This file accompanies the paper [LW] R. Laugwitz, G. Sanmarco: Finite-dimensional quantum groups of type Super A and non-semisimple modular categories, ArXiv preprint arXiv:2301.10685 and its eventual published version.

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Content: This file computes:

- the tensor products of simple modules of non-zero quantum dimension over the algebra u_q(sl_{2,I}) when q=i, the imaginary unit, a 4th root of unity.

These computations are releant to the study of the semisimplification of the representation category of this algebra in [LS, Section 6.4].

```
with(LinearAlgebra) : interface(rtablesize = 32) : with(linalg) :
with(ArrayTools) :
s := I;
```

$$s := \mathrm{I} \tag{1}$$

Test if a matrix is zero matrix:

```
IsZeroMatrix :=proc(A :: Matrix)
local v, n, m, i, j;
n, m := Size(A);
i := 1; j := 1;
v := true;
while (i < n and v = true) do
while (j < m and v = true) do
v := is(A[i, j] = 0);
j := j + 1;
end do;
i := i + 1;
end do;
return v;
end proc;
```

IsZeroMatrix := **proc**(A::Matrix) **(2)**

```
    local v, n, m, i, j;
    n, m := ArrayTools:-Size(A);
```

```
    i := 1;
    j := 1;
    v := true;
    while i < n and v = true do
        while j < m and v = true do v := is(A[i, j] = 0); j := j + 1 end do; i := i + 1
    end do;
    return v
end proc
```

Actions of y_1, y_2, x_1,x_2, k_1,k_2: Uses basis v0,v1,v1',v2, k_i=gamma_i\ov{gamma}_i.

Simple modules of non-zero dimension: A=L(1,0), B=L(2,0), C=L(3,0) Ad=A*, Bd=B*, Cd=C*.

The simple module A=L(1,0):

$ID3 := IdentityMatrix(3)$:

$Ay1 := Matrix([\langle 0, 0, 0\rangle, \langle 0, 0, 1\rangle, \langle 0, 0, 0\rangle]);$

$$Ay1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{3}$$

$Ay2 := Matrix([\langle 0, 1, 0\rangle, \langle 0, 0, 0\rangle, \langle 0, 0, 0\rangle]);$

$$Ay2 := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

$Ax1 := Matrix([\langle 0, 0, 0\rangle, \langle 0, 0, 0\rangle, \langle 0, 1 + s, 0\rangle]);$

$$Ax1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 + \mathrm{I} \\ 0 & 0 & 0 \end{bmatrix} \tag{5}$$

$Ax2 := Matrix([\langle 0,0,0\rangle, \langle 1+s^{-1},0,0\rangle, \langle 0,0,0\rangle]);$

$$Ax2 := \begin{bmatrix} 0 & 1-\mathrm{I} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$Ak1 := Matrix([\langle s,0,0\rangle, \langle 0,s,0\rangle, \langle 0,0,1\rangle]);$

$Ak2 := Matrix([\langle 1,0,0\rangle, \langle 0,s^{-1},0\rangle, \langle 0,0,s^{-1}\rangle]);$

$$Ak1 := \begin{bmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Ak2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{I} & 0 \\ 0 & 0 & -\mathrm{I} \end{bmatrix} \tag{7}$$

$Akn1 := Multiply(Ak1, Ak1) :$

$Akn2 := Multiply(Ak2, Ak2) :$

Test relations hold:

$IsZeroMatrix(simplify(Multiply(Ay1, Ax1) + Multiply(Ax1, Ay1) - ID3 + Ak2));$

$$true \tag{8}$$

$IsZeroMatrix(simplify(Multiply(Ay2, Ax2) + Multiply(Ax2, Ay2) - ID3 + Ak1));$

$$true \tag{9}$$

$IsZeroMatrix(simplify(Multiply(Ay1, Ax2) - Multiply(Ax2, Ay1)));$

$$true \tag{10}$$

$IsZeroMatrix(simplify(Multiply(Ay2, Ax1) - s \cdot Multiply(Ax1, Ay2)));$

$$true \tag{11}$$

$IsZeroMatrix(Multiply(Ak1, Ay1) - s^{-1} \cdot Multiply(Ay1, Ak1)); IsZeroMatrix(Multiply(Ak1, Ax1) - s \cdot Multiply(Ax1, Ak1));$

$IsZeroMatrix(Multiply(Ak2, Ay2) - s^{-1} \cdot Multiply(Ay2, Ak2)); IsZeroMatrix(simplify(Multiply(Ak2, Ax2) - s \cdot Multiply(Ax2, Ak2)));$

$$true \quad (12)$$

$$true \quad (12)$$

$$true \quad (12)$$

$$true \quad (12)$$

$IsZeroMatrix(Multiply(Ak1, Ay2) - Multiply(Ay2, Ak1)); IsZeroMatrix(Multiply(Ak2, Ay1) - Multiply(Ay1, Ak2));$

$$true$$

$$true \quad (13)$$

The simple module B=L(2,0):

