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Mathematical Analysis

1st Year Computer Science

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❖ Real numbers

Let us start with some standard notation: \emptyset is the empty set; $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers; $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\} = \{m - n \mid m, n \in \mathbb{N}\}$ the set of integers; $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ the set of rational numbers; \mathbb{R} the set of real numbers.

You are very much used with the real numbers. However, it is not trivial to define them in a rigorous way, see for example [1] or [2] for different methods of constructing \mathbb{R} from \mathbb{Q} . We will straightforwardly start working with the real numbers – for this reason some of their properties, e.g. [definition 1.5](#), will simply be given as definitions.

Definition 1.1. Let A be a subset of \mathbb{R} , denoted as $A \subseteq \mathbb{R}$. We define $x \in \mathbb{R}$ to be

a *lower bound* for A if $x \leq a, \forall a \in A$; an *upper bound* for A if $x \geq a, \forall a \in A$.

We define

$lb(A) := \{x \in \mathbb{R} \mid x \leq a, \forall a \in A\}$ the set of lower bounds of A ,

$ub(A) := \{x \in \mathbb{R} \mid x \geq a, \forall a \in A\}$ the set of upper bounds of A .

We define $x \in \mathbb{R}$ to be

the *minimum* of A if $x \in lb(A) \cap A$; the *maximum* of A if $x \in ub(A) \cap A$,

denoted by $\min(A)$, respectively $\max(A)$. In other words, we have that

$\min(A) \in A$ and $\min(A) \leq a, \forall a \in A$; $\max(A) \in A$ and $\max(A) \geq a, \forall a \in A$.

Note that there are sets which do not have minimum or maximum, e.g. $(0, 1)$.

Definition 1.2. A set $A \subseteq \mathbb{R}$ is defined to be

- bounded (from) below if $lb(A) \neq \emptyset$;
- bounded (from) above if $ub(A) \neq \emptyset$;
- bounded if it is both bounded below and above;
- unbounded if it is not bounded.

Definition 1.3. We say that $x \in \mathbb{R}$ is the *supremum* of $A \subseteq \mathbb{R}$, $x := \sup(A)$, if and only if:

1. $x \geq a, \forall a \in A$, that is $x \in ub(A)$.
2. if u is an upper bound for A , then $x \leq u$.

The supremum is *the least upper bound*, i.e. $\sup(A) := \min(ub(A))$.

Definition 1.4. We say that $x \in \mathbb{R}$ is the *infimum* of $A \subseteq \mathbb{R}$, $x := \inf(A)$, if and only if:

1. $x \leq a, \forall a \in A$, that is $x \in lb(A)$.
2. if u is a lower bound for A , then $x \geq u$.

The infimum is *the greatest lower bound*, i.e. $\inf(A) := \max(lb(A))$.

Definition 1.5 (Completeness Axiom). Every set $A \subseteq \mathbb{R}$ that is bounded above has a supremum. Similarly, every set $A \subseteq \mathbb{R}$ that is bounded below has an infimum.

Note that if A has a maximum, then $\sup(A) = \max(A)$. Similarly, if A has a minimum, then $\inf(A) = \min(A)$. Also, if $\sup(A) \in A$, then $\max(A) = \sup(A)$.

Example 1.6. (a) $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, $\sup(A) = 1 = \max(A)$, $\inf(A) = 0$, $\nexists \min(A)$.

(b) $A = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$, $\sup(A) = \sqrt{2}$, $\nexists \max(A)$, $\inf(A) = -\sqrt{2}$, $\nexists \min(A)$.

Theorem 1.7. Let $A \subseteq \mathbb{R}$ be a bounded set. For $\sup(A)$ and $\inf(A)$ the following are true:

$$\forall \varepsilon > 0, \exists x \in A \text{ such that } \sup(A) - \varepsilon < x,$$

$$\forall \varepsilon > 0, \exists x \in A \text{ such that } x < \inf(A) + \varepsilon.$$

Proof. By definition, for any $y < \sup(A)$, say $y = \sup(A) - \varepsilon$ with $\varepsilon > 0$, we have that $y \notin ub(A)$. Hence there exists $x \in A$ such that $y < x$. Similar proof for $\inf(A)$. \square

Proposition 1.8. Let $A \subseteq B \subseteq \mathbb{R}$ be (nonempty) bounded sets. Then

$$\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$$

and

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\},$$

$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}.$$

Proof. It follows directly from the definitions. \square

Definition 1.9. Define the *extended real line* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, where ∞ and $-\infty$ are such that

$$\forall x \in \mathbb{R}, -\infty < x < \infty.$$

If a set A is not bounded above, we define $\sup(A) := \infty$.

If a set A is not bounded below, we define $\inf(A) := -\infty$.

[Seminar] The empty set \emptyset is bounded by any number. In $\overline{\mathbb{R}}$, $\sup(\emptyset) = -\infty$, $\inf(\emptyset) = \infty$.

Definition 1.10. A set $V \subseteq \mathbb{R}$ is a *neighborhood* (vecinity) of $x \in \mathbb{R}$ if

$$\exists \varepsilon > 0 \text{ such that } (x - \varepsilon, x + \varepsilon) \subseteq V.$$

A set $V \subseteq \mathbb{R}$ is a *neighborhood* of ∞ if $\exists a \in \mathbb{R}$ such that $(a, \infty) \subseteq V$.

A set $V \subseteq \mathbb{R}$ is a *neighborhood* of $-\infty$ if $\exists a \in \mathbb{R}$ such that $(-\infty, a) \subseteq V$.

We denote all the neighborhoods of x by $\mathcal{V}(x) := \{V \subseteq \mathbb{R} \mid V \text{ is a neighborhood of } x\}$.

Definition 1.11. Let $A \subseteq \mathbb{R}$. The following set is called the *interior* of A

$$\text{int}(A) := \{x \in \mathbb{R} \mid \exists V \in \mathcal{V}(x) \text{ such that } V \subseteq A\},$$

and the following set is called the *closure* of A

$$\text{cl}(A) := \{x \in \mathbb{R} \mid \forall V \in \mathcal{V}(x), V \cap A \neq \emptyset\}.$$

Proposition 1.12. For any $A \subseteq \mathbb{R}$, it holds that $\text{int}(A) \subseteq A \subseteq \text{cl}(A)$.

Proof. To prove that $\text{int}(A) \subseteq A$ we prove that if $x \in \text{int}(A)$, then $x \in A$. Let $x \in \text{int}(A)$, then $\exists \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq A$. Since $x \in (x - \varepsilon, x + \varepsilon)$, we have that $x \in A$.

