Mathematical Analysis

1st Year Computer Science

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※ Real numbers

Let us start with some standard notation: $\emptyset$ is the empty set; $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers; $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\} = \{m - n \mid m, n \in \mathbb{N}\}$ the set of integers; $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ the set of rational numbers; $\mathbb{R}$ the set of real numbers.

You are very much used with the real numbers. However, it is not trivial to define them in a rigorous way, see for example [1] or [2] for different methods of constructing $\mathbb{R}$ from $\mathbb{Q}$. We will straightforwardly start working with the real numbers – for this reason some of their properties, e.g. definition 1.5, will simply be given as definitions.

Definition 1.1. Let A be a subset of $\mathbb{R}$, denoted as $A \subseteq \mathbb{R}$. We define $x \in \mathbb{R}$ to be

a *lower bound* for A if $x \leq a$, $\forall a \in A$; an *upper bound* for A if $x \geq a$, $\forall a \in A$.

We define

$$lb(A) := \{x \in \mathbb{R} \mid x \leq a,\ \forall a \in A\} \text{ the set of lower bounds of } A,$$

$$ub(A) := \{x \in \mathbb{R} \mid x \geq a,\ \forall a \in A\} \text{ the set of upper bounds of } A.$$

We define $x \in \mathbb{R}$ to be

the minimum of A if $x \in lb(A) \cap A$; the maximum of A if $x \in ub(A) \cap A$,

denoted by $\min(A)$, respectively $\max(A)$. In other words, we have that

$$\min(A) \in A \text{ and } \min(A) \leq a,\ \forall a \in A; \quad \max(A) \in A \text{ and } \max(A) \geq a,\ \forall a \in A.$$

Note that there are sets which do no have minimum or maximum, e.g. $(0, 1)$.

Definition 1.2. A set $A \subseteq \mathbb{R}$ is defined to be

- bounded (from) below if $lb(A) \neq \emptyset$;
- bounded (from) above if $ub(A) \neq \emptyset$;
- bounded if it is both bounded below and above;
- unbounded if it is not bounded.

Definition 1.3. We say that $x \in \mathbb{R}$ is the *supremum* of $A \subseteq \mathbb{R}$, $x := \sup(A)$, if and only if:

1. $x \geq a$, $\forall a \in A$, that is $x \in ub(A)$.
2. if u is an upper bound for A, then $x \leq u$.

The supremum is *the least upper bound*, i.e. $\sup(A) := \min(ub(A))$.

Definition 1.4. We say that $x \in \mathbb{R}$ is the *infimum* of $A \subseteq \mathbb{R}$, $x := \inf(A)$, if and only if:

1. $x \leq a$, $\forall a \in A$, that is $x \in lb(A)$.
2. if u is a lower bound for A, then $x \geq u$.

The infimum is *the greatest lower bound*, i.e. $\inf(A) := \max(lb(A))$.

Definition 1.5 (Completeness Axiom)**.** Every set $A \subseteq \mathbb{R}$ that is bounded above has a supremum. Similarly, every set $A \subseteq \mathbb{R}$ that is bounded below has an infimum.

Note that if A has a maximum, then $\sup(A) = \max(A)$. Similarly, if A has a minimum, then $\inf(A) = \min(A)$. Also, if $\sup(A) \in A$, then $\max(A) = \sup(A)$.

Example 1.6. (a) $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, $\sup(A) = 1 = \max(A)$, $\inf(A) = 0$, $\nexists \min(A)$.

(b) $A = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$, $\sup(A) = \sqrt{2}$, $\nexists \max(A)$, $\inf(A) = -\sqrt{2}$, $\nexists \min(A)$.

Theorem 1.7. Let $A \subseteq \mathbb{R}$ be a bounded set. For $\sup(A)$ and $\inf(A)$ the following are true:

$$\forall \varepsilon > 0,\ \exists x \in A \text{ such that } \sup(A) - \varepsilon < x,$$

$$\forall \varepsilon > 0,\ \exists x \in A \text{ such that } x < \inf(A) + \varepsilon.$$

Proof. By definition, for any $y < \sup(A)$, say $y = \sup(A) - \varepsilon$ with $\varepsilon > 0$, we have that $y \notin ub(A)$. Hence there exists $x \in A$ such that $y < x$. Similar proof for $\inf(A)$. □

Proposition 1.8. Let $A \subseteq B \subseteq \mathbb{R}$ be (nonempty) bounded sets. Then

$$\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$$

and

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\},$$
$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}.$$

Proof. It follows directly from the definitions. □

Definition 1.9. Define the *extended real line* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, where ∞ and $-\infty$ are such that

$$\forall x \in \mathbb{R},\ -\infty < x < \infty.$$

If a set A is not bounded above, we define $\sup(A) := \infty$.
If a set A is not bounded below, we define $\inf(A) := -\infty$.

[Seminar] The empty set $\emptyset$ is bounded by any number. In $\overline{\mathbb{R}}$, $\sup(\emptyset) = -\infty$, $\inf(\emptyset) = \infty$.

Definition 1.10. A set $V \subseteq \mathbb{R}$ is a *neighborhood (vecinity)* of $x \in \mathbb{R}$ if

$$\exists \varepsilon > 0 \text{ such that } (x - \varepsilon, x + \varepsilon) \subseteq V.$$

A set $V \subseteq \mathbb{R}$ is a *neighborhood* of ∞ if $\exists a \in \mathbb{R}$ such that $(a, \infty) \subseteq V$.

