# Course 4

## Generated subspace, linear maps

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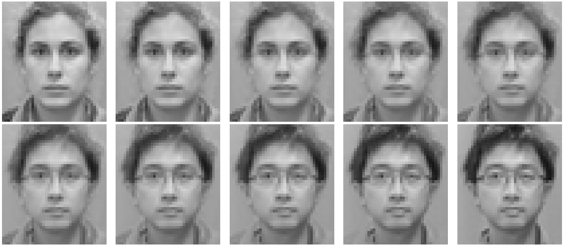
Following [Klein], we describe a way to achieve an image crossfade effect.



# Intersection of subspaces

For a vector space $V$ over $K$, we denote by $S(V)$ the set of all subspaces of $V$. Sometimes, this set is denoted by $S_K(V)$ if we like to emphasize the field $K$.

**Theorem**

*Let $V$ be a vector space over $K$ and let $(S_i)_{i\in I}$ be a family of subspaces of $V$. Then $\bigcap_{i\in I} S_i \in S(V)$.*

*Proof.* [...]

In general, the union of two subspaces of a vector space is not a subspace. For instance, $S = \{(x, 0) \mid x \in \mathbb{R}\}$ and $T = \{(0, y) \mid y \in \mathbb{R}\}$ are subspaces of the canonical real vector space $\mathbb{R}^2$, but $S \cup T$ is not a subspace of $\mathbb{R}^2$. Indeed, for instance, we have $(1, 0), (0, 1) \in S \cup T$, but $(1, 0) + (0, 1) = (1, 1) \notin S \cup T$.

Now we are interested in how to "complete" a given subset of a vector space to a subspace in a minimal way.

### Definition

Let $V$ be a vector space and let $X \subseteq V$. Then we denote

$$\langle X\rangle = \bigcap\{S \leq V \mid X \subseteq S\}$$

and we call it the *subspace generated by* $X$ or the *subspace spanned by* $X$. Here $X$ is called the *generating set* of $\langle X\rangle$. If $X = \{v_1, \dots, v_n\}$, we denote $\langle v_1, \dots, v_n\rangle = \langle\{v_1, \dots, v_n\}\rangle$.

(1) $\langle X\rangle$ is the "smallest" (with respect to inclusion) subspace of $V$ containing $X$.

(2) $\langle\emptyset\rangle = \{0\}$.

(3) If $S \leq V$, then $\langle S\rangle = S$.

# System of generators

**Definition**

A vector space $V$ over $K$ is called *finitely generated* if $\exists v_1, \ldots, v_n \in V$ $(n \in \mathbb{N})$ such that

$$V = \langle v_1, \ldots, v_n \rangle.$$

Then the set $\{v_1, \ldots, v_n\}$ is called a *system of generators for $V$*.

**Definition**

Let $V$ be a vector space over $K$ and $v_1, \ldots, v_n \in V$ $(n \in \mathbb{N})$. A finite sum of the form

$$k_1 v_1 + \cdots + k_n v_n \,,$$

where $k_i \in K$ $(i = 1, \ldots, n)$, is called a (finite) *linear combination* of the vectors $v_1, \ldots, v_n$.

# Characterization of the generated subspace

**Theorem**

*Let $V$ be a vector space over $K$ and let $\emptyset \neq X \subseteq V$. Then*

$$\langle X \rangle = \{k_1 v_1 + \cdots + k_n v_n \mid k_i \in K\,,\ v_i \in X\,, i = 1, \ldots, n\,,\ n \in \mathbb{N}^*\}\,,$$

*that is, the set of all finite linear combinations of vectors of $X$.*

*Proof.* We prove the result in 3 steps, by showing that

$$L = \{k_1 v_1 + \cdots + k_n v_n \mid k_i \in K\,,\ v_i \in X\,, i = 1, \ldots, n\,,\ n \in \mathbb{N}^*\}$$

is the smallest subspace of $V$ containing $X$ [...].

