Braided Commutative Algebras over Quantized Enveloping Algebras

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Special Session on Geometry and Representation Theory of Quantum Algebras and Related Topics

AMS Sectional Meeting, Riverside, November 9, 2019

Summary

- *Davydov* ('09), the *full center $Z(A)$*: $\mathcal{C}$ a monoidal category over a field $\Bbbk$
  A an algebra in $\mathcal{C} \Longrightarrow Z(A)$ a *commutative* algebra in $\mathcal{Z}(\mathcal{C})$
- *L.–Walton*: $\mathcal{C}$ a monoidal category relative to a braided category $\mathcal{B}$ (replacing $\mathbf{Vect}_{\Bbbk}$)
  A an algebra in $\mathcal{C} \Longrightarrow Z_{\mathcal{B}}(A)$ a *commutative* algebra in $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$
- *Application:* Given an algebra with covariant action of $\mathrm{U}_q(\mathfrak{b}_-)$, obtain a *commutative* algebra $Z_{\mathcal{B}}(A)$ with covariant action of $\mathrm{U}_q(\mathfrak{g})$

Reference: Arxiv:1901.08980

Contents

Motivation

Commutative algebras A in braided or symmetric tensor categories $\mathcal{D}$ appear in

- *Extension theory* of vertex operator algebras (Huang–Kirillov–Lepowski '15)
- Construction of bialgebroids $A \rtimes H$, when $\mathcal{C} = H$-**Mod** (J.-H. Lu '96)
- Boundary conditions in *2D rational conformal field theory* (Fuchs–Runkel–Schweigert '02, Kong–Runkel '08)
- Kirillov–Ostrik ('02) classification of commutative algebras in the *semi-simplification* of $\mathrm{U}_q(\mathfrak{sl}_2)$-**Mod** through certain Dynkin diagrams

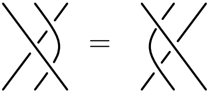
The Monoidal Center

Drinfeld, Majid, Joyal–Street:

$\mathcal{C}$ monoidal category $\Longrightarrow \mathcal{Z}(\mathcal{C})$ a *braided* monoidal category

- Objects: (V, c), $V \in \mathcal{C}$, *half-braiding* $c_W \colon V \otimes W \to W \otimes V$, natural in W

$$c_{W \otimes U} = (\mathrm{Id}_W \otimes c_U)(c_W \otimes \mathrm{Id}_U) \Longrightarrow$$



  - (V, c_V) is a solution to the Quantum Yang–Baxter Equation
- Morphisms: required to commute with the half-braidings

YETTER–DRINFELD MODULES

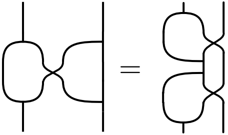
Example

For a Hopf algebra H and $\mathcal{C} = H$-**Mod**

$$\mathcal{Z}(\mathcal{C}) \simeq {}^H_H\mathbf{YD}$$

${}^H_H\mathbf{YD}$ is the category of Yetter–Drinfeld modules (V, a, δ) over H.

- $a =$: $H \otimes V \to V$ makes V an H-module
- $\delta =$: $V \to H \otimes V$ makes V an H-comodule
- Compatibility: *Yetter–Drinfeld condition*



Example

$\mathcal{C} = H$-$\mathbf{Mod}$, H — (finite-dimensional) $\Bbbk$-Hopf algebra
$\to \mathcal{C}$ monoidal category, via *coproduct* map $\Delta \colon H \to H \otimes_{\Bbbk} H$

Question: What is the center $\mathcal{Z}(\mathcal{C})$ in this case?

Answer: Modules over the Drinfeld double $\mathrm{Drin}(H)$

$\mathrm{Drin}(H) = H \otimes_{\Bbbk} H^*$ as a $\Bbbk$-vector space
H, H^* Hopf subalgebras.

Quantum Groups

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{t}) \otimes U_q(\mathfrak{n}_-)$$

$\mathfrak{g}$ — semisimple Lie algebra $\qquad$ q — a parameter

$\mathfrak{n}_\pm$ — nilpotent parts $\qquad$ $\mathfrak{t}$ — Cartan part

Theorem (Drinfeld)

$U_q(\mathfrak{g})$ *is a quotient of the double* $\mathrm{Drin}(U_q(\mathfrak{b}_-))$ *of its Borel part* $U_q(\mathfrak{b}_-)$.