$ID5 := IdentityMatrix(5):$

$By1 := Matrix([\langle 0, 0, 0, 0, 0\rangle, \langle 0, 0, 1, 0, 0\rangle, \langle 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 1\rangle, \langle 0, 0, 0, 0, 0\rangle]);$

$$By1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (14)$$

$By2 := Matrix([\langle 0, 1, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 1, 0\rangle, \langle 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0\rangle]);$

$$By2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

$Bx1 := Matrix([\langle 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0\rangle, \langle 0, I+1, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 2, 0\rangle]);$

$$Bx1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\mathrm{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

$Bx2 := Matrix\left(\left[\langle 0, 0, 0, 0, 0\rangle, \langle 2, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0\rangle, \left\langle 0, 0, 1+s^{-1}, 0, 0\right\rangle, \langle 0, 0, 0, 0, 0\rangle\right]\right);$

$$Bx2 := \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\mathrm{I} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

$Bk1 := Matrix\left(\left[\left\langle s^2, 0, 0, 0, 0\right\rangle, \left\langle 0, s^2, 0, 0, 0\right\rangle, \langle 0, 0, s, 0, 0\rangle, \langle 0, 0, 0, s, 0\rangle, \langle 0, 0, 0, 0, 1\rangle\right]\right);$
$Bk2 := Matrix\left(\left[\langle 1, 0, 0, 0, 0\rangle, \left\langle 0, s^{-1}, 0, 0, 0\right\rangle, \left\langle 0, 0, s^{-1}, 0, 0\right\rangle, \left\langle 0, 0, 0, s^{-2}, 0\right\rangle, \left\langle 0, 0, 0, 0, s^{-2}\right\rangle\right]\right);$

$$Bk1 := \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{I} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{I} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Bk2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{I} & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{I} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{18}$$

$Bkn1 := Multiply(Bk1, Bk1) :$
$Bkn2 := Multiply(Bk2, Bk2) :$

Test relations hold:

$IsZeroMatrix(simplify(Multiply(By1, Bx1) + Multiply(Bx1, By1) - ID5 + Bk2));$

$$true \tag{19}$$

$IsZeroMatrix(simplify(Multiply(By2, Bx2) + Multiply(Bx2, By2) - ID5 + Bk1));$

$$true \tag{20}$$

$IsZeroMatrix(simplify(Multiply(By1, Bx2) - Multiply(Bx2, By1)));$

$$true \tag{21}$$

$IsZeroMatrix(simplify(Multiply(By2, Bx1) - s \cdot Multiply(Bx1, By2)));$

$$true \tag{22}$$

$IsZeroMatrix(Multiply(Bx1, Bx1)); IsZeroMatrix(Multiply(Bx2, Bx2)); IsZeroMatrix(Multiply(By1, By1)); IsZeroMatrix(Multiply(By2, By2));$
$IsZeroMatrix\left(Multiply(Bk1, By1) - s^{-1} \cdot Multiply(By1, Bk1)\right); IsZeroMatrix\left(Multiply(Bk1, Bx1) - s \cdot Multiply(Bx1, Bk1)\right);$
$IsZeroMatrix\left(Multiply(Bk2, By2) - s^{-1} \cdot Multiply(By2, Bk2)\right); IsZeroMatrix\left(simplify(Multiply(Bk2, Bx2) - s \cdot Multiply(Bx2, Bk2))\right);$

$$true \tag{23}$$

$$true \tag{23}$$

$$true \tag{23}$$

$$true \tag{23}$$

$$true \tag{23}$$

$$true \tag{23}$$

$$true \tag{23}$$

$$true \tag{23}$$

$IsZeroMatrix\left(Multiply(Bk1, By2) - Multiply(By2, Bk1)\right); IsZeroMatrix\left(Multiply(Bk2, By1) - Multiply(By1, Bk2)\right);$

$$true$$

$$true \tag{24}$$

The simple module C=L(3,0):

$ID7 := IdentityMatrix(7):$

$Cy1 := Matrix([\langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 1, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 1, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 1\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle]);$

$$Cy1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{25}$$

$Cy2 := Matrix\big([\langle 0, 1, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 1, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 1, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle]\big);$

$$Cy2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

$Cx1 := Matrix\big([\langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 1 + s, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 2, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 1 + s^{-1}, 0\rangle]\big);$

$$Cx1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 + \mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 - \mathrm{I} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{27}$$

$Cx2 := Matrix\left(\left[\langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 1+s, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 2, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 1+s^{-1}, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 0\rangle\right]\right);$

$$Cx2 := \begin{bmatrix} 0 & 1+\mathrm{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\mathrm{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

$Ck1 := Matrix\left(\left[\langle s^3, 0, 0, 0, 0, 0, 0\rangle, \langle 0, s^3, 0, 0, 0, 0, 0\rangle, \langle 0, 0, s^2, 0, 0, 0, 0\rangle, \langle 0, 0, 0, s^2, 0, 0, 0\rangle, \langle 0, 0, 0, 0, s, 0, 0\rangle, \langle 0, 0, 0, 0, 0, s, 0\rangle, \langle 0, 0, 0, 0, 0, 0, 1\rangle\right]\right);$

