{3} - \frac{1}{1 \pm \cosh(\phi)}}. \quad (87)$$

3.4. Ellis wormhole null geodesics

3.4.1. Qualitative description. The Ellis wormhole is an ultrastatic solution of the Einstein equations coupled to a massless scalar field. While not necessarily physically very realistic, it has been used in the studies of gravitational lensing [6]. It has the metric

$$ds^2 = -dt^2 + dr^2 + r(r - 2M)(d\theta^2 + \sin^2 \theta d\phi^2). \quad (88)$$

Because $g_{00} = -1$, the physical spatial metric and the optical spatial metric coincide. Setting $t = 0$, $\theta = \frac{\pi}{2}$ gives the optical metric on the equatorial plane.

If we set $\sqrt{x^2 + y^2} = \sqrt{(r - M)^2 - M^2}$, we may isometrically embed into \mathbb{E}^3 with the coordinates (x, y, z) as the surface of revolution

$$\sqrt{x^2 + y^2} = M \cosh \frac{z}{M}, \quad r = M \left(1 + \sinh \frac{z}{M} \right). \quad (89)$$

Note that (89) is a *catenoid*. This may be compared with the well-known *Flamm paraboloid* which gives an isometric embedding of the physical equatorial plane geometry of the Schwarzschild metric,

$$\sqrt{x^2 + y^2} = 2M + \frac{z^2}{8M}, \quad r = \sqrt{x^2 + y^2}. \quad (90)$$

It is also possible to isometrically embed the Schwarzschild optical metric (2) into the Euclidean space but the formulae are more complicated:

$$\sqrt{x^2 + y^2} = \frac{r}{\sqrt{1 - \frac{2M}{r}}}, \quad z = \int^r \sqrt{\frac{M}{r} \left(4 - 9\frac{M}{r} \right) \left(1 - \frac{2M}{r} \right)^{-\frac{3}{2}}} dr. \quad (91)$$

If we let $u = \frac{1}{r-M}$, then the equation of null geodesic is

$$(u')^2 = (\xi - 1)M^2 u^4 + (2\xi - 1)u^2 + \frac{\xi}{M^2}, \quad (92)$$

where $\xi = M^2 E^2 / L^2$. Note that this equation does not distinguish between r and $2M - r$ for $r \in [0, M]$. Before turning to the Weierstrass functions, we give a qualitative analysis of the null geodesics. Going back to the equation for r gives

$$(r')^2 = \frac{\xi}{M^2} r^4 - \frac{4\xi}{M} r^3 + (8\xi - 1)r^2 + 2M(1 - 4\xi)r + 2M^2(2\xi - 1) \equiv f(r). \quad (93)$$

The roots of $f(r)$ have a very simple form:

$$r = (1 \pm i)M, \quad (94)$$

$$r = M \left(1 \pm \sqrt{\frac{1}{\xi} - 1} \right). \quad (95)$$

The extremal points of $f(r)$ and roots of $f'(r)$ also have a simple form:

$$r = M, \quad (96)$$

$$r = M \left(1 \pm \sqrt{\frac{1}{2\xi} - 1} \right). \quad (97)$$

From these results, it follows that if

- $\xi \in (0, 1/2)$, then $f(r)$ has one positive real root $M(1 + \sqrt{1/\xi - 1})$ and three local extrema, all with the value smaller than this root;
- $\xi \in (1/2, 1)$, then $f(r)$ has two positive real roots $M(1 \pm \sqrt{1/\xi - 1})$ and one global extremum (minimum) at $r = M$;
- $\xi \in (1, \infty)$, then $f(r)$ has no real roots and one global extremum (minimum) at $r = M$.

This shows that if

- $\xi \in (0, 1/2)$, there are *only scattering orbits* with the distance of the closest approach $M(1 + \sqrt{1/\xi - 1})$;
- $\xi \in (1/2, 1)$, there are both *scattering and trapped orbits* with the distance of the closest approach $M(1 + \sqrt{1/\xi - 1})$ and the largest attained distance $M(1 - \sqrt{1/\xi - 1})$, respectively;
- $\xi \in (1, \infty)$, there are *only absorbing orbits*, that is, orbits incoming from ∞ that hit $r = 0$.

There is an important point here. Suppose that we want to express r in terms of some Weierstrass function. The only way to convert the full quartic into cubic is to substitute $r = x + r_0$ with r_0 being a root of $f(r) = 0$ and then $s = 1/x$. If this approach is useful, we want $r_0 \in \mathbb{R}$, since otherwise, we would be looking for a complex solution of the Weierstrass equation and the imaginary part C in $\wp(\phi + C)$ would not be half-period anymore but rather some analytically incalculable number and so this approach would not be useful at all. But $f(r)$ has no real root in the case of absorbing paths and this foretells problems when treating this case.

