Global Wellposedness and long time behaviour of solutions to the 2-d nonlinear Wave equation with white noise

Robert Wegner

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0.1 Notation

Define $\mathbb{R}_+ \equiv \{x \in \mathbb{R} : x \geq 0\}$ and $B = B_1(0) \subseteq \mathbb{R}^2$. We will work with the space

$$L^2_{\text{rad}} \equiv L^2_{\text{rad}}(B) \equiv L^2_{\text{rad}}(B) \equiv \{f \in L^2(B) : f \circ T = f \text{ for all } T \in SO(2)\}.$$

We may write L^2 instead of L^2_{rad} and H^α instead of H^α_{rad}. Unless otherwise specified, the bracket $\langle \cdot, \cdot \rangle$ refers to the inner product in L^2 or evaluation of distributions, depending on the context.
We may write L^2_t, L^2_x and $L^2_{t,x}$ to refer to $L^2(\mathbb{R}_+)$, $L^2(B)$ and $L^2(\mathbb{R}_+ \times B)$ respectively.
In a context where $(e_n)_{n\in\mathbb{N}}$ is an ONB of some Hilbert space H, we introduce the notation

$$\widehat{f}(n) \equiv \langle f, e_n \rangle_H$$

and occasionally write $\mathcal{F}(f)$ for $\widehat{f}$.
For $\lambda \in \mathbb{R}$ define

$$\langle \lambda \rangle = \sqrt{1 + |\lambda|^2}$$
$$\left[\lambda\right] = \sqrt{\frac{3}{4} + |\lambda|^2}.$$

Note that $\langle \lambda \rangle \cong \left[\lambda\right]$ for large n.

For two sequences $a_n, b_n \in \mathbb{R}$ we write $a_n \sim b_n$ if both $\frac{a_n}{b_n}$ and $\frac{b_n}{a_n}$ are bounded, and $a_n \cong b_n$ if $\lim \frac{b_n}{a_n} = \lim \frac{b_n}{a_n} = 1$.

0.2 Function Spaces

We proceed as Tzvetkov does in the first section of [3] with a basis of rescalings of the zero order Bessel function

$$J_0(z) = \frac{2}{\pi} \frac{\cos(x - \pi/4)}{\sqrt{x}} + \mathcal{O}(x^{-\frac{3}{2}}).$$

The corresponding transform is also called the Hankel transform. Skipping the details, we are given an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2_{\text{rad}}(B)$ consisting of smooth functions. Crucially, the e_n are eigenfunctions of $-\Delta$ with corresponding eigenvalues

$$0 < |\lambda_1|^2 < |\lambda_2|^2 < \dots .$$

We say that $|\lambda_n|^2$ is the n-th Eigenvalue because this corresponds to λ_n being the n-th root of $x \mapsto e_1(|x|)$, the zero order Bessel function. It also keeps our notation consistent with the usage of a Fourier basis on the torus, as instances of $|n|^2$ are now replaced with $|\lambda_n|^2$. The exact formula for the e_n is

$$e_n(x) = \frac{J_0(\lambda_n |x|)}{\|J_0(\lambda_n \cdot)\|_{L^2(B)}}.$$

Note that $|\lambda_n| \cong n$ for large n. ([**Section 1.2**, 3]).
We call this basis $(e_n)_{n\in\mathbb{N}}$ the **Bessel function basis**. In this basis we can compute the Laplacian of a radial test function $g \in C_c^\infty(B)$ as

$$-\Delta g = -\Delta \sum_{n=0}^{\infty} e_n \langle g, e_n\rangle_{L^2_{\text{rad}}} = \sum_{n=0}^{\infty} |\lambda_n|^2 e_n \langle g, e_n\rangle_{L^2_{\text{rad}}}.$$

This allows us to define a scale of Sobolev spaces and corresponding fractional powers of $(1-\Delta)$. We abstractly define these spaces as subspaces of $\mathbb{R}^{\mathbb{N}}$. For $\alpha \in \mathbb{R}$ define

$$H^\alpha_{\text{rad}} \equiv \{f \in \mathbb{R}^{\mathbb{N}} : \|f\|_{H^\alpha_{\text{rad}}} < \infty\}$$

where

$$\|f\|_{H^\alpha_{\text{rad}}} \equiv \|(1-\Delta)^{\frac{\alpha}{2}} f\|_{H^0_{\text{rad}}} \equiv \sum_{n=1}^{\infty} \langle\lambda_n\rangle^{2\alpha} f_n^2.$$

and

$$\left((1-\Delta)^{\frac{\alpha}{2}} f\right)_n \equiv (1+|\lambda_n|^2)^{\frac{\alpha}{2}} f_n = \langle\lambda_n\rangle^\alpha f_n.$$

The inner product is then given by

$$\langle f, g\rangle_{H^\alpha_{\text{rad}}} = \sum_{n=1}^{\infty} \langle\lambda_n\rangle^{2\alpha} f_n g_n.$$

For $\alpha \geq 0$ the Hilbert space space H^α_{rad} is isomorphic to $W^{\alpha,2}_{0,\text{rad}}(B)$, the subspace of radial functions of the Sobolev space $W^{\alpha,2}_0(B)$, up to a possible scaling of the norm by a constant. The isomorphism is

$$f \longmapsto \sum_{n=1}^{\infty} f_n e_n.$$

This means that for all $\alpha \geq 0$ we can identify H^α_{rad} with subspaces of $L^2_{\text{rad}}(B)$. We will do so and treat their elements as equivalence classes of functions. For $\alpha < 0$ the elements of H^α_{rad} can not necessarily be represented by measurable functions on B, but there is a different representation. Let $\alpha \geq 0$. Then

$$\begin{aligned} H^{-\alpha}_{\text{rad}} &\longrightarrow (H^\alpha_{\text{rad}})^* \\ f &\longmapsto \left[g \mapsto \sum_{n=1} f_n g_n\right] \end{aligned}$$

is an isomorphism.
Under these identifications for $f \in H^\alpha_{\text{rad}}$, $\alpha \in \mathbb{R}$ we could write

$$f = \sum_{n=1}^{\infty} \langle f, e_n\rangle e_n,$$

where for $\alpha \geq 0$ the inner product is the one from L^2_{rad}, and for $\alpha < 0$ it is the dual pairing. In the latter case it just happens to be the case that the right hand side is not necessarily a.e. convergent. The norm is given by

$$\|f\|^2_{H^\alpha_{\text{rad}}} = \sum_{n=1}^{\infty} \langle\lambda_n\rangle^{2\alpha} |\hat{f}(n)|^2.$$

For the sake of clarity, let us write down explicitly some different ways of expressing the same thing

$$\begin{aligned} \alpha \geq 0 : &\qquad \hat{f}(n) = f_n = \langle f, e_n\rangle_{L^2_{\text{rad}}} = \int_B f e_n \, dx, \\ \alpha < 0 : &\qquad \hat{f}(n) = f_n = \langle f, e_n\rangle = f(e_n). \end{aligned}$$

From now on we will always write just H^α and L^2. We define the Hilbert space $\mathcal{H}^\alpha \equiv H^\alpha \times H^{\alpha-1}$ and use the following notations for such pairs of functions:

$$u = (u_1, u_2) = (\pi_1 u, \pi_2 u) = (u, u_t).$$

We can also define fractional Sobolev spaces for $p > 1$ and $\alpha \geq 0$ in this fashion via

$$\|f\|_{W^{\alpha,p}_{\mathrm{rad}}} = \|(1-\Delta)^{\frac{\alpha}{2}} f\|_{L^p_{\mathrm{rad}}} = \||(1-\Delta)^{\frac{\alpha}{2}} f|^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2_{\mathrm{rad}}}.$$

We define $\mathcal{W}^{\alpha,p} = W^{\alpha,p} \times W^{\alpha-1,p}$ for $\alpha \geq 1$.

0.3 Space-time White Noise

We equip the space of radial test functions $\mathcal{D}(\mathbb{R}_+ \times B) = C_c^\infty(\mathbb{R}_+ \times B) \cap L^2_{\mathrm{rad}}(\mathbb{R}_+ \times B)$ with the usual topology. We fix some measure space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 0.1. *(i) A* ***random space-time distribution*** *ξ is a continuous linear map*

$$\xi : \mathcal{D}(\mathbb{R}_+ \times B) \longrightarrow L^2(\Omega).$$

We analogously define a ***random space distribution*** *and* ***random time distribution****. We may also consider random distributions in several variables, i.e. test functions in $\mathcal{D}\left(\bigsqcup^k (\mathbb{R}_+ \times B)\right) \cong \mathcal{D}(\mathbb{R}_+ \times B)^2$ for $k \in \mathbb{N}$.*

(ii) A random space-time distribution ξ is called ***space-time white noise*** *if $\langle \xi, f\rangle$ is a centered Gaussian random variable and*

$$\mathbb{E}\left[\langle \xi, f\rangle\langle \xi, g\rangle\right] = \langle f, g\rangle_{L^2_{t,x}}$$

for all $f, g \in \mathcal{D}(\mathbb{R} \times B)$.

(iii) Let B be a one-dimensional Brownian motions. We define a random time distribution dW. For $f \in \mathcal{D}(\mathbb{R})$ set

$$\langle dW, f\rangle \equiv \int_{\mathbb{R}_+} f(s)\, dW(s).$$

As f is smooth and has compact support, we could choose a simple definition for the stochastic integral here:

$$\int_{\mathbb{R}_+} f(s)\, dW(s) \equiv -\int_{\mathbb{R}_+} W(s) \dot{f}(s)\, ds.$$

Lemma 0.2 (Construction of space-time white noise)**.** *Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of independent one-dimensional Brownian motions. Let e_n be any ONB of L^2_{rad} and let $f \in L^2_{t,x}$. Then*

$$\sum_{n=1}^N \langle dW_n, \langle f, e_n\rangle_{L^2_x}\rangle = \sum_{n=1}^N \int_{\mathbb{R}_+} \int_B f(t,x) e_n(x)\, dx\, dW_n(t)$$

for $N \in \mathbb{N}$ forms a Cauchy sequence in $L^2(\Omega)$. Restricting to $f \in \mathcal{D}(\mathbb{R}_+ \times B)$ and setting

$$\langle \xi, f\rangle = \lim_{N\to\infty} \sum_{n=1}^N \int_{\mathbb{R}_+} \int_B f(t,x) e_n(x)\, dx\, dW_n(t),$$

we have that ξ is space-time white noise.

Proof. Note that $\int_B f(t,x)e_n(x)\,dx$ is a test function in time and hence admissible for dW_n. For $N \leq M \in \mathbb{N}$ and any other test function g we calculate

$$\mathbb{E}\left[\sum_{n=N}^{M} \int_{\mathbb{R}_+}\int_B f(t,x)e_n(x)\,dx\,dW_n(t) \sum_{k=N}^{M}\int_{\mathbb{R}_+}\int_B g(t,x)e_k(x)\,dx\,dW_k(t)\right]$$

$$= \sum_{\substack{k,n=N\\k\neq n}}^{M} \mathbb{E}\left[\int_{\mathbb{R}_+}\int_B f(t,x)e_n(x)\,dx\,dW_n(t)\right]\mathbb{E}\left[\int_{\mathbb{R}_+}\int_B g(t,x)e_k(x)\,dx\,dW_k(t)\right]$$

$$+ \sum_{n=N}^{M}\mathbb{E}\left[\int_{\mathbb{R}_+}\int_B f(t,x)e_k(x)\,dx\,dW_k(t)\int_{\mathbb{R}_+}\int_B g(t,x)e_k(x)\,dx\,dW_k(t)\right]$$

The first sum vanishes since we are taking the expectations of local martingales starting in zero. For the second sum we use Itô isometry to get

$$= \sum_{n=N}^{M}\mathbb{E}\left[\int_{\mathbb{R}_+}\int_B f(t,x)e_n(x)\,dx\cdot\int_B g(t,x)e_n(x)\,dx\,dt\right]$$

$$= \int_{\mathbb{R}_+}\sum_{n=N}^{M}\widehat{f}(t,n)\widehat{g}(t,n)\,dt.$$

Setting $f = g$ we can follow from this that we indeed have a Cauchy sequence in $L^2(\Omega)$. As each element in the sequence depends linearly on f, the limit is also linear. Hence ξ is a random space-time distribution. Taking the limit on both sides of the computation, we see that

$$\mathbb{E}\left[\langle\xi,f\rangle\langle\xi,g\rangle\right] = \int_{\mathbb{R}_+}\langle f,g\rangle_{L^2_x}\,dt = \langle f,g\rangle_{L^2_{t,x}}.$$

We also know that time changes of Brownian Motion yield centered Gaussian random variables and hence the limit is also a centered Gaussian. Therefore ξ is space-time white noise. □

1 Local Well-posedness in $\mathcal{H}^\alpha$

In this section $(e_n)_{n\in\mathbb{N}}$ refers explicitly to the Bessel function basis unless otherwise specified.