$Ck2 := Matrix\left(\left[\langle 1, 0, 0, 0, 0, 0, 0\rangle, \langle 0, s^{-1}, 0, 0, 0, 0, 0\rangle, \langle 0, 0, s^{-1}, 0, 0, 0, 0\rangle, \langle 0, 0, 0, s^{-2}, 0, 0, 0\rangle, \langle 0, 0, 0, 0, s^{-2}, 0, 0\rangle, \langle 0, 0, 0, 0, 0, s^{-3}, 0\rangle, \langle 0, 0, 0, 0, 0, 0, s^{-3}\rangle\right]\right)$

$$Ck1 := \begin{bmatrix} -\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Ck2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} \end{bmatrix} \tag{29}$$

$Ckn1 := Multiply(Ck1, Ck1):$

$Ckn2 := Multiply(Ck2, Ck2)$:

Test relations hold:

$IsZeroMatrix(simplify(Multiply(Cy1, Cx1) + Multiply(Cx1, Cy1) - ID7 + Ck2));$
$IsZeroMatrix(simplify(Multiply(Cy2, Cx2) + Multiply(Cx2, Cy2) - ID7 + Ck1));$

$$true \tag{30}$$

$$true \tag{30}$$

$IsZeroMatrix(simplify(Multiply(Cy1, Cx2) - Multiply(Cx2, Cy1)));$

$$true \tag{31}$$

$IsZeroMatrix(simplify(Multiply(Cy2, Cx1) - s \cdot Multiply(Cx1, Cy2)));$

$$true \tag{32}$$

$IsZeroMatrix(Multiply(Cx1, Cx1)); IsZeroMatrix(Multiply(Cx2, Cx2));$
$IsZeroMatrix(Multiply(Cy1, Cy1)); IsZeroMatrix(Multiply(Cy2, Cy2));$
$IsZeroMatrix\left(Multiply(Ck1, Cy1) - s^{-1} \cdot Multiply(Cy1, Ck1)\right); IsZeroMatrix\left(Multiply(Ck1, Cx1) - s \cdot Multiply(Cx1, Ck1)\right);$
$IsZeroMatrix\left(Multiply(Ck2, Cy2) - s^{-1} \cdot Multiply(Cy2, Ck2)\right); IsZeroMatrix\left(simplify(Multiply(Ck2, Cx2) - s \cdot Multiply(Cx2, Ck2))\right);$

$$true \tag{33}$$

$$true \tag{33}$$

$$true \tag{33}$$

$$true \tag{33}$$

$$true \tag{33}$$

$$true \tag{33}$$

$$true \tag{33}$$

$$true \tag{33}$$

$IsZeroMatrix(Multiply(Ck1, Cy2) - Multiply(Cy2, Ck1)); IsZeroMatrix(Multiply(Ck2, Cy1) - Multiply(Cy1, Ck2));$

$$true$$

$$true \tag{34}$$

Arrange all action matrices in an array:

$Dim := list[3] : Dim[1] := 3 : Dim[2] := 5 : Dim[3] := 7 :$
$Y1 := Array([Ay1, By1, Cy1]) : Y2 := Array([Ay2, By2, Cy2]) :$
$X1 := Array([Ax1, Bx1, Cx1]) : X2 := Array([Ax2, Bx2, Cx2]) :$
$K1 := Array([Ak1, Bk1, Ck1]) : K2 := Array([Ak2, Bk2, Ck2]) :$
$Kn1 := Array([Akn1, Bkn1, Ckn1]) : Kn2 := Array([Akn2, Bkn2, Ckn2]) :$

Tensor product actions:

```
RepTensor :=proc(i :: integer, j :: integer) :: Array;
 description "Computes the action matrices of the tensor product L(i,0)\otimes L(j,0)";
 local d, AD;
 d := Dim[i]·Dim[j];
 AD := Array[9];
 AD[1] := d;
 AD[2] := KroneckerProduct(Y1[i], IdentityMatrix(Dim[j])) + KroneckerProduct(Kn1[i], Y1[j]);
 AD[3] := KroneckerProduct(Y2[i], IdentityMatrix(Dim[j]))
     + KroneckerProduct(Multiply(Kn2[i], K1[i]), Y2[j]);
 AD[4] := KroneckerProduct(X2[i], IdentityMatrix(Dim[j])) + KroneckerProduct(Kn2[i],
     X2[j]);
 AD[5] := KroneckerProduct(X1[i], IdentityMatrix(Dim[j]))
     + KroneckerProduct(Multiply(Kn1[i], K2[i]), X1[j]);
 AD[6] := KroneckerProduct(K1[i], K1[j]);
 AD[7] := KroneckerProduct(K2[i], K2[j]);
 AD[8] := KroneckerProduct(Kn1[i], Kn1[j]);
 AD[9] := KroneckerProduct(Kn2[i], Kn2[j]);
 return AD;
end proc;
```

```
RepTensor := proc(i::integer, j::integer)::Array;                                  (35)
    local d, AD;
    description "Computes the action matrices of the tensor product L(i,0)otimes L(j,0)";
    d := Dim[i] * Dim[j];
    AD := Array[9];
    AD[1] := d;
    AD[2] := LinearAlgebra:-KroneckerProduct(Y1[i], LinearAlgebra:-IdentityMatrix(Dim[j]))
     + LinearAlgebra:-KroneckerProduct(Kn1[i], Y1[j]);
    AD[3] := LinearAlgebra:-KroneckerProduct(Y2[i], LinearAlgebra:-IdentityMatrix(Dim[j]))
     + LinearAlgebra:-KroneckerProduct(Multiply(Kn2[i], K1[i]), Y2[j]);
    AD[4] := LinearAlgebra:-KroneckerProduct(X2[i], LinearAlgebra:-IdentityMatrix(Dim[j]))
     + LinearAlgebra:-KroneckerProduct(Kn2[i], X2[j]);
```

```
    AD[5] := LinearAlgebra:-KroneckerProduct(X1[i], LinearAlgebra:-IdentityMatrix(Dim[j]))
     + LinearAlgebra:-KroneckerProduct(Multiply(Kn1[i], K2[i]), X1[j]);
    AD[6] := LinearAlgebra:-KroneckerProduct(K1[i], K1[j]);
    AD[7] := LinearAlgebra:-KroneckerProduct(K2[i], K2[j]);
    AD[8] := LinearAlgebra:-KroneckerProduct(Kn1[i], Kn1[j]);
    AD[9] := LinearAlgebra:-KroneckerProduct(Kn2[i], Kn2[j]);
    return AD
end proc
AD := RepTensor(1, 1) : d := AD[1] :
ADy1 := AD[2] :
 ADy2 := AD[3] :
ADx2 := AD[4] :
 ADx1 := AD[5] :
ADk1 := AD[6] :
 ADk2 := AD[7] :
ADkn1 := AD[8] :
 ADkn2 := AD[9] :
```

```
IsZeroMatrix(simplify(Multiply(ADy1, ADx1) + Multiply(ADx1, ADy1) − IdentityMatrix(d)
    + ADk2));
```

