This file accompanies the paper [LW] R. Laugwitz, G. Sanmarco: Finite-dimensional quantum groups of type Super A and non-semisimple modular categories, ArXiv preprint arXiv:2301.10685 and its eventual published version.

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Content: This file computes: Content: This file computes:(for n even, s a root of unity of 2n-order)

- The action of u_q=u_q(sl_{2,I}) on W=L(n,n+1)
- The composition series of W\otimes W
- The braiding from the R-matrix
- The endomorphism ring End_{u_q}(sl_{2,I})
- The Skein relation for the braiding.

```
with(LinearAlgebra) : interface(rtablesize = 30) : with(linalg) :
with(ArrayTools) :
```

We start by computing the action of the generators of u_q(sl_{2,I}) on W=L(n,n+1).
Define identity matrices:

```
ID4 := IdentityMatrix(4) : ID16 := IdentityMatrix(16) :
```

Now need a procedure to check a matrix is zero:

```
IsZeroMatrix :=proc(A :: Matrix)
local v, n, m, i, j;
n, m := Size(A);
i := 1; j := 1;
v := true;
while (i < n and v = true) do
while (j < m and v = true) do
v := is(A[i, j] = 0);
j := j + 1;
end do;
i := i + 1;
end do;
return v;
end proc;
```

```
IsZeroMatrix := proc(A::Matrix)                                    (1)
    local v, n, m, i, j;
    n, m := ArrayTools:-Size(A);
```

```
    i := 1;
    j := 1;
    v := true;
    while i < n and v = true do
        while j < m and v = true do v := is(A[i, j] = 0); j := j + 1 end do; i := i + 1
    end do;
    return v
end proc
```

Actions of y_1, y_2, x_1,x_2, k_1,k_2: Uses basis v0,v1,v1',v2, k_i=gamma_i\ov{gamma}_i.

$$Ay1 := Matrix\left(\left[\langle 0, 1, 0, 0\rangle, \langle 0, 0, 0, 0\rangle, \left\langle 0, 0, 0, \frac{2}{1+s^{-1}}\right\rangle, \langle 0, 0, 0, 0\rangle\right]\right);$$

$$Ay1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{1+\frac{1}{s}} & 0 \end{bmatrix} \tag{2}$$

$$Ay2 := Matrix([\langle 0, 0, 1, 0\rangle, \langle 0, 0, 0, 1\rangle, \langle 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0\rangle])$$

$$Ay2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{3}$$

$$Ax1 := Matrix\left(\left[\langle 0, 0, 0, 0\rangle, \langle 1+s, 0, 0, 0\rangle, \langle 0, 0, 0, 0\rangle, \left\langle 0, 0, 1+s^{-1}, 0\right\rangle\right]\right);$$

$$Ax1 := \begin{bmatrix} 0 & 1+s & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\frac{1}{s} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

$Ax2 := Matrix\left(\left[\langle 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0\rangle, \langle 2, 0, 0, 0\rangle, \left\langle 0, 1+s^{-1}, 0, 0\right\rangle\right]\right);$

$$Ax2 := \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1+\frac{1}{s} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

$Ak1 := simplify\left(Matrix\left(\left[\langle -1, 0, 0, 0\rangle, \left\langle 0, -s^{-1}, 0, 0\right\rangle, \langle 0, 0, -1, 0\rangle, \left\langle 0, 0, 0, -s^{-1}\right\rangle\right]\right)\right);$

$Akn1 := simplify\left(Matrix\left(\left[\left\langle (-1)^n, 0, 0, 0\right\rangle, \left\langle 0, -(-1)^n, 0, 0\right\rangle, \left\langle 0, 0, (-1)^n, 0\right\rangle, \left\langle 0, 0, 0, -(-1)^n\right\rangle\right]\right)\right);$

$$Ak1 := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{s} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{1}{s} \end{bmatrix}$$

$$Akn1 := \begin{bmatrix} (-1)^n & 0 & 0 & 0 \\ 0 & -(-1)^n & 0 & 0 \\ 0 & 0 & (-1)^n & 0 \\ 0 & 0 & 0 & -(-1)^n \end{bmatrix} \tag{6}$$

$Ak2 := simplify\left(Matrix\left(\left[\langle -s, 0, 0, 0\rangle, \langle 0, -s, 0, 0\rangle, \langle 0, 0, -1, 0\rangle, \langle 0, 0, 0, -1\rangle\right]\right)\right);$

$Akn2 := simplify\left(Matrix\left(\left[\left\langle -(-1)^n, 0, 0, 0\right\rangle, \left\langle 0, -(-1)^n, 0, 0\right\rangle, \left\langle 0, 0, (-1)^n, 0\right\rangle, \left\langle 0, 0, 0, (-1)^n\right\rangle\right]\right)\right);$

$$Ak2 := \begin{bmatrix} -s & 0 & 0 & 0 \\ 0 & -s & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$Akn2 := \begin{bmatrix} -(-1)^n & 0 & 0 & 0 \\ 0 & -(-1)^n & 0 & 0 \\ 0 & 0 & (-1)^n & 0 \\ 0 & 0 & 0 & (-1)^n \end{bmatrix} \quad \textbf{(7)}$$

Test that the relations hold:

$IsZeroMatrix(simplify(Multiply(Ay1, Ax1) + Multiply(Ax1, Ay1) - ID4 + Ak2));$

$$true \quad \textbf{(8)}$$

$IsZeroMatrix(simplify(Multiply(Ay2, Ax2) + Multiply(Ax2, Ay2) - ID4 + Ak1));$

$$true \quad \textbf{(9)}$$

$IsZeroMatrix(simplify(Multiply(Ay1, Ax2) - Multiply(Ax2, Ay1)));$

$$true \quad \textbf{(10)}$$

$IsZeroMatrix(simplify(Multiply(Ay2, Ax1) - s \cdot Multiply(Ax1, Ay2)));$

$$true \quad \textbf{(11)}$$

$IsZeroMatrix(Multiply(Ax1, Ax1)); IsZeroMatrix(Multiply(Ax2, Ax2)); IsZeroMatrix(Multiply(Ay1, Ay1)); IsZeroMatrix(Multiply(Ay2, Ay2));$

$IsZeroMatrix(Multiply(Ak1, Ay1) - s^{-1} \cdot Multiply(Ay1, Ak1)); IsZeroMatrix(Multiply(Ak1, Ax1) - s \cdot Multiply(Ax1, Ak1));$

$IsZeroMatrix(Multiply(Ak2, Ay2) - s^{-1} \cdot Multiply(Ay2, Ak2)); IsZeroMatrix(simplify(Multiply(Ak2, Ax2) - s \cdot Multiply(Ax2, Ak2)));$

$$true \quad \textbf{(12)}$$

$$true \quad \textbf{(12)}$$

$$true \quad \textbf{(12)}$$

$$true \quad \textbf{(12)}$$

$$true \quad \textbf{(12)}$$

$$true \quad \textbf{(12)}$$

$$true \tag{12}$$

$$true \tag{12}$$

$IsZeroMatrix(Multiply(Ak1, Ay2) - Multiply(Ay2, Ak1)); IsZeroMatrix(Multiply(Ak2, Ay1) - Multiply(Ay1, Ak2));$

$$true$$

$$true \tag{13}$$

Hence we have implemented the action of u_q = u_q(sl_{2,I}) on W.

Concatenate matrices in order to compute intersections of kernels: This way, one can compute eigenvectors or highest weight vectors. The nullspace of simultaneous action of x_1 and x_2 is computed next.

$Ax := Matrix(Concatenate(1, Ax1, Ax2)):$
$NullSpace(Ax);$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{14}$$

Tensor product calculations:

We set n equal to a specific number to help Maple simplify signs. N needs to be even in order to obtain a knot invariant (otherwise, the twist is -Id_W), see [LW, Lemma 7.1].

$n := 1000;$

$$n := 1000 \tag{15}$$

Optional: Specify s to be a certain even root of unity of order 2n.

$s := \mathrm{I}; n := 2;$

We substitute these values in the previously defined *Akn1, Akn2:*

$Akn1 := simplify(Akn1) : Akn2 := simplify(Akn2) :$

We define the tensor product actions of the generators using the coproduct formula.

$ADy1 := KroneckerProduct(Ay1, ID4) + KroneckerProduct(Akn1, Ay1)$:
$ADy2 := KroneckerProduct(Ay2, ID4) + KroneckerProduct(Multiply(Akn2, Ak1), Ay2)$:
$ADx2 := KroneckerProduct(Ax2, ID4) + KroneckerProduct(Akn2, Ax2)$:
$ADx1 := KroneckerProduct(Ax1, ID4) + KroneckerProduct(Multiply(Akn1, Ak2), Ax1)$:
$ADk1 := KroneckerProduct(Ak1, Ak1)$:
$ADk2 := KroneckerProduct(Ak2, Ak2)$:
$ADkn1 := KroneckerProduct(Akn1, Akn1)$:
$ADkn2 := KroneckerProduct(Akn2, Akn2)$:

Test: Relations hold for tensor product action

$IsZeroMatrix(simplify(Multiply(ADy1, ADx1) + Multiply(ADx1, ADy1) - IdentityMatrix(16) + ADk2));$

$$true \quad (16)$$

$IsZeroMatrix(simplify(Multiply(ADy2, ADx2) + Multiply(ADx2, ADy2) - IdentityMatrix(16) + ADk1)); IsZeroMatrix(simplify(Multiply(ADy1, ADx2) - Multiply(ADx2, ADy1)));$
$IsZeroMatrix(simplify(Multiply(ADy2, ADx1) - s \cdot Multiply(ADx1, ADy2)));$
$IsZeroMatrix(simplify(Multiply(ADx1, ADx1))); IsZeroMatrix(simplify(Multiply(ADx2, ADx2)));$
$IsZeroMatrix(Multiply(ADy1, ADy1)); IsZeroMatrix(Multiply(ADy2, ADy2));$

$$true$$
$$true$$
$$true$$
$$true$$
$$true$$
$$true$$
$$true \quad (17)$$

Tensor product decomposition of L(n,n+1)\otimes L(n,n+1):

Finding the highest weight vectors:

$ADx := Matrix(Concatenate(1, ADx1, ADx2))$:
$Hwt := NullSpace(ADx); w1 := Hwt[1] : w2 := Hwt[2] : w3 := Hwt[3]$:

$$Hwt := \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{s} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{18}$$

Lowest weight vectors:

$ADy := Matrix(Concatenate(1, ADy1, ADy2))$:
$Lwt := NullSpace(ADy); u1 := Lwt[1] : u2 := Lwt[2] : u3 := Lwt[3] :$

$$Lwt := \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{s} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \tag{19}$$

Tensor product decomposition:

Next, we define a method *S* to compute the PBW basis of the subspace generated by a fixed highest weight vector. This is done by letting y_1 and y_2 act in alternating form.
Similarly, *T* computes a basis for a subspace generated by a lowest weight vector throught the action of x_1, x_2 in alternation.

