Problem Set 2
Random Variables

1 The loaded die

A die is loaded so that the probability of any side showing is proportional to the number on that side. If the die is rolled and you win 1 dollar for every dot showing, what is the probability distribution for X, the number of dollars won? What is the probability that X is less than 4?

Solution. The probability of any number showing is proportional to the number on that side. What this means is that an outcome of 2 is twice as likely as an outcome of 1, that 3 is three times as likely as 1, etc. Let X be the random variable for the number of dollars won (which is equal to the outcome of a single roll of the die). If the probability of getting a 1 is w, then $\mathbb{P}(X = k)$, the probability of getting any number k for $1 \leq k \leq 6$, is kw. Since probabilities must sum to one, we know that

$$\begin{aligned} 1 &= \mathbb{P}(X=1) + \mathbb{P}(X=2) + \mathbb{P}(X=3) + \mathbb{P}(X=4) + \mathbb{P}(X=5) + \mathbb{P}(X=6) = \sum_{k=1}^{6} \mathbb{P}(X=k) \\ &= \sum_{k=1}^{6} kw = w + 2w + 3w + 4w + 5w + 6w = 21w \end{aligned}$$

so that $w = \frac{1}{21}$ and thus $\mathbb{P}(X = k) = \frac{k}{21}$ for $k = 1, 2, \ldots, 6$. To calculate the expected value of the number of dollars won, we can use

$$\mathbb{E}[X] = \sum_{k=1}^{6} k\mathbb{P}(X=k) = 1 \cdot \frac{1}{21} + 2 \cdot \frac{2}{21} + 3 \cdot \frac{3}{21} + 4 \cdot \frac{4}{21} + 5 \cdot \frac{5}{21} + 6 \cdot \frac{6}{21} = \sum_{k=1}^{6} \frac{k^2}{21} = \$4.33.$$

For the last part of the problem, X being less than 4 means that $X = 1, 2$ or 3; so:

$$\mathbb{P}(X < 4) = \sum_{k=1}^{3} \mathbb{P}(X=k) = \sum_{k=1}^{3} \frac{k}{21} = \frac{2}{7}.$$

2 Interconnected parts

Parts that are connected in a system do not fail independently of one another. In particular, two interconnected parts, A and B, fail in the following pattern. On any given day there is a 20% chance that B will fail, while if A fails, there is a 0.7 probability that B will fail on the same day. The probability that both parts fail on the same day is 0.07. On any given day, what is the most likely number of parts to fail?

Solution. Let A denote the event where A fails, and let B denote the event where B fails. Thus, $\mathbb{P}(B) = 0.2$, $\mathbb{P}(B|A) = 0.7$, $\mathbb{P}(B \cap A) = 0.07$.

This problem requires that we figure out the probability distribution of the random variable X, describing how many parts fail on a given day. Either no parts can fail (neither A nor B), one part can fail (either A or B but not both) or both can fail (A and B); the last probability, $\mathbb{P}(X = 2)$ is given as 0.07.

We have that $\mathbb{P}(B) = 0.2$, but we do not know $\mathbb{P}(A)$. Rearranging the formula for conditional probability, we get

