Mathematical Analysis

1st Year Computer Science

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❊ Real numbers

Let us start with some standard notation: $\emptyset$ is the empty set; $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers; $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\} = \{m - n \mid m, n \in \mathbb{N}\}$ the set of integers; $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ the set of rational numbers; $\mathbb{R}$ the set of real numbers.

You are very much used with the real numbers. However, it is not trivial to define them in a rigorous way, see for example [1] or [2] for different methods of constructing $\mathbb{R}$ from $\mathbb{Q}$. We will straightforwardly start working with the real numbers – for this reason some of their properties, e.g. definition 1.5, will simply be given as definitions.

Definition 1.1. Let A be a subset of $\mathbb{R}$, denoted as $A \subseteq \mathbb{R}$. We define $x \in \mathbb{R}$ to be

a *lower bound* for A if $x \leq a, \forall a \in A$; an *upper bound* for A if $x \geq a, \forall a \in A$.

We define

$$lb(A) := \{x \in \mathbb{R} \mid x \leq a, \forall a \in A\} \text{ the set of lower bounds of } A,$$

$$ub(A) := \{x \in \mathbb{R} \mid x \geq a, \forall a \in A\} \text{ the set of upper bounds of } A.$$

We define $x \in \mathbb{R}$ to be

the minimum of A if $x \in lb(A) \cap A$; the maximum of A if $x \in ub(A) \cap A$,

denoted by $\min(A)$, respectively $\max(A)$. In other words, we have that

$$\min(A) \in A \text{ and } \min(A) \leq a, \forall a \in A; \quad \max(A) \in A \text{ and } \max(A) \geq a, \forall a \in A.$$

Note that there are sets which do no have minimum or maximum, e.g. $(0, 1)$.

Definition 1.2. A set $A \subseteq \mathbb{R}$ is defined to be

- bounded (from) below if $lb(A) \neq \emptyset$;
- bounded (from) above if $ub(A) \neq \emptyset$;
- bounded if it is both bounded below and above;
- unbounded if it is not bounded.

Definition 1.3. We say that $x \in \mathbb{R}$ is the *supremum* of $A \subseteq \mathbb{R}$, $x := \sup(A)$, if and only if:

1. $x \geq a, \forall a \in A$, that is $x \in ub(A)$.
2. if u is an upper bound for A, then $x \leq u$.

The supremum is *the least upper bound*, i.e. $\sup(A) := \min(ub(A))$.

Definition 1.4. We say that $x \in \mathbb{R}$ is the *infimum* of $A \subseteq \mathbb{R}$, $x := \inf(A)$, if and only if:

1. $x \leq a$, $\forall a \in A$, that is $x \in lb(A)$.
2. if u is a lower bound for A, then $x \geq u$.

The infimum is *the greatest lower bound*, i.e. $\inf(A) := \max(lb(A))$.

Definition 1.5 (Completeness Axiom)**.** Every set $A \subseteq \mathbb{R}$ that is bounded above has a supremum. Similarly, every set $A \subseteq \mathbb{R}$ that is bounded below has an infimum.

Note that if A has a maximum, then $\sup(A) = \max(A)$. Similarly, if A has a minimum, then $\inf(A) = \min(A)$. Also, if $\sup(A) \in A$, then $\max(A) = \sup(A)$.

Example 1.6. (a) $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, $\sup(A) = 1 = \max(A)$, $\inf(A) = 0$, $\nexists \min(A)$.

(b) $A = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$, $\sup(A) = \sqrt{2}$, $\nexists \max(A)$, $\inf(A) = -\sqrt{2}$, $\nexists \min(A)$.

Theorem 1.7. Let $A \subseteq \mathbb{R}$ be a bounded set. For $\sup(A)$ and $\inf(A)$ the following are true:

$$\forall \varepsilon > 0, \exists x \in A \text{ such that } \sup(A) - \varepsilon < x,$$

$$\forall \varepsilon > 0, \exists x \in A \text{ such that } x < \inf(A) + \varepsilon.$$

Proof. By definition, for any $y < \sup(A)$, say $y = \sup(A) - \varepsilon$ with $\varepsilon > 0$, we have that $y \notin ub(A)$. Hence there exists $x \in A$ such that $y < x$. Similar proof for $\inf(A)$. □

Proposition 1.8. Let $A \subseteq B \subseteq \mathbb{R}$ be (nonempty) bounded sets. Then

$$\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$$

and

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\},$$
$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}.$$

Proof. It follows directly from the definitions. □

Definition 1.9. Define the *extended real line* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, where ∞ and $-\infty$ are such that

$$\forall x \in \mathbb{R}, -\infty < x < \infty.$$

If a set A is not bounded above, we define $\sup(A) := \infty$.
If a set A is not bounded below, we define $\inf(A) := -\infty$.

