This file accompanies the paper [LW] R. Laugwitz, G. Sanmarco: Finite-dimensional quantum groups of type Super A and non-semisimple modular categories, ArXiv preprint arXiv:2301.10685 and its eventual published version.

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Content: This file computes:

- The braiding of the three-dimensional simple representation of the super type quantum group u_q(sl_{2,I}) specified in Example 5.3 of [LS].
- The structural maps (braidings, evaluations, coevaluations) for this object and its dual V*=L(0,1)
- Some knot invariants associated with this object. As the object has non-vanishing quantum dimension, these invariants are not strong, see beginning of Section 7 of [LS].

$with(LinearAlgebra) : interface(rtablesize = 30) :$

This file computes

The matrix A is the braiding on V,V, with V=L(1,0).

$$A := Matrix([[-1, 0, 0, 0, 0, 0, 0, 0, 0], [0, q^{-1} - 1, 0, -1, 0, 0, 0, 0, 0], [0, 0, -(1 - q^{-1}), 0, 0, 0, 1, 0, 0], [0, -q^{-1}, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, q^{-1}, 0, 0, 0, 0], [0, 0, 0, 0, 0, q^{-1} - 1, 0, -1, 0], [0, 0, q^{-1}, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, -q^{-1}, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, -1]]);$$

$$A := \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{q}-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q}-1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{q}-1 & 0 & -1 & 0 \\ 0 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{1}$$

$ID3 := IdentityMatrix(3) : ID9 := IdentityMatrix(9);$

$$ID9 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

The matrix B is the braiding on W, W, with W=L(0,1), the dual representation.

$B := Matrix([[-1, 0, 0, 0, 0, 0, 0, 0, 0], [0, q^{-1} - 1, 0, -q^{-1}, 0, 0, 0, 0, 0], [0, 0, q^{-1} \cdot (1 - q), 0, 0,$

$0, q^{-1}, 0, 0]$, $[0, -1, 0, 0, 0, 0, 0, 0, 0]$, $[0, 0, 0, 0, q^{-1}, 0, 0, 0, 0]$, $[0, 0, 0, 0, 0, q^{-1} - 1, 0, -q^{-1}, 0]$, $[0, 0, 1, 0, 0, 0, 0, 0, 0]$, $[0, 0, 0, 0, 0, -1, 0, 0, 0]$, $[0, 0, 0, 0, 0, 0, 0, 0, -1]]);$

$$B := \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{q} - 1 & 0 & -\frac{1}{q} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-q}{q} & 0 & 0 & 0 & \frac{1}{q} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{q} - 1 & 0 & -\frac{1}{q} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad \textbf{(3)}$$

The matrix C is the braiding on V,W.

$C := Matrix([[q, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, -q, 0, 0, 0, 0, 0], [0, 0, q^{-1} - 1, 0, q - 1, 0, -1, 0, 0], [0, -q, 0, 0, 0, 0, 0, 0, 0], [0, 0, q^{-1} - 1, 0, q, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, -1, 0], [0, 0, -1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, -1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1]]);$

$$C := \begin{bmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q}-1 & 0 & q-1 & 0 & -1 & 0 & 0 \\ 0 & -q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q}-1 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

The matrix E is the braiding on W,V.

$E := Matrix([[1, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, -1, 0, 0, 0, 0, 0], [0, 0, q\cdot(1-q), 0, (-q)\cdot(1-q), 0, -1, 0, 0], [0, -1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1-q, 0, q, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, -q, 0], [0, 0, -1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0-q, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, q]]);$

$$E := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q\,(1-q) & 0 & -q\,(1-q) & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{bmatrix} \quad (5)$$

The inverse matrices:

We compute the inverses of the above matrices A, B, C, E.