$S := w \rightarrow simplify(Matrix([w, Multiply(ADy1, w), Multiply(ADy2, Multiply(ADy1, w)),$
$Multiply(ADy1, Multiply(ADy2, Multiply(ADy1, w))), Multiply(ADy2, Multiply(ADy1,$
$Multiply(ADy2, Multiply(ADy1, w)))), Multiply(ADy1, Multiply(ADy2, Multiply(ADy1,$
$Multiply(ADy2, Multiply(ADy1, w))))), Multiply(ADy2, w), Multiply(ADy1, Multiply(ADy2,$

$w))$, $Multiply(ADy2, Multiply(ADy1, Multiply(ADy2, w)))$, $Multiply(ADy1, Multiply(ADy2, Multiply(ADy1, Multiply(ADy2, w))))$, $Multiply(ADy2, Multiply(ADy1, Multiply(ADy2, Multiply(ADy1, Multiply(ADy2, w)))))]))$;

$$S := w \mapsto simplify(Matrix([w, Multiply(ADy1, w), Multiply(ADy2, Multiply(ADy1, w)), Multiply(ADy1, Multiply(ADy2, Multiply(ADy1, w))), Multiply(ADy2, Multiply(ADy1, Multiply(ADy2, Multiply(ADy1, w)))), Multiply(ADy1, Multiply(ADy2, Multiply(ADy1, Multiply(ADy2, Multiply(ADy1, w))))), Multiply(ADy2, w), Multiply(ADy1, Multiply(ADy2, w)), Multiply(ADy2, Multiply(ADy1, Multiply(ADy2, w))), Multiply(ADy1, Multiply(ADy2, Multiply(ADy1, Multiply(ADy2, w)))), Multiply(ADy2, Multiply(ADy1, Multiply(ADy2, Multiply(ADy1, Multiply(ADy2, w)))))])) \tag{20}$$

$T := w \rightarrow simplify(Matrix([w, Multiply(ADx1, w), Multiply(ADx2, Multiply(ADx1, w)), Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, w))), Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, w)))), Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, w))))), Multiply(ADx2, w), Multiply(ADx1, Multiply(ADx2, w)), Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, w))), Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, w)))), Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, w)))))]))$;

$$T := w \mapsto simplify(Matrix([w, Multiply(ADx1, w), Multiply(ADx2, Multiply(ADx1, w)), Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, w))), Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, w)))), Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, w))))), Multiply(ADx2, w), Multiply(ADx1, Multiply(ADx2, w)), Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, w))), Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, w)))), Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, Multiply(ADx1, Multiply(ADx2, w)))))])) \tag{21}$$

First, we get an 8-dimensional indecomposable module generated by h. wt vector in degree (0,2). This is not simple: 0---> L(0,1)--->u_q(sl_{2,I})*w_3--->L(0,2)-->0

$S(w3)$;

$$\left[\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{2\,s}{1+s} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-1+s}{1+s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+s}{1+s} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{2\,s}{1+s} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-1+s}{1+s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+s}{1+s} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{(-1+s)^2}{s\,(1+s)} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{22}$$

$T(u1);$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{s^2-1}{s^2} & 0 & 0 & 0 & 0 & \frac{(1+s)^2(-1+s)}{s^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1+s}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{s^2-1}{s} & 0 & 0 & 0 & 0 & \frac{(1+s)^2(-1+s)}{s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{s^2-1}{s^2} & 0 & 0 & 0 & 0 \\
0 & 1+\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1+\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1+s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad \textbf{(23)}
$$

Next highest weight vector gives a 4-d space (highest weight is in degree (n-1,2), so this is a simple module!

$simplify(S(w2));$

$$\left[\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1+\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1-s}{s^2} & 0 & 0 & 0 & 0 & \frac{1-s}{s\,(1+s)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1+\frac{1}{s} & 0 & 0 & 0 & 0 & \frac{1-s}{1+s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{24}$$

$simplify(T(u3));$

$$\left[\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-s^2+1}{s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1+\frac{1}{s} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-s^2+1}{s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1+\frac{1}{s} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{25}$$

The next module is a submodule of the first one. It's a 3-dimensional simple module L(0,1):

$S(w1);$

$$\left[\begin{array}{ccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2 s}{1+s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1+s}{1+s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2 s}{1+s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1+s}{1+s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \textbf{(26)}$$

$T(u2);$

$$
\left[\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & -\frac{(1+s)^2(-1+s)}{s^2} & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & \frac{2s+2}{s} & 0 & 0 & 0 & 0 & \frac{(1+}{s} \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & \frac{(1+s)^2}{s^2} & 0 & 0 & 0 & 0 & \frac{(1+}{s} \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1+\frac{1}{s} & 0 \cdots \\
0 & 0 & 0 & -\frac{(1+s)^2(-1+s)}{s^2} & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & \frac{(1+s)^2}{s^2} & 0 & 0 & 0 & 0 & \frac{(1+}{s^2} \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 1+\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & \frac{2s+2}{s} & 0 & 0 & 0 & 0 & \frac{(1+}{s} \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1+\frac{1}{s} & 0 \cdots \\
0 & 1+\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots
\end{array}\right] \quad \textbf{(27)}
$$

The remaining vectors:

Modulo u_q(sl_{2.I})*v_0\otimes v_0, the vector v_1 generates a direct summand L(0,0).

