Non-Stochastic Dynamic Programming

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Note: Formal statements and proofs of the theory presented here can be found in Stokey and Lucas (1989). Further, these slides are a working draft. Please email me with any errata.

What Does DP Do?

Goal

Solve problems like

$$v^*(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \quad \text{s.t.} \tag{1}$$
$$x_{t+1} \in \Gamma(x_t), \quad \text{all } t$$

a "day at a time"

- Stokey and Lucas (1989) call (1) the "sequence problem"

"A Day at a Time"

What do I mean? It can be verified that

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) = \max_{x_1} \left\{ F(x_0, x_1) + \beta \max_{\{x_{t+1}\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \right\}$$

subject to $x_{t+1} \in \Gamma(x_t)$

- Imagine that you have solved the inner maximization
- This yields a function $V(x_1)$ which tells you the value of starting the next day in state x_1

If we knew $V()$, the outer maximization would be simple. We solve

$$\max_{x_1} \{F(x_0, x_1) + \beta V(x_1)\} \quad \text{s.t. } x_1 \in \Gamma(x_0)$$

But We Don't Know $V(x)$...

We don't know $V(x)$, but we can convince ourselves that

$$V(x) = \max_{y} \{F(x, y) + \beta V(y)\} \quad \text{s.t. } y \in \Gamma(x) \tag{2}$$

- "Bellman's Equation", i.e. "Functional Equation"
- Note the change of notation...time doesn't matter, states matter $\rightarrow$ "stationary" dynamic programming
- Solving (2),
  - Guess and verify
  - Successive approximations (or guess and update, if you will)
  - Using the Euler Equation
  - And more...

Does This Solve (1)?

- It turns out that the value function v^* in (1) solves (2) under general conditions (Stokey and Lucas, 1989, Theorem 4.2)
- But, there may be solutions v to (2) which are not v^*
  - If v satisfies a boundedness condition (Stokey and Lucas, 1989, Theorem 4.3) $\implies$ $v = v^*$

Principal of Optimality (PO)

Assume from now that we have a solution v to FE where $v = v^*$.[1] Then the PO is a characterization of *optimal* plans: i.e. sequences $\{x_t^*\}_{t=0}^{\infty}$ that attain v^*.

The PO States

- An optimal plan that attains $v^* \implies v^*(x_t) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*)$ (Necessary)
- A plan satisfying $v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*)$ is optimal (Somewhat Sufficient)
  - Only "somewhat" because need boundedness condition (Stokey and Lucas, 1989, Theorem 4.5)

$\implies$ PO is almost an equivalence theorem. If we put more structure on the problem than Stokey and Lucas (1989) Assumptions 4.1 and 4.2, PO becomes an equivalence

[1] See the discussion of assumptions 4.1 and 4.2 of Stokey and Lucas (1989). When we combine these two assumptions with the above equivalence theorems, this is a reasonable assumption (Why?)

Where to Go Now

The theory so far is, perhaps, too general. We would like to make assumptions that:

- Ensure solutions to FE are s.t. $v = v^*$
- Ensure policies which satisfy the PO achieve v^*

Some Useful Assumptions

So...what assumptions to make? Assume

- State space X is a convex subset of $\mathbb{R}^n$
- $\Gamma : X \rightrightarrows X$ is nonempty, compact-valued and continuous
- Return function F is bounded and continuous and $\beta \in (0, 1)$ (We can relax the bounded)
- F is striclty concave (link to definition)
- Γ is convex

$\therefore$ for every state, there is a unique best action (concave programming) $\rightarrow$ policy correspondence is a continuous (theorem of the max) policy function

Note

If we add strict increasingness in "states" to F and that loosening constraints only provides us with more possibilities $\rightarrow$ v is monotone

When is v Differentiable?

Important question. For example, in the one-sector growth model

$$u'\left(f(x) - g(x)\right) - \beta v'(g(x)) = 0$$

is the FOC if we knew v

- Finding optimal policy is just a standard calculus problem
- v is differentiable if we add assumption that F is continuously differentiable (Thm. 4.11 SLP)
- There are also some interiority requirements that I am leaving out

Remarks

The rest of chapter 4 deals with the case of unbounded returns.

References I

Stokey, N. L. and Lucas, R. E. (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press.