To prove that $A \subseteq \text{cl}(A)$ we show that if $x \in A$, then $x \in \text{cl}(A)$. Let $x \in A$. Then for any $V \in \mathcal{V}(x)$ it holds that $x \in V$, giving that $x \in V \cap A$. Hence $x \in \text{cl}(A)$ since $V \cap A \neq \emptyset$. \square

Definition 1.13. If $A = \text{int}(A)$, then A is called *open*. If $A = \text{cl}(A)$, then A is called *closed*.

Remark 1.14. To prove that a set A is open, it is sufficient to prove that $A \subseteq \text{int}(A)$.

To prove that a set A is closed, it is sufficient to prove that $\text{cl}(A) \subseteq A$.

Proposition 1.15. The following statements are true:

- The complement of an open set is closed.
- The complement of a closed set is open.

Proof. Let us prove the first statement, the other one being similar. Consider A an open set, i.e. $A = \text{int}(A)$, and denote by $A^c = \{x \in \mathbb{R} \mid x \notin A\}$ its complement. To prove that A^c is closed, we prove that $\text{cl}(A^c) \subseteq A^c$. Consider $x \in \text{cl}(A^c)$ and let's assume that $x \notin A^c$, i.e. $x \in A$, aiming to obtain a contradiction. Since A is open, there exists $V \in \mathcal{V}(x)$ such that $V \subseteq A$, giving that $V \cap A^c = \emptyset$: contradiction with $x \in \text{cl}(A^c)$. Hence the assumption $x \notin A^c$ is false, and we have that if $x \in \text{cl}(A^c)$, then $x \in A^c$. In other words, $\text{cl}(A^c) \subseteq A^c$. \square

Proposition 1.16. Any union of open sets is open. Any intersection of closed sets is closed. Any finite intersection of open sets is open. Any finite union of closed sets is closed.

Proof. (Optional) Left as extra homework. \square

❖ Sequences

A set $\{x_n \mid n \in \mathbb{N}\}$ is called a sequence and is denoted by $(x_n)_{n \in \mathbb{N}}$ or simply (x_n) . A sequence (x_n) is bounded above (or below) if the set $\{x_n \mid n \in \mathbb{N}\}$ is bounded above (or below). A sequence (x_n) is increasing if $x_{n+1} \geq x_n$, $\forall n \in \mathbb{N}$, and decreasing if $x_{n+1} \leq x_n$, $\forall n \in \mathbb{N}$. A sequence is monotone if it is either increasing or decreasing.

Definition 2.1. A sequence (x_n) has a limit $\ell \in \overline{\mathbb{R}}$, and we write $\lim_{n \rightarrow \infty} x_n = \ell$ or $x_n \rightarrow \ell$, if

$$\forall V \in \mathcal{V}(\ell), \exists N_V \in \mathbb{N} \text{ such that } x_n \in V, \forall n \geq N_V.$$

If $\ell \in \mathbb{R}$, we say that (x_n) converges to ℓ : $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that $|x_n - \ell| < \varepsilon$, $\forall n \geq N_\varepsilon$.

$x_n \rightarrow \infty$ if $\forall a > 0$, $\exists N_a \in \mathbb{N}$ such that $x_n > a$, $\forall n \geq N_a$.

$x_n \rightarrow -\infty$ if $\forall a < 0$, $\exists N_a \in \mathbb{N}$ such that $x_n < a$, $\forall n \geq N_a$.

Proposition 2.2. A sequence (x_n) converges to $\ell \in \mathbb{R}$ if and only if $\lim_{n \rightarrow \infty} |x_n - \ell| = 0$.

Proposition 2.3. Any convergent sequence is bounded.

Proof. TBA (left to the reader). □

Theorem 2.4 (Weierstrass). Any monotone and bounded sequence is convergent.

Proof. Assume that the sequence is increasing, for example. Let $S = \{x_n \mid n \in \mathbb{N}\}$ and consider $\sup(S) \in \mathbb{R}$ (we know that S is bounded). From [theorem 1.7](#) we have that

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ such that } \sup(S) - \varepsilon < x_{N_\varepsilon}.$$

As (x_n) is increasing, $\sup(S) - \varepsilon < x_{N_\varepsilon} \leq x_n$ $\forall n \geq N_\varepsilon$. Hence $\sup(S) - x_n < \varepsilon$, $\forall n \geq N_\varepsilon$.

The sequence converges to $\sup(S)$ by [definition 2.1](#). Similarly, a decreasing and bounded sequence converges to its infimum. □

Proposition 2.5. Any monotone sequence has a limit in $\overline{\mathbb{R}}$.

Proof. If the sequence is bounded and monotone, then it is convergent by the Weierstrass theorem. If the sequence is unbounded and monotone, then its limit will be infinite. □

Theorem 2.6 (Squeeze/Sandwich theorem). Let (x_n) , (y_n) , (z_n) be sequences for which there is an $n_0 \in \mathbb{N}$ such that

$$x_n \leq y_n \leq z_n, \forall n \geq n_0,$$

and

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n.$$

Then

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n.$$

Proof. Let $\ell := \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n$ and assume first that $\ell \in \mathbb{R}$. Let $\varepsilon > 0$. Then

$$\exists N_1 \in \mathbb{N} \text{ such that } |x_n - \ell| < \varepsilon, \forall n \geq N_1$$

and

$$\exists N_2 \in \mathbb{N} \text{ such that } |z_n - \ell| < \varepsilon, \forall n \geq N_2.$$

Taking $N_\varepsilon := \max\{N_1, N_2\}$, we have that

$$|y_n - \ell| \leq \max\{|x_n - \ell|, |z_n - \ell|\} < \varepsilon, \forall n \geq N_\varepsilon,$$

hence the conclusion. When ℓ is infinite the proof is similar. \square

Theorem 2.7 (Cantor's nested intervals). Let (a_n) be increasing and (b_n) decreasing such that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n, \forall n \in \mathbb{N}$. Consider the closed intervals $I_n := [a_n, b_n]$, with $I_{n+1} \subseteq I_n$. If $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$, then there exists $x \in \mathbb{R}$ such that

$$\bigcap_{n=1}^{\infty} I_n = \{x\}.$$

Proof. Consider the bounded sets $A := \{a_n \mid n \in \mathbb{N}\}$ and $B := \{b_n \mid n \in \mathbb{N}\}$. For any $k \in \mathbb{N}$, we have that

$$a_k \leq \sup(A) \leq b_k$$

and

$$b_k \geq \inf(B) \geq a_k.$$

Hence by the squeeze theorem we have that $\sup(A) = \inf(B)$ and $\bigcap_{n=1}^{\infty} I_n = \{\sup(A)\}$. \square

Theorem 2.8 (Bolzano-Weierstrass). Any bounded sequence has a convergent subsequence.