3.4.2. Weierstrass function approach. First we make the substitution $u^2 = 1/x$ into equation (92) which takes it into the form

$$\frac{1}{4}(x')^2 = (\xi - 1)x + (2\xi - 1)x^2 + \frac{\xi}{M^2}x^3. \quad (98)$$

Then the substitution

$$x = \frac{M^2 y}{\xi} + \frac{M^2(1 - 2\xi)}{3\xi} \quad (99)$$

takes it into the Weierstrass form

$$(y')^2 = 4y^3 - g_2 y - g_3, \quad (100)$$

where

$$g_2 = \frac{4}{3}(1 - \xi + \xi^2), \quad (101)$$

$$g_3 = \frac{4}{27}(2 - 3\xi - 3\xi^2 + 2\xi^3). \quad (102)$$

Note that $g_2 > 0 \forall \xi$ and that

$$\left(\frac{g_2}{3}\right)^3 - g_3^2 = \frac{16}{27}(\xi - 1)^2 \xi^2 > 0, \quad (103)$$

unless $\xi = 0, 1$. Setting $\xi = 0$ in equation (92) shows that this case is not possible. The case $\xi = 1$ gives two analytical solutions:

$$r_{\pm}(\phi) = M \left(1 \pm \frac{1}{\sinh \phi} \right), \quad (104)$$

where r_+ comes from ∞ , r_- comes from $r = 0$ and both are approaching $r = M$, but never reaching it. For other values of ξ , the rhs of equation (100) has three real roots $e_1 > e_2 > e_3$ where

$$e_1 = \max \left(\frac{2 - \xi}{3}, \frac{2\xi - 1}{3} \right), \quad (105)$$

$$e_2 = \min \left(\frac{2 - \xi}{3}, \frac{2\xi - 1}{3} \right), \quad (106)$$

$$e_3 = -\frac{1}{3}(1 + \xi). \quad (107)$$

Now we will analyze the separate cases. Suppose that

- $\xi \in (0, 1/2)$. Then

$$e_1 = \frac{2 - \xi}{3}, \quad (108)$$

$$e_2 = \frac{2\xi - 1}{3}, \quad (109)$$

$$e_3 = -\frac{1}{3}(1 + \xi). \quad (110)$$

As a consistency check, one can verify that plugging $y = e_1$ into the expression $r = r(y)$ indeed gives $r = M(1 + \sqrt{1/\xi - 1})$ as it should. Also, the point $r = \infty$ corresponds to the point $y = \infty$ and so the solution for the scattering orbits in this case is

$$\frac{r(\phi)}{M} = 1 + \frac{1}{\sqrt{\xi}} \sqrt{\wp(\phi) + \frac{1 - 2\xi}{3}}, \quad (111)$$

where the line $\phi = 0$ is in the direction of the ray incoming from ∞ and $\phi \in (0, 2\omega_1)$. Note that this solution always stays above $r = 2M$.

The point $r = 0$ corresponds to $y = (5\xi - 1)/3$ which in this case is in an unphysical region and so in accordance with section 1.1 we have only scattering solutions in this case.

- $\xi \in (1/2, 1)$. Then

$$e_1 = \frac{2 - \xi}{3}, \quad (112)$$

$$e_2 = \frac{2\xi - 1}{3}, \quad (113)$$

$$e_3 = -\frac{1}{3}(1 + \xi). \quad (114)$$

But now the scattering solutions penetrate into the region $M < r < 2M$ and so we have to be careful here because $r(y)$ is multivalued,

$$r = M \left(1 \pm \frac{1}{\sqrt{\xi}} \sqrt{y + \frac{1 - 2\xi}{3}} \right). \quad (115)$$

This only becomes a problem once the orbit crosses $r = 2M$ and so we did not have to worry about it in the previous case $\xi < 1/2$.

In this case, $(5\xi - 1)/3 > e_1$ and $r(5\xi - 1)/3 = 0$ or $2M$. For orbits incoming from ∞ , we clearly have to choose $r(5\xi - 1)/3 = 2M$ because $r = M$ is inaccessible.