$v1 := KroneckerProduct(\langle 0, 0, 1, 0\rangle, \langle 0, 0, 1, 0\rangle)$:

$Multiply(ADx1, v1); Multiply(ADx2, v1); simplify(Multiply(ADy1, v1)); Multiply(ADy2, v1);$

$$\left[\begin{array}{c}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{array}\right]$$

$$\begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{2 s}{1 + s} \\ 0 \\ 0 \\ \frac{2 s}{1 + s} \\ 0 \end{array}\right]$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

After quotienting by u_q(sl_{2,I})*v_0\otimes v_0, there is another direct summand of the form L(0,1), a 3-dimensional simple module.

$v2 := a \cdot KroneckerProduct(\langle 1, 0, 0, 0 \rangle, \langle 0, 0, 1, 0 \rangle) + b \cdot KroneckerProduct(\langle 0, 0, 1, 0 \rangle, \langle 1, 0, 0, 0 \rangle) :$
$Multiply(ADx1, v2) : Multiply(ADx2, v2) :$
$simplify(S(v2));$

$$\left[\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
a & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{2 s a}{1+s} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & a & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
b & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & b & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & a-b & \cdots \\
0 & 0 & \frac{2 a}{1+\frac{1}{s}}-b & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{2 b s}{1+s} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \frac{a(1+s)-2 b}{1+s} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{2 s(-1+s)(a-b)}{(1+s)^{2}} & 0 & 0 & 0 & \cdots
\end{array}\right] \quad \textbf{(29)}$$

Conclusion: We have the following extension structure:

W\otimes W=L(n,n+1)\otimes L(n,n+1) has direct summand L(2n-1,2) and a submodule M=uq*(v_0\otimes v_0) of dimension 8 with composition series 0-->L(0,1)-->M-->L(0,2)-->0.

W\otimes W/M= L(2n-1,2) + L(0,1) + L(0,0) this splits into a direct sum of simples.

Observation: W\otimes W is not generated by highest weight vectors.

Action on the dual of W:

Note: h acts by S(h)^T with respect to the dual basis.

$ASk1 := MatrixInverse(Ak1);$

$$ASk1 := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -s & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -s \end{bmatrix} \quad \textbf{(30)}$$

$ASk2 := MatrixInverse(Ak2);$

$$ASk2 := \begin{bmatrix} -\frac{1}{s} & 0 & 0 & 0 \\ 0 & -\frac{1}{s} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \textbf{(31)}$$

$ASkn1 := MatrixInverse(Akn1);$

$$ASkn1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \textbf{(32)}$$

$ASkn2 := MatrixInverse(Akn2)$

$$ASkn2 := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(33)}$$

$ASx1 := -Multiply(Multiply(ASkn1, ASk2), Ax1);$

$$ASx1 := \begin{bmatrix} 0 & \frac{1+s}{s} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\frac{1}{s} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{34}$$

$ASx2 := -Multiply(ASkn2, Ax2);$

$$ASx2 := \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1+\frac{1}{s} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

$ASy2 := -Multiply(Multiply(ASkn2, ASk1), Ay2);$

$$ASy2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \end{bmatrix} \tag{36}$$

$ASy1 := -Multiply(ASkn1, Ay1);$

$$ASy1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{1+\frac{1}{s}} & 0 \end{bmatrix} \tag{37}$$

We can test some relations:

$IsZeroMatrix(simplify(Multiply(ASx1, ASy1) + Multiply(ASy1, ASx1) - IdentityMatrix(4) + ASk2));$

$$true \tag{38}$$

Reproducing the braiding using R-matrix formula:

We define Ad0 as the action action by \delta_{n,n+1}, Ad1 as the action by action by \delta_{n-1,n+1}, Ad1dash as the action by \delta_{n,n} ,Ad2 as the action action by \delta_{n-1,n}.

$Ad0 := Matrix([\langle 1,0,0,0\rangle, \langle 0,0,0,0\rangle, \langle 0,0,0,0\rangle, \langle 0,0,0,0\rangle]) : Ad1 := Matrix([\langle 0,0,0,0\rangle, \langle 0,1,0,0\rangle, \langle 0,0,0,0\rangle, \langle 0,0,0,0\rangle]) : Ad1dash := Matrix([\langle 0,0,0,0\rangle, \langle 0,0,0,0\rangle, \langle 0,0,1,0\rangle, \langle 0,0,0,0\rangle]) : Ad2 := Matrix([\langle 0,0,0,0\rangle, \langle 0,0,0,0\rangle, \langle 0,0,0,0\rangle, \langle 0,0,0,1\rangle]) :$

Swap map:

```
Swap := Matrix(16, 16) :
for i from 1 by 1 to 4 do
 for j from 1 by 1 to 4 do
 Swap[4·(i − 1) + j, 4·(j − 1) + i] := 1 :
 end do;
 end do;
```