[Seminar] The empty set $\emptyset$ is bounded by any number. In $\overline{\mathbb{R}}$, $\sup(\emptyset) = -\infty$, $\inf(\emptyset) = \infty$.

Definition 1.10. A set $V \subseteq \mathbb{R}$ is a *neighborhood (vecinity)* of $x \in \mathbb{R}$ if

$$\exists \varepsilon > 0 \text{ such that } (x - \varepsilon, x + \varepsilon) \subseteq V.$$

A set $V \subseteq \mathbb{R}$ is a *neighborhood* of ∞ if $\exists a \in \mathbb{R}$ such that $(a, \infty) \subseteq V$.

A set $V \subseteq \mathbb{R}$ is a *neighborhood* of $-\infty$ if $\exists a \in \mathbb{R}$ such that $(-\infty, a) \subseteq V$.

We denote all the neighborhoods of x by $\mathcal{V}(x) := \{V \subseteq \mathbb{R} \mid V \text{ is a neighborhood of } x\}$.

Definition 1.11. Let $A \subseteq \mathbb{R}$. The following set is called the *interior* of A

$$\operatorname{int}(A) := \{x \in \mathbb{R} \mid \exists V \in \mathcal{V}(x) \text{ such that } V \subseteq A\},$$

and the following set is called the *closure* of A

$$\operatorname{cl}(A) := \{x \in \mathbb{R} \mid \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}.$$

Proposition 1.12. For any $A \subseteq \mathbb{R}$, it holds that $\operatorname{int}(A) \subseteq A \subseteq \operatorname{cl}(A)$.

Proof. To prove that $\operatorname{int}(A) \subseteq A$ we prove that if $x \in \operatorname{int}(A)$, then $x \in A$. Let $x \in \operatorname{int}(A)$, then $\exists \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq A$. Since $x \in (x - \varepsilon, x + \varepsilon)$, we have that $x \in A$.
To prove that $A \subseteq \operatorname{cl}(A)$ we show that if $x \in A$, then $x \in \operatorname{cl}(A)$. Let $x \in A$. Then for any $V \in \mathcal{V}(x)$ it holds that $x \in V$, giving that $x \in V \cap A$. Hence $x \in \operatorname{cl}(A)$ since $V \cap A \neq \emptyset$. □

Definition 1.13. If $A = \operatorname{int}(A)$, then A is called *open*. If $A = \operatorname{cl}(A)$, then A is called *closed*.

Remark 1.14. To prove that a set A is open, it is sufficient to prove that $A \subseteq \operatorname{int}(A)$.
To prove that a set A is closed, it is sufficient to prove that $\operatorname{cl}(A) \subseteq A$.

Proposition 1.15. The following statements are true:

- The complement of an open set is closed.
- The complement of a closed set is open.

Proof. Let us prove the first statement, the other one being similar. Consider A an open set, i.e. $A = \operatorname{int}(A)$, and denote by $A^c = \{x \in \mathbb{R} \mid x \notin A\}$ its complement. To prove that A^c is closed, we prove that $\operatorname{cl}(A^c) \subseteq A^c$. Consider $x \in \operatorname{cl}(A^c)$ and let's assume that $x \notin A^c$, i.e. $x \in A$, aiming to obtain a contradiction. Since A is open, there exists $V \in \mathcal{V}(x)$ such that $V \subseteq A$, giving that $V \cap A^c = \emptyset$: contradiction with $x \in \operatorname{cl}(A^c)$. Hence the assumption $x \notin A^c$ is false, and we have that if $x \in \operatorname{cl}(A^c)$, then $x \in A^c$. In other words, $\operatorname{cl}(A^c) \subseteq A^c$. □

Proposition 1.16. Any union of open sets is open. Any intersection of closed sets is closed. Any finite intersection of open sets is open. Any finite union of closed sets is closed.

