Seminar 8

1. Prove that for any $x, y \in \mathbb{R}^n$ the following identities hold:
   (a) $\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big)$ (the parallelogram identity).
   (b) $\langle x, y\rangle = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big)$.

2. ★ For $x, y \in \mathbb{R}^n$ prove that the following statements are equivalent:
   (a) $\langle x, y\rangle = 0$. (b) $\|x+y\| = \|x-y\|$. (c) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

3. Find the orthogonal projection of a vector $v \in \mathbb{R}^2$ onto a vector $a \in \mathbb{R}^2$.

4. Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix with angle θ in $\mathbb{R}^2$.

5. Consider the p-norm $\|x\|_p := \big(|x_1|^p + \ldots + |x_n|^p\big)^{\frac{1}{p}}$, $p \geq 1$ and $\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$. Draw the unit ball in $\mathbb{R}^2$ for the p-norm with $p \in \{1, 2, \infty\}$.

6. ★ [Python] Represent the unit ball in $\mathbb{R}^2$ for the p-norm with $p \in \{1.25, 1.5, 3, 8\}$ by random sampling and plotting the points that are inside the unit ball.

7. Find the interior, the closure and the boundary for each of the following sets:
   (a) $[0, 1) \times (1, 2]$.
   (b) $\{(x, y) \in \mathbb{R}^2 \mid |x| < |y|\}$.
   (c) $\{(x, y) \in \mathbb{R}^2 \mid |x| + |y| < 1\}$.
   (d) $\{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1,\ x \leq 1\}$.

8. Draw the level sets $L_c = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$ and sketch the graph for the following.
   (a) $f(x, y) = x + y$ and $c \in \{0, \pm 1\}$.
   (b) $f(x, y) = 2x^2 + y^2$ and $c \in \{0, 1, 4\}$.
   (c) $f(x, y) = \sqrt{x^2 + y^2}$ and $c \in \{0, 1, 2\}$.
   (d) $f(x, y) = x^2 - y^2$ and $c \in \{0, \pm 1\}$.

Homework questions are marked with ★.