$$true \tag{36}$$

```
IsZeroMatrix(simplify(Multiply(ADy2, ADx2) + Multiply(ADx2, ADy2) − IdentityMatrix(d)
    + ADk1)); IsZeroMatrix(simplify( Multiply(ADy1, ADx2) − Multiply(ADx2, ADy1)));
IsZeroMatrix(simplify( Multiply(ADy2, ADx1) − s·Multiply(ADx1, ADy2)));
 IsZeroMatrix(simplify(Multiply(ADx1, ADx1))); IsZeroMatrix(simplify(Multiply(ADx2, ADx2)));
    IsZeroMatrix(Multiply(ADy1, ADy1)); IsZeroMatrix(Multiply(ADy2, ADy2));
```

true

true

true

true

true

true

$$true \tag{37}$$

Computing Tensor Product decompositions:

Finding the highest weight vectors:

```
Hweight :=proc(AD1 :: Matrix, AD2 :: Matrix) :: Matrix;
 local A, H;
  A := Matrix(Concatenate(1, AD1, AD2)) :
H := NullSpace(A);
 return H;
 end proc
```

```
Hweight := proc(AD1::Matrix, AD2::Matrix)::Matrix;                    (38)
    local A, H;
    A := Matrix(ArrayTools:-Concatenate(1, AD1, AD2));
    H := LinearAlgebra:-NullSpace(A);
    return H
end proc
```

Function that looks at the generated submodule by a given highest weight vector:

```
Span :=proc(w :: Vector) :: Array;

    description "Fills up an array of vectors spanned by a given vector under action of ...y1,y2,y1 and
    then y2,y1,y2";
 local S, v, u, i, j, k, l, t, a, b, r, MatrixS;
i := 1; j := 0; v := w; u := w;
 S[1] := w;
 while i < Dimension(w) do
v := Multiply(AD[2], v);
 S[2· i] := v;
v := Multiply(AD[3], v);
S[2 ·i + 1] := v;
i := i + 1;
 end do;
 while j < Dimension(w) − 1 do
 u := Multiply(AD[3], u);
 S[2 ·i + 2· j] := u;
u := Multiply(AD[2], u);
S[2 i + 2·j + 1] := u;
```

```
j := j + 1;
 end do;
 a := 0;
 for b from 1 to 2·i + 2·j + 1 do
 if IsZero(S[b]) = false then
 a := a + 1;
 end if; end do;
 l := 1;
 MatrixS := Matrix(Dimension(w), a);
for k from 1 to 2 ·i + 2·j − 1 do
 if IsZero(S[k]) = false then
 for t from 1 to Dimension(S[1]) do MatrixS[t, l] := S[k][t] end do;
l := l + 1;
 end if ;
 end do;
 return simplify(MatrixS);
 end proc;
```

(39)

```
Span := proc(w::Vector)::Array;
    local S, v, u, i, j, k, l, t, a, b, r, MatrixS;
    description
    "Fills up an array of vectors spanned by a given vector under action of ...y1,y2,y1 and then y2,
    y1,y2";
    i := 1;
    j := 0;
    v := w;
    u := w;
    S[1] := w;
    while i < LinearAlgebra:-Dimension(w) do
        v := Multiply(AD[2], v); S[2 * i] := v; v := Multiply(AD[3], v); S[2 * i + 1
        ] := v;
        i := i + 1
    end do;
    while j < LinearAlgebra:-Dimension(w) − 1 do
        u := Multiply(AD[3], u);
        S[2 * i + 2 * j] := u;
        u := Multiply(AD[2], u);
        S[2 * i + 2 * j + 1] := u;
```

```
        j := j + 1
    end do;
    a := 0;
    for b to 2*i + 2*j + 1 do
        if ArrayTools:-IsZero(S[b]) = false then a := a + 1 end if
    end do;
    l := 1;
    MatrixS := Matrix(LinearAlgebra:-Dimension(w), a);
    for k to 2*i + 2*j - 1 do
        if ArrayTools:-IsZero(S[k]) = false then
            for t to LinearAlgebra:-Dimension(S[1]) do MatrixS[t, l] := S[k][t] end do; l := l
            + 1
        end if
    end do;
    return simplify(MatrixS)
end proc
```