Proof. Consider the bounded set $A := \{x_n \mid n \in \mathbb{N}\}$. Let $a_1 := \inf(A)$ and $b_1 := \sup(A)$, and define $I_1 := [a_1, b_1]$. Bisect I_1 and notice that at least one of the two halves must contain infinitely many terms from the sequence. Take $I_2 := [a_2, b_2]$ to be the half that does. Continuing this procedure we obtain for each $k \in \mathbb{N}$ an interval $I_k := [a_k, b_k]$ containing (at least) a term $x_{n_k} \in A$, such that $I_{k+1} \subseteq I_k$ and $b_k - a_k \rightarrow 0$.

From Cantor's nested intervals [theorem 2.7](#) we have that there exists $x \in \mathbb{R}$ such that $\bigcap_{n=1}^{\infty} I_n = \{x\}$, and hence the subsequence (x_{n_k}) converges to x . \square

Definition 2.9. For a sequence (x_n) we define the set of its *limit points* by

$$\text{LIM}(x_n) := \{x \in \overline{\mathbb{R}} \mid \text{there exists a subsequence } (x_{n_k}) \text{ s.t. } x_{n_k} \rightarrow x\},$$

and

$$\liminf_{n \rightarrow \infty} x_n := \inf (\text{LIM}(x_n)),$$

$$\limsup_{n \rightarrow \infty} x_n := \sup (\text{LIM}(x_n)).$$

Example 2.10. For $x_n = \frac{(-1)^n n}{n+1}$, $\text{LIM}(x_n) = \{-1, 1\}$, $\liminf_{n \rightarrow \infty} x_n = -1$, $\limsup_{n \rightarrow \infty} x_n = 1$.

Proposition 2.11. $\lim_{n \rightarrow \infty} x_n = \ell \in \overline{\mathbb{R}}$ if and only if $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = \ell$.

Definition 2.12 (Cauchy sequence). A sequence (x_n) is called *Cauchy* (or *fundamental*) if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ such that } |x_m - x_n| < \varepsilon, \forall m, n \geq N_\varepsilon.$$

Proposition 2.13. Any Cauchy sequence is bounded.

Proof. For $\varepsilon = 1$, there exists $N_1 \in \mathbb{N}$ such that $|x_m - x_n| < 1, \forall m, n \geq N_1$. In particular, $|x_n - x_{N_1}| < 1, \forall n \geq N_1$, hence the terms after index N_1 are bounded. The terms before index N_1 are also bounded since there is a finite number of them. We thus conclude that the entire sequence is bounded. \square

Theorem 2.14. A sequence is convergent if and only if it is Cauchy.

Proof. Let's consider first a convergent sequence (x_n) with $x_n \rightarrow \ell$. For any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_n - \ell| < \frac{\varepsilon}{2}$, for any $n \geq N_\varepsilon$. Then $|x_m - x_n| \leq |x_m - \ell| + |x_n - \ell| < \varepsilon$, for any $n \geq N_\varepsilon$. Hence the sequence (x_n) is Cauchy.

Assume now that (x_n) is a Cauchy sequence. From the previous proposition we have that (x_n) must be bounded, and thus it has a convergent subsequence $(x_{n_k}), x_{n_k} \rightarrow x \in \mathbb{R}$. Let $\varepsilon > 0$. There exists thus $K_\varepsilon \in \mathbb{N}$ such that $|x_{n_k} - x| < \varepsilon, \forall k \geq K_\varepsilon$. Also, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon, \forall m, n \geq N_\varepsilon$. In particular, $|x_{n_k} - x_n| < \varepsilon, \forall k, n \geq N_\varepsilon$. Hence $|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < 2\varepsilon, \forall n \geq \max\{K_\varepsilon, N_\varepsilon\}$, meaning that $x_n \rightarrow x$. \square

Example 2.15. The sequence defined by $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ is not convergent. Indeed, one can see, for example, that

$$x_{2n} - x_n = \frac{1}{n+1} + \dots + \frac{1}{2n} > \frac{n}{2n},$$

hence $x_{2n} - x_n > \frac{1}{2}$ for any $n \in \mathbb{N}$. Thus (x_n) is not convergent since it is not Cauchy.

❖ Series. Power series

For a sequence (x_n) , the sum $\sum_{n=1}^{\infty} x_n$ is called a *series* and $s_n := \sum_{k=1}^n x_k$ is called the *partial sum of the series*. The summation in a series can start from any index, not necessarily 0 or 1. A series is also often written as $\sum_{n \geq 1} x_n$.

Definition 3.1. The series $\sum_{n=1}^{\infty} x_n$ converges iff the sequence of partial sums (s_n) converges.

Example 3.2. The *geometric series* $\sum_{n=0}^{\infty} q^n$ converges iff $|q| < 1$, with sum $\frac{1}{1-q}$.

Example 3.3. The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ diverges since (s_n) is not a Cauchy sequence.

Example 3.4 (Euler's number). $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Proof. (Optional) Let $s_n = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!}$. Start from $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ and expand

$$\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \dots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdot \dots \cdot \left(1 - \frac{n-1}{n}\right) \leq s_n.$$

We have that

$$\left(1 + \frac{1}{n}\right)^n \leq s_n.$$

Consider now an index $k \geq n$. We have that

$$\left(1 + \frac{1}{k}\right)^k \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{k}\right) + \dots + \frac{1}{n!}\left(1 - \frac{1}{k}\right)\left(1 - \frac{2}{k}\right) \cdot \dots \cdot \left(1 - \frac{n-1}{k}\right)$$

and taking $k \rightarrow \infty$ we obtain that $e \geq s_n$. We conclude with the squeeze theorem for

$$\left(1 + \frac{1}{n}\right)^n \leq s_n \leq e,$$

obtaining that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges and its sum is e . □

Proposition 3.5. If the series $\sum_{n=1}^{\infty} x_n$ is convergent, then $\lim_{n \rightarrow \infty} x_n = 0$.

Proof. Consider the partial sum s_n . We have that $x_n = s_n - s_{n-1}$, hence the conclusion. □

It thus follows that if $\lim_{n \rightarrow \infty} x_n \neq 0$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

Example 3.6. Series like $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1$ are called *telescoping series*. The partial sum of a telescoping series can be easily computed since after cancellations the only remaining terms are the first one and the last one.

If the sequence (x_n) has only nonnegative terms $x_n \geq 0$, then the sequence of partial sums (s_n) is increasing. The series $\sum_{n=1}^{\infty} x_n$ then converges iff (s_n) is bounded.