Now we define the action by divided powers:

$Ay12 := simplify\left((1-s)^{-1} \cdot (Multiply(Ay1, Ay2) - s \cdot Multiply(Ay2, Ay1))\right);$

$$Ay12 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{s}{1+s} & 0 & 0 & 0 \end{bmatrix} \tag{39}$$

$Ax21 := simplify(Multiply(Ax2, Ax1) - Multiply(Ax1, Ax2));$

$$Ax21 := \begin{bmatrix} 0 & 0 & 0 & \frac{-s^2+1}{s} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

Braiding:

These formulas are simifications of the R-matrix given in [LW,Cor. 5.12] valid only when n is even. The group elements have been simplified taking into account that Akni^n=1 since n is even, for i=1,2.

```
Hv0 := simplify( Multiply( Swap, Multiply( KroneckerProduct( Multiply( Multiply( Akn1, Akn2 ), Ak1 ),
    ID4 ), Multiply( KroneckerProduct( ID4, ID4 ), KroneckerProduct( ID4, Ad0 ) ) ) +
Multiply( KroneckerProduct( Multiply( Multiply( Akn1, Akn2 ), Ak1 ), ID4 ),
    Multiply( KroneckerProduct( Ay1, Ax1 ), KroneckerProduct( ID4, Ad0 ) ) ) +
Multiply( KroneckerProduct( Multiply( Multiply( Akn1, Akn2 ), Ak1 ), ID4 ),
    Multiply( KroneckerProduct( Ay2, Ax2 ), KroneckerProduct( ID4, Ad0 ) ) ) +
Multiply( KroneckerProduct( Multiply( Multiply( Akn1, Akn2 ), Ak1 ), ID4 ),
    Multiply( KroneckerProduct( Ay12, Ax21 ), KroneckerProduct( ID4, Ad0 ) ) ) +
Multiply( KroneckerProduct( Multiply( Multiply( Akn1, Akn2 ), Ak1 ), ID4 ),
    Multiply( KroneckerProduct( Multiply( Ay2, Ay1 ), Multiply( Ax2, Ax1 ) ), KroneckerProduct( ID4,
    Ad0 ) ) ) ) ) :
 Hv1 := simplify( Multiply( Swap, Multiply( KroneckerProduct( Multiply( Ak1, Akn2 ), ID4 ),
    Multiply( KroneckerProduct( ID4, ID4 ), KroneckerProduct( ID4, Ad1 ) ) ) +
Multiply( KroneckerProduct( Multiply( Ak1, Akn2 ), ID4 ), Multiply( KroneckerProduct( Ay1, Ax1 ),
    KroneckerProduct( ID4, Ad1 ) ) ) +
Multiply( KroneckerProduct( Multiply( Ak1, Akn2 ), ID4 ), Multiply( KroneckerProduct( Ay2, Ax2 ),
    KroneckerProduct( ID4, Ad1 ) ) ) +
Multiply( KroneckerProduct( Multiply( Ak1, Akn2 ), ID4 ), Multiply( KroneckerProduct( Ay12, Ax21 ),
    KroneckerProduct( ID4, Ad1 ) ) ) + Multiply( KroneckerProduct( Multiply( Ak1, Akn2 ), ID4 ),
    Multiply( KroneckerProduct( Multiply( Ay2, Ay1 ), Multiply( Ax1, Ax2 ) ), KroneckerProduct( ID4,
    Ad1 ) ) ) ) ) :
Hv1dash := simplify( Multiply( Swap, Multiply( KroneckerProduct( Akn1, ID4 ),
    Multiply( KroneckerProduct( ID4, ID4 ), KroneckerProduct( ID4, Ad1dash ) ) ) +
Multiply( KroneckerProduct( Akn1, ID4 ), Multiply( KroneckerProduct( Ay1, Ax1 ),
    KroneckerProduct( ID4, Ad1dash ) ) ) +
Multiply( KroneckerProduct( Akn1, ID4 ), Multiply( KroneckerProduct( Ay2, Ax2 ),
    KroneckerProduct( ID4, Ad1dash ) ) ) +
Multiply( KroneckerProduct( Akn1, ID4 ), Multiply( KroneckerProduct( Ay12, Ax21 ),
    KroneckerProduct( ID4, Ad1dash ) ) ) + Multiply( KroneckerProduct( Akn1, ID4 ),
    Multiply( KroneckerProduct( Multiply( Ay2, Ay1 ), Multiply( Ax1, Ax2 ) ), KroneckerProduct( ID4,
    Ad1dash ) ) ) ) ) :
 Hv2 := simplify( Multiply( Swap, Multiply( KroneckerProduct( ID4, ID4 ),
    Multiply( KroneckerProduct( ID4, ID4 ), KroneckerProduct( ID4, Ad2 ) ) ) +
Multiply( KroneckerProduct( ID4, ID4 ), Multiply( KroneckerProduct( Ay1, Ax1 ),
    KroneckerProduct( ID4, Ad2 ) ) ) +
Multiply( KroneckerProduct( ID4, ID4 ), Multiply( KroneckerProduct( Ay2, Ax2 ),
    KroneckerProduct( ID4, Ad2 ) ) ) +
Multiply( KroneckerProduct( ID4, ID4 ), Multiply( KroneckerProduct( Ay12, Ax21 ),
```