To get Local well-posedness we will decompose the problem into three easier problems.

1. For a random initial data $w_0 \in \mathcal{H}^\alpha$ we solve the linear problem

$$\begin{aligned} w_{tt} + w_t + (1-\Delta)w &= 0\\ w(0) &= w_0\\ w_t(0) &= (w_0)_t. \end{aligned}$$

2. For a space-time white noise ξ we construct a mild solution to the linear problem with inhomogeneits $\sqrt{2}\xi$

$$\begin{aligned} \psi_{tt} + \psi_t + (1-\Delta)\psi &= \sqrt{2}\xi\\ \psi(0) &= 0\\ \psi_t(0) &= 0. \end{aligned}$$

This is called the stochastic convolution

3. Given w and ψ we now solve the deterministic and homogeneous nonlinear problem

$$\begin{aligned} v_{tt} + v_t + (1-\Delta)v + (w+\psi+v)^3 &= 0 \\ v(0) &= 0 \\ v_t(0) &= 0. \end{aligned}$$

Then $u = w + \psi + v$ solves the initial problem.

1.1 The Linear Problem with Random Initial Data

Let $\{g_n\}_{n\in\mathbb{N}}$, $\{h_n\}_{n\in\mathbb{N}}$ be families of standard Gaussian random variables $\sim \mathcal{N}(0,1)$ which are all independent (also g_n from h_m). We consider random intial data of the form

$$w_0 = \sum_{n=1}^{\infty} \begin{pmatrix} a_n g_n e_n \\ b_n h_n e_n \end{pmatrix}$$

where a_n, b_n are two sequences of real numbers.

Corollary 1.1. *For all $s \geq 0$ the following are equivalent:*

(i) $w_0 \in \mathcal{H}^\alpha$ *a.s.*

(ii) $\mathbb{P}(w_0 \in \mathcal{H}^\alpha) > 0$,

(iii) $\mathbb{E}\left[\|w_0\|^2_{\mathcal{H}^\alpha}\right] < \infty$,

(iv) $\left(\langle\lambda_n\rangle^\alpha a_n\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ *and* $\left(\langle\lambda_n\rangle^{\alpha-1} b_n\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$.

This is an easy consequence of the following lemma applied to $\pi_1 w_0$ and $\pi_2 w_0$ with $t = 1$ and f being constant.

Lemma 1.2. *Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of independent one-dimensional Brownian motions. Let e_n be any ONB of L^2_{rad}. Consider a stochastic process ψ of the form*

$$\psi(t) = \sum_{n=0}^{\infty} \int_0^t f_n(s)\, dW_n(s) \cdot e_n$$

for functions $f_n \in C(\mathbb{R}_+)$. Let $\alpha \geq 0$ and $T \geq 0$. Then the following are equivalent:

(i) $\psi \in C([0,T], H^\alpha)$ *a.s.*

(ii) $\mathbb{P}(\psi \in C([0,T], H^\alpha)) > 0$,

(iii) $\mathbb{E}\left[\|\psi(t)\|^2_{C([0,T],H^\alpha)}\right] < \infty$,

(iv) $\left(\langle\lambda_n\rangle^\alpha \|f_n\|_{L^2([0,T])}\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$.

In particular we have the estimate

$$\mathbb{E}\left[\|\psi(t)\|^2_{C([0,T],H^\alpha)}\right] \leq 2\left\|\left(\langle\lambda_n\rangle^\alpha \|f_n\|_{L^2([0,T])}\right)_{n\in\mathbb{N}}\right\|_{\ell^2(\mathbb{N})} = 2\,\mathbb{E}\left[\|\psi(T)\|^2_{H^\alpha}\right].$$

Proof. We start by showing $(iii) \iff (iv)$ and the estimate.
Observe that we can trivially move the supremum inside the sum, and the sum out of the integral by Fubini-Tonelli:

$$\mathbb{E}\left[\|\psi(t)\|^2_{C([0,T],H^\alpha)}\right] = \mathbb{E}\left[\sup_{0\leq t\leq T} \sum_{n=0}^{\infty} \left|\int_0^t f_n(s)\, dW_n(s)\right|^2 \langle\lambda_n\rangle^{2\alpha}\right]$$

$$\leq \sum_{n=0}^{\infty} \langle \lambda_n \rangle^{2\alpha} \, \mathbb{E}\left[\left| \sup_{0\leq t\leq T} \int_0^t f_n(s)\, dW_n(s) \right|^2 \right].$$

Since $\int_0^t f_n(s)\, dW_n(s)$ is a square integrable martingale we can use Doob's L^2 inequality and then Itô-isometry:

$$\leq \sum_{n=0}^{\infty} \langle \lambda_n \rangle^{2\alpha} \cdot 2 \sup_{0\leq t\leq T} \mathbb{E}\left[\left| \int_0^t f_n(s)\, dW_n(s) \right|^2 \right]$$

$$= 2 \sum_{n=0}^{\infty} \langle \lambda_n \rangle^{2\alpha} \int_0^T |f_n(s)|^2 \, ds.$$

For the reverse inequality we do the same steps except that the supremum is now absent and hence no inequalities are needed.
Clearly $(i) \implies (ii)$. Regarding $(iii) \implies (i)$, observe that for $t_1 < t_2 \in [0, T]$ we can calculate

$$\|\psi(t_2) - \psi(t_1)\|_{H^\alpha}^2 = \sum_{n=0}^{\infty} \langle \lambda_n \rangle^{2\alpha} \int_{t_1}^{t_2} |f_n(s)|^2 \, ds,$$

and since (iv) states that

$$\sum_{n=0}^{\infty} \langle \lambda_n \rangle^{2\alpha} \int_0^T |f_n(s)|^2 \, ds < \infty$$

we get that ψ is both continuous and bounded with respect to $\| \cdot \|_{H^\alpha}$. It now only remains to show $(ii) \implies (iii)$, or equivalently

$$\mathbb{E}\left[\|\psi\|_{C([0,T],H^\alpha)}^2 \right] = \infty \quad \implies \quad \mathbb{P}(\psi(t) \in C([0,T], H^\alpha)) = 0.$$

Let $\mathbb{E}\left[\|\psi\|_{C([0,T],H^\alpha)}^2 \right] = \infty$. Then our main inequality implies $\mathbb{E}\left[\|\psi(T)\|_{H^\alpha}^2 \right] = \infty$. We will can now conclude the proof by showing that $\mathbb{P}(\psi(T) \in H^s) = 0$.
Set $\phi = \psi(T)$ and note that

$$\phi = \sum_{n=1}^{\infty} \widehat{\phi}(n) e_n$$

where $\widehat{\phi}(n)$ are either independent Gaussian random variables or 0 (in which case we can leave them out). We want to show that

$$\mathbb{P}\left(\sum_{n=1} \langle \lambda_n \rangle^{2\alpha} |\widehat{\phi}(n)|^2 \text{ diverges } = 1 \right).$$

This follows from the subsequent lemma with

$$a_n \equiv \langle \lambda_n \rangle^{2\alpha} \, \mathbb{E}\left[|\widehat{\phi}(n)|^2 \right] \text{ and } X_n \equiv \frac{|\widehat{\phi}(n)|^2}{\mathbb{E}\left[|\widehat{\phi}(n)|^2 \right]}$$

since

$$\sum_{n=1}^{\infty} \langle \lambda_n \rangle^{2\alpha} \, \mathbb{E}\left[|\widehat{\phi}(n)|^2 \right] = \infty.$$

□

The following Lemma is a special case of Kolmogorov's three-series theorem.

Lemma 1.3. *Let X_n be a sequence of independent, non-zero and non-negative random variables with $\mathbb{E}[X_n] = 1$ and $\mathbb{E}\left[X_n^4\right] \leq C < \infty$ for all $n \in \mathbb{N}$. Let $a_n \geq 0$. Then*

$$\sum_{n=1}^{\infty} a_n X_n \text{ converges a.s.} \quad \Longleftrightarrow \sum_{n=1}^{\infty} a_n < \infty.$$

Proof. For the direction "$\Longleftarrow$" note that

$$\sum_{n=1}^{N} a_n = \mathbb{E}\left[\sum_{n=1}^{N} a_n X_n\right].$$

If this converges then it can't be the case that on a set of positive measure $\sum_{n=1}^{N} a_n X_n$ diverges.
We show "$\Longrightarrow$" by contradiction. Suppose that the sum on the right diverges. We assume for now that a_n is bounded. The other case will then be easy. We want to apply Lyapunov's Central Limit Theorem to

$$\sum_{n=0}^{N} a_n (X_n - 1).$$

For this we verify Lyapunov's condition for $\delta = 2$. It states that

$$\lim_{N\to\infty} \frac{1}{(s_N^2)^2} \sum_{n=1}^{N} \mathbb{E}\left[(a_n^2 (X_n - 1)^2)^2\right] = 0,$$

where s_N^2 is the sum of the variances:

$$s_N^2 = \sum_{n=1}^{N} a_n^2 \, \mathbb{E}\left[(X_n - 1)^2\right].$$

Let $\#$ be the counting measure on $\mathbb{N}$. We rewrite the above:

$$\frac{1}{(s_N^2)^2} \sum_{n=1}^{N} \mathbb{E}\left[(a_n^2 (X_n - 1)^2)^2\right] = \frac{1}{N} \frac{\frac{1}{N}\int_{\Omega\times\{1,\ldots,N\}} \left(a_n^2 (X_n - 1)^2\right)^2 \, d\,\mathbb{P} \times d\#}{\left(\frac{1}{N}\int_{\Omega\times\{1,\ldots,N\}} a_n^2 (X_n - 1)^2 \, d\,\mathbb{P} \times d\#\right)^2}.$$

Here the difference between the numerator and the denominator on the right is precisely the variance of the random variable $a_n^2 (X_n - 1)^2$ in the probability space $\Omega \times \{1, \ldots, N\}$ with measure $\mathbb{P} \times \#$. Since

$$\mathrm{Var}_{\Omega\times\{1,\ldots,N\}}(a_n^2 (X_n - 1)^2) \lesssim \sup_{n\in\mathbb{N}} a_n^4 \cdot \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}\left[X_n^4\right] \leq C \|a\|_\infty$$

we have a uniform bound in N for the fraction and so

$$\lim_{N\to\infty} = \frac{1}{N} \frac{\frac{1}{N}\int_{\Omega\times\{1,\ldots,N\}} \left(a_n^2 (X_n - 1)^2\right)^2 \, d\,\mathbb{P} \times d\#}{\left(\frac{1}{N}\int_{\Omega\times\{1,\ldots,N\}} a_n^2 (X_n - 1)^2 \, d\,\mathbb{P} \times d\#\right)^2} = 0.$$

We can therefore apply Lyapunov's CLT. Let $G \sim \mathcal{N}(0, 1)$ and consider the following calculation:

$$1 - \mathbb{P}\left(\sum_{n=1}^{\infty} a_n (X_n - 1) = \infty\right) \geq \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{K\geq N} \sum_{n=1}^{K} a_n (X_n - 1) \leq 0\right)$$

$$= \lim_{N\to\infty} \mathbb{P}\left(\bigcup_{K\geq N} \sum_{n=1}^{K} a_n (X_n - 1) \leq 0\right) \geq \lim_{N\to\infty} \mathbb{P}\left(\sum_{n=1}^{N} a_n (X_n - 1) \leq 0\right)$$

$$= \lim_{N\to\infty} \mathbb{P}\left(\frac{1}{s_N}\sum_{n=1}^{N} a_n(X_n-1) \le 0\right) = \mathbb{P}(G \le 0) = \frac{1}{2} > 0.$$

Since the event $\sum_{n=1}^{\infty} a_n(X_n-1) = \infty$ is a tail event for the sequence of independent random variables X_n, it must by Kolmogorov's 0-1 law have a probability of zero or one. The estimate above then implies

$$\mathbb{P}\left(\sum_{n=1}^{\infty} a_n(X_n-1) = \infty\right) = 0.$$

The same argument with ∞ replaced by $-\infty$ shows that

$$\mathbb{P}\left(\sum_{n=1}^{\infty} a_n(X_n-1) = -\infty\right) = 0$$

and hence

$$\mathbb{P}\left(\exists\, R : \left|\sum_{n=1}^{N} a_n X_n - \sum_{n=1}^{N} a_n\right| \le R \text{ for infinitely many } N\right) = 1.$$

Since $\sum_{n=1}^{\infty} a_n$ diverges we get that $\sum_{n=1}^{N} a_n X_n$ a.s. has no upper bound and hence a.s. diverges.
It remains to deal with the case where a_n is unbounded. Here we can just set $b_n \equiv b_n \wedge 1$ and apply the previous result. Then $\sum_{n=1}^{N} b_n X_n \ge \sum_{n=1}^{N} (a_n \wedge 1) X_n$ a.s. diverges. □