Theorem 3.7 (Comparison test). Let $\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n$ be series with nonnegative terms. If there is an $n_0 \in \mathbb{N}$ such that

$$x_n \leq y_n, \forall n \geq n_0, \text{ then}$$

(a) If $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ also converges.

(b) If $\sum_{n=1}^{\infty} x_n$ diverges, then $\sum_{n=1}^{\infty} y_n$ also diverges.

Proof. Consider the sequences of partial sums. In case (a), both sequences are bounded. In case (b), both sequences are unbounded. \square

Example 3.8. If $p \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$ since $\frac{1}{n^p} \geq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. E.g. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$.

Theorem 3.9. Let $\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n$ be series with nonnegative terms. If

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \ell, \text{ then}$$

- if $\ell \in (0, \infty)$, then the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ have the same nature.
- if $\ell = 0$, then if the series $\sum_{n=1}^{\infty} y_n$ converges, the series $\sum_{n=1}^{\infty} x_n$ also converges.
- if $\ell = \infty$, then if the series $\sum_{n=1}^{\infty} y_n$ diverges, the series $\sum_{n=1}^{\infty} x_n$ also diverges.

Theorem 3.10 (Ratio test). Let $\sum_{n=1}^{\infty} x_n$ be a series with nonnegative terms such that

$$\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \ell.$$

- If $\ell < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.
- If $\ell > 1$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

The test is *inconclusive* when $\ell = 1$.

Proof. The idea is that $\sum_{n \geq 1} x_n$ behaves like a geometric series with ratio ℓ . We will only give a proof when $\ell < 1$, the other case being similar.

Take $\varepsilon > 0$ such that $q := \ell + \varepsilon < 1$. There exists $N \in \mathbb{N}$ such that

$$\frac{x_{n+1}}{x_n} - \ell < \varepsilon, \forall n \geq N,$$

giving that $x_{n+1} < x_n \cdot q$, $\forall n \geq N$. Hence $x_n < q^{n-N} x_N$, that is $x_n < q^n \frac{x_N}{q^N}$. Since $q < 1$, the series converges by comparison with the geometric series $\sum_{n \geq 1} q^n$. \square

Theorem 3.11 (Root test). Let $\sum_{n=1}^{\infty} x_n$ be a series with nonnegative terms such that

$$\lim_{n \rightarrow \infty} \sqrt[n]{x_n} = \ell.$$

- If $\ell < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.
- If $\ell > 1$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

The test is *inconclusive* when $\ell = 1$.

Proof. Idea: $\sum_{n \geq 1} x_n$ behaves like a geometric series with ratio ℓ , as in the ratio test. \square

Example 3.12. The series $\sum_{n \geq 0} \frac{x^n}{n!}$ converges for any $x \in \mathbb{R}$. We will see later that $\sum_{n \geq 0} \frac{x^n}{n!} = e^x$. We have that $\frac{x_{n+1}}{x_n} = \frac{x}{n+1} \rightarrow 0 < 1$, hence the series converges by the ratio test.

Theorem 3.13 (Cauchy condensation test). Let (x_n) be a decreasing sequence with $x_n > 0$. Then the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=0}^{\infty} 2^n x_{2^n}$ have the same nature.

Proof. Let $S_n = x_1 + x_2 + \dots + x_n$ and $T_n = x_1 + 2x_2 + \dots + 2^n x_n$. Since $x_n > 0$, the two series will have the same nature if and only if S_n and T_n are both bounded/unbounded.

For any $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ s.t. $2^k \leq n \leq 2^{k+1} - 1$. Since (x_n) is decreasing and positive, we can group the terms in the following ways

$$\begin{aligned} S_n &= x_1 + x_2 + \dots + x_n \leq x_1 + x_2 + \dots + x_{2^{k+1}-1} \\ &\leq x_1 + (x_2 + x_3) + \dots + (x_{2^k} + \dots + x_{2^{k+1}-1}) \\ &\leq T_k, \end{aligned}$$

and

$$\begin{aligned} S_n &= x_1 + x_2 + \dots + x_n \geq x_1 + x_2 + \dots + x_{2^k} \\ &\geq x_1 + x_2 + (x_3 + x_4) + \dots + (x_{2^{k-1}+1} + \dots + x_{2^k}) \\ &\geq \frac{x_1}{2} + \frac{1}{2}T_k. \end{aligned}$$

We obtained that $0 \leq \frac{1}{2}T_k \leq S_n \leq T_k$, hence (S_n) bounded if and only if (T_n) is bounded. \square

Example 3.14. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Proof. By the Cauchy condensation test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ has the same nature as $\sum_{n=1}^{\infty} \frac{2^n}{2^{np}} = \sum_{n=1}^{\infty} (2^{1-p})^n$, which converges if and only if $2^{1-p} < 1$, i.e for $p > 1$. \square

Theorem 3.15 (Kummer's test). Let (x_n) be a positive sequence and consider another positive sequence (c_n) .

(a) If

$$\lim_{n \rightarrow \infty} \left(c_n \frac{x_n}{x_{n+1}} - c_{n+1} \right) > 0,$$

then $\sum_{n=1}^{\infty} x_n$ is convergent.

(b) If $\sum_{n=1}^{\infty} \frac{1}{c_n} = \infty$ and

$$\lim_{n \rightarrow \infty} \left(c_n \frac{x_n}{x_{n+1}} - c_{n+1} \right) < 0,$$

then $\sum_{n=1}^{\infty} x_n$ is divergent.

Proof. Let us start with (a). Since that limit is positive, there exist $r > 0$ and $n_0 \in \mathbb{N}$ such that

$$c_n x_n - c_{n+1} x_{n+1} \geq r x_{n+1}, \quad \forall n \geq n_0.$$

Denote by $s_n = x_1 + \dots + x_n$. Adding all these inequalities for $k \in \{n_0, \dots, n\}$ we have that

$$c_{n_0} x_{n_0} - c_{n+1} x_{n+1} \geq r(s_{n+1} - s_{n_0}),$$

which gives $s_{n+1} \leq s_{n_0} + \frac{1}{r} c_{n_0} x_{n_0}$. Hence (s_n) is bounded and the series converges.

Let us now consider (b). Since the limit is negative, there exists $n_0 \in \mathbb{N}$ such that

$$c_n x_n < c_{n+1} x_{n+1}, \quad \forall n \geq n_0.$$

Hence for $n > n_0$, we have that $c_{n_0} x_{n_0} < c_n x_n$, which gives

$$\frac{1}{c_n} < \frac{1}{c_{n_0} x_{n_0}} x_n.$$

Since $\sum_{n \geq 1} \frac{1}{c_n} = \infty$, we conclude that $\sum_{n \geq 1} x_n = \infty$. \square

Many convergence tests can be obtained by taking particular sequences in Kummer's test. We will restrict to the following one.