A set $V \subseteq \mathbb{R}$ is a *neighborhood* of $-\infty$ if $\exists a \in \mathbb{R}$ such that $(-\infty, a) \subseteq V$.

We denote all the neighborhoods of x by $\mathcal{V}(x) := \{V \subseteq \mathbb{R} \mid V \text{ is a neighborhood of } x\}$.

Definition 1.11. Let $A \subseteq \mathbb{R}$. The following set is called the *interior* of A

$$\text{int}(A) := \{x \in \mathbb{R} \mid \exists V \in \mathcal{V}(x) \text{ such that } V \subseteq A\},$$

and the following set is called the *closure* of A

$$\text{cl}(A) := \{x \in \mathbb{R} \mid \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}.$$

Proposition 1.12. For any $A \subseteq \mathbb{R}$, it holds that $\text{int}(A) \subseteq A \subseteq \text{cl}(A)$.

Proof. To prove that $\text{int}(A) \subseteq A$ we prove that if $x \in \text{int}(A)$, then $x \in A$. Let $x \in \text{int}(A)$, then $\exists \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq A$. Since $x \in (x - \varepsilon, x + \varepsilon)$, we have that $x \in A$.
To prove that $A \subseteq \text{cl}(A)$ we show that if $x \in A$, then $x \in \text{cl}(A)$. Let $x \in A$. Then for any $V \in \mathcal{V}(x)$ it holds that $x \in V$, giving that $x \in V \cap A$. Hence $x \in \text{cl}(A)$ since $V \cap A \neq \emptyset$. □

Definition 1.13. If $A = \text{int}(A)$, then A is called *open*. If $A = \text{cl}(A)$, then A is called *closed*.

Remark 1.14. To prove that a set A is open, it is sufficient to prove that $A \subseteq \text{int}(A)$.
To prove that a set A is closed, it is sufficient to prove that $\text{cl}(A) \subseteq A$.

Proposition 1.15. The following statements are true:

- The complement of an open set is closed.
- The complement of a closed set is open.

Proof. Let us prove the first statement, the other one being similar. Consider A an open set, i.e. $A = \text{int}(A)$, and denote by $A^c = \{x \in \mathbb{R} \mid x \notin A\}$ its complement. To prove that A^c is closed, we prove that $\text{cl}(A^c) \subseteq A^c$. Consider $x \in \text{cl}(A^c)$ and let's assume that $x \notin A^c$, i.e. $x \in A$, aiming to obtain a contradiction. Since A is open, there exists $V \in \mathcal{V}(x)$ such that $V \subseteq A$, giving that $V \cap A^c = \emptyset$: contradiction with $x \in \text{cl}(A^c)$. Hence the assumption $x \notin A^c$ is false, and we have that if $x \in \text{cl}(A^c)$, then $x \in A^c$. In other words, $\text{cl}(A^c) \subseteq A^c$. □

Proposition 1.16. Any union of open sets is open. Any intersection of closed sets is closed. Any finite intersection of open sets is open. Any finite union of closed sets is closed.

Proof. (Optional) Left as extra homework. □

❊ Sequences

A set $\{x_n \mid n \in \mathbb{N}\}$ is called a sequence and is denoted by $(x_n)_{n\in\mathbb{N}}$ or simply (x_n). A sequence (x_n) is bounded above (or below) if the set $\{x_n \mid n \in \mathbb{N}\}$ is bounded above (or below). A sequence (x_n) is increasing if $x_{n+1} \geq x_n$, $\forall n \in \mathbb{N}$, and decreasing if $x_{n+1} \leq x_n$, $\forall n \in \mathbb{N}$. A sequence is monotone if it is either increasing or decreasing.

Definition 2.1. A sequence (x_n) *has a limit* $\ell \in \overline{\mathbb{R}}$, and we write $\lim\limits_{n\to\infty} x_n = \ell$ or $x_n \longrightarrow \ell$, if

$$\forall V \in \mathcal{V}(\ell),\ \exists N_V \in \mathbb{N} \text{ such that } x_n \in V,\ \forall n \geq N_V.$$

If $\ell \in \mathbb{R}$, we say that (x_n) *converges* to ℓ: $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that $|x_n - \ell| < \varepsilon$, $\forall n \geq N_\varepsilon$.
$x_n \longrightarrow \infty$ if $\forall a > 0$, $\exists N_a \in \mathbb{N}$ such that $x_n > a$, $\forall n \geq N_a$.
$x_n \longrightarrow -\infty$ if $\forall a < 0$, $\exists N_a \in \mathbb{N}$ such that $x_n < a$, $\forall n \geq N_a$.

Proposition 2.2. A sequence (x_n) converges to $\ell \in \mathbb{R}$ if and only if $\lim\limits_{n\to\infty} |x_n - \ell| = 0$.

Proposition 2.3. Any convergent sequence is bounded.

Proof. TBA (left to the reader). □

Theorem 2.4 (Weierstrass)**.** Any monotone and bounded sequence is convergent.

