Seminar 4

1. Study if the following series are convergent or divergent:

   (a) $\sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}$.

   (b) ★ $\sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}\cdot\frac{1}{n^2}$.

   (c) $\sum_{n\geq 1} a^{\ln n}$, $a>0$.

   (d) $\sum_{n\geq 1} \frac{a^n n!}{n^n}$ $a>0$.

2. Study the convergence and the absolute convergence of the following series:

   (a) $\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$. (b) $\sum_{n\geq 1} (-1)^n \sin\frac{1}{n}$. (c) $\sum_{n\geq 1} \frac{\sin n}{n^2}$.

3. Prove by differentiating the geometric series that, for $|x|<1$,

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \quad \sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}.$$

4. Prove by integrating the geometric series that, for $|x|<1$,

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x), \quad \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = \ln(1+x).$$

5. Prove that $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots = \arctan x$, for $x\in[-1,1]$.

6. Find the radius of convergence and the convergence set for each of the following series:

   (a) $\sum_{n\geq 1} \frac{(x-2)^n}{(n+1)3^n}$. (b) $\sum_{n\geq 1} \frac{(x-1)^n}{n^p}$, $p>0$. (c) ★ $\sum_{n\geq 1} \frac{nx^n}{2^n}$.

7. ★ [Python] Show numerically that $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = \ln 2$. Change the order of summation in this series – for example by first adding p positive terms, then q negative terms, and so on – and show numerically that the rearrangement gives a different sum (depending on p, q).

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.