We convert the equation to a Hilbert space valued ODE of degree 1 and then solve it with an operator exponential. We have to solve

$$\partial_t \begin{pmatrix} \pi_1 w_0 \\ \pi_2 w_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -1 \end{pmatrix} \begin{pmatrix} \pi_1 w_0 \\ \pi_2 w_0 \end{pmatrix}$$

We know that we can solve this by using the operator exponential to get a family of operators

$$S(t) = \exp\left(t \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -1 \end{pmatrix}\right).$$

Then $w(t) = S(t)w_0$ should be solution.
To find the family operators $S(t)$, note that

$$\widehat{S(t)w_0}(n) = \exp\left(t \begin{pmatrix} 0 & 1 \\ -1-|\lambda_n|^2 & -1 \end{pmatrix}\right) \widehat{w_0}(n).$$

We therefore have to only compute for $c > 0$ the matrix exponential of

$$\begin{pmatrix} 0 & 1 \\ -1-c & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ a_1 & a_2 \end{pmatrix}^{-1} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ a_1 & a_2 \end{pmatrix}.$$

Here

$$a_{1,2} = -\frac{1}{2} \pm i\sqrt{c + \frac{3}{4}}.$$

We get

$$\begin{aligned} \exp\left(t \begin{pmatrix} 0 & 1 \\ -1-c & -1 \end{pmatrix}\right) &= \begin{pmatrix} 1 & 1 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} e^{ta_1} & 0 \\ 0 & e^{ta_2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ a_1 & a_2 \end{pmatrix}^{-1} \\ &= \frac{1}{a_2 - a_1} \begin{pmatrix} e^{ta_1} & e^{ta_2} \\ a_1 e^{ta_1} & a_2 e^{ta_2} \end{pmatrix} \begin{pmatrix} a_2 & -1 \\ -a_1 & 1 \end{pmatrix} \\ &= \frac{1}{a_2 - a_1} \begin{pmatrix} e^{ta_1} a_2 - e^{ta_2} a_1 & e^{ta_2} - e^{ta_1} \\ e^{ta_1} a_1 a_2 - e^{ta_2} a_1 a_2 & e^{ta_2} a_2 - e^{ta_1} a_1 \end{pmatrix}. \end{aligned}$$

Set $b = \sqrt{c + \frac{3}{4}} = [\sqrt{c}]$. The above is equal to

$$
\begin{aligned}
&\frac{e^{-t\frac{1}{2}}}{-2ib} \begin{pmatrix} i\frac{e^{-itb}-e^{itb}}{2i} - 2ib\frac{e^{itb}+e^{-itb}}{2} & -2i\frac{e^{itb}-e^{-itb}}{2i} \\ 2i(c+1)\frac{e^{itb}-e^{-itb}}{2i} & i\frac{e^{itb}-e^{-itb}}{2i} - 2ib\frac{e^{-itb}+e^{itb}}{2} \end{pmatrix} \\
&= \frac{e^{-t\frac{1}{2}}}{-2b} \begin{pmatrix} -\sin(tb) - 2b\cos(tb) & -2\sin(tb) \\ 2(c+1)\sin(tb) & \sin(tb) - 2b\cos(tb) \end{pmatrix}
\end{aligned}
$$

Using the notations $\left[\lambda_n\right] = \sqrt{|\lambda_n|^2 + \frac{3}{4}}$ and $\left[\nabla\right] = \sqrt{-\Delta + \frac{3}{4}}$, we ultimately get that S conjugated with $\mathcal{F}$ in the n-th coordinate is given by the matrix

$$
T_n(t) = e^{-t\frac{1}{2}} \begin{pmatrix} \frac{1}{2\left[\lambda_n\right]} \sin\left(t\left[\lambda_n\right]\right) + \cos\left(t\left[\lambda_n\right]\right) & \frac{1}{\left[\lambda_n\right]} \sin\left(t\left[\lambda_n\right]\right) \\ -\left(\frac{1}{4\left[\lambda_n\right]} + \left[\lambda_n\right]\right) \sin\left(t\left[\lambda_n\right]\right) & -\frac{1}{2\left[\lambda_n\right]} \sin\left(t\left[\lambda_n\right]\right) + \cos\left(t\left[\lambda_n\right]\right) \end{pmatrix}.
$$

This can be written in shorthand as

$$
S(t) = e^{-t\frac{1}{2}} \begin{pmatrix} \frac{1}{2\left[\nabla\right]} \sin\left(t\left[\nabla\right]\right) + \cos\left(t\left[\nabla\right]\right) & \frac{1}{\left[\nabla\right]} \sin\left(t\left[\nabla\right]\right) \\ -\left(\frac{1}{4\left[\nabla\right]} + \left[\nabla\right]\right) \sin\left(t\left[\nabla\right]\right) & -\frac{1}{2\left[\nabla\right]} \sin\left(t\left[\nabla\right]\right) + \cos\left(t\left[\nabla\right]\right) \end{pmatrix}.
$$

Take note of the componentwise estimate for $T_n(t)$ by the leading order term of $\langle \lambda_n \rangle$ in each entry:

$$
T_n(t) \lesssim \begin{pmatrix} 1 & \langle \lambda_n \rangle^{-1} \\ \langle \lambda_n \rangle & 1 \end{pmatrix}.
$$

Lemma 1.4. *For all $\alpha \in \mathbb{R}$, $S \in C([0,\infty), L(\mathcal{H}^\alpha, \mathcal{H}^\alpha))$. For $u \in \mathcal{H}^\alpha$,*

$$
\|S(t)u\|_{\mathcal{H}^\alpha} \lesssim e^{-\frac{t}{2}} \|u\|_{\mathcal{H}^\alpha}.
$$

Proof. We first show the inequality.

$$
\begin{aligned}
e^t \|S(t)u\|_{\mathcal{H}^\alpha}^2 &= e^t \|(1-\Delta)^{\frac{\alpha}{2}} \left(S(t)u\right)_1\|_{L^2}^2 + e^t \|(1-\Delta)^{\frac{\alpha-1}{2}} \left(S(t)u\right)_2\|_{L^2}^2 \\
&= \sum_{n=1}^\infty |\langle (1-\Delta)^{\frac{\alpha}{2}} e^{\frac{t}{2}} \left(S(t)u\right)_1, e_n \rangle|^2 + \sum_{n=1}^\infty |\langle (1-\Delta)^{\frac{\alpha-1}{2}} e^{\frac{t}{2}} \left(S(t)u\right)_2, e_n \rangle|^2 \\
&= \sum_{n=1}^\infty \langle \lambda_n \rangle^{\alpha 2} \left| \left(T_n(t)\right)_{1,1} \right|^2 |\widehat{u_1}(n)|^2 + \sum_{n=1}^\infty \langle \lambda_n \rangle^{\alpha 2} \left| \left(T_n(t)\right)_{1,2} \right|^2 |\widehat{u_2}(n)|^2 \\
&+ \sum_{n=1}^\infty \langle \lambda_n \rangle^{\alpha 2 - 2} \left| \left(T_n(t)\right)_{2,1} \right|^2 |\widehat{u_1}(n)|^2 + \sum_{n=1}^\infty \langle \lambda_n \rangle^{\alpha 2 - 2} \left| \left(T_n(t)\right)_{2,2} \right|^2 |\widehat{u_1}(n)|^2
\end{aligned}
$$

$$
\lesssim \sum_{n=1}^\infty \left(\langle \lambda_n \rangle^{\alpha 2} + \langle \lambda_n \rangle^{\alpha 2 - 2} \langle \lambda_n \rangle^2 \right) |\widehat{u_1}(n)|^2 + \sum_{n=1}^\infty \left(\langle \lambda_n \rangle^{\alpha 2} \langle \lambda_n \rangle^{-2} + \langle \lambda_n \rangle^{\alpha 2 - 2} \right) |\widehat{u_2}(n)|^2 \lesssim \|u\|_{\mathcal{H}^\alpha}^2.
$$

To see that S is a continuous family of operators it suffices to show that $\|S(h) - \mathrm{Id}\,\|_{L(\mathcal{H}^\alpha, \mathcal{H}^\alpha)} \to 0$ as $h \to 0$ by the semigroup property. Now for the Fourier multipliers we have $T_n(h) - \mathrm{Id} \to 0$ locally uniformly in h for all n, and with the above estimates we can use dominated convergence in $\ell^2(\mathbb{N})$. ☐

Lemma 1.5. *For all $\alpha \in \mathbb{R}$, $S \in C^1([0,\infty), L(\mathcal{H}^\alpha, \mathcal{H}^{\alpha-1}))$ with*

$$
\partial_t S(t) = LS(t).
$$

where

$$
L = \begin{pmatrix} 0 & 1 \\ \Delta - 1 & -1 \end{pmatrix} \in L(\mathcal{H}^\alpha, \mathcal{H}^{\alpha-1}).
$$

Proof. We only have to show existence and the formula of the derivative at $t = 0$, as the other cases follow from the flow property of S. We first compute $\partial_t T_n(t)$:

$$\partial_t T_n(t) = e^{-t\frac{1}{2}} \begin{pmatrix} \frac{1}{2}\cos\left(t\left[\lambda_n\right]\right) - \left[\lambda_n\right]\sin\left(t\left[\lambda_n\right]\right) & \cos\left(t\left[\lambda_n\right]\right) \\ -\left(\frac{1}{4} + \left[\lambda_n\right]^2\right)\cos\left(t\left[\lambda_n\right]\right) & -\frac{1}{2}\cos\left(t\left[\lambda_n\right]\right) - \left[\lambda_n\right]\sin\left(t\left[\lambda_n\right]\right) \end{pmatrix}$$

$$-\frac{1}{2}e^{-t\frac{1}{2}} \begin{pmatrix} \frac{1}{2\left[\lambda_n\right]}\sin\left(t\left[\lambda_n\right]\right) + \cos\left(t\left[\lambda_n\right]\right) & \frac{1}{\left[\lambda_n\right]}\sin\left(t\left[\lambda_n\right]\right) \\ -\left(\frac{1}{4\left[\lambda_n\right]} + \left[\lambda_n\right]\right)\sin\left(t\left[\lambda_n\right]\right) & -\frac{1}{2\left[\lambda_n\right]}\sin\left(t\left[\lambda_n\right]\right) + \cos\left(t\left[\lambda_n\right]\right) \end{pmatrix}$$

$$= e^{-t\frac{1}{2}} \begin{pmatrix} -\left(\frac{1}{4\left[\lambda_n\right]} + \left[\lambda_n\right]\right)\sin\left(t\left[\lambda_n\right]\right) & -\frac{1}{2\left[\lambda_n\right]}\sin\left(t\left[\lambda_n\right]\right) + \cos\left(t\left[\lambda_n\right]\right) \\ -\left(\frac{1}{4} + \left[\lambda_n\right]^2\right)\left(-\frac{1}{2}\frac{1}{\left[\lambda_n\right]}\sin\left(t\left[\lambda_n\right]\right) + \cos\left(t\left[\lambda_n\right]\right)\right) & -\left(\frac{1}{4\left[\lambda_n\right]} + \left[\lambda_n\right]\right)\sin\left(t\left[\lambda_n\right]\right) - \cos\left(t\left[\lambda_n\right]\right) \end{pmatrix}.$$

Then

$$\partial_t T_n(0) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{4} - \left[\lambda_n\right]^2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 - \langle\lambda_n\rangle^2 & -1 \end{pmatrix}$$

Furthermore we can bound the distance of the difference quotient to the derivative uniformly in time t. For convenience of notation we use $\leq$ and $|\cdot|$ in the following, with the meaning that this holds for each individual component of $T_n(t)$. We write $T_n(t) = e^{-\frac{1}{2}t}e^{\frac{1}{2}t}T_n(t)$ and use the product formula for difference quotients

$$\left|\frac{T_n(t) - \mathrm{Id}}{t} - \partial_t T_n(0)\right|$$

$$= \left| e^{-\frac{t}{2}} \begin{pmatrix} \frac{\frac{1}{2\left[\lambda_n\right]}\sin\left(t\left[\lambda_n\right]\right) + \cos\left(t\left[\lambda_n\right]\right) - 1}{t} - \frac{1}{2} & \frac{\frac{1}{\left[\lambda_n\right]}\sin\left(t\left[\lambda_n\right]\right)}{t} - 1 \\ \frac{-\left(\frac{1}{4\left[\lambda_n\right]} + \left[\lambda_n\right]\right)\sin\left(t\left[\lambda_n\right]\right)}{t} + \frac{1}{4} + \left[\lambda_n\right]^2 & \frac{-\frac{1}{2\left[\lambda_n\right]}\sin\left(t\left[\lambda_n\right]\right) + \cos\left(t\left[\lambda_n\right]\right) - 1}{t} + \frac{1}{2} \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} \frac{e^{-\frac{t}{2}} - 1}{t} + \frac{1}{2} & 0 \\ 0 & \frac{e^{-\frac{t}{2}} - 1}{t} + \frac{1}{2} \end{pmatrix} \right|$$