Also $r(e_1) = M(1 \pm \sqrt{1/\xi - 1})$ and for the same reason we have to choose $+$ for orbits incoming from ∞ . Hence, as before

$$\frac{r(\phi)}{M} = 1 + \frac{1}{\sqrt{\xi}} \sqrt{\wp(\phi) + \frac{1 - 2\xi}{3}}, \quad (116)$$

where again the line $\phi = 0$ is in the direction of the ray incoming from ∞ and $\phi \in (0, 2\omega_1)$. What is left are orbits trapped in the region $r < M(1 - \sqrt{1/\xi - 1})$. For these, we need to choose minus signs in the above equations and so we obtain

$$\frac{r(\phi)}{M} = 1 - \frac{1}{\sqrt{\xi}} \sqrt{\wp(\phi + \omega_1) + \frac{1 - 2\xi}{3}}, \quad (117)$$

where now the additive constant in the argument of the Weierstrass function is necessary. This choice corresponds to setting the line $\phi = 0$ to be the axis of symmetry and $\phi \in (-\beta, \beta)$ where

$$\beta = \omega_1 - \int_{(5\xi-1)/3}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}}. \quad (118)$$

- $\xi \in (1, \infty)$. Then

$$e_1 = \frac{2\xi - 1}{3}, \quad (119)$$

$$e_2 = \frac{2 - \xi}{3}, \quad (120)$$

$$e_3 = -\frac{1}{3}(1 + \xi). \quad (121)$$

We know that in this case all orbits are incoming from ∞ and reach $r = 0$. Both $y = e_2$ and $y = e_3$ correspond to unphysical (complex) r and this time, $y = e_1$ corresponds to $r = M$ without any ambiguity. Suppose we have an orbit starting at ∞ . $(5\xi - 1)/3 > e_1$ and so we need to choose a plus sign in the relation $r(y)$.

Thus, $r(y = \infty) = \infty$; then $r(y = (5\xi - 1)/3) = 2M$ and finally we reach $r(y = e_1) = M$. But if we continued the same Weierstrass function solution now, r would begin to increase again, which we know is unphysical. Therefore, we need to switch the branches and continue with minus sign in the relation $r(y)$ so that we reach $r(y = (5\xi - 1)/3) = 0$. Now there is no way of continuing the solution and we need to start a new one, first using a minus sign and then a plus sign on its journey from $r = 0$ to $r = \infty$. Therefore, an orbit going from ∞ to $r = 0$ travels a total angle $\omega_1 + \beta$ and satisfies

$$\frac{r(\phi)}{M} = 1 + \frac{1}{\sqrt{\xi}} \sqrt{\wp(\phi) + \frac{1 - 2\xi}{3}} \quad \text{for } \phi \in (0, \omega_1), \quad (122)$$

$$\frac{r(\phi)}{M} = 1 - \frac{1}{\sqrt{\xi}} \sqrt{\wp(\phi) + \frac{1 - 2\xi}{3}} \quad \text{for } \phi \in (\omega_1, \beta), \quad (123)$$

where again the line $\phi = 0$ is in the direction of the ray incoming from ∞ .

General remarks

(i) Note that the scattering solutions depend directly on $\wp(\phi)$ and so the addition formula for Weierstrass functions can be applied directly.

(ii) The same is true for the absorbing one; however, we need to be careful to stay in the region $\phi \in (0, \omega_1)$ or $\phi \in (\omega_1, \beta)$ when applying it.

3.4.3. *Angle of deflection in the scattering case.* The equation for u can be factorized as

$$(u')^2 = (1 + M^2 u^2) \left(\frac{\xi}{M^2} + (\xi - 1)u^2 \right). \quad (124)$$

Now, the distance of the closest approach is $r_0 = M + M\sqrt{1/\xi - 1}$, which corresponds to

$$u_0 = \frac{1}{M} \sqrt{\frac{\xi}{1 - \xi}}. \quad (125)$$

Let I be half of the angle ϕ traveled by the light:

$$I = \int_0^{u_0} \frac{du}{\sqrt{1 + M^2 u^2} \sqrt{\xi/M^2 + (\xi - 1)u^2}}. \quad (126)$$

Making the substitution $u = u_0 t$, we have

$$I = \frac{1}{M} \sqrt{\frac{\xi}{1 - \xi}} \int_0^1 \frac{dt}{\sqrt{1 + t^2 \frac{\xi}{1 - \xi}} \sqrt{\frac{\xi}{M^2} + \frac{\xi}{M^2} t^2}} = \int_0^1 \frac{dt}{\sqrt{1 - t^2} \sqrt{1 - (1 - t^2)\xi}}. \quad (127)$$

Write $f(t, \xi)$ for the final integrand above. It is straightforward to differentiate f n times w.r.t. ξ and the result is

$$\frac{\partial^n f}{\partial \xi^n} = (1 - t^2)^{\frac{2n-1}{2}} \frac{(2n-1)!!}{2^n} \frac{1}{(1 - (1 - t^2)\xi)^{\frac{2n+1}{2}}}. \quad (128)$$

Therefore, we can expand f as

$$f(t, \xi) = \sum_{n=0}^{\infty} (1 - t^2)^{\frac{2n-1}{2}} \frac{(2n-1)!!}{n!} 2^{-n} \xi^n. \quad (129)$$