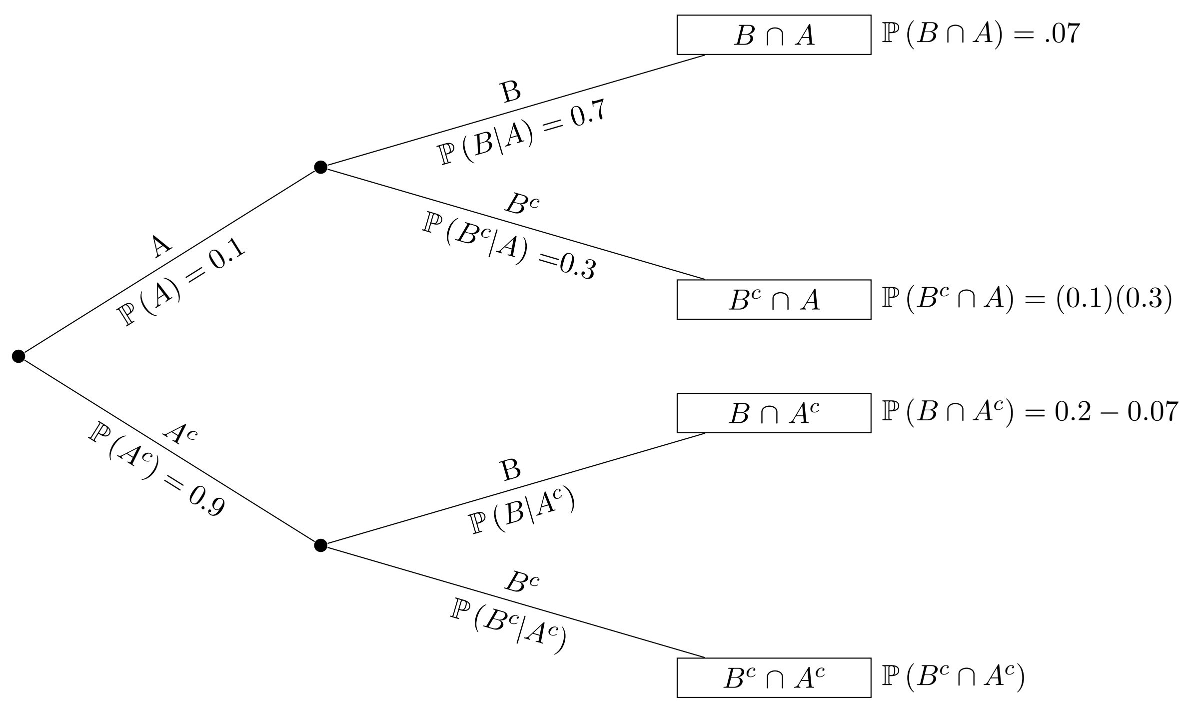
$$\mathbb{P}(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B|A)} = \frac{0.07}{0.7} = 0.1.$$

The probability of getting neither A nor B, $\mathbb{P}(X = 0)$, is

$$\begin{aligned} \mathbb{P}(X = 0) = 1 - \mathbb{P}(A \cup B) &= 1 - [\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)] \\ &= 1 - [(0.1) + (0.2) - (0.07)] \\ &= 0.77. \end{aligned}$$

Since probabilities must sum to one, we have $\mathbb{P}(X = 0) = 0.77$, $\mathbb{P}(X = 1) = 0.16$ and $\mathbb{P}(X = 2) = 0.07$, and the most likely number of parts to fail is 0.

We can obtain the same result by constructing a probability tree and a probability table. This construction follows next.



	B	B^c	
A	$\mathbb{P}(A \cap B)$	$\mathbb{P}(A \cap B^c)$	0.1
A^c	$\mathbb{P}(A^c \cap B)$	$\mathbb{P}(A^c \cap B^c)$	0.9
	0.20	0.80	1

We can fill the gaps in the tree and then in the table, and we obtain:

	B	B^c	
A	$\mathbb{P}(X=2)$ 0.07	$\mathbb{P}(X=1)$ 0.03	0.10
A^c	$\mathbb{P}(X=1)$ 0.13	$\mathbb{P}(X=0)$ 0.77	0.90
	0.20	0.80	1

3 Flexible manufacturing system

In a particular FMS (flexible manufacturing system) setup, it can take either 1, 2, 3, or 4 hours to change over a production line. If a changeover takes at least 3 hours, there is a 90% probability that it will take 4 hours. If a changeover takes at least 2 hours, there is a 0.5 probability that it will take at least 3 hours. The probability that a changeover is completed in 1 hour is 0.20. Let X be the number of hours required for a changeover. What is the probability distribution for X?

Solution. $\mathbb{P}(X=1) = 0.2$ so, $\mathbb{P}(X \geq 2) = 0.8$. Furthermore, $\mathbb{P}(X \geq 3 | X \geq 2) = 0.5$ and $\mathbb{P}(X = 4 | X \geq 3) = 0.9$.

Note that, because we are dealing with discrete numbers of hours, a statement like $\mathbb{P}(X \geq 2)$ really means $\mathbb{P}(X = 2, 3, \text{ or } 4)$. To figure out the probability distribution is to figure out each of the individual probabilities $\mathbb{P}(X=1)$ through $\mathbb{P}(X=4)$.