Proof. Assume that the sequence is increasing, for example. Let $S = \{x_n \mid n \in \mathbb{N}\}$ and consider $\sup(S) \in \mathbb{R}$ (we know that S is bounded). From theorem 1.7 we have that

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ such that } \sup(S) - \varepsilon < x_{N_\varepsilon}.$$

As (x_n) is increasing, $\sup(S) - \varepsilon < x_{N_\varepsilon} \leq x_n\, \forall n \geq N_\varepsilon$. Hence $\sup(S) - x_n < \varepsilon$, $\forall n \geq N_\varepsilon$. The sequence converges to $\sup(S)$ by definition 2.1. Similarly, a decreasing and bounded sequence converges to its infimum. □

Proposition 2.5. Any monotone sequence has a limit in $\overline{\mathbb{R}}$.

Proof. If the sequence is bounded and monotone, then it is convergent by the Weierstrass theorem. If the sequence is unbounded and monotone, then its limit will be infinite. □

Theorem 2.6 (Squeeze/Sandwich theorem)**.** Let (x_n), (y_n), (z_n) be sequences for which there is an $n_0 \in \mathbb{N}$ such that

$$x_n \leq y_n \leq z_n,\ \forall n \geq n_0,$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n.$$

Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

Proof. Let $\ell := \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n$ and assume first that $\ell \in \mathbb{R}$. Let $\varepsilon > 0$. Then

$$\exists N_1 \in \mathbb{N} \text{ such that } |x_n - \ell| < \varepsilon, \forall n \geq N_1$$

and

$$\exists N_2 \in \mathbb{N} \text{ such that } |z_n - \ell| < \varepsilon, \forall n \geq N_2.$$

Taking $N_\varepsilon := \max\{N_1, N_2\}$, we have that

$$|y_n - \ell| \leq \max\{|x_n - \ell|, |z_n - \ell|\} < \varepsilon, \forall n \geq N_\varepsilon,$$

hence the conclusion. When ℓ is infinite the proof is similar. □

Theorem 2.7 (Cantor's nested intervals)**.** Let (a_n) be increasing and (b_n) decreasing such that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$, $\forall n \in \mathbb{N}$. Consider the closed intervals $I_n := [a_n, b_n]$, with $I_{n+1} \subseteq I_n$. If $\lim\limits_{n\to\infty}(b_n - a_n) = 0$, then there exists $x \in \mathbb{R}$ such that

$$\bigcap_{n=1}^{\infty} I_n = \{x\}.$$

Proof. Consider the bounded sets $A := \{a_n \mid n \in \mathbb{N}\}$ and $B := \{b_n \mid n \in \mathbb{N}\}$. For any $k \in \mathbb{N}$, we have that

$$a_k \leq \sup(A) \leq b_k$$

and

$$b_k \geq \inf(B) \geq a_k.$$

Hence by the squeeze theorem we have that $\sup(A) = \inf(B)$ and $\bigcap\limits_{n=1}^{\infty} I_n = \{\sup(A)\}$. □

Theorem 2.8 (Bolzano-Weierstrass)**.** Any bounded sequence has a convergent subsequence.

Proof. Consider the bounded set $A := \{x_n \mid n \in \mathbb{N}\}$. Let $a_1 := \inf(A)$ and $b_1 := \sup(A)$, and define $I_1 := [a_1, b_1]$. Bisect I_1 and notice that at least one of the two halves must contain infinitely many terms from the sequence. Take $I_2 := [a_2, b_2]$ to be the half that does. Continuing this procedure we obtain for each $k \in \mathbb{N}$ an interval $I_k := [a_k, b_k]$ containing (at least) a term $x_{n_k} \in A$, such that $I_{k+1} \subseteq I_k$ and $b_k - a_k \to 0$.

From Cantor's nested intervals theorem 2.7 we have that there exists $x \in \mathbb{R}$ such that $\bigcap\limits_{n=1}^{\infty} I_n = \{x\}$, and hence the subsequence (x_{n_k}) converges to x. □

Definition 2.9. For a sequence (x_n) we define the set of its *limit points* by

$$\mathrm{LIM}(x_n) := \{x \in \overline{\mathbb{R}} \mid \text{there exists a subsequence } (x_{n_k}) \text{ s.t. } x_{n_k} \to x\},$$

and

$$\liminf_{n\to\infty} x_n := \inf\big(\mathrm{LIM}(x_n)\big),$$
$$\limsup_{n\to\infty} x_n := \sup\big(\mathrm{LIM}(x_n)\big).$$

Example 2.10. For $x_n = \frac{(-1)^n n}{n+1}$, $\mathrm{LIM}(x_n) = \{-1, 1\}$, $\liminf_{n\to\infty} x_n = -1$, $\limsup_{n\to\infty} x_n = 1$.

Proposition 2.11. $\lim_{n\to\infty} x_n = \ell \in \overline{\mathbb{R}}$ if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \ell$.

Definition 2.12 (Cauchy sequence)**.** A sequence (x_n) is called *Cauchy (or fundamental)* if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ such that } |x_m - x_n| < \varepsilon,\ \forall m, n \geq N_\varepsilon.$$

Proposition 2.13. Any Cauchy sequence is bounded.