Theorem 3.16 (Raabe-Duhamel). Let $\sum_{n \geq 1} x_n$ be a series with positive terms such that

$$\lim_{n \rightarrow \infty} n \left(\frac{x_n}{x_{n+1}} - 1 \right) = R.$$

- If $R > 1$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.
- If $R < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

Proof. Take $c_n = n$ in Kummer's test ([theorem 3.15](#)). \square

Example 3.17. Study the convergence of the series $\sum_{n \geq 0} \frac{n!}{a(a+1)\dots(a+n)}$, with $a > 0$.

Proof. The ratio test is inconclusive since $\frac{x_{n+1}}{x_n} = \frac{n+1}{a+n+1} \rightarrow 1$. Let us then try the Raabe-Duhamel test:

$$\lim_{n \rightarrow \infty} n \left(\frac{x_n}{x_{n+1}} - 1 \right) = \lim_{n \rightarrow \infty} n \left(\frac{a+n+1}{n+1} - 1 \right) = a.$$

Hence if $a > 1$ the series converges; and if $a < 1$ the series diverges. When $a = 1$ the series is $\sum_{n \geq 0} \frac{1}{n+1} = \infty$. \square

A series $\sum_{n \geq 1} x_n$ is called an *alternating series* if $x_n x_{n+1} \leq 0$, $\forall n \in \mathbb{N}$. A fundamental class of alternating series are series of the form $\sum_{n \geq 1} (-1)^n a_n$ or $\sum_{n \geq 1} (-1)^{n+1} a_n$, with $a_n > 0$.

Example 3.18. The series $\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges to $\ln 2$.

Proof. Let us prove convergence by considering the partial sums $s_n = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}$. Notice that $s_{2k+2} - s_{2k} = \frac{1}{2k+1} - \frac{1}{2k+2} > 0$ and that $s_{2k+3} - s_{2k+1} = \frac{1}{2k+3} - \frac{1}{2k+2} < 0$. This means that the subsequence (s_{2k}) is increasing, while the subsequence (s_{2k+1}) is decreasing. Notice also that $s_{2k+1} - s_{2k} = \frac{1}{2k+1}$ and $s_{2k} < s_{2k+1}$, so both subsequences are also bounded and converge to the same limit. To find the sum of the alternating series, recall (from the seminar) that

$$\begin{aligned} \lim_{n \rightarrow \infty} 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n &= \gamma \in (0, 1), \text{ hence} \\ s_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{2n} = 1 + \frac{1}{2} + \dots + \frac{1}{2n} - 2\left(\frac{1}{2} + \dots + \frac{1}{2n}\right) \\ &= \underbrace{1 + \frac{1}{2} + \dots + \frac{1}{2n} - \ln(2n)}_{\rightarrow \gamma} - \underbrace{\left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n\right)}_{\rightarrow \gamma} + \ln 2 \rightarrow \ln 2. \end{aligned}$$

□

Theorem 3.19 (Leibniz test). Let (x_n) be a decreasing sequence with $x_n \rightarrow 0$. Then the series $\sum_{n \geq 1} (-1)^n x_n$ is convergent.

Proof. Consider the partial sum $s_n = \sum_{k=1}^n (-1)^k x_k$. We will prove that (s_n) is convergent by showing that it is a Cauchy sequence. For $n, p \in \mathbb{N}$ consider

$$\begin{aligned} |s_{n+p} - s_n| &= |(-1)^{n+1} x_{n+1} + \dots + (-1)^{n+p} x_{n+p}| \\ &= \underbrace{|x_{n+1} - x_{n+2}|}_{\geq 0} + \underbrace{|x_{n+3} - x_{n+4}|}_{\geq 0} + \dots + (-1)^{p-2} x_{n+p-1} + (-1)^{p-1} x_{n+p} \\ &= x_{n+1} - \underbrace{x_{n+2} + x_{n+3} - x_{n+4}}_{\leq 0} + \dots \pm x_{n+p-1} \mp x_{n+p} \\ &\leq x_{n+1}. \end{aligned}$$

Since $x_n \rightarrow 0$, $|s_{n+p} - s_n|$ can be made arbitrarily small, so (s_n) is Cauchy. □

Definition 3.20. A series $\sum_{n \geq 1} x_n$ is called *absolutely convergent* if $\sum_{n \geq 1} |x_n|$ is convergent.

Proposition 3.21. Any absolutely convergent series is also convergent.

Proof. If $\sum_{k=1}^n |x_k|$ gives a Cauchy sequence, then $\sum_{k=1}^n x_k$ also gives a Cauchy sequence. \square

Theorem 3.22 (Cauchy). Let $\sum_{n \geq 1} x_n$ be an *absolutely convergent series* and let $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. Then $\sum_{n \geq 1} x_{\sigma(n)}$ is also absolutely convergent and $\sum_{n \geq 1} x_{\sigma(n)} = \sum_{n \geq 1} x_n$. In other words, any rearrangement of an absolutely convergent series has the same sum.

Proof. (Optional) See [2][Theorem 7.4.3]. \square

Definition 3.23. A series $\sum_{n \geq 1} x_n$ is called *conditionally convergent* (or semi-convergent) if $\sum_{n \geq 1} x_n$ converges, but $\sum_{n \geq 1} |x_n|$ diverges.

Theorem 3.24 (Riemann). Let $\sum_{n \geq 1} x_n$ be a *conditionally convergent series* and let $x \in \overline{\mathbb{R}}$. Then there exists a bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sum_{n \geq 1} x_{\sigma(n)} = x$. In other words, a conditionally convergent series can be rearranged to converge to any value or diverge to $\pm\infty$.

Proof. (Optional) See [2][Theorem 8.2.8]. \square

Example 3.25. Rearranging the terms in the alternating harmonic series one can obtain a different sum. Indeed, consider $\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2$, and reorder the terms in the following way: one positive, two negative. Then

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots = \frac{1}{2} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \right) = \frac{1}{2} \ln 2.$$

Definition 3.26. Let (a_n) be a sequence of real numbers and let $c \in \mathbb{R}$. The series

$$\sum_{n=0}^{\infty} a_n(x - c)^n$$

is called a *power series* centered at c .

Theorem 3.27. Consider the power series $\sum_{n=0}^{\infty} a_n(x - c)^n$. There exists a unique $R \in [0, \infty]$, called the *radius of convergence* of the power series, such that the power series

- converges absolutely when $|x - c| < R$.
- diverges when $|x - c| > R$.