**Corollary**

*Let $V$ be a vector space over $K$ and let $x_1, \ldots, x_n \in V$. Then*

$$\langle x_1, \ldots, x_n \rangle = \{k_1 x_1 + \cdots + k_n x_n \mid k_i \in K\,,\ x_i \in X\,, i = 1, \ldots, n\}\,.$$

($a$) Consider the canonical real vector space $\mathbb{R}^3$. Then

$$\begin{aligned}
&\langle (1,0,0),(0,1,0),(0,0,1)\rangle \\
&= \{k_1(1,0,0)+k_2(0,1,0)+k_3(0,0,1) \mid k_1,k_2,k_3\in\mathbb{R}\} \\
&= \{(k_1,0,0)+(0,k_2,0)+(0,0,k_3) \mid k_1,k_2,k_3\in\mathbb{R}\} \\
&= \{(k_1,k_2,k_3) \mid k_1,k_2,k_3\in\mathbb{R}\} = \mathbb{R}^3 .
\end{aligned}$$

Hence $\mathbb{R}^3$ is generated by the three vectors $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, and thus it is finitely generated.

($b$) Consider the canonical vector space $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$. Similarly as above, we have:

$$\langle (\widehat{1},\widehat{0},\widehat{0}),(\widehat{0},\widehat{1},\widehat{0})\rangle = \{(k_1,k_2,\widehat{0}) \mid k_1,k_2\in\mathbb{Z}_2\} \neq \mathbb{Z}_2^3 .$$

Hence $\mathbb{Z}_2^3$ is not generated by the two vectors $(\widehat{1}, \widehat{0}, \widehat{0})$ and $(\widehat{0}, \widehat{1}, \widehat{0})$. But it is generated by $(\widehat{1}, \widehat{0}, \widehat{0})$, $(\widehat{0}, \widehat{1}, \widehat{0})$ and $(\widehat{0}, \widehat{0}, \widehat{1})$, hence it is finitely generated.

$(c)$ Consider the subspace

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x - y - z = 0\}$$

of the canonical real vector space $\mathbb{R}^3$. Let us write it as a generated subspace. Expressing $x = y + z$, we have:

$$\begin{aligned} S &= \{(y+z, y, z) \mid y, z \in \mathbb{R}\} = \{(y, y, 0) + (z, 0, z) \mid y, z \in \mathbb{R}\} \\ &= \{y(1,1,0) + z(1,0,1) \mid y, z \in \mathbb{R}\} = \langle (1,1,0), (1,0,1) \rangle. \end{aligned}$$

Alternatively, one may express $y$ or $z$ by using the other two components and get other writings of $S$ as a generated subspace, namely $S = \langle (1,1,0), (0,-1,1) \rangle = \langle (1,0,1), (0,1,-1) \rangle$. We see that $S$ is finitely generated.

# Sum of subspaces

In what follows we shall be interested in "decomposing" a vector space into subspaces.

**Definition**

Let $V$ be a vector space over $K$ and let $S, T \leq V$.
We define the *sum* of the subspaces $S$ and $T$ as the set

$$S + T = \{s + t \mid s \in S\,,\ t \in T\}\,.$$

**Theorem**

*Let $V$ be a vector space over $K$ and $S, T \leq V$. Then $S + T = \langle S \cup T \rangle$, hence $S + T \leq V$.*

*Proof.* We prove the equality by double inclusion [...].

# Direct sum of subspaces

## Definition

Let $V$ be a vector space over $K$ and let $S, T \le V$.
If $S \cap T = \{0\}$, then $S + T$ is denoted by $S \oplus T$ and is called the *direct sum* of the subspaces $S$ and $T$.

## Theorem

*Let $V$ be a vector space over $K$ and let $S, T \le V$. Then*

$$V = S \oplus T \iff \forall v \in V\,, \exists! s \in S\,, t \in T\ : v = s + t\,.$$

*Proof.* [...]

## Example

Consider the canonical real vector space $\mathbb{R}^2$. Then $\mathbb{R}^2 = S \oplus T$, where $S = \{(x, 0) \mid x \in \mathbb{R}\}$ and $T = \{(0, y) \mid y \in \mathbb{R}\}$.