Scattering orbits exist for $\xi \in (0, 1)$ and for this range of values of ξ , the sum converges uniformly (for example, by the straightforward application of the Weierstrass M -test) and therefore we can write

$$I = \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{n!} 2^{-n} \xi^n \int_0^1 (1 - t^2)^{\frac{2n-1}{2}} dt \right). \quad (130)$$

The integral in this sum can be computed by hand; one way is as follows. The volume V_n of an n -dimensional ball is given as

$$V_n = \frac{\pi^{n/2}}{\Gamma(1 + \frac{n}{2})}. \quad (131)$$

Therefore,

$$\int_0^1 (1 - t^2)^{\frac{2n-1}{2}} dt = \frac{V_{2n}}{2V_{2n-1}} = \frac{1}{2} \frac{\pi^n}{\Gamma(n+1)} \frac{\Gamma(n + \frac{1}{2})}{\pi^{n-1/2}} = \frac{\sqrt{\pi}}{2n!} \Gamma\left(n + \frac{1}{2}\right). \quad (132)$$

Now using the identity

$$\Gamma\left(n + \frac{1}{2}\right) = (2n-1)!! 2^{-n} \sqrt{\pi} \quad (133)$$

gives

$$I = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{n!} \right)^2 2^{-2n} \xi^n. \quad (134)$$

This can be further simplified using the identity $(2n-1)!!n! = (2n)!2^{-n}$ to give

$$I = \frac{\pi}{2} \sum_{n=0}^{\infty} \binom{2n}{n}^2 2^{-4n} \xi^n. \quad (135)$$

Now the angle of deflection $\delta\phi$ is given by $\delta\phi = \pi - 2I$ and so

$$\delta\phi = \pi - \pi \sum_{n=0}^{\infty} \binom{2n}{n}^2 2^{-4n} \xi^n. \quad (136)$$

The first few terms of this expansion are

$$\delta\phi = -\frac{\pi}{4}\xi - \frac{9\pi}{64}\xi^2 - \frac{25\pi}{256}\xi^3 - \frac{1225\pi}{16384}\xi^4 - \frac{3969\pi}{65536}\xi^5 - \frac{53361\pi}{1048576}\xi^6 - \dots \quad (137)$$

with $\xi = (M/b)^2$.

We have also tried expanding the deflection angle in terms of $\mu = M/r_0$ following [6]. Substituting

$$\xi = \frac{1}{1 + \left(\frac{1}{\mu} - 1\right)^2} \quad (138)$$

into integral (127) and expanding in the powers of μ , using Mathematica, the first few terms are

$$\begin{aligned} \frac{\delta\phi}{\pi} = & -\frac{1}{4}\mu^2 - \frac{1}{2}\mu^3 - \frac{41}{64}\mu^4 - \frac{9}{16}\mu^5 - \frac{25}{256}\mu^6 + \frac{37}{128}\mu^7 + \frac{11959}{16384}\mu^8 + \frac{1591}{2048}\mu^9 \\ & + \frac{13311}{65536}\mu^{10} - \frac{29477}{32768}\mu^{11} - \dots \end{aligned} \quad (139)$$

This expansion is not very useful, since the coefficients do not seem to be decreasing very fast; the coefficient of μ^{11} is almost 1.

Note that this expansion is completely different from the one given in [6].

4. Conclusion

In this paper, we used Weierstrass elliptic functions to give a full description and classification of null geodesics in Schwarzschild spacetime. We then used this description to derive some analytic formulae connecting three points on these geodesics and found a second-order expansion of the deflection angle in the scattering case. Finally, we derived some properties of light triangles in this spacetime and used the Gauss–Bonnet theorem to derive a quantity which gives the same answer when integrated along a scattering geodesic, independently of the geodesic in question.

We then showed that the Weierstrass elliptic function formalism can also be used to describe other more exotic cases such as Reissner–Nordström null geodesics and Schwarzschild null geodesics in spacetimes with spatial dimensions 4 and 6. In all these cases, the elliptic function approach allows one to find the special cases when explicit analytic solutions are available with ease (simply by looking at the values of parameters for which the elliptic function in question collapses into a periodic one).

Finally, we applied the formalism to describe the null geodesics of the Ellis wormhole and found an expansion for the angle of deflection in this case.

After the appearance of the first version of this paper on the archive, Betti Hartmann pointed out to us that our results may be easily extended to the case of a Schwarzschild black hole pierced by an infinitely cosmic string studied in [17]. One needs only to replace the variable ϕ by $\delta\phi$, where $0 < \delta \leq 1$ is the deficit parameter. Similar remarks apply to the other metrics studied in this paper.

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