We define

$$\delta(x) \equiv \max\left\{\left|\frac{\sin(x)}{x} - 1\right|, \left|\frac{\cos(x) - 1}{x}\right|\right\} \qquad \gamma(t) = \left|\frac{e^{-\frac{t}{2}} - 1}{t} + \frac{1}{2}\right|.$$

and use it to estimate the above as

$$\leq \left|\begin{pmatrix} \frac{1}{2}\delta\left(t\left[\lambda_n\right]\right) + \left[\lambda_n\right]\delta\left(t\left[\lambda_n\right]\right) & \delta\left(t\left[\lambda_n\right]\right) \\ \frac{1}{4}\delta\left(t\left[\lambda_n\right]\right) + \left[\lambda_n\right]^2\delta\left(t\left[\lambda_n\right]\right) & \frac{1}{2}\delta\left(t\left[\lambda_n\right]\right) + \left[\lambda_n\right]\delta\left(t\left[\lambda_n\right]\right) \end{pmatrix}\right| + \begin{pmatrix} \gamma(t) & 0 \\ 0 & \gamma(t) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \left[\lambda_n\right] & 1 \\ \frac{1}{4} + \left[\lambda_n\right]^2 & \frac{1}{2} + \left[\lambda_n\right] \end{pmatrix}\delta\left(t\left[\lambda_n\right]\right) + \begin{pmatrix} \gamma(t) & 0 \\ 0 & \gamma(t) \end{pmatrix}.$$

Now let $t \neq 0$. Then

$$\begin{aligned} &\sup\nolimits_{0\neq u\in\mathcal{H}^\alpha} && \|u\|_{\mathcal{H}^\alpha}^{-2}\left\|\tfrac{S(t)-\mathrm{Id}}{t}u - \begin{pmatrix} 0 & 1 \\ \Delta - 1 & -1 \end{pmatrix}u\right\|_{\mathcal{H}^{\alpha-1}}^2 \\ \lesssim\ &\sup\nolimits_{0\neq u\in\mathcal{H}^\alpha} && \|u\|_{\mathcal{H}^\alpha}^{-2}\textstyle\sum_{n=1}^\infty\langle\lambda_n\rangle^{\alpha 2-2}\delta\left(t\left[\lambda_n\right]\right)^2\left(\left|\left[\lambda_n\right]\widehat{u_1}(n)\right|^2 + \left|\widehat{u_2}(n)\right|^2\right) \\ &+ && \|u\|_{\mathcal{H}^\alpha}^{-2}\textstyle\sum_{n=1}^\infty\langle\lambda_n\rangle^{\alpha 2-4}\delta\left(t\left[\lambda_n\right]\right)^2\left(\left|\left[\lambda_n\right]^2\widehat{u_1}(n)\right|^2 + \left|\left[\lambda_n\right]\widehat{u_2}(n)\right|^2\right) \\ &+ && \|u\|_{\mathcal{H}^\alpha}^{-2}\textstyle\sum_{n=1}^\infty\langle\lambda_n\rangle^{\alpha 2-2}\left|\gamma(t)\widehat{u_1}(n)\right|^2 + \langle\lambda_n\rangle^{\alpha 2-4}\left|\gamma(t)\widehat{u_2}(n)\right|^2 \end{aligned}$$

Now we let $\epsilon > 0$ be arbitrarily small and in particular small enough so that δ is monotonous on $(0, \epsilon)$. We get

$$\begin{aligned} &\leq \sup_{0\neq u\in\mathcal{H}^\alpha} && \|u\|_{\mathcal{H}^\alpha}^{-2} \sum_{n=1}^\infty \delta\left(|t|\left[\lambda_n\right]\right)^2 \left(\langle\lambda_n\rangle^{2\alpha}|\widehat{u_1}(n)|^2 + \langle\lambda_n\rangle^{2-2\alpha}|\widehat{u_2}(n)|^2\right) + \gamma(t) \\ &\lesssim \sup_{0\neq u\in\mathcal{H}^\alpha} && \delta(\epsilon)^2 \\ &\quad + && \|u\|_{\mathcal{H}^\alpha}^{-2} \sum_{\substack{n=1 \\ |t|\left[\lambda_n\right]>\epsilon}}^\infty \left(\sup_x \delta(x)\right)^2 \left(\langle\lambda_n\rangle^{2\alpha}|\widehat{u_1}(n)|^2 + \langle\lambda_n\rangle^{2-2\alpha}|\widehat{u_2}(n)|^2\right) \\ &\quad + && \gamma(t). \end{aligned}$$

We can choose ϵ so that $\delta(\epsilon)^2$ is arbitrarily small. Then in the second line we are only summing over those n where $n \sim \left[\lambda_n\right] > \frac{\epsilon}{|t|}$, so the second line vanishes as $t \to 0$ (note that $\sup_x \delta(x) \leq 2$). Lastly $\gamma(t) \to 0$. $\square$

Theorem 1.6 (Global well-posedness for linear problem with random initial data)**.** *Let $\alpha \geq 0$ and w_0 be initial data of the aforementioned form so that $w_0 \in \mathcal{H}^\alpha$ a.s. Then $w(t) = S(t)w_0$ and the following holds almost surely:*

(i) $w \in C([0,\infty), \mathcal{H}^\alpha) \cap C^1([0,\infty), \mathcal{H}^{\alpha-1})$,

(ii) w *solves* $w(0) = w_0$ *and*

$$\partial_t w(t) = \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -1 \end{pmatrix} w(t).$$

(iii) We have estimates

$$\|w(t)\|_{\mathcal{H}^\alpha} \lesssim e^{-\frac{t}{2}}\|w_0\|_{\mathcal{H}^\alpha} \qquad \|\partial_t w(t)\|_{\mathcal{H}^{\alpha-1}} \lesssim e^{-\frac{t}{2}}\|w_0\|_{\mathcal{H}^\alpha}.$$

Proof. By the estimate $\|S(t)w_0\|_{\mathcal{H}^\alpha} \leq e^{-\frac{t}{2}}\|w_0\|_{\mathcal{H}^\alpha}$ we get that $w \in C([0,\infty, \mathcal{H}^\alpha)$. We see that

$$\begin{aligned} &\left\|\frac{w(t+h)-w(t)}{h} - \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -1 \end{pmatrix} w(t)\right\|_{\mathcal{H}^{\alpha-1}} \\ \leq &\left\|\frac{S(t+h)-S(t)}{h} - \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -1 \end{pmatrix} S(t)\right\|_{L(\mathcal{H}^\alpha,\mathcal{H}^{\alpha-1})} \|w_0\|_{\mathcal{H}^\alpha}. \\ \leq &\left\|\frac{S(h)-\mathrm{Id}}{h} - L\right\|_{L(\mathcal{H}^\alpha,\mathcal{H}^{\alpha-1})} \|S(t)\|_{L(\mathcal{H}^\alpha,\mathcal{H}^\alpha)}\|w_0\|_{\mathcal{H}^\alpha}. \end{aligned}$$

We know that the first norm vanishes as $h \searrow 0$ and the second norm decays with $e^{-\frac{t}{2}}$ in t, so $w(t)$ is differentiable in $\mathcal{H}^{\alpha-1}$ with bounded, continuous derivatives and $\partial_t w(t) = LS(t)w_0 = Lw(t)$. $\square$

1.2 The linear Problem with White noise Inhomogeneity

Written in terms of pairs of functions, Ψ is supposed to solve

$$\partial_t \begin{pmatrix} \pi_1\Psi \\ \pi_2\Psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -1 \end{pmatrix} \begin{pmatrix} \pi_1\Psi \\ \pi_2\Psi \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{2}\xi \end{pmatrix}$$

with initial data $\Psi_0 = 0$. Since we know the solution operator $S(t)$ to the homogeneous problem, we can construct a mild solution to this problem as the stochastic convolution:

$$\Psi(t) = \int_0^t S(t-s) \cdot \begin{pmatrix} 0 \\ \sqrt{2}\xi(s) \end{pmatrix} ds$$

It is not entirely obvious how this has to be interpreted, so let us define it

Definition 1.7. *For $t \geq 0$ define a random space distribution in two variables which we call the* ***Stochastic convolution at time*** t *by*

$$\langle \Psi_t, f \rangle \equiv \left\langle \int_0^t S(t-s) \begin{pmatrix} 0 \\ \sqrt{2}\xi(s) \end{pmatrix} ds, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle \equiv \left\langle \xi, \mathbb{1}_{[0,t]}(s)\sqrt{2} \cdot \pi_2 S^*(t-s) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle$$

for $f = (f_1, f_2) \in \mathcal{D}(B)^2$. Here π_1 and π_2 are the projections onto the first and second function. We can unfold the definition to

$$\langle \Psi_t, f \rangle = \left\langle \xi, \mathbb{1}_{[0,t]}(s)\sqrt{2}e^{-\frac{t-s}{2}} \cdot \left(\frac{1}{[\nabla]} \sin\left((t-s)[\nabla]\right) f_1 + \left(-\frac{1}{2[\nabla]} \sin\left((t-s)[\nabla]\right) + \cos\left((t-s)[\nabla]\right) \right) f_2 \right) \right\rangle$$

Using that ξ is space-time white noise, we can compute for $f, g \in \mathcal{D}(B)^2$ that

$$\mathbb{E}\left[\langle \Psi_t, f \rangle \langle \Psi_t, g \rangle\right] = \left\langle \mathbb{1}_{[0,t]}(s)\sqrt{2}e^{-\frac{t-s}{2}} \cdot \pi_2 S^*(t-s) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \mathbb{1}_{[0,t]}(s)\sqrt{2}e^{-\frac{t-s}{2}} \cdot \pi_2 S^*(t-s) \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle_{L^2_{t,x}}$$

$$\begin{aligned} = \int_0^t \sum_{n=1}^{\infty} 2e^{s-t} & \left(\frac{1}{[\lambda_n]} \sin\left((t-s)[\lambda_n]\right) \widehat{f_1}(n) + \left(-\frac{1}{2[\lambda_n]} \sin\left((t-s)[\lambda_n]\right) + \cos\left((t-s)[\lambda_n]\right) \right) \widehat{f_2}(n) \right) \\ & \cdot \left(\frac{1}{[\lambda_n]} \sin\left((t-s)[\lambda_n]\right) \widehat{g_1}(n) + \left(-\frac{1}{2[\lambda_n]} \sin\left((t-s)[\lambda_n]\right) + \cos\left((t-s)[\lambda_n]\right) \right) \widehat{g_2}(n) \right) ds. \end{aligned}$$

Furthermore $\langle \Psi_t, f \rangle$ is a centered Gaussian random variable.
In the case that $f = (e_n, 0)$ and $g = (e_m, 0)$ we get that

$$\mathbb{E}\left[\langle \Psi_t, (e_n, 0) \rangle \langle \Psi_t, (e_m, 0) \rangle\right] = \begin{cases} \int_0^t \frac{2e^{s-t}}{[\lambda_n]^2} \sin\left((t-s)[\lambda_n]\right)^2 \, ds & , n = m \\ 0 & , n \neq m \end{cases}.$$

For $f = (0, e_n)$ and $g = (0, e_m)$ we have

$$\mathbb{E}\left[\langle \Psi_t, (0, e_n) \rangle \langle \Psi_t, (0, e_m) \rangle\right] = \begin{cases} \int_0^t 2e^{s-t} \left(\frac{1}{4[\lambda_n]^2} \sin\left((t-s)[\lambda_n]\right)^2 - \frac{1}{2}\frac{1}{[\lambda_n]} \sin\left(2(t-s)[\lambda_n]\right) + \frac{1}{2}\cos\left(2(t-s)[\lambda_n]\right) + \frac{1}{2} \right) ds & , \\ 0 & , \end{cases}$$

We can calculate the integrals:

$$\int_0^t \frac{2e^{s-t}}{[\lambda_n]^2} \sin\left((t-s)[\lambda_n]\right)^2 \, ds = 2[\lambda_n]^{-3} \int_{-t[\lambda_n]}^0 e^{\frac{\tau}{[\lambda_n]}} \sin(-\tau)^2 \, d\tau$$

$$= 2[\lambda_n]^{-3} \left[\frac{[\lambda_n] e^{\frac{\tau}{[\lambda_n]}}}{8[\lambda_n]^2 + 2} \left(4[\lambda_n]^2 - 2[\lambda_n]\sin(2\tau) - \cos(2\tau) + 1 \right) \right]_{-t[\lambda_n]}^0$$

$$= \frac{1}{4\langle \lambda_n \rangle^2} \left(4 - e^{-t}\left(4 + 2[\lambda_n]^{-1} \sin(2t[\lambda_n]) - [\lambda_n]^{-2} \cos(2t[\lambda_n]) + [\lambda_n]^{-2} \right) \right)$$

We define

$$c_n^2(t) = \frac{\Theta_n(t)}{\langle \lambda_n \rangle^2} \quad \text{where} \quad \Theta_n(t) = 1 - e^{-t}\left(1 + \frac{1}{2}[\lambda_n]^{-1} \sin(2t[\lambda_n]) + \frac{1}{4}[\lambda_n]^{-2} \left(1 - \cos(2t[\lambda_n])\right) \right).$$

The previous results can be summarized as

$$\langle \pi_1 \Psi(t), e_n \rangle \sim \mathcal{N}(0, c_n^2(t))$$

and they are independent in n. Observe that

$$\Theta_n(0) = 0 \qquad \lim_{t\to\infty} \Theta_n(t) = 1 \qquad \exists\, K > 0 : \forall\, n \in \mathbb{N}\ \forall\, t \geq 0, \quad 0 \leq \Theta_n(t) \leq K.$$

and that there exists $N \in \mathbb{N}$ so that for all $n \geq N$ and $t \geq 0$

$$1 - 2e^{-t} \leq \Theta_n(t) \leq 1 - \frac{1}{2}e^{-t}.$$

Hence for large n and fixed $t > 0$ we have $c_n^2(t) \sim \langle\lambda_n\rangle^{-2}$.