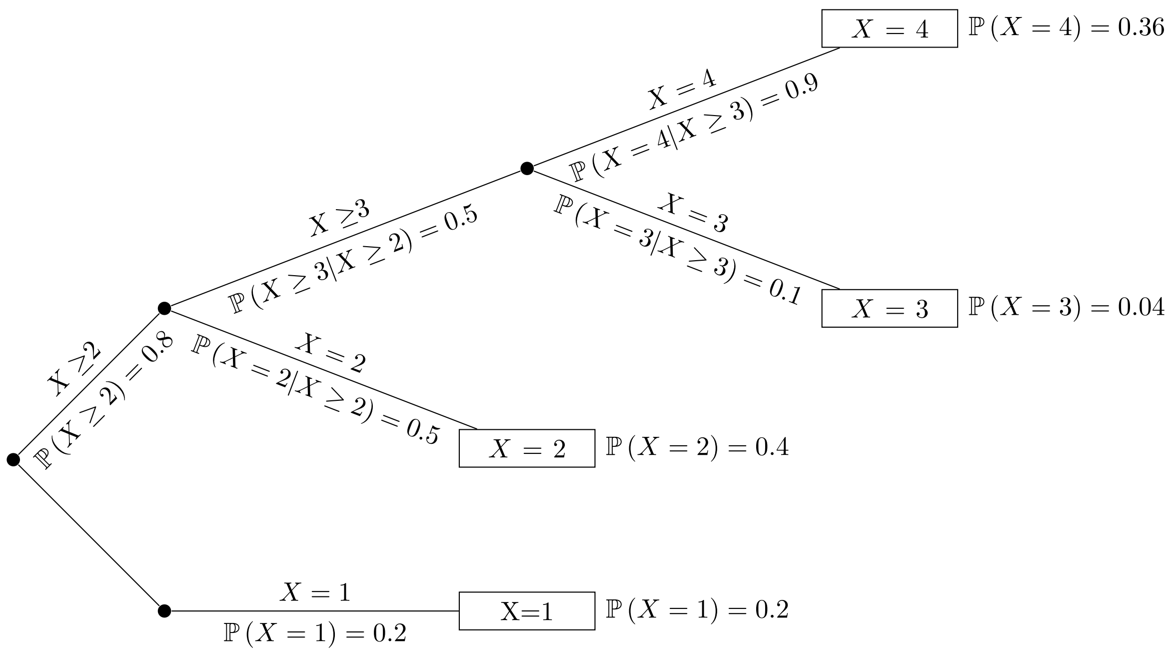
We can use the given information as follows:

$$\begin{aligned} \mathbb{P}(X \geq 3) &= \mathbb{P}(X \geq 3 | X \geq 2)\,\mathbb{P}(X \geq 2) = 0.5(0.8) = 0.4 \\ \mathbb{P}(X = 4) &= \mathbb{P}(X = 4 | X \geq 3)\,\mathbb{P}(X \geq 3) = 0.9(0.4) = 0.36 \end{aligned}$$

So, $\mathbb{P}(X=4) = 0.36$, and $\mathbb{P}(X=3) = 0.04$. Since probabilities must sum to 1 and we are given that $\mathbb{P}(X=1) = 0.2$, we have $\mathbb{P}(X=2) = 0.4$. Thus, the distribution for X is

$$\mathbb{P}(X=1) = 0.2, \qquad \mathbb{P}(X=2) = 0.4, \qquad \mathbb{P}(X=3) = 0.04, \qquad \mathbb{P}(X=4) = 0.36.$$

We could also have seen this using a tree:



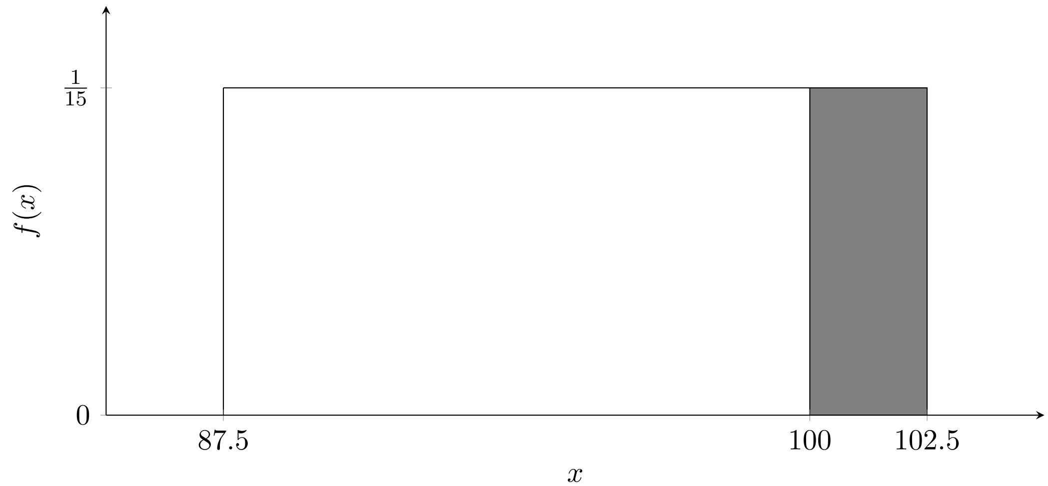
4 Durham temperatures

A continuous distribution with a constant density is called uniform (see textbook). For instance, suppose that the daily high temperature in Durham in July is uniformly distributed between 87.5 degrees and 102.5 degrees. Then the probability density function for the daily high temperature X is given by