Note

$\mathrm{Drin}(U_q(\mathfrak{b}_-))$ is defined on $U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{t}) \otimes U_q(\mathfrak{t})^* \otimes U_q(\mathfrak{n}_+)$

$$\Longrightarrow \mathcal{Z}(U_q(\mathfrak{b}_-)\text{-}\mathbf{Mod}) \text{ is too large}$$

Solution: *Use a relative version* $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ *of the monoidal center.*

BRAIDED HOPF ALGEBRAS

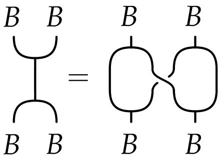
Idea: Take the Drinfeld double of $U_q(\mathfrak{n}_-) \subseteq U_q(\mathfrak{b}_-)$

Problem: $U_q(\mathfrak{n}_-)$ is *not* a Hopf algebra in $\mathbf{Vect}_{\Bbbk}$
Solution: $U_q(\mathfrak{n}_-)$ is a *braided* Hopf algebra in $\mathbf{Vect}_{\Bbbk}^{q}$

$\mathbf{Vect}_{\Bbbk}^{q}$: $\mathbb{Z}$-graded vector spaces with braiding

$$\Psi_{V,W}(v \otimes w) = q^{\deg(v)\deg(w)} w \otimes v$$

Bialgebra condition involves braiding



The Relative Monoidal Center

Assumption: $\mathcal{C}$ is $\mathcal{B}$-augmented if there exist:

- a braided category $\mathcal{B}$
- a forgetful functor $\mathrm{F}\colon \mathcal{C} \to \mathcal{B}$
- a functor $\mathrm{T}\colon \mathcal{B} \to \mathcal{C}$ which is a section to F

Recall: $\mathcal{Z}(\mathcal{C})$ objects $\longleftrightarrow$ pairs (V, c)

$$V \in \mathcal{C} \text{ object}, \qquad c_M\colon V \otimes M \xrightarrow{\sim} M \otimes V \text{ natural in } M$$

$$(V, c) \in \mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \quad \Longleftrightarrow \quad \mathrm{F}(c_{\mathrm{T}(B)}) = \Psi_{\mathrm{F}(V),B}$$

braiding of $\mathcal{B}$

$\rightsquigarrow$ isomorphisms $c_{\mathrm{T}(B)}$ on objects B of $\mathcal{B}$ descent to braiding

Note: $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ can be understood as a Mügers centralizer

Yetter–Drinfeld modules in $\mathcal{B}$

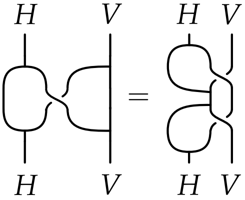
Let H be a *braided* Hopf algebra in $\mathcal{B}$ and $\mathcal{C} = H\text{-}\mathbf{Mod}(\mathcal{B})$

Proposition (L.)

The relative monoidal center $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ is equivalent as a braided category to the category ${}^{H}_{H}\mathbf{YD}(\mathcal{B})$ of Yetter–Drinfeld modules over H in $\mathcal{B}$.

${}^{H}_{H}\mathbf{YD}(\mathcal{B})$ consists of objects (V, a, δ) over H.

- $a =$: $H \otimes V \to V$ makes V an H-module in $\mathcal{B}$
- $\delta =$: $V \to H \otimes V$ makes V an H-comodule in $\mathcal{B}$
- Compatibility: *Yetter–Drinfeld condition*



QUANTUM GROUPS EXAMPLE

$$\mathcal{C} = \mathrm{U}_q(\mathfrak{n}_-)\text{-}\mathbf{Mod}(\mathcal{B}), \qquad \mathcal{B} = \mathbf{Vect}_{\Bbbk}^{q}$$

$\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ is equivalent to the category of $\mathrm{U}_q(\mathfrak{g})$-modules V satisfying

- V is a weight module, i.e. $V = \bigoplus_{i\in\mathbb{Z}} V_i$, where $K_j \cdot v_i = q^{i\cdot j} v_i$ for any $v_i \in V_i$
- V is locally finite, i.e. $\dim(\mathrm{U}_q(\mathfrak{n}_+) \cdot v) < \infty$ for any $v \in V$

$\rightsquigarrow$ $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ contains an analogue of category $\mathcal{O}$: $\mathcal{O}_q$ defined by Andersen–Mazorchuk

$\rightsquigarrow$ braided monoidal category

Braided Commutative Algebras

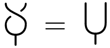
Definition

An *algebra* $\mathcal{C}$ is an object A together with morphisms

$$m\colon A \otimes A \to A, \qquad u\colon \mathbf{1} \to A$$

satisfying associativity and unitary axioms.
We say A is *commutative* in $\mathcal{C}$ if

$$m \circ \Psi_{A,A} = m \qquad \iff$$



Goal

Construct *commutative algebras* in $\mathrm{U}_q(\mathfrak{g})$-$\mathbf{Mod}$ and $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$

The $\mathcal{B}$-center

Definition

Let $\mathcal{C}$ be a $\mathcal{B}$-augmented monoidal category, A an algebra in $\mathcal{C}$. The *$\mathcal{B}$-center* $(Z_{\mathcal{B}}(A), \zeta_A)$ of A is the *terminal* pair (Z, ζ), where Z is an object in $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ with half-braiding $c_{Z,A}$ and $\zeta\colon Z \to A$ is a morphism in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
Z \otimes A \arrow[rr, "c_{Z,A}"] \arrow[d, "\zeta\otimes\mathrm{Id}"] & & A \otimes Z \arrow[d, "\mathrm{Id}\otimes\zeta"] \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

Davydov's full center: the case $\mathcal{B} = \mathbf{Vect}_{\Bbbk}$

THE $\mathcal{B}$-CENTER

- $Z_{\mathcal{B}}(A)$ is a *commutative algebra* in $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$
- $\zeta_A \colon Z_{\mathcal{B}}(A) \to A$ is an algebra morphism
- The $\mathcal{B}$-center extends to an *invariant of $\mathcal{C}$-module categories*

Question: When does $Z_{\mathcal{B}}(A)$ exist and how to compute it?