Proof. For $\varepsilon = 1$, there exists $N_1 \in \mathbb{N}$ such that $|x_m - x_n| < 1$, $\forall m, n \geq N_1$. In particular, $|x_n - x_{N_1}| < 1$, $\forall n \geq N_1$, hence the terms after index N_1 are bounded. The terms before index N_1 are also bounded since there is a finite number of them. We thus conclude that the entire sequence is bounded. □

Theorem 2.14. A sequence is convergent if and only if it is Cauchy.

Proof. Let's consider first a convergent sequence (x_n) with $x_n \to \ell$. For any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_n - \ell| < \frac{\varepsilon}{2}$, for any $n \geq N_\varepsilon$. Then $|x_m - x_n| \leq |x_m - \ell| + |x_n - \ell| < \varepsilon$, for any $n \geq N_\varepsilon$. Hence the sequence (x_n) is Cauchy.

Assume now that (x_n) is a Cauchy sequence. From the previous proposition we have that (x_n) must be bounded, and thus it has a convergent subsequence (x_{n_k}), $x_{n_k} \to x \in \mathbb{R}$. Let $\varepsilon > 0$. There exists thus $K_\varepsilon \in \mathbb{N}$ such that $|x_{n_k} - x| < \varepsilon$, $\forall k \geq K_\varepsilon$. Also, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$, $\forall m, n \geq N_\varepsilon$. In particular, $|x_{n_k} - x_n| < \varepsilon$, $\forall k, n \geq N_\varepsilon$. Hence $|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < 2\varepsilon$, $\forall n \geq \max\{K_\varepsilon, N_\varepsilon\}$, meaning that $x_n \to x$. □

Example 2.15. The sequence defined by $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is not convergent. Indeed, one can see, for example, that

$$x_{2n} - x_n = \frac{1}{n+1} + \ldots + \ldots + \frac{1}{2n} > \frac{n}{2n},$$

hence $x_{2n} - x_n > \frac{1}{2}$ for any $n \in \mathbb{N}$. Thus (x_n) is not convergent since it is not Cauchy.

※ Series. Power series

For a sequence (x_n), the sum $\sum_{n=1}^{\infty} x_n$ is called a *series* and $s_n := \sum_{k=1}^{n} x_k$ is called the *partial sum of the series*. The summation in a series can start from any index, not necessarily 0 or 1. A series is also often written as $\sum_{n\geq 1} x_n$.

Definition 3.1. The series $\sum_{n=1}^{\infty} x_n$ converges iff the sequence of partial sums (s_n) converges.

Example 3.2. The *geometric series* $\sum_{n=0}^{\infty} q^n$ converges iff $|q| < 1$, with sum $\frac{1}{1-q}$.

Example 3.3. The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ diverges since (s_n) is not a Cauchy sequence.

Example 3.4 (Euler's number)**.** $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Proof. (Optional) Let $s_n = 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}$. Start from $\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$ and expand

$$(1 + \frac{1}{n})^n = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \ldots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdot \ldots \cdot (1 - \frac{n-1}{n}) \leq s_n.$$

We have that

$$(1 + \frac{1}{n})^n \leq s_n.$$

Consider now an index $k \geq n$. We have that

$$(1 + \frac{1}{k})^k \geq 1 + 1 + \frac{1}{2!}(1 - \frac{1}{k}) + \ldots + \frac{1}{n!}(1 - \frac{1}{k})(1 - \frac{2}{k}) \cdot \ldots \cdot (1 - \frac{n-1}{k})$$

and taking $k \to \infty$ we obtain that $e \geq s_n$. We conclude with the squeeze theorem for

$$(1 + \frac{1}{n})^n \leq s_n \leq e,$$

obtaining that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges and its sum is e. □

Proposition 3.5. If the series $\sum_{n=1}^{\infty} x_n$ is convergent, then $\lim_{n\to\infty} x_n = 0$.

Proof. Consider the partial sum s_n. We have that $x_n = s_n - s_{n-1}$, hence the conclusion. □

It thus follows that if $\lim_{n\to\infty} x_n \neq 0$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

Example 3.6. Series like $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = 1$ are called *telescoping series*. The partial sum of a telescoping series can be easily computed since after cancellations the only remaining terms are the first one and the last one.

If the sequence (x_n) has only nonnegative terms $x_n \geq 0$, then the sequence of partial sums (s_n) is increasing. The series $\sum_{n=1}^{\infty} x_n$ then converges iff (s_n) is bounded.

Theorem 3.7 (Comparison test)**.** Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ be series with nonnegative terms. If there is an $n_0 \in \mathbb{N}$ such that

$$x_n \leq y_n,\ \forall n \geq n_0,\ \text{then}$$

(a) If $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ also converges.

(b) If $\sum_{n=1}^{\infty} x_n$ diverges, then $\sum_{n=1}^{\infty} y_n$ also diverges.

Proof. Consider the sequences of partial sums. In case (a), both sequences are bounded. In case (b), both sequences are unbounded. □

Example 3.8. If $p \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$ since $\frac{1}{n^p} \geq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. E.g. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$.

Theorem 3.9. Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ be series with nonnegative terms. If

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \ell,\ \text{then}$$

- if $\ell \in (0, \infty)$, then the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ have the same nature.
- if $\ell = 0$, then if the series $\sum_{n=1}^{\infty} y_n$ converges, the series $\sum_{n=1}^{\infty} x_n$ also converges.
- if $\ell = \infty$, then if the series $\sum_{n=1}^{\infty} y_n$ diverges, the series $\sum_{n=1}^{\infty} x_n$ also diverges.