Theorem 3.28. If the limit

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L \in [0, \infty]$$

exists, then the power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ has the radius of convergence

$$R = \begin{cases} \frac{1}{L}, & \text{if } L \in (0, \infty) \\ 0, & \text{if } L = \infty \\ \infty, & \text{if } L = 0. \end{cases}$$

Proof. It follows from the root test for series with positive terms. □

Corollary 3.29. If the limit

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L \in [0, \infty]$$

exists, then the power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ has the radius of convergence

$$R = \begin{cases} \frac{1}{L}, & \text{if } L \in (0, \infty) \\ 0, & \text{if } L = \infty \\ \infty, & \text{if } L = 0. \end{cases}$$

Proof. It follows from $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$. □

Definition 3.30. The convergence set of a power series is

$$C := \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n(x - c)^n \text{ converges}\}.$$

Remark 3.31. The convergence set C contains the open interval $(c - R, c + R)$ and possibly the endpoints $\{c - R, c + R\}$.

Example 3.32. The power series $\sum_{n \geq 0} x^n$ has radius of convergence $R = 1$, it converges absolutely for $|x| < 1$ and diverges when $|x| > 1$ (by the root test or the ratio test). The convergence set is $(-1, 1)$ and for $x \in (-1, 1)$ we have that

$$\sum_{n \geq 0} x^n = \frac{1}{1 - x}, \quad \sum_{n \geq 0} (-x)^n = \frac{1}{1 + x}.$$

Example 3.33. The power series $\sum_{n \geq 1} \frac{x^n}{n}$ has radius of convergence $R = 1$, it converges absolutely for $|x| < 1$ and diverges when $|x| > 1$ (by the root test or the ratio test). Moreover, the series converges for $x = -1$ (alternating harmonic series) and diverges for $x = 1$ (harmonic series), hence its convergence set is $C = [-1, 1)$.

Theorem 3.34. Consider a power series with radius of convergence R , given by

$$s(x) = \sum_{n=0}^{\infty} a_n (x - c)^n.$$

Then for any $x \in (c - R, c + R)$, the power series can be differentiated term by term and

$$s'(x) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1},$$

and for any $t \in (c - R, c + R)$ the power series can be integrated term by term

$$\int_c^t s(x) dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (t - c)^{n+1}.$$

Example 3.35. The power series $\sum_{n \geq 1} \frac{x^n}{n!}$ converges absolutely for any $x \in \mathbb{R}$ (ratio test). Let

$\exp(x) := \sum_{n \geq 1} \frac{x^n}{n!}$ and differentiate term by term, then $\exp'(x) = \exp(x)$ and $\exp(0) = 1$.

❖ Limits, continuity, differentiability

Definition 4.1. Let $A \subseteq \mathbb{R}$. We say that $x_0 \in \overline{\mathbb{R}}$ is an *accumulation point* (or *cluster point*) if

$$\forall V \in \mathcal{V}(x_0), V \cap (A \setminus \{x_0\}) \neq \emptyset.$$

We denote by A' the set of the accumulation points of A . We say that $x_0 \in A$ is an *isolated point* if $x_0 \in A \setminus A'$.

Remark 4.2. $\text{cl}(A) = A' \cup \{\text{isolated points}\}$.

Proposition 4.3. Let $A \subseteq \mathbb{R}$ and $x_0 \in \overline{\mathbb{R}}$, then $x_0 \in A'$ if and only if there exists a sequence (x_n) in $A \setminus \{x_0\}$ such that $\lim_{n \rightarrow \infty} x_n = x_0$.

Proof. Assume that $x_0 \in A'$, with $x_0 \in \mathbb{R}$, and consider the neighborhoods $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$. Then each neighborhood must contain an $x_n \in A \setminus \{x_0\}$ with $|x_n - x_0| < \frac{1}{n}$, hence $x_n \rightarrow x_0$. If x_0 is infinite, the neighborhoods can be taken $(-\infty, -n)$ or (n, ∞) , respectively.

Assume now that there exists a sequence (x_n) in $A \setminus \{x_0\}$ such that $\lim_{n \rightarrow \infty} x_n = x_0$. Then for any $V \in \mathcal{V}(x_0)$, there exists $N_V \in \mathbb{N}$ such that $x_n \in V$, for any $n \geq N_V$. In particular, $x_{N_V} \in V \cap (A \setminus \{x_0\})$, for any $V \in \mathcal{V}(x_0)$, hence $x_0 \in A'$. \square

Example 4.4. For $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, each element $x \in A$ is an isolated point and $A' = \{0\}$.

Definition 4.5. Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$ and $x_0 \in A'$. We say that $\lim_{x \rightarrow x_0} f(x) = \ell \in \overline{\mathbb{R}}$ if

$$\forall V \in \mathcal{V}(\ell), \exists U \in \mathcal{V}(x_0) \text{ s.t. } f(x) \in V, \forall x \in U \cap (A \setminus \{x_0\}).$$

Remark 4.6 (ε - δ). Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$ and $x_0 \in A'$ finite. If $\lim_{x \rightarrow x_0} f(x) = \ell \in \mathbb{R}$, then

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - \ell| < \varepsilon, \forall x \in A \text{ with } |x - x_0| < \delta.$$

Theorem 4.7. Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$ and $x_0 \in A'$. Then $\lim_{x \rightarrow x_0} f(x) = \ell \in \overline{\mathbb{R}}$ iff

for any sequence (x_n) in $A \setminus \{x_0\}$ with $\lim_{n \rightarrow \infty} x_n = x_0$, we have that $\lim_{n \rightarrow \infty} f(x_n) = \ell$.

Theorem 4.8. Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$, $x_0 \in \mathbb{R}$ s.t. $x_0 \in (A \cap (-\infty, x_0))'$ and $x_0 \in (A \cap (x_0, \infty))'$. Then

$$\lim_{x \rightarrow x_0} f(x) = \ell \text{ iff } \lim_{\substack{x \rightarrow x_0 \\ x < x_0}} f(x) = \lim_{\substack{x \rightarrow x_0 \\ x > x_0}} f(x) = \ell.$$

Example 4.9. (a) $\text{sgn} : \mathbb{R} \rightarrow \mathbb{R}$, $\text{sgn}(x) = \begin{cases} +1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0. \end{cases}$ has no limit at 0.

(b) $f : \mathbb{R}^* \rightarrow \mathbb{R}$, $f(x) = \sin(\frac{1}{x})$ has no limit at 0 since $f(\frac{1}{2n\pi}) = 0$, $f(\frac{1}{2n\pi + \pi/2}) = 1$.

(c) $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ has no limit at any $x \in \mathbb{R}$.