Theorem (Davydov, L.-Walton)

Assume the monoidal functor $\mathrm{F} \colon \mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \to \mathcal{C}$ *has a right adjoint* R *such that the counit of the adjunction is an epimorphism.*
Then $\mathrm{R}(A)$ *is an algebra in* $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ *and the $\mathcal{B}$-center* $Z_{\mathcal{B}}(A)$ *is the left center of* $\mathrm{R}(A)$.

Note: The assumptions hold for $\mathcal{C} = H\text{-}\mathbf{Mod}(\mathcal{B})$, categories of modules over a braided Hopf algebra H in $\mathcal{B}$.

In this case, $Z_{\mathcal{B}}(A)$ exists e.g. for $\mathcal{B} = K\text{-}\mathbf{Mod}$ (K a quasitriangular Hopf algebra).

The case $\mathcal{C} = H\text{-}\mathbf{Mod}(\mathcal{C})$

Given H a Hopf algebra in $\mathcal{B}$ and A a H-module algebra.

Theorem (Davydov, L.-Walton)

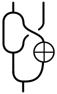
The $\mathcal{B}$-center $Z_{\mathcal{B}}(A)$ is isomorphic as an algebra in $\mathcal{B}$ to the (Ψ^{-1}-opposite algebra) of the (left) centralizer $\mathrm{Cent}^{l}_{A \rtimes H}(A)$ of A inside of the (braided) smash product $A \rtimes H$.

$\Longrightarrow$ This helps to compute $Z_{\mathcal{B}}(A)$ in examples

FIRST EXAMPLE

Let H be a braided Hopf algebra in $\mathcal{B}$.

Take $A = \mathbf{1}$, the tensor unit in $\mathcal{C} = H\text{-}\mathbf{Mod}(\mathcal{B})$. Then $Z_{\mathcal{B}}(\mathbf{1}) = H$ with action a and coaction δ given by

$a =$  , $\delta = \Delta =$ 

Note: These are the *adjoint action* and *regular coaction*.

Corollary

With this structure H is commutative in ${}^{H}_{H}\mathbf{YD}(\mathcal{B})$.

This generalizes the same result for H a $\Bbbk$-Hopf algebra (i.e. $\mathcal{B} = \mathbf{Vect}_{\Bbbk}$) of Cohen–Fischman–Montgomery ('99) to general $\mathcal{B}$.

A FAMILY OF $\mathrm{u}_q(\mathfrak{sl}_2)$-EXAMPLES

- Let $\mathcal{C} = \mathrm{u}_q(\mathfrak{sl}_2^+)\text{-}\mathbf{Mod}(\mathbf{Vect}_{\Bbbk}^q)$, $q^{2n} = 1$, then $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \simeq \mathrm{u}_q(\mathfrak{sl}_2)\text{-}\mathbf{Mod}$, with $\mathrm{u}_q(\mathfrak{sl}_2)$ generators $\mathbf{k}, \mathbf{e}, \mathbf{f}$
- For $\gamma \in \Bbbk$, $A_\gamma = \Bbbk[\mathbf{u}]$ is an algebra in $\mathcal{C}$ with

$$\mathbf{k} \cdot \mathbf{u} = q^2 \mathbf{u} \qquad \mathbf{f} \cdot \mathbf{u} = \gamma 1$$

The $\mathcal{B}$-center of A_γ is

$$Z_{\mathcal{B}}(A_\gamma) = \begin{cases} H \otimes \Bbbk[\mathbf{u}^n], & \text{for } \gamma = 0 \\ \Bbbk[\mathbf{z}], & \text{for } \gamma \neq 0, \end{cases} \quad \subseteq \quad A_\gamma \rtimes \mathrm{u}_q(\mathfrak{sl}_2^+)$$

$$\text{with} \quad \mathbf{z} := \sum_{i=0}^{n-1} \gamma^{-i} q^{-2\left(\binom{i+1}{2}+i\right)} (1-q^2)^i (\mathbf{f}^i \otimes \mathbf{u}^{i+1}).$$

For $\gamma \neq 0$, $Z_{\mathcal{B}}(A_\gamma)$ is a commutative algebra in $u_q(\mathfrak{sl}_2)$-**Mod** via

$$\mathbf{k} \cdot \mathbf{z} = q^2 \mathbf{z}, \qquad \mathbf{f} \cdot \mathbf{z} = \gamma 1, \qquad \mathbf{e} \cdot \mathbf{z} = -q\gamma^{-1}\mathbf{z}^2.$$

Thank you very much for your attention!

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