$KroneckerProduct(ID4, Ad2))) + Multiply(KroneckerProduct(ID4, ID4),$
$Multiply(KroneckerProduct(Multiply(Ay2, Ay1), Multiply(Ax1, Ax2)), KroneckerProduct(ID4,$
$Ad2))))):$

$Hnew := Hv0 + Hv1 + Hv1dash + Hv2;$

$$Hnew := \left[\begin{array}{ccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1+s}{s} & 0 & 0 & -\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{3s+1}{s} & 0 & 0 & -2 & 0 & 0 & -2 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1+\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+\frac{1}{s} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1+\frac{1}{s} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right] \quad (41)$$

We now define the action of the brading on the first two (respectively, last two) tensor factors and check the QYBE:

$Hnew12 := KroneckerProduct(Hnew, ID4):$
$Hnew23 := KroneckerProduct(ID4, Hnew):$

$simplify(Multiply(Hnew23, Multiply(Hnew12, Hnew23)) - Multiply(Hnew12, Multiply(Hnew23, Hnew12))) :$

We can check that Hnew (the braiding) is indeed a **morphism of uq-modules** W\otimes W-->W\otimes W: For instance, we compute:

$IsZeroMatrix(simplify(Multiply(ADk2, Hnew) - Multiply(Hnew, ADk2)));$

$$true \tag{42}$$

Endomorphisms of W\otimes W:

As the u_q-module W is not projective, it is necessary to compute the endomorphisms of W\otimes W to check the existence of a generalized trace for [LW, Section 7].

The following method can check if a given 16x16 matrix commutes with the action of a 16x16 matrix.

$A := Matrix(16, 16, symbol = a) : M := Matrix(16, 16, symbol = m) :$
$M := simplify(ADx1) : A := Hnew :$

$va := convert(A, Vector) :$
$MM := DiagonalMatrix([M, M, M, M, M, M, M, M, M, M, M, M, M, M, M, M]) : Multiply(MM, va) :$

$MMdual := Matrix(256, 256) :$
for i **from** 1 **to** 16 **do**
for j **from** 1 **to** 16 **do**
$MMdual[16\cdot(i-1)+1..16\cdot(i-1)+16, 16\cdot(j-1)+1..16\cdot(j-1)+16] := M[j, i]\cdot ID16 :$
end do
end do;
$MMdual :$

$simplify(Multiply(MM, va) - Multiply(MMdual, va)) :$

Systematic listing of the commutation operator matrices for the action of x1,x2,y1,y2,k1,k2:

$Ox1 := MM - MMdual :$
$M := ADx2 : MM := DiagonalMatrix([M, M, M, M, M, M, M, M, M, M, M, M, M, M, M, M]) :$
$MMdual := Matrix(256, 256) :$
for i **from** 1 **to** 16 **do**
for j **from** 1 **to** 16 **do**

$MMdual[16\cdot(i-1)+1..16\cdot(i-1)+16, 16\cdot(j-1)+1..16\cdot(j-1)+16] := M[j,i]$
$\cdot ID16$:
end do
end do;
$Ox2 := MM - MMdual$:

$M := ADy1 : MM := DiagonalMatrix([M, M, M, M, M, M, M, M, M, M, M, M, M, M, M, M])$:
$MMdual := Matrix(256, 256)$:
for i **from** 1 **to** 16 **do**
for j **from** 1 **to** 16 **do**
$MMdual[16\cdot(i-1)+1..16\cdot(i-1)+16, 16\cdot(j-1)+1..16\cdot(j-1)+16] := M[j,i]$
$\cdot ID16$:
end do
end do;
$Oy1 := MM - MMdual$:

$M := ADy2 : MM := DiagonalMatrix([M, M, M, M, M, M, M, M, M, M, M, M, M, M, M, M])$:
$MMdual := Matrix(256, 256)$:
for i **from** 1 **to** 16 **do**
for j **from** 1 **to** 16 **do**
$MMdual[16\cdot(i-1)+1..16\cdot(i-1)+16, 16\cdot(j-1)+1..16\cdot(j-1)+16] := M[j,i]$
$\cdot ID16$:
end do
end do;
$Oy2 := MM - MMdual$:
$M := ADk1 : MM := DiagonalMatrix([M, M, M, M, M, M, M, M, M, M, M, M, M, M, M, M])$:
$MMdual := Matrix(256, 256)$:
for i **from** 1 **to** 16 **do**
for j **from** 1 **to** 16 **do**
$MMdual[16\cdot(i-1)+1..16\cdot(i-1)+16, 16\cdot(j-1)+1..16\cdot(j-1)+16] := M[j,i]$
$\cdot ID16$:
end do
end do;
$Ok1 := MM - MMdual$:

$M := ADk2 : MM := DiagonalMatrix([M, M, M, M, M, M, M, M, M, M, M, M, M, M, M, M])$:
$MMdual := Matrix(256, 256)$:
for i **from** 1 **to** 16 **do**

```
for j from 1 to 16 do
MMdual[16·(i − 1) + 1..16·(i − 1) + 16, 16·(j − 1) + 1..16·(j − 1) + 16] := M[j, i]
    ·ID16:
end do
end do;
Ok2 := MM − MMdual:
```

Test: We test that the brading commutes with these actions (we can replace Ok2 by the other cummutation operator matrices, corresponding to the actions of x_1,y_1, etc. and see if that the vector is zero)

```
simplify(Multiply(Ox2, va)):
```

Doing the above for all generators shows that the braiding is a morphism of u_q-modules.