1.) Tensor Product L(1,0)\otimes L(1,0):

$AD := RepTensor(1, 1) : d := AD[1];$

$$d := 9 \tag{40}$$

$Hwt := Hweight(AD[4], AD[5]);$

$$Hwt := \left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{41}$$

$S1 := Span(Hwt[1]); AD[6][2, 2]; AD[7][2, 2];$

$$S1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-\mathrm{I} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1+\mathrm{I} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1-\mathrm{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-1$$

$$-\mathrm{I} \tag{42}$$

Highest weight module of shape L(2,3).

$S2 := Span(Hwt[2]); AD[6][1, 1]; AD[7][1, 1];$

$$S2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1-I & 0 & 0 \end{bmatrix}$$

$$-1$$

$$1 \qquad \textbf{(43)}$$

Highest weight module of shape L (2,0)
Conclusion: **L(1,0)\otimes L(1,0) = L(2,3) \oplus L(2,0).**

2.) Tensor Product L(1,0)\otimes L(2,0):

$AD := RepTensor(1,2) : d := AD[1];$

$$d := 15 \qquad \textbf{(44)}$$

$Hwt := Hweight(AD[4], AD[5]);$

$$Hwt := \left\{ \begin{bmatrix} 0 \\ -\frac{1}{2}+\frac{I}{2} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{45}$$

$S1 := Span(Hwt[1]); Rank(S1); AD[6][2, 2]; AD[7][2, 2];$

$$S1 := \tag{46}$$

$$\left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
-\frac{1}{2}+\frac{I}{2} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{2}-\frac{I}{2} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \frac{1}{2}+\frac{I}{2} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -\frac{1}{2}-\frac{I}{2} & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}- & \cdots \\
0 & 0 & \frac{1}{2}-\frac{I}{2} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -\frac{1}{2}-\frac{I}{2} & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -\frac{1}{2}-\frac{I}{2} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \frac{1}{2}+\frac{I}{2} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]$$

$$8 \tag{46}$$

$$-I \tag{46}$$

$$-I \tag{46}$$

Highest weight module of weight (3,3), dimension 8.

$S2 := Span(Hwt[2]); Rank(S2); AD[6][1,1]; AD[7][1,1];$

$$S2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1-I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}$$

$$7$$

$$-I$$

$$1 \tag{47}$$

Highest weight module of weight (3,0), dimension 7.

Conclusion: **L(1,0)\otimes L(2,0) = L(3,3) \oplus L(3,0).**

3.) Tensor Product L(1,0)\otimes L(3,0):

$AD := RepTensor(1,3) : d := AD[1];$

$$d := 21 \tag{48}$$

$Hwt := Hweight(AD[4], AD[5]);$

$$Hwt := \left\{ \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \textbf{(49)}$$

$S2 := Span(Hwt[2]); Rank(S2); AD[6][1, 1]; AD[7][1, 1];$

$$S2 := \left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1-I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1+I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \qquad \textbf{(50)}$$

$$8 \qquad \textbf{(50)}$$

$$1 \qquad \textbf{(50)}$$

$$1 \qquad \textbf{(50)}$$

This module is 8 dimensional, generated in degree (0,0).

$S2 := Span(Hwt[1]); Rank(S2); AD[6][2,2]; AD[7][2,2];$

$$S2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$7$$

$$1$$

$$-I \tag{51}$$

This module is 7 dimensional, generated in degree (0,3), so it is isomophic to L(0,3).

The second module is a submodule of the first one. So we have an extension 0--> L(0,3)--> V1:=u^-\cdot v_0\otimes v_0 --> L(0,0)-->0.

There are still 13 dimensions unaccounted for!

The tensor product **L(1,0)\otimes L(3,0)** must be indecomposable (every direct summand would contain a highest weight vector).
The module has non-zero quantum dimension:

$$Trace(MatrixPower(Multiply(AD[6], AD[7]), 2));$$

$$1 \tag{52}$$

The top vector not contained in V1 is
$w := convert(a \cdot KroneckerProduct(\langle 1, 0, 0\rangle, \langle 0, 1, 0, 0, 0, 0, 0\rangle) + b \cdot KroneckerProduct(\langle 0, 1, 0\rangle, \langle 1, 0, 0, 0, 0, 0, 0\rangle), Vector);$

$$w := \begin{bmatrix} 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{53}$$

requiring that a is not equal to I*b. Consider the submodule this generates:

$S3 := Span(w); Rank(S3); AD[6][2,2]; AD[7][2,2];$

$S3 :=$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -a & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -\mathrm{I}\,a & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \mathrm{I}\,a & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -a & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a & 0 & \cdots \\ b & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -a & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mathrm{I}\,a & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & a & \cdots \\ 0 & b & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -b & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -a-b & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a+b & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & (1+\mathrm{I})\,a+b & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -(1+\mathrm{I})\,a-b & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{I}\cdots \end{bmatrix}$$

$$14$$

$$1$$

$$-\mathrm{I} \tag{54}$$

While a is not I*b we get the lowest weight vector of degree (0,0) at the socket. We have a quotient module of dimension 13. Now all dimensions are accounted for.