$$f(x) = \begin{cases} \frac{1}{15} & \text{if } 87.5 \leq x \leq 102.5 \\ 0 & \text{elsewhere} \end{cases}$$

Draw this probability density function and verify that the area under the curve defined by the density function equals one. What is the probability that the daily high temperature on July 7 is greater than 100? What is the probability that the daily high temperature on July 7 is greater than or equal to 100?

Solution. Below on the left is the probability distribution (or probability density function (pdf)) in question, a uniform distribution extending from 87.5 to 102.5. Probabilities of events are given by the associated areas. Note that the entire area is that of a box of length 15 and height $\frac{1}{15}$, or 1, as it should be. The probability of the event "the daily high temperature on July 7 is greater than 100" is the area that lies beyond 100; this is a rectangle of length 2.5 and height $\frac{1}{15}$, showing the probability to be $\frac{1}{6}$. The probability that the temperature is greater than or equal to 100 is also $\frac{1}{6}$, as the temperature being exactly equal to 100 (when temperature is considered a continuous random variable) is a zero probability event.



5 Oil drilling

An oil wildcatter owns drilling rights at two widely separated locations. After consulting a geologist, he feels that at each location the odds against discovering oil if a well is drilled are 9 to 1. A well costs \$100,000 to drill, and this is a total loss if no oil is found. On the other hand, if oil is discovered, rights to the oil can be sold for \$1,600,000. The wildcatter has \$100,000 available for drilling expenses. Find the mean and standard deviation of the wildcatter's profit:

a) if the \$100,000 is used to drill a single well,

(Hint: Work with profits in units of \$100,000 to simplify calculations.)

Solution. The probability of finding oil in a specific well is given here as 0.1. If oil is found, profit (X) is given by 15 (in units of \$100,000); the probability of this event is 0.1. If oil is not found, profit is given by -1; the probability of this event is $(1 - 0.1) = 0.9$. We must find the mean and standard deviation of profit, X:

$$\begin{aligned}
\mathbb{E}[X] &= (-1)\mathbb{P}(X = -1) + 15\mathbb{P}(X = 15) = -1(0.9) + 15(0.1) = 0.6 \\
\text{Var}[X] &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 = (-1)^2(0.9) + (15)^2(0.1) - (0.6)^2 = 23.04 \\
\sqrt{\text{Var}[X]} &= \sqrt{23.04} = 4.80.
\end{aligned}$$

b) if the wildcatter finds a partner to share costs and profits equally (each will receive $\frac{1}{2}$ of the final profit, positive or negative) and their pooled funds are used to drill wells in two different locations.

Solution. The wildcatter can also take a partner. In this case, when two wells are drilled, there are four mutually exclusive events that can take place: neither well will have oil with probability equal to $(0.9)(0.9) = 0.81$; the first will have oil and the second will not with probability equal to $(0.1)(0.9) = 0.09$; the first won't have oil and the second will with probability equal to $(0.9)(0.1) = 0.09$; both will have oil with probability $(0.1)(0.1) = 0.01$. The profit to the wildcatter associated with each of these cases is merely the total profit divided in half, or -1, 7, 7, and 15 respectively.

We have all the information necessary to calculate the mean and variance of profit, X:

$$\begin{aligned}
\mathbb{E}\left[X\right] &= (-1)\mathbb{P}\left(X=-1\right) + 7\mathbb{P}\left(X=7\right) + 15\mathbb{P}\left(X=15\right) = -1(0.81) + 7(0.18) + 15(0.01) = 0.6 \\
\mathrm{Var}\left[X\right] &= \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2 = (-1)^2(0.81) + (7)^2(0.18) + (15)^2(0.1) - (0.6)^2 = 11.52 \\
\sqrt{\mathrm{Var}\left[X\right]} &= \sqrt{11.52} = 3.39.
\end{aligned}$$

Notice that, by the averaging of two independent identical agents, the mean profit stayed the same, while the variance was halved, reducing the standard deviation from 4.80 to 3.39.