Seminar 5

1. Find the accumulation points for each of the following sets: $[0,1) \cup \{2\}$, $\mathbb{Z}$, $\{0.1, 0.11, \ldots\}$.

2. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous everywhere, with $|f|$ continuous everywhere.

3. If $f : [a,b] \to [a,b]$ is continuous, then it has at least one fixed point x^* s.t. $x^* = f(x^*)$.

4. Study the continuity and the differentiability for f and f', where $f : \mathbb{R} \to \mathbb{R}$,
$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

5. Prove (from scratch) that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ and then that $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$.

6. Compute the following limits:

   (a) $\lim\limits_{x\to\infty} \dfrac{\lfloor x \rfloor}{x}$.

   (b) $\lim\limits_{x\to\infty} x\big(\ln(x+2) - \ln(x+1)\big)$.

   (c) $\lim\limits_{x\to 0} \big(\cos x\big)^{\frac{1}{x^2}}$.

   (d) $\lim\limits_{\substack{x\to 0\\ x>0}} x^x$.

   (e) $\lim\limits_{\substack{x\to 0\\ x>0}} (\sin x)^x$.

   (f) $\lim\limits_{x\to\infty} x\big((1+\frac{1}{x})^x - e\big)$.

7. Find the n^{th} derivative of the following functions:

   (a) $f : (-1,\infty) \to \mathbb{R},\ f(x) = \ln(1+x)$.

   (b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$.

   (c) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 \sin x$.

   (d) $f : \mathbb{R} \to \mathbb{R},\ f(x) = e^{2x} x^3$.

8. ★ Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. To minimize f, consider the *gradient descent* method
$$x_{n+1} = x_n - \eta f'(x_n),$$
where $x_1 \in \mathbb{R}$ and $\eta > 0$ (learning rate). Use Python (numerics or graphics) for the following:
(a) Take a convex f and show that for small η the method converges to the minimum of f.
(b) Show that by increasing η the method can converge faster (in fewer steps).
(c) Show that taking η too large might lead to the divergence of the method.
(d) Take a nonconvex f and show that the method can get stuck in a local minimum.

Homework questions are marked with ★.