Endomorphism ring computation:

We are now ready to compute the endormophism space End(W\otimes W).
We stack the cummutation operators on top of one another. Then compute the Nullspace of this combined operator that checks commuting with the generators.

```
CommOp := simplify(Matrix(Concatenate(1, Ox1, Ox2, Oy1, Oy2, Ok1, Ok2))):
```

```
Endo := simplify(NullSpace(CommOp)): E1 := simplify(Reshape( convert(Endo[1], Array), [16,
    16])); E2 := simplify(Reshape( convert(Endo[2], Array), [16, 16])); E3 :=
    simplify(Reshape( convert(Endo[3], Array), [16, 16]));
E1 :=
```

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1+s}{s} & 0 & 0 & -\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{3s+1}{s} & 0 & 0 & -2 & 0 & 0 & -2 & 0 & 0 & -\cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1+s}{s} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+s}{s} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1+s}{s} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{bmatrix}
$$

$E2 :=$

$$
E3 := \left[\begin{array}{ccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -\frac{2\,s}{-1+s} & 0 & 0 & \frac{2\,s}{-1+s} & 0 & 0 & \frac{2\,s}{-1+s} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{-1-s}{-1+s} & 0 & 0 & \frac{1+s}{-1+s} & 0 & 0 & \frac{1+s}{-1+s} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{-1-s}{-1+s} & 0 & 0 & \frac{1+s}{-1+s} & 0 & 0 & \frac{1+s}{-1+s} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots \\
0 & 0 & 0 & -\frac{2\,s}{-1+s} & 0 & 0 & \frac{2\,s}{-1+s} & 0 & 0 & \frac{2\,s}{-1+s} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right] \qquad \textbf{(43)}
$$

$$
\left[\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & -\frac{1}{s} & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1+s}{s\,(-1+s)} & 0 & 0 & -\frac{2}{-1+s} & 0 & 0 & -\cdots \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \frac{-1+s}{s} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1+s}{s\,(-1+s)} & 0 & 0 & -\frac{2}{-1+s} & 0 & 0 & -\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{s} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1+s}{s\,(-1+s)} & 0 & 0 & -\frac{2}{-1+s} & 0 & 0 & -\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1+s}{-1+s} & 0 & 0 & -\frac{2\,s}{-1+s} & 0 & 0 & -\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
$$

$nops(Endo);$

$$3 \tag{44}$$

The last number computes the dimension of End_{u_q}(W\otimes W). The three three linearly independent endomorphisms of W\otimes W are displayed as matrices.

Check that all endomorphisms of W\otimes W commute with the braiding:

$IsZeroMatrix(simplify(Multiply(E1, Hnew) - Multiply(Hnew, E1)));$
$IsZeroMatrix(simplify(Multiply(E2, Hnew) - Multiply(Hnew, E2)));$
$IsZeroMatrix(simplify(Multiply(E3, Hnew) - Multiply(Hnew, E3)));$

$$true$$
$$true$$
$$true \tag{45}$$

The above commutators are all zero, showing that all endomorphisms of W\otimes W commute with the braiding. This means that a generalized trace exist for W.

Skein relation:

We aim to verify the Skein relation for the braiding Hnew.

First, we create a better basis and Note that E1+E2+E3=Id_{W\otimes W}, E1=Psi.

We define E to be the matrix with the three basis vectors of End(W\otimes W) as columns, and then solve for the braiding to be expressed in this basis. Recall that the vector form of the braiding is defined as the variable *va*. We also express the identity in this basis:

$E := Matrix([Endo[1], Endo[2], Endo[3]]) : LinearSolve(E, va);$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{46}$$

$LinearSolve(E, convert(ID16, Vector));$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{47}$$

Now we can also express the inverse braiding in terms of the given basis.

$Hnewinv := MatrixInverse(Hnew) :$
$LinearSolve(E, convert(Hnewinv, Vector));$

$$\begin{bmatrix} 1 \\ 2 \\ 1+s \end{bmatrix} \tag{48}$$

We can also consider a basis consising of Psi, Id, Psi^{-1}. This is summarized in the matrix Ebraid.

$Ebraid := Matrix([convert(Hnew, Vector), convert(ID16, Vector), convert(Hnewinv, Vector)])$:
$Rank(Ebraid)$;

$$3 \tag{49}$$

Then we can express Psi^2 (and, to check, also Psi^{-2}) in these bases and obtain a minimal relationship.

$LinearSolve(Ebraid, convert(simplify(Multiply(Hnew, Hnew)), Vector))$;

$$\begin{bmatrix} \frac{2s+1}{s} \\ -\frac{s+2}{s} \\ \frac{1}{s} \end{bmatrix} \tag{50}$$

$LinearSolve(Ebraid, convert(simplify(Multiply(Hnewinv, Hnewinv)), Vector))$;

$$\begin{bmatrix} s \\ -2s-1 \\ s+2 \end{bmatrix} \tag{51}$$

Multiplying by Psi, the first relation says that

Psi^3+(2+q^{-1})*Psi^2 +(2q^{-1}+1) *Psi + q^{-1}*Id = 0

The second relation says, multiplying by Psi^2, that

q*Psi^3 +(1+2q)*Psi^2 + (q+2)*Psi + Id= 0

These are the same monic polynomial.