Definition 4.10. Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$ and $x_0 \in A$. We say that f is *continuous* at x_0 if

$$\forall V \in \mathcal{V}(f(x_0)), \exists U \in \mathcal{V}(x_0) \text{ s.t. } f(x) \in V, \forall x \in U \cap A.$$

Remark 4.11. If $x_0 \in A \cap A'$ is an accumulation point, then f is continuous at x_0 if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

Remark 4.12. If x_0 is an isolated point of A , then $\exists U \in \mathcal{V}(x_0)$ with $U \cap A = \{x_0\}$, and since $f(x_0) \in V$, $\forall V \in \mathcal{V}(f(x_0))$, we have that f is continuous at x_0 .

Theorem 4.13. Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$ and $x_0 \in A \cap A'$. The following are equivalent:

- (a) f is continuous at x_0 .
- (b) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$, $\forall x \in A$ with $|x - x_0| < \delta$.
- (c) for any sequence (x_n) in A with $\lim_{n \rightarrow \infty} x_n = x_0$, we have that $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$.

Remark 4.14. Elementary operations – e.g. sums, products or compositions – of continuous functions are continuous (when defined).

Definition 4.15. For $f : A \rightarrow \mathbb{R}$ denote by $f(A) := \{f(x) \mid x \in A\}$ the image of A . We say that f is *bounded* if $f(A)$ is *bounded*, i.e. $\inf(f(A)), \sup(f(A))$ are finite.

Theorem 4.16 (Weierstrass). Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then f is bounded and it attains its bounds, i.e. there exist $\min(f(A)), \max(f(A))$.

Proof. Let us first prove that f is bounded. Assuming that this is not the case, we have that for any $n \in \mathbb{N}$ there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. Since the sequence (x_n) is bounded, we have that it has a convergent subsequence (x_{n_k}) , see [theorem 2.8](#); denote its limit by x . We have that $x_{n_k} \rightarrow x$ and f is continuous, hence $f(x_{n_k}) \rightarrow f(x)$. But $|f(x_{n_k})| > n_k \rightarrow \infty$, contradiction. Hence f is bounded on $[a, b]$.

To prove that f attains its bounds, let's consider the upper bound and show that there exists $x_M \in [a, b]$ such that $f(x_M) = \sup(f(A))$, i.e. $f(x_M) = \max(f(A)) = \sup(f(A))$. From [theorem 1.7](#), we obtain a sequence (x_n) in $[a, b]$ such that $f(x_n) \rightarrow \sup(f(A))$. Since the sequence (x_n) is bounded, it has a convergent subsequence (x_{n_k}) ; let's call its limit $x_M \in [a, b]$. Since f is continuous, it follows that $f(x_{n_k}) \rightarrow f(x_M)$, but we know that $f(x_{n_k}) \rightarrow \sup(f(A))$, hence $f(x_M) = \sup(f(A))$ and f reaches its upper bound. \square

Theorem 4.17 (Intermediate value property). Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then f has the intermediate value property, i.e. if $y \in \mathbb{R}$ is in between $f(a)$ and $f(b)$, there exists $c \in (a, b)$ such that $f(c) = y$.

Proof. Assume that $f(a) < y < f(b)$ and consider the set $S := \{x \in [a, b] \mid f(x) \leq y\}$. Take

$$c := \sup(S)$$

Let $\varepsilon > 0$, then $\exists \delta > 0$ such that $|f(x) - f(c)| < \varepsilon$, whenever $|x - c| < \delta$. Since $c = \sup(S)$, we have from [theorem 1.7](#) that there exists $x_1 \in S$ such that $c - \delta < x_1 \leq c$. From continuity we have that $f(c) < f(x_1) + \varepsilon \leq y + \varepsilon$. Also, for $x_2 \in (c, c + \delta)$, we have from continuity that $f(c) > f(x_2) - \varepsilon$. From the definition of the supremum, $x_2 \notin S$ hence $f(x_2) > y$ and $f(c) > y - \varepsilon$. We conclude that $y - \varepsilon < f(c) < y + \varepsilon$, for any $\varepsilon > 0$. Hence $f(c) = y$. \square

Definition 4.18. Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$ and $x_0 \in A \cap A'$. The *derivative* of f at x_0 is

$$f'(x_0) := \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \overline{\mathbb{R}}$$

If $f'(x_0) \in \mathbb{R}$ (finite) we say that f is *differentiable* at x_0 .

Remark 4.19. $f'(x_0)$ represents the gradient of the tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$. The equation of the tangent is $f(x) - f(x_0) = f'(x_0)(x - x_0)$.

Theorem 4.20. Let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$ and $x_0 \in A \cap A'$. If f is differentiable at x_0 , then f is continuous at x_0 .

Proof. Since $f(x) = f(x_0) + \frac{f(x) - f(x_0)}{x - x_0}(x - x_0)$, we have that $\lim_{x \rightarrow x_0} f(x) = f(x_0) + 0 = f(x_0)$. \square

Example 4.21. $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = |x|$ is not differentiable in 0 since $\nexists \lim_{x \rightarrow 0} \frac{|x|}{x}$.

Theorem 4.22 (Calculus Rules).

- $(cf)'(x) = cf'(x)$, for any constant $c \in \mathbb{R}$.
- $(f + g)'(x) = f'(x) + g'(x)$.
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$. (Product Rule)
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$. (Quotient Rule)
- $(f \circ g)'(x) = f'(g(x))g'(x)$. (Chain Rule)

Proposition 4.23 (l'Hôpital's rule). Let I be an open interval, $x_0 \in \overline{\mathbb{R}}$ and $f, g : I \setminus \{x_0\} \rightarrow \mathbb{R}$ differentiable. If $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0$ or $\pm\infty$, and $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} \in \overline{\mathbb{R}}$, then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}.$$

Definition 4.24. $f : A \rightarrow \mathbb{R}$ has a local extremum (minimum or maximum) at $x_0 \in A$ if

$$\exists V \in \mathcal{V}(x_0) \text{ s.t. } f(x_0) \leq f(x) \text{ or } f(x_0) \geq f(x), \forall x \in V \cap A.$$

Theorem 4.25 (Fermat). Let $f : (a, b) \rightarrow \mathbb{R}$ and $x_0 \in (a, b)$. If f is differentiable at x_0 and x_0 is a local extremum, then $f'(x_0) = 0$.

Proof. The lateral derivatives at x_0 are equal. Since x_0 is a local extremum, one of them is ≥ 0 , the other ≤ 0 . Hence $f'(x_0) = 0$. \square

Theorem 4.26 (Rolle). Let $f : (a, b) \rightarrow \mathbb{R}$ with $f(a) = f(b)$. If f is continuous on $[a, b]$ and differentiable on (a, b) , then there exists $c \in (a, b)$ s.t. $f'(c) = 0$.