Proof. (Optional) Left as extra homework. □

※ Sequences

A set $\{x_n \mid n \in \mathbb{N}\}$ is called a sequence and is denoted by $(x_n)_{n\in\mathbb{N}}$ or simply (x_n). A sequence (x_n) is bounded above (or below) if the set $\{x_n \mid n \in \mathbb{N}\}$ is bounded above (or below). A sequence (x_n) is increasing if $x_{n+1} \geq x_n$, $\forall n \in \mathbb{N}$, and decreasing if $x_{n+1} \leq x_n$, $\forall n \in \mathbb{N}$. A sequence is monotone if it is either increasing or decreasing.

Definition 2.1. A sequence (x_n) *has a limit* $\ell \in \overline{\mathbb{R}}$, and we write $\lim_{n\to\infty} x_n = \ell$ or $x_n \to \ell$, if

$$\forall V \in \mathcal{V}(\ell),\ \exists N_V \in \mathbb{N} \text{ such that } x_n \in V,\ \forall n \geq N_V.$$

If $\ell \in \mathbb{R}$, we say that (x_n) *converges* to ℓ: $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that $|x_n - \ell| < \varepsilon$, $\forall n \geq N_\varepsilon$.
$x_n \to \infty$ if $\forall a > 0$, $\exists N_a \in \mathbb{N}$ such that $x_n > a$, $\forall n \geq N_a$.
$x_n \to -\infty$ if $\forall a < 0$, $\exists N_a \in \mathbb{N}$ such that $x_n < a$, $\forall n \geq N_a$.

Proposition 2.2. A sequence (x_n) converges to $\ell \in \mathbb{R}$ if and only if $\lim_{n\to\infty} |x_n - \ell| = 0$.

Proposition 2.3. Any convergent sequence is bounded.

Proof. TBA (left to the reader). □

Theorem 2.4 (Weierstrass)**.** Any monotone and bounded sequence is convergent.

Proof. Assume that the sequence is increasing, for example. Let $S = \{x_n \mid n \in \mathbb{N}\}$ and consider $\sup(S) \in \mathbb{R}$ (we know that S is bounded). From theorem 1.7 we have that

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ such that } \sup(S) - \varepsilon < x_{N_\varepsilon}.$$

As (x_n) is increasing, $\sup(S) - \varepsilon < x_{N_\varepsilon} \leq x_n\ \forall n \geq N_\varepsilon$. Hence $\sup(S) - x_n < \varepsilon$, $\forall n \geq N_\varepsilon$. The sequence converges to $\sup(S)$ by definition 2.1. Similarly, a decreasing and bounded sequence converges to its infimum. □

Proposition 2.5. Any monotone sequence has a limit in $\overline{\mathbb{R}}$.

Proof. If the sequence is bounded and monotone, then it is convergent by the Weierstrass theorem. If the sequence is unbounded and monotone, then its limit will be infinite. □

Theorem 2.6 (Squeeze/Sandwich theorem)**.** Let (x_n), (y_n), (z_n) be sequences for which there is an $n_0 \in \mathbb{N}$ such that

$$x_n \leq y_n \leq z_n,\ \forall n \geq n_0,$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n.$$

Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

Proof. Let $\ell := \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$ and assume first that $\ell \in \mathbb{R}$. Let $\varepsilon > 0$. Then

$$\exists N_1 \in \mathbb{N} \text{ such that } |x_n - \ell| < \varepsilon,\ \forall n \geq N_1$$

and

$$\exists N_2 \in \mathbb{N} \text{ such that } |z_n - \ell| < \varepsilon,\ \forall n \geq N_2.$$

Taking $N_\varepsilon := \max\{N_1, N_2\}$, we have that

$$|y_n - \ell| \leq \max\{|x_n - \ell|, |z_n - \ell|\} < \varepsilon,\ \forall n \geq N_\varepsilon,$$

hence the conclusion. When ℓ is infinite the proof is similar. □

Theorem 2.7 (Cantor's nested intervals)**.** Let (a_n) be increasing and (b_n) decreasing such that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$, $\forall n \in \mathbb{N}$. Consider the closed intervals $I_n := [a_n, b_n]$, with $I_{n+1} \subseteq I_n$. If $\lim_{n\to\infty}(b_n - a_n) = 0$, then there exists $x \in \mathbb{R}$ such that

$$\bigcap_{n=1}^{\infty} I_n = \{x\}.$$

Proof. Consider the bounded sets $A := \{a_n \mid n \in \mathbb{N}\}$ and $B := \{b_n \mid n \in \mathbb{N}\}$. For any $k \in \mathbb{N}$, we have that

$$a_k \leq \sup(A) \leq b_k$$

and

$$b_k \geq \inf(B) \geq a_k.$$

Hence by the squeeze theorem we have that $\sup(A) = \inf(B)$ and $\bigcap_{n=1}^{\infty} I_n = \{\sup(A)\}$. □

Theorem 2.8 (Bolzano-Weierstrass)**.** Any bounded sequence has a convergent subsequence.