Theorem 3.10 (Ratio test)**.** Let $\sum_{n=1}^{\infty} x_n$ be a series with nonnegative terms such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell.$$

- If $\ell < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.
- If $\ell > 1$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

The test is *inconclusive* when $\ell = 1$.

Proof. The idea is that $\sum_{n\geq 1} x_n$ behaves like a geometric series with ratio ℓ. We will only give a proof when $\ell < 1$, the other case being similar.

Take $\varepsilon > 0$ such that $q := \ell + \varepsilon < 1$. There exists $N \in \mathbb{N}$ such that

$$\frac{x_{n+1}}{x_n} - \ell < \varepsilon, \ \forall n \geq N,$$

giving that $x_{n+1} < x_n \cdot q$, $\forall n \geq N$. Hence $x_n < q^{n-N} x_N$, that is $x_n < q^n \frac{x_N}{q^N}$. Since $q < 1$, the series converges by comparison with the geometric series $\sum_{n\geq 1} q^n$. □

Theorem 3.11 (Root test)**.** Let $\sum_{n=1}^{\infty} x_n$ be a series with nonnegative terms such that

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \ell.$$

- If $\ell < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.
- If $\ell > 1$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

The test is *inconclusive* when $\ell = 1$.

Proof. Idea: $\sum_{n\geq 1} x_n$ behaves like a geometric series with ratio ℓ, as in the ratio test. □

Example 3.12. The series $\sum_{n\geq 0} \frac{x^n}{n!}$ converges for any $x \in \mathbb{R}$. We will see later that $\sum_{n\geq 0} \frac{x^n}{n!} = e^x$. We have that $\frac{x_{n+1}}{x_n} = \frac{x}{n+1} \to 0 < 1$, hence the series converges by the ratio test.

Theorem 3.13 (Cauchy condensation test)**.** Let (x_n) be a decreasing sequence with $x_n > 0$. Then the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=0}^{\infty} 2^n x_{2^n}$ have the same nature.

Proof. Let $S_n = x_1 + x_2 + \ldots + x_n$ and $T_n = x_1 + 2x_2 + \ldots + 2^n x_n$. Since $x_n > 0$, the two series will have the same nature if and only if S_n and T_n are both bounded/unbounded.

For any $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ s.t. $2^k \leq n \leq 2^{k+1} - 1$. Since (x_n) is decreasing and positive, we can group the terms in the following ways

$$\begin{aligned} S_n = x_1 + x_2 + \ldots + x_n &\leq x_1 + x_2 + \ldots + x_{2^{k+1}-1} \\ &\leq x_1 + (x_2 + x_3) + \ldots + (x_{2^k} + \ldots + x_{2^{k+1}-1}) \\ &\leq T_k, \end{aligned}$$

and

$$\begin{aligned} S_n = x_1 + x_2 + \ldots + x_n &\geq x_1 + x_2 + \ldots + x_{2^k} \\ &\geq x_1 + x_2 + (x_3 + x_4) + \ldots + (x_{2^{k-1}+1} + \ldots + x_{2^k}) \\ &\geq \frac{x_1}{2} + \frac{1}{2} T_k. \end{aligned}$$

We obtained that $0 \leq \frac{1}{2} T_k \leq S_n \leq T_k$, hence (S_n) bounded if and only if (T_n) is bounded. □

Example 3.14. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Proof. By the Cauchy condensation test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ has the same nature as $\sum_{n=1}^{\infty} \frac{2^n}{2^{np}} = \sum_{n=1}^{\infty} (2^{1-p})^n$, which converges if and only if $2^{1-p} < 1$, i.e for $p > 1$. □

Theorem 3.15 (Kummer's test)**.** Let (x_n) be a positive sequence and consider another positive sequence (c_n).

(a) If

$$\lim_{n\to\infty} \left(c_n \frac{x_n}{x_{n+1}} - c_{n+1}\right) > 0,$$

then $\sum_{n\geq 1} x_n$ is convergent.

(b) If $\sum_{n\geq 1} \frac{1}{c_n} = \infty$ and

$$\lim_{n\to\infty} \left(c_n \frac{x_n}{x_{n+1}} - c_{n+1}\right) < 0,$$

then $\sum_{n\geq 1} x_n$ is divergent.

Proof. Let us start with (a). Since that limit is positive, there exist $r > 0$ and $n_0 \in \mathbb{N}$ such that

$$c_n x_n - c_{n+1} x_{n+1} \geq r x_{n+1}, \quad \forall n \geq n_0.$$

Denote by $s_n = x_1 + \ldots + x_n$. Adding all these inequalities for $k \in \{n_0, \ldots, n\}$ we have that

$$c_{n_0} x_{n_0} - c_{n+1} x_{n+1} \geq r(s_{n+1} - s_{n_0}),$$

which gives $s_{n+1} \leq s_{n_0} + \frac{1}{r} c_{n_0} x_{n_0}$. Hence (s_n) is bounded and the series converges.