For the other integral we compute

$$2\big[\lambda_n\big]^{-1} \int_{-t[\lambda_n]}^{0} e^{\frac{\tau}{[\lambda_n]}} \left(-\frac{1}{2[\lambda_n]}\sin(-2\tau) + \frac{1}{2}\cos(-2\tau) + \frac{1}{2}\right) d\tau$$

$$= 2\big[\lambda_n\big]^{-1} \left[\frac{e^{\frac{\tau}{[\lambda_n]}}}{8[\lambda_n]^2 + 2}\Big(4\big[\lambda_n\big]^3 + (2\big[\lambda_n\big]^2 + 1)\sin(2\tau) - \big[\lambda_n\big](3\cos(2\tau) + 1)\Big)\right]_{-t[\lambda_n]}^{0}$$

$$= \frac{\big[\lambda_n\big]^2}{4\langle\lambda_n\rangle^2}\Big(4 - 4\big[\lambda_n\big]^{-2} - e^{-t}\Big(4 - \Big(2\big[\lambda_n\big]^{-2} + \big[\lambda_n\big]^{-3}\Big)\sin(2t[\lambda_n]) - \big[\lambda_n\big]^{-2}3\cos(2t[\lambda_n]) - \big[\lambda_n\big]^{-2}\Big)\Big).$$

We define

$$d_n^2(t) = \frac{\frac{1}{4}\Theta_n(t) + \big[\lambda_n\big]^2\Xi_n(t)}{\langle\lambda_n\rangle^2}$$

where

$$\Xi_n(t) = 1 - \big[\lambda_n\big]^{-2} - e^{-t}\left(1 - \big[\lambda_n\big]^{-2}\left(\frac{1}{2}\sin(2t[\lambda_n]) + \frac{3}{4}\cos(2t[\lambda_n]) + \frac{1}{4}\right) - \frac{1}{4}\big[\lambda_n\big]^{-3}\sin(2t[\lambda_n])\right)$$

We then have that

$$\langle\pi_2\Psi(t), e_n\rangle \sim \mathcal{N}(0, d_n^2(t))$$

and they are independent in n. Observe that

$$\Xi_n(0) = 0 \qquad \lim_{t\to\infty} \Xi_n(t) = 1 - \big[\lambda_n\big]^{-2} \qquad \exists\, K > 0 : \forall\, n \in \mathbb{N}\ \forall\, t \geq 0, \quad 0 \leq \Theta_n(t) \leq K.$$

and that there exists $N \in \mathbb{N}$ so that for all $n \geq N$ and $t \geq 0$

$$1 - \big[\lambda_n\big]^{-2} - 2e^{-t} \leq \Xi_n(t) \leq 1 - \big[\lambda_n\big]^{-2} + \frac{1}{2}e^{-t}.$$

Lemma 1.8. *For $t > 0$ we have $\pi_1\Psi_t \in \mathcal{H}^\alpha$ a.s. if and only if $\alpha < \frac{1}{2}$. More precisely, only in this case is*

$$\Psi_t^N \equiv \sum_{n=1}^{N} \langle\Psi_t, (e_n, 0)\rangle e_n$$

a Cauchy sequence in $L^2(\Omega, \mathcal{H}^\alpha)$.

Proof. Let $N \leq M \in \mathbb{N}$. Then

$$\mathbb{E}\left[\|\pi_1\Psi_t^N - \pi_1\Psi_t^M\|_{H^\alpha}^2\right] = \sum_{n=N}^{M} \langle\lambda_n\rangle^{2\alpha} c_n^2(t) \underset{N,M \text{ large}}{\sim} \sum_{n=N}^{M} n^{2\alpha-2}.$$

Here we use that $c_n^2(t) \sim \langle\lambda_n\rangle \sim |\lambda_n| \sim n$. The power series at the end converges if and only if $\alpha < \frac{1}{2}$. $\square$

This lemma tells us that $\pi_1\Psi(t)$ does exist not only as a random distribution, but a.s. as a function in $L^2(B)$.

We write $\psi(t,x) = \pi_1\Psi(t,x)$. Let us explicitly remark that for a fixed $t > 0$ there exist iid. random variables $g_n(t) \sim \mathcal{N}(0,1)$ so that

$$\psi(t,x) = \sum_{n=1}^{\infty} c_n(t) e_n(x) g_n(t).$$

Lemma 1.9. *There exists $C > 0$ so that for all $0 \le \alpha < \frac{1}{2}$ and $\epsilon \in (1-2\alpha, 1)$ we have for almost all $x \in B$ the estimate*

$$\sum_{n=1}^{\infty} \langle\lambda_n\rangle^{2\alpha-2} e_n(x)^2 \le C \begin{cases} (1-\epsilon)^{-1} + \epsilon^{-1}|\log(|x|)|, & \alpha = 0 \\ (2\alpha+\epsilon-1)^{-1} + (2\alpha)^{-1}|x|^{-\frac{2\alpha}{\epsilon}}, & \alpha > 0. \end{cases}$$

Proof. Let $N \in \mathbb{N}$. Recall that $\langle\lambda_n\rangle \sim n$ and that

$$e_n(x) = \|J_0(\lambda_n n|\cdot|)\|_{L^2(B)}^{-1} J_0(\lambda_n|x|) \text{ with } \|J_0(\lambda_n n|\cdot|)\|_{L^2(B)} \sim n^{-\frac{1}{2}}.$$

We split the sum into two parts for an $\epsilon \in (0,1)$ to be chosen later:

$$\sum_{n=1}^{N} \langle\lambda_n\rangle^{2\alpha-2} e_n(x)^2 \lesssim \sum_{\substack{n=1 \\ n^\epsilon|x|\ge 1}}^{N} n^{2\alpha-2} n J_0(\lambda_n|x|)^2 + \sum_{\substack{n=1 \\ n^\epsilon|x|<1}}^{N} n^{2\alpha-2} n J_0(\lambda_n|x|)^2.$$

On the first part we that estimate $J_0(y) \in \mathcal{O}(y^{-\frac{1}{2}})$ for large y and use $|x|^{-1} \le n^\epsilon$. On the second part we use $J_0 \in L^\infty$:

$$\lesssim \sum_{n=N\wedge\lceil|x|^{-\frac{1}{\epsilon}}\rceil}^{N} n^{2\alpha+\epsilon-2} + \sum_{n=1}^{N\wedge\lfloor|x|^{-\frac{1}{\epsilon}}\rfloor} n^{2\alpha-1}.$$

Now we carefully estimate the sums by the corresponding integrals

$$\le 2 + \int_{N\wedge\lceil|x|^{-\frac{1}{\epsilon}}\rceil}^{N} s^{2\alpha+\epsilon-2} + \int_{1}^{N\wedge|x|^{-\frac{1}{\epsilon}}} s^{2\alpha-1}\,ds = (\star).$$

In the case $\alpha = 0$ we get

$$(\star) = 2 + \frac{1}{\epsilon-1}\left(N^{\epsilon-1} - \left(N\wedge\lceil|x|^{-\frac{1}{\epsilon}}\rceil\right)^{\epsilon-1}\right) + \left(\ln(N\wedge|x|^{-\frac{1}{\epsilon}} - 0\right).$$

For large N this becomes

$$\lesssim 2 + (1-\epsilon)^{-1} + \epsilon^{-1}|\ln(|x|)|.$$

In the case $\alpha > 0$ we make sure to choose $\epsilon < 1-2\alpha$ and get

$$(\star) = 2 + \frac{1}{2\alpha+\epsilon-1}\left(N^{2\alpha+\epsilon-1} - \left(N\wedge\lceil|x|^{-\frac{1}{\epsilon}}\rceil\right)^{2\alpha+\epsilon-1}\right) + \frac{1}{2\alpha}\left(\left(N\wedge|x|^{-\frac{1}{\epsilon}}\right)^{2\alpha} - 1\right).$$

For large N this becomes

$$\lesssim 2 + (1-2\alpha+\epsilon)^2 + \frac{1}{2\alpha}|x|^{-\frac{2\alpha}{\epsilon}}.$$

$\square$

We can use the previous lemma to show that ψ is in Lebesgue and Sobolev spaces. We critically use the inequality

$$\mathbb{E}\left[|X|^p\right] \le C(p)\,\mathbb{E}\left[|X|^2\right]^{\frac{p}{2}}$$

for a gaussian random variable X.

Lemma 1.10. *Let $1 \leq p < \infty$ and $T > 0$ The following hold:*

(i) *$\psi(t,x)$ is a centered Gaussian random variable for almost all $t \in [0,T]$ and $x \in B$. We have an estimate*
$$\mathbb{E}\left[|\psi(t,x)|^2\right] \leq C(T)(1 + |\log(|x|)|).$$

(ii) *For almost all $t \in [0,T]$, $\psi(t,\cdot) \in L^p(B)$ a.s, $\psi \in L^p([0,T], L^p(B))$ a.s. and*
$$\mathbb{E}\left[\int_B |\psi(t,x)|^p \, dx\right] < C(p) \qquad \mathbb{E}\left[\int_0^T \int_B |\psi(t,x)|^p \, dx \, dt\right] < TC(p).$$

(iii) *Let $0 < \alpha < \frac{1}{2}$ so that $p\alpha < 1$. Then $\langle \nabla \rangle^\alpha \psi(t,x)$ is a centered Gaussian random variable for almost all $t \in [0,T]$ and $x \in B$. For every $\epsilon \in (1-2\alpha, 1)$ we have an estimate*
$$\mathbb{E}\left[|\langle \nabla \rangle^\alpha \psi(t,x)|^2\right] \leq C(\alpha,\epsilon)|x|^{-\frac{2\alpha}{\epsilon}}.$$

(iv) *For almost all $t \in [0,T]$, $\psi(t,\cdot) \in W^{\alpha,p}(B)$ a.s., $\psi \in L^p([0,T], W^{\alpha,p}(B))$ and*
$$\mathbb{E}\left[\int_B |\langle \nabla \rangle^\alpha \psi(t,x)|^p \, dx\right] < C(\alpha,p) \qquad \mathbb{E}\left[\int_0^T \int_B |\langle \nabla \rangle^\alpha \psi(t,x)|^p \, dx \, dt\right] < TC(\alpha,p).$$

For such p we now find that $\psi(t,\cdot) \in W^{\alpha,p}(B)$. For $p \leq \frac{1}{\alpha} - 2$ we use that $W^{\alpha,p}(B) \subseteq W^{\alpha,q}(B)$ for $q \geq p$.