We get that the tensor product V=L(1,0)\otimes L(3,0) is an extensions

0-->V1-->V-->V2-->0,

where V2 is a 13-dimensional module generated in degree (0,3). It has to be indecomposable.
0--> L(0,2) -->V2-->L(0,3).

$AD[6][9,9]; AD[7][9,9];$

$$1$$

$$-1 \tag{55}$$

The degree of the submodule generator is (0,2).

Altogether we have:

[L(1,0)]*[L(3,0)]= 2L(0,3)+L(0,2)+2[L(0,0)].

4.) Tensor Product L(2,0)\otimes L(2,0):

$AD := RepTensor(2,2) : d := AD[1];$

$$d := 25 \tag{56}$$

$Hwt := Hweight(AD[4], AD[5]);$

$$Hwt := \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} - \frac{I}{2} \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} - \frac{I}{2} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{57}$$

$S1 := Span(Hwt[3]); Rank(S1); AD[6][1,1]; AD[7][1,1];$

$$S1 := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1+\mathrm{I} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1+\mathrm{I} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{I} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1+\mathrm{I} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{I} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$8$$

$$1$$

$$1 \tag{58}$$

This is a submodule V1 of dimension 8, generated in degree (0,0). Contains second highest weight vector. We again get an extension
0-->L(3,0)-->u^-\cdot v0\otimes v0-->L(0,0)-->0.

$S2 := Span(Hwt[1]); Rank(S2); AD[6][4, 4]; AD[7][4, 4];$

$$
S2 := \left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2}-\frac{I}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2}+\frac{I}{2} & 0 & 0 & 0 \\
0 & 0 & I & 0 & -\frac{1}{2}+\frac{I}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}-\frac{I}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & -\frac{1}{2}+\frac{I}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}+\frac{I}{2} & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -\frac{1}{2}-\frac{I}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \frac{1}{2}+\frac{I}{2} & 0 & 0
\end{array}\right.
$$

$$\begin{array}{c} 4 \\ -\mathrm{I} \\ -1 \end{array} \tag{59}$$

This gives a submodule L(3,2), which is 4-dimensional, denoted by V2. This has to be a direct summand.

Left over dimensions: 13

The highest degree vector not in the submodule is

$w := convert\big(a \cdot KroneckerProduct(\langle 1, 0, 0, 0, 0\rangle, \langle 0, 1, 0, 0, 0\rangle) + b \cdot KroneckerProduct(\langle 0, 1, 0, 0, 0\rangle, \langle 1, 0, 0, 0, 0\rangle), Vector\big);$

$$w := \begin{bmatrix} 0 \\ a \\ 0 \\ 0 \\ 0 \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(60)}$$

as long as a is not -b. It generates the following module.

$S3 := Span(w); Rank(S3); AD[6][2, 2]; AD[7][2, 2];$

$S3 :=$

$$\left[\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
a & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & a & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -a & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -a & 0 & 0 & 0 & \cdots \\
b & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & a & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -a & 0 & 0 & \cdots \\
0 & b & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -\mathrm{I}\, b & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & a + \mathrm{I}\, b & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -\mathrm{I}\, a + b & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & (-1 + \mathrm{I})\, a - b & 0 & \cdots \\
0 & 0 & b & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & a + \mathrm{I}\, b & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & (-1 + \mathrm{I})\, a - b & \cdots \\
0 & 0 & 0 & b & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & b & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & (1 + \mathrm{I})\, b + a & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & (1 + \mathrm{I})\, b + a & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I}\cdots
\end{array}\right]$$

$$14$$

$$1$$

$$-I \tag{61}$$

As long as a is not -b we get a quotient module V3 of dimension 14, generated in h.wt. (0,3). This necessarily has a submodule L(0,2) and composition series.

0-->L(0,0)-->L(0,2)-->V3-->L(0,3)-->0.

Overall, we get the following symbol in the Grothendieck ring:

[L(0,2)]*[L(0,2)]= 2[L(0,3)]+[L(0,2)]+2[L(0,0)]+[L(3,2)].

Question: Do we have L(0,1)\otimes L(0,3) as a direct summand in this?

Check what happens when acting with x_1, x_2 on w.

$Multiply(AD[4], w)$; $Multiply(AD[5], w)$;

$$\begin{bmatrix} 2\,a + 2\,b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{62}$$

Get cleanly the highest weight vector.

5.) Tensor Product L(2,0)\otimes L(3,0):

$AD := RepTensor(2, 3) : d := AD[1];$

$$d := 35 \tag{63}$$

$Hwt := Hweight(AD[4], AD[5]);$

$$Hwt := \left\{ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\mathrm{I} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ -1+\mathrm{I} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \right\} \qquad \textbf{(64)}$$

$S1 := Span(Hwt[3]); Rank(S1); AD[6][1,1]; AD[7][1,1];$

$$
S1 := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1+\mathrm{I} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1+\mathrm{I} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{I} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1+\mathrm{I} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{I} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$8$$

$$I$$

$$1 \tag{65}$$

8-dimensional submodule generated in h. wt (1,0).
Contains first h.wt vector.