$Ainv := MatrixInverse(A) : Binv := MatrixInverse(B) : Cinv := MatrixInverse(C) : Einv := MatrixInverse(E) :$

$Einv - q^{-1} \cdot C;$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\frac{1}{q}-1}{q} & 0 & -\frac{q-1}{q} & 0 & \frac{1}{q}-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\frac{1}{q}-1}{q} & 0 & \frac{1}{q}-1 & 0 & -\frac{q-1}{q} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q}-1 & 0 & q-1 & 0 & q-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \textbf{(6)}$$

Skein relations:

We check that the braiding satisfies the following Skein relations:

q*A - A^{-1}=(1-q)Id_{V,V}

$simplify(q \cdot A - Ainv);$

$$\begin{bmatrix} 1-q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-q \end{bmatrix} \tag{7}$$

q*B - B^{-1}=(1-q)Id_{V^*,V^*}

$simplify(q \cdot B - Binv);$

$$\begin{bmatrix} 1-q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-q \end{bmatrix} \tag{8}$$

EC + q*C^{-1}E^{-1} = (q+1)Id_{V,V^*}

$simplify(Multiply(E, C) + q \cdot Multiply(Cinv, Einv));$

$$\left[\begin{array}{ccccccccc} q+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q+1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q+1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q+1 \end{array}\right] \quad \textbf{(9)}$$

CE + q*E^{-1}C^{-1} = (q+1)Id_{V^*,V}

$simplify(Multiply(C, E) + q \cdot Multiply(Einv, Cinv));$

$$\left[\begin{array}{ccccccccc} q+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q+1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q+1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q+1 \end{array}\right] \quad \textbf{(10)}$$

Change to dual bases, evaluations and Coevaluations, Twist, and Quantum Dimension:

T is base change from the basis {w0,w1,w2} to the basis {v2*,v1*,v0*}

$T := Matrix([[1, 0, 0], [0, -1, 0], [0, 0, -q]]); Tinv := MatrixInverse(T) : TT := KroneckerProduct(T, T) : T1 := KroneckerProduct(T, ID3) : T2 := KroneckerProduct(ID3, T) : TinvTinv := KroneckerProduct(Tinv, Tinv) : Tinv1 := KroneckerProduct(Tinv, ID3) : Tinv2 := KroneckerProduct(ID3, Tinv) :$

$$T := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -q \end{bmatrix} \tag{11}$$

We can display the coevaluation and evaluation maps explicitly with these bases:

$coev := Multiply(Tinv2, Matrix(\langle 0, 0, 1, 0, 1, 0, 1, 0, 0\rangle))$

$$coev := \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{q} \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

$ev := Multiply(Matrix([0, 0, 1, 0, 1, 0, 1, 0, 0]), T1); Multiply(KroneckerProduct(Matrix([0, 0, 1, 0, 1, 0, 1, 0, 0]), ID3), KroneckerProduct(ID3, Matrix(\langle 0, 0, 1, 0, 1, 0, 1, 0, 0\rangle))) :$
$Multiply(KroneckerProduct(ev, ID3), KroneckerProduct(ID3, coev)) :$

$$ev := \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 & -q & 0 & 0 \end{bmatrix} \tag{13}$$

Twist computation on V:

We now compute the ribbon twist on V=L(1,0). For this we need *Signs,* the action of the group-like element k_1^nk_2^n on V. The twist turns out to be trivial, so we have a knot invariant.

$Signs := Matrix([[-1, 0, 0], [0, 1, 0], [0, 0, -1]]) :$

$Multiply(Matrix([0, 0, 1, 0, 1, 0, 1, 0, 0]), Multiply(KroneckerProduct(Signs, ID3), Matrix(\langle 0, 0, 1, 0, 1, 0, 1, 0, 0\rangle))) :$

We can now compute the right evaluation and coevaluation for the right dual of L(1,0).