Proof. We show the statements for the pointwise in time cases. For the cases with an integral in time, simply swap the integral with the expectation and apply the pointwise in time case. We begin with (i). We see that for any $N \in \mathbb{N}$ by independence of the g_n
$$\mathbb{E}\left[\left|\sum_{n=1}^N \langle \lambda_n \rangle^{2\alpha} c_n(t) e_n(x) g_n(t)\right|^2\right] = \sum_{n=1}^N c_n^2(t) e_n(x)^2.$$

Since $c_n^2(t) \lesssim \langle \lambda_n \rangle^{-2}$ and by the estimate from the previous Lemma we know that this is finite and so $\langle \nabla \rangle^\alpha \psi(t,x)$ is a centered Gaussian r.v. in $L^2(\Omega)$. Using that the g_n are independent and standard, we get
$$\mathbb{E}\left[\int_B |\langle \nabla \rangle^\alpha \psi(t,x)|^p \, dx\right] \lesssim \int_B \mathbb{E}\left[|\langle \nabla \rangle^\alpha \psi(t,x)|^2\right]^{\frac{p}{2}} \, dx \lesssim \int_B \left(\sum_{n=1}^\infty \langle \lambda_n \rangle^{2\alpha-2} e_n(x)^2\right)^{\frac{p}{2}} \, dx.$$

We now apply the estimates from the previous lemma. For $\alpha = 0$ we can choose any ϵ and get
$$\lesssim \int_B \left((1-\epsilon)^{-1} + \epsilon^{-1}|\log(|x|)|\right)^{\frac{p}{2}} \, dx < \infty.$$

For $\alpha > 0$ we get
$$\lesssim \int_B \left(2\alpha + \epsilon - 1\right)^{-1} + (2\alpha)^{-1}|x|^{-\frac{2\alpha}{\epsilon}}\big)^{\frac{p}{2}} \, dx.$$

In order for this to be integrable we need that $\frac{2\alpha}{\epsilon}\frac{p}{2} < 2 \iff \epsilon > \alpha p$. Such an $\epsilon \in (1-2\alpha, 1)$ can only exist if
$$1 - 2\alpha < \alpha p < 1 \iff \frac{1}{\alpha} - 2 < p < \frac{1}{\alpha}.$$

□

We can furthermore find continuity in certain Sobolev spaces via the Kolmogorov continuity theorem:

Lemma 1.11. *Let $T > 0$, $\alpha < \frac{1}{2}$, $1 \le p < \infty$ so that $\alpha p < 1$ and let $0 < \gamma < \frac{1}{p} - \alpha$. Then $\psi \in C^{0,\gamma}([0,T], W^{\alpha,p})$ a.s.*

Proof. We will use the Kolmogorov continuity theorem. From the previous lemma we already know that $\psi(t,\cdot) \in W^{\alpha,p}$ a.s., so to apply the theorem it remains to find a $\beta > 0$ so that for all $s < t \le T$

$$\mathbb{E}\left[\int_B |\langle\nabla\rangle^\alpha(\psi(t,x) - \psi(s,x))|^p\, dx\right] \le C(T,p,s)|t-s|^{1+\beta}.$$

We already know that $\nabla^\alpha\psi(t,x) - \nabla^\alpha\psi(t,s)$ is a centered Gaussian r.v., so

$$\int_B \mathbb{E}\left[|\langle\nabla\rangle^\alpha(\psi(t,x) - \psi(s,x))|^p\right] dx \lesssim \int_B \mathbb{E}\left[|\langle\nabla\rangle^\alpha(\psi(t,x) - \psi(s,x))|^2\right]^{\frac{p}{2}} dx.$$

Now

$$\mathbb{E}\left[|\langle\nabla\rangle^\alpha(\psi(t,x) - \psi(s,x))|^2\right] = \mathbb{E}\left[\sum_{n=1}^{\infty}\langle\lambda_n\rangle^{2\alpha}(c_n(t)g_n(t) - c_n(s)g_n(s))^2 e_n(x)^2\right]$$

and for all $N \ge 0$

$$\mathbb{E}\left[\sum_{n=1}^{N}\langle\lambda_n\rangle^{2\alpha}(c_n(t)g_n(t) - c_n(s)g_n(s))^2 e_n(x)^2\right] = \sum_{n=1}^{N}\langle\lambda_n\rangle^{2\alpha}\,\mathbb{E}\left[(c_n(t)g_n(t) - c_n(s)g_n(s))^2\right] e_n(x)^2$$

The new term that we have to analyze here is the covariance:

$$\mathbb{E}\left[c_n(t)g_n(t)c_n(s)g_n(s)\right] = \mathbb{E}\left[\langle\psi(t), e_n\rangle\langle\psi(s), e_n\rangle\right]$$

$$= \left\langle \mathbb{1}_{[0,t]}(\tau)\sqrt{2}e^{-\frac{t-\tau}{2}}\cdot\pi_2 S^*(t-\tau)\begin{pmatrix} e_n \\ 0\end{pmatrix}, \mathbb{1}_{[0,s]}(\tau)\sqrt{2}e^{-\frac{s-\tau}{2}}\cdot\pi_2 S^*(s-\tau)\begin{pmatrix} e_n \\ 0\end{pmatrix}\right\rangle_{L^2_{t,x}}$$

$$= \int_0^s 2e^{\frac{t-\tau}{2}}e^{\frac{s-\tau}{2}}\frac{1}{\lceil\lambda_n\rceil}\sin\left((t-\tau)\lceil\lambda_n\rceil\right)\cdot\frac{1}{\lceil\lambda_n\rceil}\sin\left((s-\tau)\lceil\lambda_n\rceil\right)\, d\tau$$

Then

$$\mathbb{E}\left[c_n^2(t)g_n(t)^2 - 2c_n(t)g_n(t)c_n(s)g_n(s) + c_n^2(s)g_n(s)^2\right] = \mathbb{E}\left[|\langle\psi(t), e_n\rangle - \langle\psi(s), e_n\rangle|^2\right]$$

$$= \frac{2}{\lceil\lambda_n\rceil^2}\Bigg(\int_0^s\left(e^{\frac{t-\tau}{2}}\sin\left((t-\tau)\lceil\lambda_n\rceil\right) - e^{\frac{s-\tau}{2}}\sin\left((s-\tau)\lceil\lambda_n\rceil\right)\right)^2 d\tau$$
$$+ \int_s^t e^{t-\tau}\sin\left((t-\tau)\lceil\lambda_n\rceil\right)^2\, d\tau\Bigg).$$

We can easily estimate the second term by $C(T)\,|t-s|$. For the first term we can not avoid introducting a factor that is some power of $\langle\lambda_n\rangle$.

$$\int_0^s\left(e^{\frac{h+r}{2}}\sin\left((h+r)\lceil\lambda_n\rceil\right) - e^{\frac{r}{2}}\sin\left(r\lceil\lambda_n\rceil\right)\right)^2 dr$$

$$\lesssim \int_0^s (e^{\frac{h+r}{2}} - e^{\frac{r}{2}})^2\, dr + \frac{e^{\frac{r}{2}}}{\lceil\lambda_n\rceil}\int_0^{s\lceil\lambda_n\rceil}\left(\sin\left(h\lceil\lambda_n\rceil + r\right) - \sin(r)\right)^2 dr$$

$$\lesssim (e^{\frac{h}{2}} - 1)^2(e^s - 1) + C(T,\epsilon)\frac{e^{\frac{r}{2}}}{\lceil\lambda_n\rceil}s\lceil\lambda_n\rceil(h\lceil\lambda_n\rceil)^\epsilon$$

$$\le C_1(T)h + C_2(T,\epsilon)\left(h\lceil\lambda_n\rceil\right)^\epsilon \lesssim C(T,\epsilon)\,|t-s|^\epsilon\langle\lambda_n\rangle^\epsilon.$$

Since we have introduced an additional factor of $\langle\lambda_n\rangle^\epsilon$, we can continue as in the previous lemma but have to replace α by $\alpha+\frac{\epsilon}{2}$. We choose an $\epsilon>0$ so that $(\alpha+\epsilon/2)p<2$ still holds, but also $\frac{\epsilon}{2}p>1$. Then

$$\mathbb{E}\left[|\langle\nabla\rangle^\alpha(\psi(t,x)-\psi(s,x))|^2\right]\leq C(T)|t-s|^\epsilon\begin{cases}(1-\log(|x|))\,, & \alpha+\frac{\epsilon}{2}=0\\ \frac{1}{|x|^{2\alpha+\epsilon}} & \alpha+\frac{\epsilon}{2}>0\end{cases}$$

It follows that

$$\mathbb{E}\left[\|\psi(t,\cdot)-\psi(s,\cdot)\|^p_{W^{\alpha,p}}\,dx\right]\leq C(T,p,\alpha)|t-s|^{1+\frac{\epsilon p-2}{2}}$$

Finally by the Kolmogorov continuity theorem $\psi\in C^{0,\gamma}([0,T],W^{\alpha,p})$ for all $0<\gamma<\frac{\epsilon p-2}{2p}=\frac{\epsilon}{2}-\frac{1}{p}$. Since this works for any ϵ so that $(\alpha+\epsilon/2)p<2$, we make the largest possible choice and get that the result holds for all $0<\gamma<\frac{2}{p}-\alpha-\frac{1}{p}=\frac{1}{p}-\alpha$. □

1.3 Local well-posedness for the complete problem

Given w and ψ from the previous sections (abusing notation and not writing $\pi_1 w$ and $\pi_1\psi$), we are now looking for a v so that

$$v_{tt}+v_t+(1-\Delta)v+(w+\Psi+v)^3=0.$$

In terms of pairs of functions this is

$$\partial_t\begin{pmatrix}\pi_1 v\\ \pi_2 v\end{pmatrix}=\begin{pmatrix}0 & 1\\ -(1-\Delta) & -1\end{pmatrix}\begin{pmatrix}\pi_1 v\\ \pi_2 v\end{pmatrix}+\begin{pmatrix}0\\ (\pi_1 w+\pi_1\Psi+\pi_1 v)^3\end{pmatrix}$$

A solution to this should then fulfill

$$u(t)=\int_0^t S(t-s)\begin{pmatrix}0\\ (\pi_1 w+\pi_1\Psi+\pi_1 v)^3(s)\end{pmatrix}\,ds.$$

With

$$H(v)=t\longmapsto\int_0^t S(t-s)\begin{pmatrix}0\\ (\pi_1 w+\pi_1\Psi+\pi_1 v)^3(s)\end{pmatrix}\,ds.$$

we can phrase this as a fixed point problem. We choose the space $X=L^\infty([0,T],\mathcal{H}^\alpha)$. we can phrase this as a fixed point problem. We choose the space $X=L^\infty([0,T],\mathcal{H}^\alpha)$ where $\alpha\in\left(0,\frac{4}{3}\right)$.

We will need some inequalities.

Lemma 1.12 (fractional Leibnitz inequality / Fractional Leibnitz rule [2]). *For* $\frac{1}{r}=\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{p_2}$, $\alpha\in(0,1)$ *and* $f,g\in L^2(\mathbb{R}^n)$

$$\|\langle\nabla\rangle^\alpha(fg)\|_{L^r}\lesssim\|\langle\nabla\rangle^\alpha f\|_{L^{p_1}}\|g\|_{L^{q_1}}+\|\langle\nabla\rangle^\alpha g\|_{L^{p_2}}\|f\|_{L^{q_2}}.$$

In particular

$$\|\langle\nabla\rangle^\alpha(f^3)\|_{L^2}\lesssim\|\langle\nabla\rangle^\alpha f\|_{L^6}\|f\|^2_{L^6}.$$

Proof. A proof can be found in [2]. The second inequality follows from it by an application with $\frac{1}{2}=\frac{1}{6}+\frac{1}{3}=\frac{1}{3}+\frac{1}{6}$:

$$\|\langle\nabla\rangle^\alpha(f^3)\|_{L^2}\lesssim\|\langle\nabla\rangle^\alpha f\|_{L^6}\|f^2\|_{L^3}+\|\langle\nabla\rangle^\alpha(f^2)\|_{L^3}\|f\|_{L^6}$$

and another one with $\frac{1}{3}=\frac{1}{6}+\frac{1}{6}$:

$$\|\langle\nabla\rangle^\alpha(f^2)\|_{L^3}\lesssim\|\langle\nabla\rangle^\alpha f\|_{L^6}\|f\|_{L^6}.$$

□

Let us also Recall the fractional Gagliardo-Nirenberg interpolation inequality.

Lemma 1.13 (Gagliardo-Nirenberg interpolation inequality)**.** *Let $\theta \in (0,1)$, $p_1, p_2 \in [1,\infty]$, $\alpha_1, \alpha_2 \in [0,\infty)$ so that*

$$\alpha = \theta\alpha_1 + (1-\theta)\alpha_2 \qquad\qquad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

and the following does not hold:

$$1 \le \alpha_2 \in \mathbb{Z} \text{ and } p_2 = 1 \text{ and } \alpha_2 - \alpha_1 \le 1 - \frac{1}{p_1}.$$

Then

$$\|u\|_{W^{\alpha,p}} \lesssim \|u\|_{W^{\alpha_1,p_1}}^{\theta} \|u\|_{W^{\alpha_2,p_2}}^{1-\theta}.$$

We apply this in the following way: Given some $\alpha \in (0, \frac{4}{3})$ we choose $\alpha_2 = 0$, $\theta = \frac{1}{4}$, $\alpha_1 = \theta\alpha$ and $p_1 = 2$, $p_2 = 18$ so that

$$\frac{\theta}{p_1} + \frac{1-\theta}{p_2} = \frac{1}{8} + \frac{3}{4 \cdot 18} = \frac{1}{6}$$

and then since $\alpha - 1 < \frac{1}{4}\alpha = \theta\alpha$ we get

$$\|u\|_{W^{\alpha-1},6} \lesssim \|u\|_{W^{\theta\alpha,6}} \lesssim \|u\|_{H^\alpha}^{\frac{1}{4}} \|u\|_{L^{18}}^{\frac{3}{4}}.$$

If $\alpha > 1$ then there exists $C(\alpha)$ so that $\|fg\|_{H^\alpha} \le C(\alpha)\|f\|_{H^\alpha}\|g\|_{H^\alpha}$ for $\alpha > 1$. Then for any even $1 \le q < \infty$

$$\|u\|_{L^q} = \||u|^{\frac{q}{2}}\|_{L^2}^{\frac{2}{q}} \le C(\alpha)^{\frac{q}{2}-1}\|u\|_{H^\alpha}.$$

As a result

$$\|u\|_{W^{\alpha-1,6}} \lesssim \|u\|_{H^\alpha}.$$

Even though we previously considered the problem for initial data $w_0 \in \mathcal{H}^\alpha$ a.s., we will now take it to be a.s. in a certain function space X^α so that it has precisely the properties we need.