$S2 := Span(Hwt[1]); Rank(S2); AD[6][4,4]; AD[7][4,4];$

$$S2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1+\mathrm{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1+\mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$5$$

$$1$$

$$-1 \tag{66}$$

The first h.wt vector generates 5-d module L(0,2). Thus, we have an 8d extension
0-->L(0,2)--> u^-\cdot v0\otimes v0 --> L(1,0)-->0.

$S3 := Span(Hwt[2]); Rank(S3); AD[6][2,2]; AD[7][2,2];$

$$
S3 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
-1+\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & -1+\mathrm{I} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1-\mathrm{I} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1-\mathrm{I} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -1+\mathrm{I} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & -1+\mathrm{I} & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -1+\mathrm{I} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1-\mathrm{I} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -1+\mathrm{I} & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -\mathrm{I} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -1+2\,\mathrm{I} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 2+\mathrm{I} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & -1-2\,\mathrm{I} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -2+\mathrm{I} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -1+2\,\mathrm{I} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -1-2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{bmatrix}
$$

$$16$$

$$I$$

$$-I$$

$$Matrix(\%id = 36893490731990939156) \tag{67}$$

This highest weight vector generates the 16-dimensional standard module M(1,3)=L(1,3) which is also simple. This has quantum dimension zero though.

So far, we have used 24 dimensions. There are 11 dimensions remaining.

The highest degree vector not in the submodule is

$w := convert(a \cdot KroneckerProduct(\langle 1, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 1, 0, 0, 0\rangle) + b \cdot KroneckerProduct(\langle 0, 1, 0, 0, 0\rangle, \langle 0, 0, 1, 0, 0, 0, 0\rangle), Vector) :$

$S4 := Span(w); Rank(S4); AD[6][4, 4]; AD[7][4, 4]; Multiply(AD[2], w); Multiply(AD[3], w);$

$$
S4 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & -a & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & -a & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & -a & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & b & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & -\mathrm{I}\,b & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & a+\mathrm{I}\,b & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & -\mathrm{I}\,a+b & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & (-1+\mathrm{I})\,a-b & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & b & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & a+\mathrm{I}\,b & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots
\end{bmatrix}
$$

14

1

-1

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a+b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \tag{68}$$

The requirement on a,b is that a+b is not zero. We can see (by checking which vectors of degree (0,1) and (-1,2) were already contained in S1 and S3, that in the quotient, for any choice of a+b non-zero, we have y_1*w and y_2*w non-zero in the quotient. Thus, the vector w generates a submodule of dimension at least 8 in the quotient. Potentially, there are three dimensions remaining, in degrees (1,0), (1,-1), (0,-1). These degrees could not have been hit by the submodules S1, S3. The above shows that these are also contained in the quotient module generated by w.

Thus, w generates an 11-dimensional module V_4 in the quotient. It has a socle isomorphic to L(1,0).

0--L(1,0)-->V_4--> V_5-->0.

and

0-->L(0,1)--V_5--> L(2,0).

The key question is what we get when we act on w by x_1 or x_2.

$Multiply(AD[4], w)$; $Multiply(AD[5], w)$;

$$\left[\begin{array}{c} 0 \\ 0 \\ 2\,a+2\,b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

$$\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ (1 - \mathrm{I})\, b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \tag{69}$$

Acting by x_1 gives x_1*w in

6.) Tensor Product L(3,0)\otimes L(3,0):

$AD := RepTensor(3, 3) : d := AD[1];$

$$d := 49 \tag{70}$$

$Hwt := Hweight(AD[4], AD[5]);$

$$
Hwt := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathrm{I} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1+\mathrm{I} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1+\mathrm{I} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\mathrm{I} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1+\mathrm{I} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1-\mathrm{I} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \textbf{(71)}
$$

$S1 := Span(Hwt[1]);\ Rank(S1);\ AD[6][1, 1];\ AD[7][1, 1];$

$$S1 := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1+I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1+I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1+I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$8$$

$$-1$$

$$1 \tag{72}$$

This 8d module, generated in degree (2,0) also contains the first h.wt vector

$S2 := Span(Hwt[2]); Rank(S2); AD[6][6,6]; AD[7][6,6];$

$$
S2 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathrm{I} & 0 & 0 & 0 & 0 \\
0 & -\mathrm{I} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\mathrm{I} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1+\mathrm{I} & 0 & 0 & 0 & 0 \\
0 & \mathrm{I} & 0 & 0 & 0 \\
0 & 0 & \mathrm{I} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1+\mathrm{I} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathrm{I} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$3$$
$$1$$
$$I \quad (73)$$

This is the simple module L(0,1). Thus, we have an 8-dimensional extension
0-->L(0,1)-->u^-\cdot v0\otimes v0 --> L(2,0)--> 0.