Let us now consider (b). Since the limit is negative, there exists $n_0 \in \mathbb{N}$ such that

$$c_n x_n < c_{n+1} x_{n+1}, \quad \forall n \geq n_0.$$

Hence for $n > n_0$, we have that $c_{n_0} x_{n_0} < c_n x_n$, which gives

$$\frac{1}{c_n} < \frac{1}{c_{n_0} x_{n_0}} x_n.$$

Since $\displaystyle\sum_{n\geq 1} \frac{1}{c_n} = \infty$, we conclude that $\displaystyle\sum_{n\geq 1} x_n = \infty$. □

Many convergence tests can be obtained by taking particular sequences in Kummer's test. We will restrict to the following one.

Theorem 3.16 (Raabe-Duhamel)**.** Let $\displaystyle\sum_{n\geq 1} x_n$ be a series with positive terms such that

$$\lim_{n\to\infty} n\Big(\frac{x_n}{x_{n+1}} - 1\Big) = R.$$

- If $R > 1$, then the series $\displaystyle\sum_{n=1}^{\infty} x_n$ is convergent.
- If $R < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} x_n$ is divergent.

Proof. Take $c_n = n$ in Kummer's test (theorem 3.15). □

Example 3.17. Study the convergence of the series $\displaystyle\sum_{n\geq 0} \frac{n!}{a(a+1)\ldots(a+n)}$, with $a > 0$.

Proof. The ratio test is inconclusive since $\frac{x_{n+1}}{x_n} = \frac{n+1}{a+n+1} \to 1$. Let us then try the Raabe-Duhamel test:

$$\lim_{n\to\infty} n\Big(\frac{x_n}{x_{n+1}} - 1\Big) = \lim_{n\to\infty} n\Big(\frac{a+n+1}{n+1} - 1\Big) = a.$$

Hence if $a > 1$ the series converges; and if $a < 1$ the series diverges. When $a = 1$ the series is $\displaystyle\sum_{n\geq 0} \frac{1}{n+1} = \infty$. □

A series $\sum_{n\geq 1} x_n$ is called an *alternating series* if $x_n x_{n+1} \leq 0$, $\forall n \in \mathbb{N}$. A fundamental class of alternating series are series of the form $\sum_{n\geq 1}(-1)^n a_n$ or $\sum_{n\geq 1}(-1)^{n+1}a_n$, with $a_n > 0$.

Example 3.18. The series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges to $\ln 2$.

Proof. Let us prove convergence by considering the partial sums $s_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. Notice that $s_{2k+2} - s_{2k} = \frac{1}{2k+1} - \frac{1}{2k+2} > 0$ and that $s_{2k+3} - s_{2k+1} = \frac{1}{2k+3} - \frac{1}{2k+2} < 0$. This means that the subsequence (s_{2k}) is increasing, while the subsequence (s_{2k+1}) is decreasing. Notice also that $s_{2k+1} - s_{2k} = \frac{1}{2k+1}$ and $s_{2k} < s_{2k+1}$, so both subsequences are also bounded and converge to the same limit. To find the sum of the alternating series, recall (from the seminar) that

$$\lim_{n\to\infty} 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n = \gamma \in (0,1), \text{ hence}$$

$$\begin{aligned} s_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{2n} = 1 + \frac{1}{2} + \ldots + \frac{1}{2n} - 2\left(\frac{1}{2} + \ldots + \frac{1}{2n}\right) \\ &= \underbrace{1 + \frac{1}{2} + \ldots + \frac{1}{2n} - \ln(2n)}_{\to\gamma} - \underbrace{\left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n\right)}_{\to\gamma} + \ln 2 \longrightarrow \ln 2. \end{aligned}$$

□

Theorem 3.19 (Leibniz test)**.** Let (x_n) be a decreasing sequence with $x_n \longrightarrow 0$. Then the series $\sum_{n\geq 1}(-1)^n x_n$ is convergent.

Proof. Consider the partial sum $s_n = \sum_{k=1}^{n}(-1)^k x_k$. We will prove that (s_n) is convergent by showing that it is a Cauchy sequence. For $n, p \in \mathbb{N}$ consider

$$\begin{aligned} |s_{n+p} - s_n| &= |(-1)^{n+1}x_{n+1} + \ldots + (-1)^{n+p}x_{n+p}| \\ &= |\underbrace{x_{n+1} - x_{n+2}}_{\geq 0} + \underbrace{x_{n+3} - x_{n+4}}_{\geq 0} + \ldots + (-1)^{p-2}x_{n+p-1} + (-1)^{p-1}x_{n+p}| \\ &= x_{n+1} \underbrace{-x_{n+2} + x_{n+3}}_{\leq 0} - x_{n+4} + \ldots \pm x_{n+p-1} \mp x_{n+p} \\ &\leq x_{n+1}. \end{aligned}$$

Since $x_n \longrightarrow 0$, $|s_{n+p} - s_n|$ can be made arbitrarily small, so (s_n) is Cauchy. □

Definition 3.20. A series $\sum_{n\geq 1} x_n$ is called *absolutely convergent* if $\sum_{n\geq 1} |x_n|$ is convergent.

Proposition 3.21. Any absolutely convergent series is also convergent.