Definition 1.14. *For an initial data $w_0 \in L^2 \times L^2$ and $\alpha \in \mathbb{R}$. we define a norm*

$$\|w_0\|_{X^\alpha} = \|S(t)w_0\|_{L^6_{t,x}(\mathbb{R}_+ \times (B \times B))} + \|S(t)w_0\|_{L^6_t \mathcal{W}^{\alpha,6}_x}$$

We write $w = S(t)w_0$.

Theorem 1.15. *Let $\alpha < \frac{4}{3}$ and $w_0 \in X^\alpha$ a.s. be an intial data. Then there exists a $T > 0$ and $v \in L^\infty([0,T], \mathcal{H}^\alpha)$ so that*

$$v = H(v) = t \longmapsto \int_0^t S(t-s) \begin{pmatrix} 0 \\ -(\pi_1 w + \psi + v)^3 \end{pmatrix} ds.$$

Proof. It suffices to prove this for $1 < \alpha < \frac{4}{3}$. Let $R > 0$ be chosen later and $\|v\|_{L^\infty([0,T],\mathcal{H}^\alpha)} \le R$. Then since $\|S(t)u\|_{\mathcal{H}^\alpha} \le e^{-\frac{t}{2}}\|u\|_{\mathcal{H}^\alpha}$,

$$\|H(v)(t)\|_{\mathcal{H}^\alpha} \le \int_0^t \|(\pi_1 w + \psi + v)^3\|_{H^{\alpha-1}}\, ds.$$

We use the fractional Leibnitz inequality

$$\lesssim \int_0^t \|\langle\nabla\rangle^{\alpha-1} - (\pi_1 w + \psi + v)\|_{L^6}\|\pi_1 w + \psi + v\|_{L^6}^2\, ds$$

and Hölder with $\frac{1}{2} + \frac{1}{6} + \frac{1}{3} = 1$

$$\begin{aligned}
&\lesssim T^{\frac{1}{2}}\|S(s)w_0 + \psi + v\|_{L^6([0,T],W^{\alpha-1,6})}\|\pi_1 w + \psi + v\|_{L^6([0,T],L^6)} \\
&\lesssim T^{\frac{1}{2}}\big(\|\pi_1 w\|^2_{L^6([0,T],W^{\alpha-1,6})} + \|\psi\|^2_{L^6([0,T],W^{\alpha-1,6})} + \|v\|^2_{L^6([0,T],W^{\alpha-1,6})} \\
&\quad + \|\pi_1 w\|^2_{L^6([0,T],L^6)} + \|\psi\|^2_{L^6([0,T],L^6)} + \|v\|^2_{L^6([0,T],L^6)}\big)
\end{aligned}$$

We use

$$\|v\|_{L^6([0,T],L^6)} \lesssim \|v\|_{L^6([0,T],H^\alpha)}$$
$$\|v\|_{L^6([0,T],W^{\alpha-1,6})} \lesssim \|v\|_{L^6([0,T],H^\alpha)}$$

and get

$$\|H(v)\|_{L^\infty([0,T],\mathcal{H}^\alpha)} \lesssim T^{\frac{1}{2}} \big(\underbrace{\|w_0\|^2_{X^\alpha} + \|\psi\|^2_{L^6([0,T],W^{\alpha-1,6})} + \|\psi\|^2_{L^6([0,T],L^6)}}_{\equiv A(w_0,\psi)} + \|v\|^2_{L^6([0,T],\mathcal{H}^\alpha)}\big).$$
$$\le T^{\frac{1}{2}}(A(w_0,\psi) + T^{\frac{1}{3}}R^2).$$

Here $A(w_0,\psi)$ is finite by assumption on w_0 and our regularity results for ψ, given that $6(\alpha-1) < 2 \iff \alpha < \frac{4}{3}$. From this we can see that if T is small enoguh we get a selfmap

$$H : B_R^{L^\infty([0,T],\mathcal{H}^\alpha)} \longrightarrow B_R^{L^\infty([0,T],\mathcal{H}^\alpha)}.$$

We want to show that H is a contraction on B_R, so let $v_1, v_2 \in B_R$.

$$\begin{aligned}
\|H(v_1) - H(v_2)\|_{L^\infty([0,T],\mathcal{H}^\alpha)} &\le \int_0^t \|(\pi_1 w + \psi + v_1)^3 - (\pi_1 w + \psi + v_2)^3\|_{H^{\alpha-1}}\, ds \\
&\le 3\int_0^t \|(\pi_1 w + \psi)^2 (v_1 - v_2)\|_{H^{\alpha-1}}\, ds \\
&+ 3\int_0^t \|(\pi_1 w + \psi)(v_1 - v_2)^2\|_{H^{\alpha-1}}\, ds \\
&+ \int_0^t \|(v_1 - v_2)^3\|_{H^{\alpha-1}}\, ds \\
&= (I) + (II) + (III)
\end{aligned}$$

We apply the fractional Leibnitz inequality several times.

$$\begin{aligned}
(I) &\lesssim \int_0^t \|(\pi_1 w + \psi)^2\|_{W^{\alpha-1,3}} \|v_1 - v_2\|_{L^6} + \|(\pi_1 w + \psi)^2\|_{L^3} \|v_1 - v_2\|_{W^{\alpha-1,6}}\, ds \\
&\lesssim \int_0^t \big(\|\pi_1 w + \psi\|_{W^{\alpha-1,6}} \|\pi_1 w + \psi\|_{L^6} + \|\pi_1 w + \psi\|^2_{L^6}\big) \|v_1 - v_2\|_{W^{\alpha-1,6}}\, ds \\
&\lesssim T^{\frac{1}{2}} \|v_1 - v_2\|_{L^6([0,T],\mathcal{H}^\alpha)} \|\pi_1 w + \psi\|_{L^6([0,T],W^{\alpha-1,6})}.
\end{aligned}$$

$$\begin{aligned}
(II) &\lesssim \int_0^t \|\pi_1 w + \psi\|_{W^{\alpha-1,6}} \|(v_1 - v_2)^2\|_{L^3} + \|\pi_1 w + \psi\|_{L^6} \|(v_1 - v_2)^2\|_{W^{\alpha-1,3}}\, ds \\
&\lesssim \int_0^t \big(\|\pi_1 w + \psi\|_{W^{\alpha-1,6}} + \|\pi_1 w + \psi\|_{L^6}\big) \|v_1 - v_2\|^2_{W^{\alpha-1,6}}\, ds \\
&\lesssim T^{\frac{1}{2}} \|v_1 - v_2\|^2_{L^6([0,T],\mathcal{H}^\alpha)} \|\pi_1 w + \psi\|_{L^6([0,T],W^{\alpha-1,6})}.
\end{aligned}$$

$$(III) \lesssim \int_0^t \|v_1 - v_2\|^3_{W^{\alpha-1,6}}\, ds \lesssim T^{\frac{1}{2}} \|v_1 - v_2\|^3_{L^6([0,T],\mathcal{H}^\alpha)}.$$

Again we can choose T small enough (depending on $A(w_0,\psi)$ so that H is a contraction. By the Banach fixed-point theorem there exists a $v \in B_R$ so that $H(v) = v$. $\square$

Lemma 1.16. *Let $\alpha < \frac{4}{3}$, $v \in L^\infty([0,T],\mathcal{H}^\alpha)$ solve $H(v) = v$ as before. Then*

$$v \in C([0,T],\mathcal{H}^\alpha) \cap C^1([0,T],\mathcal{H}^{\alpha-1})$$

and the derivative in the latter space is

$$\partial_t v = Lv + \begin{pmatrix} 0 \\ -(\pi_1 w + \psi + v)^3 \end{pmatrix}.$$

Proof. Let $0 \leq t \leq t+h \leq T$. Then

$$\|v(t+h)-v(t)\|_{\mathcal{H}^\alpha} \leq \int_0^t \left\| (S(t+h-s)-S(t-s)) \begin{pmatrix} 0 \\ -(\pi_1 w + \psi + v)^3 \end{pmatrix} \right\|_{\mathcal{H}^\alpha} ds + \int_t^{t+h} \left\| S(t+h-s) \begin{pmatrix} 0 \\ -(\pi_1 w + \psi + v)^3 \end{pmatrix} \right\|_{\mathcal{H}^\alpha} ds.$$

In the proof of the local well-posedness we have seen that $\|(S(t)w_0 + \psi + v)^3\|_{L^\infty([0,T],H^{\alpha-1})} < \infty$ and so the above vanishes as $h \to 0$. Thos proves the continuity. For the differentiability, recall that we have shown $S \in C^1([0,T], L(\mathcal{H}^\alpha, \mathcal{H}^{\alpha-1})$. We estimate

$$\begin{aligned}
&\left\| \frac{v(t+h)-v(t)}{h} - Lv(t) - \begin{pmatrix} 0 \\ -(\pi_1 w + \psi + v)^3 \end{pmatrix} \right\|_{\mathcal{H}^{\alpha-1}} \\
\leq &\left\| \int_0^t \left(\frac{S(t+h-s)-S(t-s)}{h} - LS(t-s) \right) \begin{pmatrix} 0 \\ -(\pi_1 w + \psi + v)^3 \end{pmatrix} ds \right\|_{\mathcal{H}^{\alpha-1}} \\
+ &\left\| \int_t^{t+h} S(t+h-s) \begin{pmatrix} 0 \\ -(\pi_1 w + \psi + v)^3 \end{pmatrix} ds - \begin{pmatrix} 0 \\ -(\pi_1 w + \psi + v)^3 \end{pmatrix} \right\|_{\mathcal{H}^{\alpha-1}}.
\end{aligned}$$

From what we alrady know about S it follows that this vanishes as $h \to 0$. $\square$

2 Global Well-posedness

2.1 Controlling the growth of $\|\psi\|_{L^p}$

The energy estimates we will derive to get global well-posedness involve L^p-norms of ψ for large p, so we have to control its growth. What we know so far is that

$$\mathbb{E}\left[\int |\psi(t,x)|^p \, dx\right] \leq p^{\frac{p}{2}} \int \mathbb{E}\left[|\psi(t,x)|^2\right]^{\frac{p}{2}} dx \leq C^p p^{\frac{p}{2}} \int (1 + |\ln(|x|)|)^{\frac{p}{2}} dx.$$

The size of the integral on the right is important, so let us get the following calculation out of the way.

Lemma 2.1. *There exists $C > 0$ so that for all $p > 1$*

$$\int_B (1 + |\ln|x||)^p \, dx \leq Cp^p.$$

Proof.

$$\begin{aligned}
\int_B (1 + |\ln|x||)^p \, dx &\leq \sum_{k=0}^\infty (k+1)^p \mathcal{L}^2(\{x \in B : k < 1 - \ln|x| \leq k+1\}) \\
&\leq \sum_{k=0}^\infty (k+1)^p \mathcal{L}^2(\{x \in B : e^{1-k} > |x| \geq e^{-k}\}) \\
&\leq \mathcal{L}^2(B) \sum_{k=0}^\infty (k+1)^p e^{2-2k} \, dx \\
&\lesssim \left| \int_0^\infty (s+1)^p e^{-2s} \, dx \right| \\
&= \left| 1 - \int_0^\infty p(s+1)^{p-1}(-2)^{-1} e^{-2s} \right| \\
&\leq 1 + p\frac{1}{2} \left| \int_0^\infty (s+1)^{p-1} e^{-2s} \, ds \right|
\end{aligned}$$

Now apply integration by parts repeatedly:

$$\leq 1 + 2^{-1}p + 2^{-2}p\,(p-1) + \cdots + 2^{-\lfloor p \rfloor -1}p\,(p-1)\ldots(p-\lfloor p \rfloor)$$
$$+ \left| \int_0^\infty (s+1)^{p-\lfloor p \rfloor} e^{-2s}\, ds \right|$$
$$\leq 2p^{\lfloor p \rfloor}\,(p - \lfloor p \rfloor) + \int_0^\infty (s+1)^2 e^{-2s}\, ds \leq 4p^p.$$

□

Using this estimate we have

$$\mathbb{E}\left[\int_B |\psi(t,x)|^p\, dx\right] \leq C_1^p p^p.$$

and with Jensen's inequality we can find $C_2 > 0$ so that

$$\mathbb{E}\left[\left(\int_B |\psi(t,x)|^p\, dx\right)^2\right] \leq C_2^p p^p p^p$$

and

$$\operatorname{Var}\left(\int_B |\psi(t,x)|^p\, dx\right) \leq C_2^p p^p p^p.$$

What we need is to prove the existence of such a kind of growth estimate **almost surely**. We do this via an argument with the Borel-Cantelli lemma and the Chebyshev inequality.