Proof. Since f is continuous, it is bounded and it attains its bounds. Denote by x_m and x_M the minimum and maximum points of f on $[a, b]$. If at least one of x_m and x_M belongs to (a, b) , then $f'(x_m) = 0$ or $f'(x_M) = 0$. Otherwise, $x_m, x_M \in \{a, b\}$ and $f(x_m) = f(x_M)$, hence the function is constant and its derivative is zero on (a, b) . \square

Theorem 4.27 (Mean value theorem). Let $f : (a, b) \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Then there exists $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Consider the function $g : (a, b) \rightarrow \mathbb{R}$, $g(x) := f(x) - x \frac{f(b) - f(a)}{b - a}$. Since $g(a) = g(b)$, the conclusion follows from Rolle's theorem. \square

Theorem 4.28 (Monotony). Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable on (a, b) . Then

$$f \text{ is increasing iff } f' \geq 0,$$

$$f \text{ is decreasing iff } f' \leq 0.$$

Proof. \Rightarrow follows from the definition of the derivative; \Leftarrow from the mean value theorem. \square

❖ Taylor series

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \rightarrow \mathbb{R}$ and $x_0 \in I$ a point where f is differentiable n times ($n \in \mathbb{N}$). Does there exist a polynomial $P : \mathbb{R} \rightarrow \mathbb{R}$ that matches the function f and all its derivatives up to order n at the point x_0 ? That is

$$\begin{aligned} P(x_0) &= f(x_0) \\ P'(x_0) &= f'(x_0) \\ P''(x_0) &= f''(x_0) \\ &\vdots \\ P^{(n)}(x_0) &= f^{(n)}(x_0). \end{aligned}$$

Let us look for P of degree at most n of the form

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n.$$

By imposing the conditions at x_0 and differentiating P we have that

$$P(x_0) = a_0 = f(x_0), P'(x_0) = a_1 = f'(x_0), P''(x_0) = 2a_2 = f''(x_0), \dots, P^{(n)}(x_0) = n!a_n = f^{(n)}(x_0).$$

We thus see that there exists a unique such polynomial P of degree at most n given by

$$P(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

that matches the function f and all its derivatives up to order n at the point x_0 .

Definition 5.1. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \rightarrow \mathbb{R}$ and $x_0 \in I$ where f is differentiable n times. The polynomial $T_n : \mathbb{R} \rightarrow \mathbb{R}$,

$$T_n(x) := f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the *Taylor polynomial* of degree n centered around x_0 . □

The Taylor polynomial T_n gives a good approximation of f around x_0 , i.e. when $x \approx x_0$,

$$f(x) \approx T_n(x).$$

The simplest approximations are: the *linear approximation* of f around x_0 given by T_1 , i.e.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

and the *quadratic approximation* of f around x_0 given by T_2 , i.e.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

The closer x is to x_0 and the higher the degree of T_n is, the better $T_n(x)$ approximates $f(x)$.

Example 5.2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = e^x$ and $x_0 = 0$. Then $f(0) = f'(0) = \dots = f^{(n)}(0) = 1$ and

$$T_n(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!}.$$

Definition 5.3. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \rightarrow \mathbb{R}$ and $x_0 \in I$ where f is differentiable n times. We define $R_n : \mathbb{R} \rightarrow \mathbb{R}$ to be the remainder when approximating f by T_n ,

$$R_n(x) := f(x) - T_n(x).$$

Theorem 5.4 (Taylor-Lagrange). Let $I \subseteq \mathbb{R}$ be an interval, $f : I \rightarrow \mathbb{R}$ differentiable $n + 1$ times. Then for any $x, x_0 \in I$, there exists $c \in (x_0, x)$ or $c \in (x, x_0)$ such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1},$$

called *the remainder in Lagrange's form*. Taylor's formula with Lagrange remainder is

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}.$$

Remark 5.5. There exist other forms of the remainder, but we will only use this one. Its main advantage is that assuming that all the derivatives of f are bounded by $M > 0$,

$$|f(x) - T_n(x)| = |R_n(x)| \leq \frac{M}{(n+1)!} |x - x_0|^{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Definition 5.6. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be infinitely differentiable. For $x_0 \in I$ and $x \in \mathbb{R}$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

is called the *Taylor series* of f around x_0 . If the series converges and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n,$$

we say that f can be expanded in a Taylor series around x_0 (also called Taylor expansion).

Remark 5.7. The partial sum of a Taylor series is the Taylor polynomial $T_n(x)$. A Taylor series converges to $f(x)$ if and only if the remainder $f(x) - T_n(x) = R_n(x) \rightarrow 0$ as $n \rightarrow \infty$.

Remark 5.8. The Taylor series around 0 is called the *MacLaurin series*,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Example 5.9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = e^x$ and $x_0 = 0$. Then

$$T_n(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!}.$$

Consider Taylor's formula $f(x) = T_n(x) + R_n(x)$ with the Lagrange remainder, for which there exists c in between 0 and x such that

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1} \rightarrow 0$$

since $\frac{|x|^n}{n!} \rightarrow 0$ as $n \rightarrow \infty$. It follows that e^x can be expanded as a Taylor series around 0:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots, \forall x \in \mathbb{R}.$$

Example 5.10. The functions \sin and \cos can be expanded in a Taylor series around 0.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Example 5.11. The function $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is infinitely differentiable at 0, but f is not expandable in a Taylor series around 0.

Example 5.12 (Convex/concave). Let $f : I \rightarrow \mathbb{R}$ be two times differentiable, with a critical point at x_0 , i.e. $f'(x_0) = 0$. Then from Taylor's formula we have that

$$f(x) = f(x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + R_2(x).$$

When x is very close to x_0 , the quadratic approximation is very accurate and the remainder $R_2(x)$ is very small. Thus the behaviour of $f(x)$ around x_0 is dictated by the quadratic term $f''(x_0)(x - x_0)^2$ and we see that:

- If $f''(x_0) > 0$ (convexity), then $f(x) > f(x_0)$ and x_0 is a local minimum.
- If $f''(x_0) < 0$ (concavity), then $f(x) < f(x_0)$ and x_0 is a local maximum.

Theorem 5.13 (Local optimality conditions). Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \rightarrow \mathbb{R}$ and $x_0 \in I$ a point where f is differentiable n times and

$$f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0 \text{ and } f^{(n)}(x_0) \neq 0.$$

1. If n is even and $f^{(n)}(x_0) > 0$, then x_0 is a *local minimum* of f .
2. If n is even and $f^{(n)}(x_0) < 0$, then x_0 is a *local maximum* of f .
3. If n is odd, then x_0 is not a local extremum point of f .

Proof. It follows from the Taylor approximation $f(x) = f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$. \square

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