Proof. If $\sum_{k=1}^{n} |x_k|$ gives a Cauchy sequence, then $\sum_{k=1}^{n} x_k$ also gives a Cauchy sequence. □

Theorem 3.22 (Cauchy)**.** Let $\sum_{n\geq 1} x_n$ be an *absolutely convergent series* and let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n\geq 1} x_{\sigma(n)}$ is also absolutely convergent and $\sum_{n\geq 1} x_{\sigma(n)} = \sum_{n\geq 1} x_n$. In other words, any rearrangement of an absolutely convergent series has the same sum.

Proof. (Optional) See [2][Theorem 7.4.3]. □

Definition 3.23. A series $\sum_{n\geq 1} x_n$ is called *conditionally convergent* (or semi-convergent) if $\sum_{n\geq 1} x_n$ converges, but $\sum_{n\geq 1} |x_n|$ diverges.

Theorem 3.24 (Riemann)**.** Let $\sum_{n\geq 1} x_n$ be a *conditionally convergent series* and let $x \in \overline{\mathbb{R}}$. Then there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n\geq 1} x_{\sigma(n)} = x$. In other words, a conditionally convergent series can be rearranged to converge to any value or diverge to $\pm\infty$.

Proof. (Optional) See [2][Theorem 8.2.8]. □

Example 3.25. Rearranging the terms in the alternating harmonic series one can obtain a different sum. Indeed, consider $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2$, and reorder the terms in the following way: one positive, two negative. Then

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \ldots = \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots) = \frac{1}{2}\ln 2.$$

Definition 3.26. Let (a_n) be a sequence of real numbers and let $c \in \mathbb{R}$. The series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

is called a *power series* centered at c.

Theorem 3.27. Consider the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$. There exists a unique $R \in [0, \infty]$, called the *radius of convergence* of the power series, such that the power series

- converges absolutely when $|x-c| < R$.
- diverges when $|x-c| > R$.

Theorem 3.28. If the limit

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \in [0, \infty]$$

exists, then the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ has the radius of convergence

$$R = \begin{cases} \frac{1}{L}, & \text{if } L \in (0, \infty) \\ 0, & \text{if } L = \infty \\ \infty, & \text{if } L = 0. \end{cases}$$

Proof. It follows from the root test for series with positive terms. □

Corollary 3.29. If the limit

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \in [0, \infty]$$

exists, then the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ has the radius of convergence

$$R = \begin{cases} \frac{1}{L}, & \text{if } L \in (0, \infty) \\ 0, & \text{if } L = \infty \\ \infty, & \text{if } L = 0. \end{cases}$$

Proof. It follows from $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$. □

Definition 3.30. The convergence set of a power series is

$$C := \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n (x-c)^n \text{ converges}\}.$$

Remark 3.31. The convergence set C contains the open interval $(c-R, c+R)$ and possibly the endpoints $\{c-R, c+R\}$.

Example 3.32. The power series $\sum_{n\geq 0} x^n$ has radius of convergence $R = 1$, it converges absolutely for $|x| < 1$ and diverges when $|x| > 1$ (by the root test or the ratio test). The convergence set is $(-1, 1)$ and for $x \in (-1, 1)$ we have that

$$\sum_{n\geq 0} x^n = \frac{1}{1-x}, \quad \sum_{n\geq 0} (-x)^n = \frac{1}{1+x}.$$

Example 3.33. The power series $\sum_{n\geq 1} \frac{x^n}{n}$ has radius of convergence $R = 1$, it converges absolutely for $|x| < 1$ and diverges when $|x| > 1$ (by the root test or the ratio test). Moreover, the series converges for $x = -1$ (alternating harmonic series) and diverges for $x = 1$ (harmonic series), hence its convergence set is $C = [-1, 1)$.

Theorem 3.34. Consider a power series with radius of convergence R, given by

$$s(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

Then for any $x \in (c-R, c+R)$, the power series can be differentiated term by term and

$$s'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1},$$

and for any $t \in (c-R, c+R)$ the power series can be integrated term by term

$$\int_c^t s(x)\mathrm{dx} = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (t-c)^{n+1}.$$

Example 3.35. The power series $\sum_{n\geq 1} \frac{x^n}{n!}$ converges absolutely for any $x \in \mathbb{R}$ (ratio test). Let $\exp(x) := \sum_{n\geq 1} \frac{x^n}{n!}$ and differentiate term by term, then $\exp'(x) = \exp(x)$ and $\exp(0) = 1$.

※ Limits, continuity, differentiability

Definition 4.1. Let $A \subseteq \mathbb{R}$. We say that $x_0 \in \overline{\mathbb{R}}$ is an *accumulation point* (or cluster point) if

$$\forall V \in \mathcal{V}(x_0),\ V \cap \big(A \setminus \{x_0\}\big) \neq \emptyset.$$

We denote by A' the set of the accumulation points of A. We say that $x_0 \in A$ is an *isolated point* if $x_0 \in A \setminus A'$.

Remark 4.2. $\mathrm{cl}(A) = A' \cup \{\text{isolated points}\}$.

Proposition 4.3. Let $A \subseteq \mathbb{R}$ and $x_0 \in \overline{\mathbb{R}}$, then $x_0 \in A'$ if and only if there exists a sequence (x_n) in $A \setminus \{x_0\}$ such that $\lim\limits_{n\to\infty} x_n = x_0$.