Projection onto direct summands:

There is a third basis for End_{u_q}(W\otimes W), which is given by projections onto the direct summands of W\otimes W = L(2n-1,2) \oplus M, see [LW, Lemma 7.5].
We can compute the projection onto these submodules by looking for idempotent endomorphisms: The general ansatz is that we are looking for E = x*E_1 + y*E_2+ z*E_3 to be an idempotent.

$Idem := LinearSolve(E, simplify(convert(Multiply(x \cdot E1 + y \cdot E2 + z \cdot E3, x \cdot E1 + y \cdot E2 + z \cdot E3), Vector)));$

$$Idem := \begin{bmatrix} x^2 \\ -x^2 + 2\,x\,y \\ \dfrac{s\,z^2 - x^2 + 2\,x\,z - z^2}{s} \end{bmatrix} \tag{52}$$

$Vars := Vector(3, \{1 = x, 2 = y, 3 = z\});$

$$Vars := \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{53}$$

We solve for the above expression in x,y,z to equal the vector [x,y,z].

$solve(\{Idem[1] - Vars[1], Idem[2] - Vars[2], Idem[3] - Vars[3]\}, \{x, y, z\});$

$$\{x = 0, y = 0, z = 0\}, \left\{x = 0, y = 0, z = \frac{s}{-1 + s}\right\}, \{x = 1, y = 1, z = 1\}, \left\{x = 1, y = 1, z = -\frac{1}{-1 + s}\right\} \tag{54}$$

There are three solutions, the identity (x,y,z)=(1,1,1), and the non-trivial solutions (x,y,z)=(0,0,q/(q-1)) and (x,y,z)= (1,1,1/(1-q)).

We define the corresponding projection matrices Pr1, Pr2:

$Pr1 := convert\left(simplify\left(E1 + E2 + \dfrac{1}{(1 - s)} \cdot E3\right), Matrix\right);$

$$Pr1 := \tag{55}$$

$$\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
0 & \frac{s}{-1+s} & 0 & 0 & -\frac{1}{-1+s} & 0 & 0 & 0\cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\cdots \\
0 & 0 & 0 & \frac{s(s-3)}{(-1+s)^2} & 0 & 0 & \frac{2s}{(-1+s)^2} & 0\cdots \\
0 & \frac{s}{-1+s} & 0 & 0 & \frac{1}{1-s} & 0 & 0 & 0\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
0 & 0 & 0 & \frac{-1-s}{(-1+s)^2} & 0 & 0 & \frac{s^2+1}{(-1+s)^2} & 0\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{s}{-1}\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
0 & 0 & 0 & \frac{-1-s}{(-1+s)^2} & 0 & 0 & \frac{2s}{(-1+s)^2} & 0\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
0 & 0 & 0 & -\frac{s(1+s)}{(-1+s)^2} & 0 & 0 & \frac{2s^2}{(-1+s)^2} & 0\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{s}{-1}\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots
\end{array}\right]$$

We check that this matrix is indeed idempotent and compute its rank:

$IsZeroMatrix(simplify(Multiply(Pr1, Pr1) - Pr1)); Rank(Pr1);$

$$true$$

$$12 \tag{56}$$

This shows that Pr1 is the projection onto a 12-dimensional module, the module M.

$Pr2 := convert(simplify(E3), Matrix);$

$$Pr2 := \left[\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & -\frac{1}{s} & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1+s}{s\,(-1+s)} & 0 & 0 & -\frac{2}{-1+s} & 0 & 0 & -\cdots \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \frac{-1+s}{s} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1+s}{s\,(-1+s)} & 0 & 0 & -\frac{2}{-1+s} & 0 & 0 & -\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{s} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1+s}{s\,(-1+s)} & 0 & 0 & -\frac{2}{-1+s} & 0 & 0 & -\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1+s}{-1+s} & 0 & 0 & -\frac{2\,s}{-1+s} & 0 & 0 & -\cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right] \quad (57)$$

We check that this matrix is indeed idempotent and compute its rank:

$IsZeroMatrix(simplify(Multiply(Pr2, Pr2) - Pr2)); Rank(Pr2);$

$$true$$

$$4 \quad (58)$$

This shows that Pr2 is the projection onto a 4-dimensional module, the 4-dimensional simple module L (2n-1,2).

Now we can define the new basis *Enew* for the endomorphism space: Pr2, ID16, Psi.

$$Enew := Matrix([convert(ID16, Vector), convert(Pr2, Vector), convert(Hnew, Vector)]) :$$
$$Rank(Enew);$$

$$3 \tag{59}$$

To find the alternative Skein relation from [LW, Equation (7.2.1)], we express the inverse braiding Psi^{-1}, stored in the matrix Hnewinv in terms of this basis Enew:

$$LinearSolve(Enew, convert(Hnewinv, Vector));$$

$$\begin{bmatrix} 2 \\ -1+s \\ -1 \end{bmatrix} \tag{60}$$

This verifies the following Skein relation:

Psi+Psi^{-1} +(-1)^n 2Id+ (q-1)Pr_2=0