$coevr := Multiply(Einv, coev); evr := Multiply(ev, C); simplify(Multiply(KroneckerProduct(evr, ID3), KroneckerProduct(ID3, coevr))) :$

$$coevr := \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ -\frac{1}{q} - \frac{q-1}{q} \\ 0 \\ \frac{1}{q} \\ 0 \\ 0 \end{bmatrix}$$

$$evr := \begin{bmatrix} 0 & 0 & q & 0 & -1 & 0 & -1 & 0 & 0 \end{bmatrix} \tag{14}$$

Finally, we confirm that the twist is, indeed, trivial:

$TwistV := simplify(Multiply(KroneckerProduct(Multiply(Matrix([0, 0, 1, 0, 1, 0, 1, 0, 0]), T2), Signs), KroneckerProduct(ID3, coevr)));$

$$TwistV := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

Quantum Dimension:

We can now compute the quantum dimension of V. This can be computed in two ways:

$qdim := Multiply(evr, coev);\ simplify(Multiply(ev, coevr));$

$$qdim := \begin{bmatrix} -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \end{bmatrix} \tag{16}$$

Some knot invariants:

We can now evaluate knot invariants for specific knots using the simple object V=L(1,0). As a consistency check, we compute different versions of the same knot or link.

Six versions of the ***Hopf link***:

$HL1 := simplify(Multiply(KroneckerProduct(evr, evr), Multiply(KroneckerProduct(ID3, KroneckerProduct(Multiply(C, E), ID3)), KroneckerProduct(coev, coev))));$

$$HL1 := \left[\begin{array}{c} 1 \end{array}\right] \tag{17}$$

$HL2 := simplify(Multiply(KroneckerProduct(evr, evr), Multiply(KroneckerProduct(ID3, KroneckerProduct(Multiply(Einv, Cinv), ID3)), KroneckerProduct(coev, coev))));$

$$HL2 := \left[\begin{array}{c} 1 \end{array}\right] \tag{18}$$

$HL3 := simplify(Multiply(KroneckerProduct(evr, ev), Multiply(KroneckerProduct(ID3, KroneckerProduct(Multiply(B, B), ID3)), KroneckerProduct(coev, coevr))));$

$$HL3 := \left[\begin{array}{c} 1 \end{array}\right] \tag{19}$$

$HL4 := simplify(Multiply(KroneckerProduct(evr, ev), Multiply(KroneckerProduct(ID3, KroneckerProduct(Multiply(Binv, Binv), ID3)), KroneckerProduct(coev, coevr))));$

$$HL4 := \left[\begin{array}{c} 1 \end{array}\right] \tag{20}$$

$HL5 := simplify(Multiply(KroneckerProduct(ev, ev), Multiply(KroneckerProduct(ID3, KroneckerProduct(Multiply(E, C), ID3)), KroneckerProduct(coevr, coevr))));$

$$HL5 := \left[\begin{array}{c} 1 \end{array}\right] \tag{21}$$

$HL5 := simplify(Multiply(KroneckerProduct(ev, ev), Multiply(KroneckerProduct(ID3, KroneckerProduct(Multiply(Cinv, Einv), ID3)), KroneckerProduct(coevr, coevr))))$

$$HL5 := \left[\begin{array}{c} 1 \end{array}\right] \tag{22}$$

Several versions of the ***trefoil knot:***

$TF1 := simplify(Multiply(evr, Multiply(KroneckerProduct(ID3, KroneckerProduct(evr, ID3)), Multiply(KroneckerProduct(Ainv, Binv), Multiply(KroneckerProduct(ID3, KroneckerProduct(E, ID3)), KroneckerProduct(coev, coev))))));$

$$TF1 := \left[\begin{array}{c} -1 \end{array}\right] \tag{23}$$

$TF2 := simplify(Multiply(KroneckerProduct(evr, evr), Multiply(KroneckerProduct(ID3, KroneckerProduct(Multiply(B, Multiply(B, B)), ID3)), KroneckerProduct(coev, coev))));$

$$TF2 := \begin{bmatrix} -1 \end{bmatrix} \tag{24}$$