## Definition

Let $V$ and $V'$ be vector spaces over the same field $K$. A function $f : V \to V'$ is called:

(1) *$(K$-$)$linear map* (or *(vector space) homomorphism* or *linear transformation*) if

$$f(v_1 + v_2) = f(v_1) + f(v_2), \quad \forall v_1, v_2 \in V,$$
$$f(kv) = kf(v), \quad \forall k \in K, \forall v \in V.$$

(2) *isomorphism* if it is a bijective $K$-linear map.
(3) *endomorphism* if it is a $K$-linear map and $V = V'$.
(4) *automorphism* if it is a bijective $K$-linear map and $V = V'$.

# Properties of linear maps

If $f : V \to V'$ is a $K$-linear map, then the first condition from its definition tells us that $f$ is a group homomorphism between the groups $(V, +)$ and $(V', +)$. Then we have $f(0) = 0'$ and $f(-v) = -f(v)$, $\forall v \in V$.

We denote by $V \simeq V'$ the fact that two vector spaces $V$ and $V'$ are isomorphic. We also denote

$$\begin{aligned}
\mathrm{Hom}_K(V, V') &= \{f : V \to V' \mid f \text{ is } K\text{-linear}\}\,, \\
\mathrm{End}_K(V) &= \{f : V \to V \mid f \text{ is } K\text{-linear}\}\,, \\
\mathrm{Aut}_K(V) &= \{f : V \to V \mid f \text{ is bijective } K\text{-linear}\}\,.
\end{aligned}$$

**Theorem**

*Let $V$ and $V'$ be vector spaces over $K$ and $f : V \to V'$. Then $f$ is a $K$-linear map $\Longleftrightarrow f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2), \forall k_1, k_2 \in K, \forall v_1, v_2 \in V$.*

*Proof.* [...]

# Examples I

$(a)$ Let $V$ and $V'$ be vector spaces over $K$ and let $f : V \to V'$ be defined by $f(v) = 0'$, $\forall v \in V$. Then $f$ is a $K$-linear map, called the *trivial linear map*.

$(b)$ Let $V$ be a vector space over $K$. Then the identity map $1_V : V \to V$ is an automorphism of $V$.

$(c)$ Let $V$ be a vector space and $S \leq V$. Define $i : S \to V$ by $i(v) = v$, $\forall v \in S$. Then $i$ is a $K$-linear map, called the *inclusion linear map*.

$(d)$ Let $V$ be a vector space over $K$ and $a \in K$. Define $t_a : V \to V$ by $t_a(v) = av$, $\forall v \in V$. Then $t_a$ is an endomorphism of $V$.

(e) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x + y$. Then $f$ is an $\mathbb{R}$-linear map, because we have

$$\begin{aligned} f(k_1(x_1, y_1) + k_2(x_2, y_2)) &= f(k_1x_1 + k_2x_2, k_1y_1 + k_2y_2) \\ &= (k_1x_1 + k_2x_2) + (k_1y_1 + k_2y_2) \\ &= k_1(x_1 + y_1) + k_2(x_2 + y_2) \\ &= k_1f(x_1, y_1) + k_2f(x_2, y_2) \end{aligned}$$

for every $k_1, k_2 \in K$ and for every $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.
On the other hand, $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy$ is not an $\mathbb{R}$-linear map, because, for instance, we have

$$f((1, 0) + (0, 1)) = f(1, 1) = 1 \neq 0 = f(1, 0) + f(0, 1).$$

(f) Let $\theta \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta),$$

which is the counterclockwise rotation of angle $\theta$ about the origin in the plane. Then $f$ is an $\mathbb{R}$-linear map. In particular, for $\theta = \frac{\pi}{2}$, we have $f(x, y) = (-y, x)$.

$(g)$ For an interval $I = [a, b] \subseteq \mathbb{R}$ we considered the real vector space

$$\mathbb{R}^I = \{f \mid f : I \to \mathbb{R}\}$$

and its subspaces

$$C(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ continuous on } I\},$$

$$D(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ derivable on } I\}.$$

Then

$$F : D(I, \mathbb{R}) \to \mathbb{R}^I, \quad F(f) = f',$$

$$G : C(I, \mathbb{R}) \to \mathbb{R}, \quad G(f) = \int_a^b f(t)\mathrm{d}t,$$

are $\mathbb{R}$-linear maps.