Proof. Consider the bounded set $A := \{x_n \mid n \in \mathbb{N}\}$. Let $a_1 := \inf(A)$ and $b_1 := \sup(A)$, and define $I_1 := [a_1, b_1]$. Bisect I_1 and notice that at least one of the two halves must contain infinitely many terms from the sequence. Take $I_2 := [a_2, b_2]$ to be the half that does. Continuing this procedure we obtain for each $k \in \mathbb{N}$ an interval $I_k := [a_k, b_k]$ containing (at least) a term $x_{n_k} \in A$, such that $I_{k+1} \subseteq I_k$ and $b_k - a_k \to 0$.

From Cantor's nested intervals theorem 2.7 we have that there exists $x \in \mathbb{R}$ such that $\bigcap_{n=1}^{\infty} I_n = \{x\}$, and hence the subsequence (x_{n_k}) converges to x. □

Definition 2.9. For a sequence (x_n) we define the set of its *limit points* by

$$\mathrm{LIM}(x_n) := \{x \in \overline{\mathbb{R}} \mid \text{there exists a subsequence } (x_{n_k}) \text{ s.t. } x_{n_k} \to x\},$$

and

$$\liminf_{n\to\infty} x_n := \inf\big(\mathrm{LIM}(x_n)\big),$$
$$\limsup_{n\to\infty} x_n := \sup\big(\mathrm{LIM}(x_n)\big).$$

Example 2.10. For $x_n = \frac{(-1)^n n}{n+1}$, $\mathrm{LIM}(x_n) = \{-1, 1\}$, $\liminf_{n\to\infty} x_n = -1$, $\limsup_{n\to\infty} x_n = 1$.

Proposition 2.11. $\lim_{n\to\infty} x_n = \ell \in \overline{\mathbb{R}}$ if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \ell$.

Definition 2.12 (Cauchy sequence)**.** A sequence (x_n) is called *Cauchy (or fundamental)* if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ such that } |x_m - x_n| < \varepsilon,\ \forall m, n \geq N_\varepsilon.$$

Proposition 2.13. Any Cauchy sequence is bounded.

Proof. For $\varepsilon = 1$, there exists $N_1 \in \mathbb{N}$ such that $|x_m - x_n| < 1$, $\forall m, n \geq N_1$. In particular, $|x_n - x_{N_1}| < 1$, $\forall n \geq N_1$, hence the terms after index N_1 are bounded. The terms before index N_1 are also bounded since there is a finite number of them. We thus conclude that the entire sequence is bounded. □

Theorem 2.14. A sequence is convergent if and only if it is Cauchy.

Proof. Let's consider first a convergent sequence (x_n) with $x_n \to \ell$. For any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_n - \ell| < \frac{\varepsilon}{2}$, for any $n \geq N_\varepsilon$. Then $|x_m - x_n| \leq |x_m - \ell| + |x_n - \ell| < \varepsilon$, for any $n \geq N_\varepsilon$. Hence the sequence (x_n) is Cauchy.

Assume now that (x_n) is a Cauchy sequence. From the previous proposition we have that (x_n) must be bounded, and thus it has a convergent subsequence (x_{n_k}), $x_{n_k} \to x \in \mathbb{R}$. Let $\varepsilon > 0$. There exists thus $K_\varepsilon \in \mathbb{N}$ such that $|x_{n_k} - x| < \varepsilon$, $\forall k \geq K_\varepsilon$. Also, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$, $\forall m, n \geq N_\varepsilon$. In particular, $|x_{n_k} - x_n| < \varepsilon$, $\forall k, n \geq N_\varepsilon$. Hence $|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < 2\varepsilon$, $\forall n \geq \max\{K_\varepsilon, N_\varepsilon\}$, meaning that $x_n \to x$. □

Example 2.15. The sequence defined by $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is not convergent. Indeed, one can see, for example, that

$$x_{2n} - x_n = \frac{1}{n+1} + \ldots + \ldots + \frac{1}{2n} > \frac{n}{2n},$$

hence $x_{2n} - x_n > \frac{1}{2}$ for any $n \in \mathbb{N}$. Thus (x_n) is not convergent since it is not Cauchy.