$S3 := Span(Hwt[3]); Rank(S3); AD[6][4,4]; AD[7][4,4];$

$$S3 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
-\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1+\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots \\
0 & 0 & \mathrm{I} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} & 0 & \cdots \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -\mathrm{I} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1-\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 2+\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 2+\mathrm{I} & 0 & 0 & 0 & 0 & 0 & \mathrm{I} & \cdots \\
0 & 0 & 0 & 0 & 0 & 1+\mathrm{I} & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{I} & 0 & 0 & 0 & \cdots \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -\mathrm{I} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\mathrm{I} & 0 & \cdots \\
0 & 0 & 0 & 0 & 2+\mathrm{I} & 0 & 0 & 0 & 0 & 0 & -1+\cdots & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{bmatrix}$$

$$12$$

$$I$$

$$-1$$

$$Matrix(\%id = 36893490731936236596) \tag{74}$$

We get a 12-dimensional h.wt submodule generated in degree (1,2). This has to be the simple module L (1,2).

$S4 := Span(Hwt[4]); Rank(S4); AD[6][2,2]; AD[7][2,2];$

$$
S4 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\mathrm{I} & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\mathrm{I} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\mathrm{I} & \cdots \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 2+\mathrm{I} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 2+\mathrm{I} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+\mathrm{I} & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 2+\mathrm{I} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{bmatrix}
$$

$$16$$
$$-1$$
$$-I \tag{75}$$

This module is a 16-dimensional module generated in degree (2,3). It contains the former module.
Thus, we have an extension
0-->L(1,2)-->u^- (-v0\otimes v1+v1\otimes v0)--> L(2,3)-->0.

These two submodule generated by h.wt, vectors account for 16+8=24 of the 49 dimensions.

$w := convert\left(a \cdot KroneckerProduct\left(\langle 1, 0, 0, 0, 0, 0, 0 \rangle, \langle 0, 0, 0, 1, 0, 0, 0 \rangle \right) + 0 \cdot KroneckerProduct\left(\langle 0, 1, 0, 0, 0, 0, 0 \rangle, \langle 0, 0, 1, 0, 0, 0, 0 \rangle \right) + 0 \cdot KroneckerProduct\left(\langle 0, 0, 1, 0, 0, 0, 0 \rangle, \langle 0, 1, 0, 0, 0, 0, 0 \rangle \right) + e \cdot KroneckerProduct\left(\langle 0, 0, 0, 1, 0, 0, 0 \rangle, \langle 1, 0, 0, 0, 0, 0, 0 \rangle \right), Vector \right) :$

$S5 := Span(w); Rank(S5); AD[6][4, 4]; AD[7][4, 4]; Multiply(AD[4], w); Multiply(AD[5], w);$

$$S5 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \mathrm{I}\,a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -\mathrm{I}\,a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & \cdots \\
0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -\mathrm{I}\,a & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & (-1-\mathrm{I})\,a & 0 & 0 & 0 & 0 & 0 & \cdots \\
e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{bmatrix}$$

16

I

-1

$$\left[\begin{array}{c} 0 \\ 0 \\ 2\,a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2\,e \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

$$\left[\begin{array}{c}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{array}\right] \quad \textbf{(76)}$$

$u := convert\left(KroneckerProduct\left(\langle 1, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 1, 0, 0, 0\rangle\right) + KroneckerProduct\left(\langle 0, 0, 0, 1, 0, 0, 0\rangle, \langle 1, 0, 0, 0, 0, 0, 0\rangle\right), Vector\right) : S5 := Span(u); Rank(S5); AD[6][4, 4]; AD[7][4, 4]; Multiply(AD[4], u); Multiply(AD[5], u);$

$$
S5 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{I} & 0 & \cdots \\
0 & 0 & 0 & 0 & -\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots \\
0 & 0 & 0 & 0 & 0 & -1-\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & \cdots \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{bmatrix}
$$

16

I

−1

$$\begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{77}$$

Now, if we quotient out by S_1, S_4 and S_5 we are left with 9 dimensions. The highest degree is (0,1). This weight space is 7-dimensional containing the 7,13,19,25,31,43 th basis vectors. We need to find the one dimensional subspace of vectors not contained in S_1+S_4+S_5.

Try a vector u

$$s := convert\big(0 \cdot KroneckerProduct(\langle 1, 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 0, 1, 0\rangle) + 1 \cdot KroneckerProduct(\langle 0, 1, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 1, 0, 0\rangle) + (-1) \cdot KroneckerProduct(\langle 0, 0, 1, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 1, 0, 0, 0\rangle) + (-I) \cdot KroneckerProduct(\langle 0, 0, 0, 1, 0, 0, 0\rangle, \langle 0, 0, 1, 0, 0, 0, 0\rangle) + (-I) \cdot KroneckerProduct(\langle 0, 0, 0, 0, 1, 0, 0\rangle, \langle 0, 1, 0, 0, 0, 0, 0\rangle) + 0 \cdot KroneckerProduct(\langle 0, 0, 0, 0, 0, 1, 0\rangle, \langle 1, 0, 0, 0, 0, 0, 0\rangle), Vector\big) : S6 := Span(s); Rank(S6); AD[6][6, 6]; AD[7][6, 6]; Multiply(AD[4], s); Multiply(AD[5], s);$$

$$S6 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1+\mathrm{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1+\mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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I

$$\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1+\mathrm{I} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2-2\,\mathrm{I} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

$$\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1+\mathrm{I} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1-\mathrm{I} \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \tag{78}$$