# Properties of linear maps

**Theorem**

*(i) Let $f : V \to V'$ be an isomorphism of vector spaces over $K$. Then $f^{-1} : V' \to V$ is again an isomorphism of vector spaces over $K$.*
*(ii) Let $f : V \to V'$ and $g : V' \to V''$ be $K$-linear maps. Then $g \circ f : V \to V''$ is a $K$-linear map.*

*Proof.* Homework.

# Kernel and image of a linear map

**Definition**

Let $f : V \to V'$ be a $K$-linear map. Then the set

$$\operatorname{Ker} f = \{v \in V \mid f(v) = 0'\}$$

is called the *kernel* (or the *null space*) of the $K$-linear map $f$ and the set

$$\operatorname{Im} f = \{f(v) \mid v \in V\}$$

is called the *image* (or the *range space*) of the $K$-linear map $f$.

# Kernel and image are subspaces

**Theorem**

*Let $f : V \to V'$ be a $K$-linear map. Then*

$$\mathrm{Ker} f \leq V \text{ and } \mathrm{Im} f \leq V' .$$

*Proof.* [...]

**Theorem**

*Let $f : V \to V'$ be a $K$-linear map. Then*

$$\mathrm{Ker}\, f = \{0\} \iff f \text{ is injective} .$$

*Proof.* [...]

**Theorem**

*Let $f : V \to V'$ be a $K$-linear map and let $X \subseteq V$. Then*

$$f(\langle X \rangle) = \langle f(X) \rangle .$$

*Proof.* If $X = \emptyset$, then we have:

$$f(\langle \emptyset \rangle) = f(\{0\}) = \{f(0)\} = \{0'\} = \langle \emptyset \rangle = \langle f(\emptyset) \rangle.$$

If $X \neq \emptyset$, use

$\langle X \rangle = \{k_1 v_1 + \cdots + k_n v_n \mid k_i \in K\,,\ v_i \in X\,, i = 1, \ldots, n\,,\ n \in \mathbb{N}^*\}$ [...].

# The vector space of linear maps

**Theorem**

*Let $V$ and $V'$ be vector spaces over $K$. Consider on $\mathrm{Hom}_K(V, V')$ the operations: $\forall f, g \in \mathrm{Hom}_K(V, V')$ and $\forall k \in K$, $f + g, k \cdot f \in \mathrm{Hom}_K(V, V')$, where*

$$(f+g)(v) = f(v) + g(v),$$
$$(kf)(v) = kf(v)$$

*$\forall v \in V$. Then $\mathrm{Hom}_K(V, V')$ is a vector space over $K$.*

**Corollary**

*Let $V$ be a vector space over $K$. Then $\mathrm{End}_K(V)$ is a vector space over $K$.*

A black-and-white image of (say)

$$n = 1024 \times 768$$

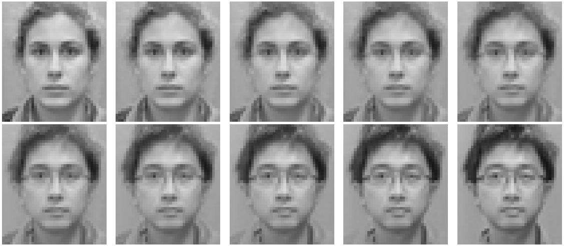
pixels can be viewed as a vector in the real canonical vector space $\mathbb{R}^n$, where each component of the vector is the intensity of the corresponding pixel.

Let us consider two vectors representing images:

$v_1 =$ , $v_2 =$ .

Now consider the following intermediate images:

# Extra: Image crossfade II



The vectors corresponding to the above images are the following linear combinations of the vectors $v_1$ and $v_2$:

$$v_1, \quad \frac{8}{9}v_1 + \frac{1}{9}v_2, \quad \frac{7}{9}v_1 + \frac{2}{9}v_2, \quad \frac{6}{9}v_1 + \frac{3}{9}v_2, \quad \frac{5}{9}v_1 + \frac{4}{9}v_2,$$
$$\frac{4}{9}v_1 + \frac{5}{9}v_2, \quad \frac{3}{9}v_1 + \frac{6}{9}v_2, \quad \frac{2}{9}v_1 + \frac{7}{9}v_2, \quad \frac{1}{9}v_1 + \frac{8}{9}v_2, \quad v_2.$$

One may use these images as frames in a video in order to get a crossfade effect.