※ Series. Power series

For a sequence (x_n), the sum $\sum_{n=1}^{\infty} x_n$ is called a *series* and $s_n := \sum_{k=1}^{n} x_k$ is called the *partial sum of the series*. The summation in a series can start from any index, not necessarily 0 or 1. A series is also often written as $\sum_{n\geq 1} x_n$.

Definition 3.1. The series $\sum_{n=1}^{\infty} x_n$ converges iff the sequence of partial sums (s_n) converges.

Example 3.2. The *geometric series* $\sum_{n=0}^{\infty} q^n$ converges iff $|q| < 1$, with sum $\frac{1}{1-q}$.

Example 3.3. The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ diverges since (s_n) is not a Cauchy sequence.

Example 3.4 (Euler's number)**.** $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Proof. (Optional) Let $s_n = 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}$. Start from $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$ and expand

$$\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{n}\right) \leq s_n.$$

We have that

$$\left(1+\frac{1}{n}\right)^n \leq s_n.$$

Consider now an index $k \geq n$. We have that

$$\left(1+\frac{1}{k}\right)^k \geq 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{k}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{k}\right)$$

and taking $k \to \infty$ we obtain that $e \geq s_n$. We conclude with the squeeze theorem for

$$\left(1+\frac{1}{n}\right)^n \leq s_n \leq e,$$

obtaining that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges and its sum is e. □

Proposition 3.5. If the series $\sum_{n=1}^{\infty} x_n$ is convergent, then $\lim_{n\to\infty} x_n = 0$.

Proof. Consider the partial sum s_n. We have that $x_n = s_n - s_{n-1}$, hence the conclusion. □

It thus follows that if $\lim_{n\to\infty} x_n \neq 0$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

Example 3.6. Series like $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1}) = 1$ are called *telescoping series*. The partial sum of a telescoping series can be easily computed since after cancellations the only remaining terms are the first one and the last one.

If the sequence (x_n) has only nonnegative terms $x_n \geq 0$, then the sequence of partial sums (s_n) is increasing. The series $\sum_{n=1}^{\infty} x_n$ then converges iff (s_n) is bounded.

Theorem 3.7 (Comparison test)**.** Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ be series with nonnegative terms. If there is an $n_0 \in \mathbb{N}$ such that

$$x_n \leq y_n,\ \forall n \geq n_0,\ \text{then}$$

(a) If $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ also converges.

(b) If $\sum_{n=1}^{\infty} x_n$ diverges, then $\sum_{n=1}^{\infty} y_n$ also diverges.

Proof. Consider the sequences of partial sums. In case (a), both sequences are bounded. In case (b), both sequences are unbounded. □

Example 3.8. If $p \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$ since $\frac{1}{n^p} \geq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. E.g. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$.

Theorem 3.9. Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ be series with nonnegative terms. If

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \ell,\ \text{then}$$

- if $\ell \in (0, \infty)$, then the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ have the same nature.
- if $\ell = 0$, then if the series $\sum_{n=1}^{\infty} y_n$ converges, the series $\sum_{n=1}^{\infty} x_n$ also converges.
- if $\ell = \infty$, then if the series $\sum_{n=1}^{\infty} y_n$ diverges, the series $\sum_{n=1}^{\infty} x_n$ also diverges.

Theorem 3.10 (Ratio test)**.** Let $\sum_{n=1}^{\infty} x_n$ be a series with nonnegative terms such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell.$$

- If $\ell < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.
- If $\ell > 1$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

The test is *inconclusive* when $\ell = 1$.

Proof. The idea is that $\sum_{n\geq 1} x_n$ behaves like a geometric series with ratio ℓ. We will only give a proof when $\ell < 1$, the other case being similar.

Take $\varepsilon > 0$ such that $q := \ell + \varepsilon < 1$. There exists $N \in \mathbb{N}$ such that

$$\frac{x_{n+1}}{x_n} - \ell < \varepsilon,\ \forall n \geq N,$$

giving that $x_{n+1} < x_n \cdot q$, $\forall n \geq N$. Hence $x_n < q^{n-N} x_N$, that is $x_n < q^n \frac{x_N}{q^N}$. Since $q < 1$, the series converges by comparison with the geometric series $\sum_{n\geq 1} q^n$. □

Theorem 3.11 (Root test)**.** Let $\sum_{n=1}^{\infty} x_n$ be a series with nonnegative terms such that

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \ell.$$

- If $\ell < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.
- If $\ell > 1$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

The test is *inconclusive* when $\ell = 1$.