Lemma 2.2. *There exists a $K > 0$ so that for all $t \geq s \geq 0$*

(i) $\exists p_0 > 0 : \forall p \geq p_0, \|\psi(t)\|_{L^p} \leq Kp.$

(ii) $\exists p_0 > 0 : \forall p \geq p_0, \|\psi\|_{L^p([s,t]\times B)} \leq (t-s)^{\frac{1}{p}} Kp.$

(iii) $\forall p \in \mathbb{N}, \|\psi(t)\|_{L^p} \leq Q(t)^{\frac{1}{p}} Kp$ *where* $Q(t) = \sum_{p=1}^\infty \frac{\|\psi(t)\|_{L^p}^p}{K^p p^p}$.

(iv) $\forall p \in \mathbb{N}, \|\psi\|_{L^p([s,t]\times B)} \leq (t-s)^{\frac{1}{p}} Q(s,t)^{\frac{1}{p}} Kp$ *where* $Q(s,t) = \sum_{p=1}^\infty \frac{\|\psi\|_{L^p([s,t]\times B)}^p}{K^p p^p}$.

Proof. We prove with (i) and (iii) as including the time integral for (ii) and (iv) causes no complications. As we are on a bounded domain it suffices to show this for $p \in \mathbb{N}$. Observe that for a large $K > 0$

$$\sum_{p=1}^\infty \mathbb{P}\left(\int_B |\psi(t,x)|^p\, dx > 2K^p p^p\right) \leq \sum_{p=1}^\infty \mathbb{P}\left(\left|\int_B |\psi(t,x)|^p - \mathbb{E}\left[\int_B |\psi(t,x)|^p\, dx\right]\right| > K^p p^p\right)$$
$$+ \sum_{p=1}^\infty \mathbb{P}\left(\mathbb{E}\left[\int_B |\psi(t,x)|^p\, dx\right] > K^p p^p\right)$$
$$= (I) + (II).$$

The sum (II) is zero is $K > C_1$. For (I) we use Chebyshev's inequality:

$$(I) \leq \sum_{p=1}^\infty \frac{\operatorname{Var}\left(\int_B |\psi(t,x)|^p\, dx\right)}{K^{2p} p^{2p}} \leq \sum_{p=1}^\infty \left(\frac{C_2}{K^2}\right)^p \frac{p^p p^p}{p^p p^p} < \infty,$$

where the result at the end is certainly summable given $K^2 > C_2$. Then the Borel-Cantelli lemma now implies

$$\mathbb{P}\left(\int_B |\psi(t,x)|^p\, dx > 2K^p p^p \text{ for infinitely many } p \in \mathbb{N}\right) = 0.$$

This finishes the proof of (i). While this is a good result, we would like to have the estimate **for all** p as opposed to just large p. In return we give up that the constant in the estimate is deterministic. Statement (iii) is an alternative to using Borel-Cantelli that was suggested to the author by their advisor L. Tolomeo, who himself attributed it to M. Gubinelli. Quantitatively, what we have really shown is that

$$\mathrm{Var}(Q) < \infty \text{ where } Q = \sum_{p=1}^{\infty} \frac{\|\psi\|_{L^p}}{K^p p^p}.$$

Therefore $Q < \infty$ a.s. and for all $p \in \mathbb{N}$

$$\|\psi\|_{L^p} \leq \frac{Q}{\sum_{p=1}^{\infty} K^{-p} p^{-p}} \leq Q K^p p^p.$$

$\square$

Lemma 2.3. *The process Q is a.s. continuous in time.*

2.2 Energy Estimates

Let $v \in \mathcal{H}^\alpha$, $\alpha > 1$ and write $(v, v_t) \equiv (\pi_1 v, \pi_2 v) = v$. we define an energy $E : \mathcal{H}^\alpha \longrightarrow [0, \infty)$ by

$$E(v) = \int_B |v|^2 + |v_t|^2 + |\nabla v|^2 + \frac{1}{2}|v|^4 \, dx.$$

Recall that $H^1 \overset{L^p}{\hookrightarrow}$ for all p, in particular $p = 4$ so the above is well-defined. This energy has regularity $E \in C^1(\mathcal{H}^1, \mathbb{R})$ with Fréchet derivative

$$DE(v)(f) = \int_B 2vf + 2v_t f_t + \nabla v \nabla f + 2v^3 f \, dx.$$

Our solutions only have regularity $C^1([0,T], \mathcal{H}^{\alpha-1})$ with $\alpha - 1 < 1$, so we can not simply differentiate the energy straight away.Instead we mollify our solutions $v(t)$ to improve their regularity. Then we take a limit to show that $E(v(t))$ is absolutely continuous in time.

Definition 2.4. *For $N \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $f \in H^\beta$ define*

$$P_N f = \sum_{n=1}^{N} |\hat{f}(n)|^2 e_n.$$

Lemma 2.5. *For all $k \in \mathbb{N}$ and $\beta_1, \beta_2 \in \mathbb{R}$,*

$$P_N : C^k([0,T], H^{\beta_1}) \longrightarrow C^k([0,T], H^{\beta_2}).$$

Proof. Let $f \in C^k([0,T], H^{\beta_1})$. We write down the proof only for $k = 1$:

$$\left\| \frac{P_N f(t+h) - P_N f(t)}{h} - P_N \partial_t f(t) \right\|^2_{H^{\beta_1}}$$

$$= \sum_{n=1}^{N} \underbrace{\langle \lambda_n \rangle^{2\frac{\beta_1}{\beta_2}}}_{\lesssim N^{|\frac{\beta_1}{\beta_2}|}} \langle \lambda_n \rangle^{2\beta_2} \left\langle \frac{f(t+h) - f(t)}{h} - \partial_t f(t), e_n \right\rangle.$$

$$\leq C(N, \beta_1, \beta_2) \left\| \frac{f(t+h) - f(t)}{h} - \partial_t f(t) \right\|^2_{H^{\beta_2}}$$

$\square$

We now derive a global energy estimate.

Lemma 2.6. *Suppose* $v \in C^0([0,T),\mathcal{H}^\alpha) \cap C^1([0,T),\mathcal{H}^{\alpha-1})$ *solves* $v_0 = 0$ *and*

$$\partial_t \begin{pmatrix} v \\ v_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(1-\Delta) & -1 \end{pmatrix} \begin{pmatrix} v \\ v_t \end{pmatrix} - \begin{pmatrix} 0 \\ (\pi_1 w + \pi_1 \Psi + v)^3 \end{pmatrix}.$$

Then there is an estimate

Proof. Since $P_N v \in C^1([0,T],\mathcal{H}^1)$ we can compute

$$\begin{aligned}
\frac{d}{dt}E(P_N v(t)) &= DE(P_N v(t))(\partial_t P_N v(t)) \\
&= \int_B 2P_N v P_N v_t + P_N v_t P_N v_{tt} + \nabla P_N v \nabla P_N v_t + (P_N v)^3 P_N v_t \, dx \\
&= \int_B 2P_N v_t (v - \Delta v + (P_N v)^3 + v_{tt}) \, dx
\end{aligned}$$

.

This implies that for any $0 \leq t_1 < t_2 \leq T$

$$E(P_N v(t_2)) - E(P_N v(t_1)) = \int_{t_1}^{t_2} \int_B 2P_N v_t (v - \Delta v + (P_N v)^3 + v_{tt}) \, dx \, ds.$$

Now we let $N \longrightarrow \infty$ on both sides using dominated convergence. A majorant is found by applying Cauchy Schwartz. As an example, for the nonlinear-term we get

$$\left| \int_B P_N v_t (P_N v)^3 \, dx \right| \leq \|P_N v\|_{L^2} \|P_N v\|_{L^6}^3 \leq \|P_N v\|_{H^1}^4 \leq \|v\|_{H^1}^4.$$

In the limit we get that $E(v(t))$ is absolutely continuous in time with derivative given by

$$\begin{aligned}
\frac{d}{dt}E(v(t)) &= 2\int_B v_t (v - \Delta v + v^3 + v_{tt}) \, dx \\
&= 2\int_B -|v_t|^2 + v^3 - (\pi_1 w + \psi + v)^3 \, dx \\
&\leq 2\int_B -3v_t v (\pi_1 w + \psi)^2 - 3v_t v^2 (\pi_1 w + \psi) - v_t (\pi_1 w + \psi)^3 \, dx \\
&= (I) + (II) + (III).
\end{aligned}$$

For (I) we apply Hölder with $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$:

$$|(I)| \lesssim \|v_t\|_{L^2} \|v\|_{L^4} \|\pi_1 w + \psi\|_{L^8}^2 \lesssim E^{\frac{3}{4}} \|\pi_1 w + \psi\|_{L^8}^2$$

For (III) we apply Hölder with $\frac{1}{2} + \frac{1}{2} = 1$:

$$|(III)| \lesssim \|v_t\|_{L^2} \|\pi_1 w + \psi\|_{L^6}^3 \lesssim E^{\frac{1}{2}} \|\pi_1 w + \psi\|_{L^6}^2.$$

For (II) we apply Hölder with $\frac{1}{2} + \frac{1}{q} + \frac{1}{p}$ where $p > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$.

$$(II) \leq \|v_t\|_{L^2} \|v\|_{L^{2q}}^2 \|\pi_1 w + \psi\|_{L^p}.$$

Now we apply a specical case of the Gagliardo-Nirenberg-Sobolev inequality, also known as a generalization of Ladyzhenskaya's inequality:

$$\|u\|_{L^{2q}} \lesssim \|u\|_{L^4}^{\frac{2}{q}} \|u\|_{H^1}^{1-\frac{2}{q}}.$$

The Gagliardo-Nirenberg-Sobolev inequality is obtained, as the name suggests, by combining the Sobolev inequality and the Gagliardo-Nirenberg interpolation inequality. A detailed but transparent study of this inequality can be found at [1]. From lemma 2.2 we know that $\|\pi_1 w(t) + \psi(t)\|_{L^p} \lesssim Q(t)^{\frac{1}{p}} Kp \leq (Q(t) \wedge 1)Kp$, so we get

$$(II) \lesssim Q^{\frac{1}{p}} Kp \|v_t\|_{L^2} \|u\|_{L^4}^{\frac{4}{q}} \|u\|_{H^1}^{2-\frac{4}{q}} \lesssim Q^{\frac{1}{p}} KpE^{\frac{1}{2}+\frac{1}{q}+1-\frac{2}{q}} = \tilde{Q}(t)KpE^{1+\frac{1}{p}}.$$

where $\tilde{Q}(t) = Q(t) \vee 1$. Assuming we have $t_0 < t_1$ so that $E(t) \geq 1$ for all $t \in [t_0, t_1]$, we choose $p(s) = \ln(E(s))$ and consider the integral form of the inequality:

$$\ln(E(t)) - \ln(E(t_0)) = \int_{t_0}^{t} \frac{E'(s)}{E(s)}\, ds \lesssim Ke \int_0^t \tilde{Q}(s) \ln(E(s))\, ds.$$

Then by Grönwall's inequality, for all $t \in [t_0, t_1]$

$$\begin{aligned} \ln(E(t)) &\lesssim \ln(E(t_0)) + \int_{t_0}^{t} \ln(E(t_0))\tilde{Q}(s)Ke \cdot e^{\int_s^t \tilde{Q(r)} Ke\, dr}\, ds \\ &= \ln(E(t_0)) \left(1 + Ke \int_{t_0}^{t} \tilde{Q}(s) \cdot e^{Ke \int_s^t \tilde{Q(r)}\, dr}\, ds\right) \\ &\leq \ln(E(t_0)) \left(1 + Ke^{1+Ke\int_{t_0}^{t_1} \tilde{Q}} \int_{t_0}^{t_1} \tilde{Q}\, ds\right) \end{aligned}$$

Apply the exponential on both sides to get

$$E(t_1) \lesssim E(t_0)^{1+Ke^{1+Ke\int_{t_0}^{t_1}\tilde{Q}} \int_{t_0}^{t_1} \tilde{Q}\, ds}$$

□

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