Seminar 10

1. For $f : \mathbb{R}^2 \to R, f(x, y) = x^2 + xy$ find:

   (a) the gradient of f and the direction of steepest descent at the point $(1, 0)$.

   (b) the directional derivative at the point $(1, 0)$ in the direction of $\vec{i} + \vec{j}$.

   (c) the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 0, 1)$.

2. Find the equation of the tangent line to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at an arbitrary point (x_0, y_0).

3. Let $f : \mathbb{R}^n \to \mathbb{R}, f(x) = \frac{1}{2}\|x\|^2$. Find the gradient of f. Find the directional derivative $D_v f(x)$ in two ways: using the definition and using the gradient.

4. Let $D = \operatorname{diag}(d_1, \ldots, d_n)$ be a diagonal $n \times n$ matrix and consider the quadratic function $f : \mathbb{R}^n \to R$, $f(x) = \frac{1}{2} x^T D x$. Prove that $\nabla f(x) = Dx$ and $H(x) = D$.

5. For each of the following functions, compute $\dfrac{\mathrm{d}f}{\mathrm{d}t}$ directly and using the chain rule:

   (a) $f(x, y) = \ln(x^2 + y^2)$,
   $x = t$, $y = t^2$.

   (b) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$,
   $x = \cos t$, $y = \sin t$, $z = t > 0$.

6. Let $f : \mathbb{R}^2 \to R$ and $(x, y) = (g_1(u, v), g_2(u, v)) = g(u, v)$, $f(x, y) = (f \circ g)(u, v)$. Prove that

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

Homework questions will be given in a separate document.