Proof. Idea: $\sum_{n\geq 1} x_n$ behaves like a geometric series with ratio ℓ, as in the ratio test. □

Example 3.12. The series $\sum_{n\geq 0} \frac{x^n}{n!}$ converges for any $x \in \mathbb{R}$. We will see later that $\sum_{n\geq 0} \frac{x^n}{n!} = e^x$. We have that $\frac{x_{n+1}}{x_n} = \frac{x}{n+1} \to 0 < 1$, hence the series converges by the ratio test.

Theorem 3.13 (Cauchy condensation test)**.** Let (x_n) be a decreasing sequence with $x_n > 0$. Then the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=0}^{\infty} 2^n x_{2^n}$ have the same nature.

Proof. Let $S_n = x_1 + x_2 + \ldots + x_n$ and $T_n = x_1 + 2x_2 + \ldots + 2^n x_n$. Since $x_n > 0$, the two series will have the same nature if and only if S_n and T_n are both bounded/unbounded.

For any $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ s.t. $2^k \le n \le 2^{k+1} - 1$. Since (x_n) is decreasing and positive, we can group the terms in the following ways

$$\begin{aligned} S_n = x_1 + x_2 + \ldots + x_n &\le x_1 + x_2 + \ldots + x_{2^{k+1}-1} \\ &\le x_1 + (x_2 + x_3) + \ldots + (x_{2^k} + \ldots + x_{2^{k+1}-1}) \\ &\le T_k, \end{aligned}$$

and

$$\begin{aligned} S_n = x_1 + x_2 + \ldots + x_n &\ge x_1 + x_2 + \ldots + x_{2^k} \\ &\ge x_1 + x_2 + (x_3 + x_4) + \ldots + (x_{2^{k-1}+1} + \ldots + x_{2^k}) \\ &\ge \frac{x_1}{2} + \frac{1}{2} T_k. \end{aligned}$$

We obtained that $0 \le \frac{1}{2} T_k \le S_n \le T_k$, hence (S_n) bounded if and only if (T_n) is bounded. $\square$

Example 3.14. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Proof. By the Cauchy condensation test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ has the same nature as $\sum_{n=1}^{\infty} \frac{2^n}{2^{np}} = \sum_{n=1}^{\infty} (2^{1-p})^n$, which converges if and only if $2^{1-p} < 1$, i.e for $p > 1$. $\square$

Theorem 3.15 (Kummer's test)**.** Let (x_n) be a positive sequence and consider another positive sequence (c_n).

(a) If
$$\lim_{n\to\infty} \left(c_n \frac{x_n}{x_{n+1}} - c_{n+1}\right) > 0,$$
then $\sum_{n\ge1} x_n$ is convergent.

(b) If $\sum_{n\ge1} \frac{1}{c_n} = \infty$ and
$$\lim_{n\to\infty} \left(c_n \frac{x_n}{x_{n+1}} - c_{n+1}\right) < 0,$$
then $\sum_{n\ge1} x_n$ is divergent.

Proof. Let us start with (a). Since that limit is positive, there exist $r > 0$ and $n_0 \in \mathbb{N}$ such that

$$c_n x_n - c_{n+1} x_{n+1} \geq r x_{n+1}, \quad \forall n \geq n_0.$$

Denote by $s_n = x_1 + \ldots + x_n$. Adding all these inequalities for $k \in \{n_0, \ldots, n\}$ we have that

$$c_{n_0} x_{n_0} - c_{n+1} x_{n+1} \geq r(s_{n+1} - s_{n_0}),$$

which gives $s_{n+1} \leq s_{n_0} + \frac{1}{r} c_{n_0} x_{n_0}$. Hence (s_n) is bounded and the series converges.

Let us now consider (b). Since the limit is negative, there exists $n_0 \in \mathbb{N}$ such that

$$c_n x_n < c_{n+1} x_{n+1}, \quad \forall n \geq n_0.$$

Hence for $n > n_0$, we have that $c_{n_0} x_{n_0} < c_n x_n$, which gives

$$\frac{1}{c_n} < \frac{1}{c_{n_0} x_{n_0}} x_n.$$

Since $\displaystyle\sum_{n\geq 1} \frac{1}{c_n} = \infty$, we conclude that $\displaystyle\sum_{n\geq 1} x_n = \infty$. □

Many convergence tests can be obtained by taking particular sequences in Kummer's test. We will restrict to the following one.

Theorem 3.16 (Raabe-Duhamel)**.** Let $\displaystyle\sum_{n\geq 1} x_n$ be a series with positive terms such that

$$\lim_{n\to\infty} n\Big(\frac{x_n}{x_{n+1}} - 1\Big) = R.$$

- If $R > 1$, then the series $\displaystyle\sum_{n=1}^{\infty} x_n$ is convergent.
- If $R < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} x_n$ is divergent.

Proof. Take $c_n = n$ in Kummer's test (theorem 3.15). □

Example 3.17. Study the convergence of the series $\displaystyle\sum_{n\geq 0} \frac{n!}{a(a+1)\ldots(a+n)}$, with $a > 0$.

Proof. The ratio test is inconclusive since $\frac{x_{n+1}}{x_n} = \frac{n+1}{a+n+1} \to 1$. Let us then try the Raabe-Duhamel test:

$$\lim_{n\to\infty} n\Big(\frac{x_n}{x_{n+1}} - 1\Big) = \lim_{n\to\infty} n\Big(\frac{a+n+1}{n+1} - 1\Big) = a.$$

Hence if $a > 1$ the series converges; and if $a < 1$ the series diverges. When $a = 1$ the series is $\displaystyle\sum_{n\geq 0} \frac{1}{n+1} = \infty$. □

A series $\sum_{n\geq 1} x_n$ is called an *alternating series* if $x_n x_{n+1} \leq 0$, $\forall n \in \mathbb{N}$. A fundamental class of alternating series are series of the form $\sum_{n\geq 1}(-1)^n a_n$ or $\sum_{n\geq 1}(-1)^{n+1} a_n$, with $a_n > 0$.

Example 3.18. The series $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges to $\ln 2$.

Proof. Let us prove convergence by considering the partial sums $s_n = \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. Notice that $s_{2k+2} - s_{2k} = \frac{1}{2k+1} - \frac{1}{2k+2} > 0$ and that $s_{2k+3} - s_{2k+1} = \frac{1}{2k+3} - \frac{1}{2k+2} < 0$. This means that the subsequence (s_{2k}) is increasing, while the subsequence (s_{2k+1}) is decreasing. Notice also that $s_{2k+1} - s_{2k} = \frac{1}{2k+1}$ and $s_{2k} < s_{2k+1}$, so both subsequences are also bounded and converge to the same limit. To find the sum of the alternating series, recall (from the seminar) that

$$\lim_{n\to\infty} 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n = \gamma \in (0,1), \text{ hence}$$

$$\begin{aligned} s_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{2n} = 1 + \frac{1}{2} + \ldots + \frac{1}{2n} - 2\left(\frac{1}{2} + \ldots + \frac{1}{2n}\right) \\ &= \underbrace{1 + \frac{1}{2} + \ldots + \frac{1}{2n} - \ln(2n)}_{\to\gamma} - \underbrace{\left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n\right)}_{\to\gamma} + \ln 2 \longrightarrow \ln 2. \end{aligned}$$

□

Theorem 3.19 (Leibniz test)**.** Let (x_n) be a decreasing sequence with $x_n \longrightarrow 0$. Then the series $\sum_{n\geq 1}(-1)^n x_n$ is convergent.

Proof. Consider the partial sum $s_n = \displaystyle\sum_{k=1}^{n}(-1)^k x_k$. We will prove that (s_n) is convergent by showing that it is a Cauchy sequence. For $n, p \in \mathbb{N}$ consider

$$\begin{aligned} |s_{n+p} - s_n| &= |(-1)^{n+1} x_{n+1} + \ldots + (-1)^{n+p} x_{n+p}| \\ &= |\underbrace{x_{n+1} - x_{n+2}}_{\geq 0} + \underbrace{x_{n+3} - x_{n+4}}_{\geq 0} + \ldots + (-1)^{p-2} x_{n+p-1} + (-1)^{p-1} x_{n+p}| \\ &= x_{n+1} \underbrace{-x_{n+2} + x_{n+3}}_{\leq 0} - x_{n+4} + \ldots \pm x_{n+p-1} \mp x_{n+p} \\ &\leq x_{n+1}. \end{aligned}$$

Since $x_n \longrightarrow 0$, $|s_{n+p} - s_n|$ can be made arbitrarily small, so (s_n) is Cauchy. □

Definition 3.20. A series $\sum_{n\geq 1} x_n$ is called *absolutely convergent* if $\sum_{n\geq 1} |x_n|$ is convergent.

Proposition 3.21. Any absolutely convergent series is also convergent.

Proof. If $\sum_{k=1}^{n} |x_k|$ gives a Cauchy sequence, then $\sum_{k=1}^{n} x_k$ also gives a Cauchy sequence. □

Theorem 3.22 (Cauchy)**.** Let $\sum_{n\geq 1} x_n$ be an *absolutely convergent series* and let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n\geq 1} x_{\sigma(n)}$ is also absolutely convergent and $\sum_{n\geq 1} x_{\sigma(n)} = \sum_{n\geq 1} x_n$. In other words, any rearrangement of an absolutely convergent series has the same sum.

Proof. (Optional) See [2][Theorem 7.4.3]. □

Definition 3.23. A series $\sum_{n\geq 1} x_n$ is called *conditionally convergent* (or semi-convergent) if $\sum_{n\geq 1} x_n$ converges, but $\sum_{n\geq 1} |x_n|$ diverges.

Theorem 3.24 (Riemann)**.** Let $\sum_{n\geq 1} x_n$ be a *conditionally convergent series* and let $x \in \overline{\mathbb{R}}$. Then there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n\geq 1} x_{\sigma(n)} = x$. In other words, a conditionally convergent series can be rearranged to converge to any value or diverge to $\pm\infty$.

Proof. (Optional) See [2][Theorem 8.2.8]. □

Example 3.25. Rearranging the terms in the alternating harmonic series one can obtain a different sum. Indeed, consider $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2$, and reorder the terms in the following way: one positive, two negative. Then

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \ldots = \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots) = \frac{1}{2}\ln 2.$$

Definition 3.26. Let (a_n) be a sequence of real numbers and let $c \in \mathbb{R}$. The series

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is called a *power series* centered at c.

Theorem 3.27. Consider the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$. There exists a unique $R \in [0,\infty]$, called the *radius of convergence* of the power series, such that the power series

- converges absolutely when $|x-c| < R$.
- diverges when $|x-c| > R$.

Theorem 3.28. If the limit

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \in [0,\infty]$$

exists, then the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has the radius of convergence

$$R = \begin{cases} \frac{1}{L}, & \text{if } L \in (0,\infty) \\ 0, & \text{if } L = \infty \\ \infty, & \text{if } L = 0. \end{cases}$$

Proof. It follows from the root test for series with positive terms. □

Corollary 3.29. If the limit

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \in [0,\infty]$$

exists, then the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has the radius of convergence

$$R = \begin{cases} \frac{1}{L}, & \text{if } L \in (0,\infty) \\ 0, & \text{if } L = \infty \\ \infty, & \text{if } L = 0. \end{cases}$$

Proof. It follows from $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$. □

Definition 3.30. The convergence set of a power series is

$$C := \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n(x-c)^n \text{ converges}\}.$$

Remark 3.31. The convergence set C contains the open interval $(c-R, c+R)$ and possibly the endpoints $\{c-R, c+R\}$.

Example 3.32. The power series $\sum_{n\geq 0} x^n$ has radius of convergence $R = 1$, it converges absolutely for $|x| < 1$ and diverges when $|x| > 1$ (by the root test or the ratio test). The convergence set is $(-1, 1)$ and for $x \in (-1, 1)$ we have that

$$\sum_{n\geq 0} x^n = \frac{1}{1-x}, \quad \sum_{n\geq 0} (-x)^n = \frac{1}{1+x}.$$

Example 3.33. The power series $\sum_{n\geq 1} \frac{x^n}{n}$ has radius of convergence $R = 1$, it converges absolutely for $|x| < 1$ and diverges when $|x| > 1$ (by the root test or the ratio test). Moreover, the series converges for $x = -1$ (alternating harmonic series) and diverges for $x = 1$ (harmonic series), hence its convergence set is $C = [-1, 1)$.

Theorem 3.34. Consider a power series with radius of convergence R, given by

$$s(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

Then for any $x \in (c-R, c+R)$, the power series can be differentiated term by term and

$$s'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1},$$

and for any $t \in (c-R, c+R)$ the power series can be integrated term by term

$$\int_c^t s(x)\mathrm{dx} = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (t-c)^{n+1}.$$

Example 3.35. The power series $\sum_{n\geq 1} \frac{x^n}{n!}$ converges absolutely for any $x \in \mathbb{R}$ (ratio test). Let $\exp(x) := \sum_{n\geq 1} \frac{x^n}{n!}$ and differentiate term by term, then $\exp'(x) = \exp(x)$ and $\exp(0) = 1$.

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