Proof. Assume that $x_0 \in A'$, with $x_0 \in \mathbb{R}$, and consider the neighborhoods $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$. Then each neighborhood must contain an $x_n \in A \setminus \{x_0\}$ with $|x_n - x_0| < \frac{1}{n}$, hence $x_n \to x_0$. If x_0 is infinite, the neighborhoods can be taken $(-\infty, -n)$ or (n, ∞), respectively.

Assume now that there exists a sequence (x_n) in $A \setminus \{x_0\}$ such that $\lim\limits_{n\to\infty} x_n = x_0$. Then for any $V \in \mathcal{V}(x_0)$, there exists $N_V \in N$ such that $x_n \in V$, for any $n \geq N_V$. In particular, $x_{N_V} \in V \cap \big(A \setminus \{x_0\}\big)$, for any $V \in \mathcal{V}(x_0)$, hence $x_0 \in A'$. □

Example 4.4. For $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, each element $x \in A$ in an isolated point and $A' = \{0\}$.

Definition 4.5. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A'$. We say that $\lim\limits_{x\to x_0} f(x) = \ell \in \overline{\mathbb{R}}$ if

$$\forall V \in \mathcal{V}(\ell),\ \exists U \in \mathcal{V}(x_0) \text{ s.t. } f(x) \in V,\ \forall x \in U \cap \big(A \setminus \{x_0\}\big).$$

Remark 4.6 (ε–δ)**.** Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A'$ finite. If $\lim\limits_{x\to x_0} f(x) = \ell \in \mathbb{R}$, then

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - \ell| < \varepsilon,\ \forall x \in A \text{ with } |x - x_0| < \delta.$$

Theorem 4.7. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A'$. Then $\lim\limits_{x\to x_0} f(x) = \ell \in \overline{\mathbb{R}}$ iff

$$\text{for any sequence } (x_n) \text{ in } A \setminus \{x_0\} \text{ with } \lim_{n\to\infty} x_n = x_0, \text{ we have that } \lim_{n\to\infty} f(x_n) = \ell.$$

Theorem 4.8. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $x_0 \in \mathbb{R}$ s.t. $x_0 \in \big(A \cap (-\infty, x_0)\big)'$ and $x_0 \in \big(A \cap (x_0, \infty)\big)'$. Then

$$\lim_{x\to x_0} f(x) = \ell \text{ iff } \lim_{\substack{x\to x_0\\ x<x_0}} f(x) = \lim_{\substack{x\to x_0\\ x>x_0}} f(x) = \ell.$$

Example 4.9. (a) $\mathrm{sgn} : \mathbb{R} \to \mathbb{R}$, $\mathrm{sgn}(x) = \begin{cases} +1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0. \end{cases}$ has no limit at 0.

(b) $f : \mathbb{R}^* \to \mathbb{R}$, $f(x) = \sin(\frac{1}{x})$ has no limit at 0 since $f(\frac{1}{2n\pi}) = 0$, $f(\frac{1}{2n\pi+\pi/2}) = 1$.

(c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ has no limit at any $x \in R$.

Definition 4.10. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A$. We say that f is *continuous* at x_0 if

$$\forall V \in \mathcal{V}(f(x_0)),\ \exists U \in \mathcal{V}(x_0) \text{ s.t. } f(x) \in V,\ \forall x \in U \cap A.$$

Remark 4.11. If $x_0 \in A \cap A'$ is an accumulation point, then f is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Remark 4.12. If x_0 is an isolated point of A, then $\exists U \in \mathcal{V}(x_0)$ with $U \cap A = \{x_0\}$, and since $f(x_0) \in V$, $\forall V \in \mathcal{V}(f(x_0))$, we have that f is continuous at x_0.

Theorem 4.13. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$. The following are equivalent:

(a) f is continuous at x_0.

(b) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$, $\forall x \in A$ with $|x - x_0| < \delta$.

(c) for any sequence (x_n) in A with $\lim_{n\to\infty} x_n = x_0$, we have that $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Remark 4.14. Elementary operations – e.g. sums, products or compositions – of continuous functions are continuous (when defined).

Definition 4.15. For $f : A \to \mathbb{R}$ denote by $f(A) := \{f(x) \mid x \in A\}$ the image of A. We say that f is *bounded* if $f(A)$ is *bounded*, i.e. $\inf\big(f(A)\big), \sup\big(f(A)\big)$ are finite.

Theorem 4.16 (Weierstrass)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded and it attains its bounds, i.e. there exist $\min\big(f(A)\big), \max\big(f(A)\big)$.

Proof. Let us first prove that f is bounded. Assuming that this is not the case, we have that for any $n \in \mathbb{N}$ there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. Since the sequence (x_n) is bounded, we have that it has a convergent subsequence (x_{n_k}), see theorem 2.8; denote its limit by x. We have that $x_{n_k} \to x$ and f is continuous, hence $f(x_{n_k}) \to f(x)$. But $|f(x_{n_k})| > n_k \to \infty$, contradiction. Hence f is bounded on $[a, b]$.

To prove that f attains its bounds, let's consider the upper bound and show that there exists $x_M \in [a, b]$ such that $f(x_M) = \sup\big(f(A)\big)$, i.e. $f(x_M) = \max\big(f(A)\big) = \sup\big(f(A)\big)$. From theorem 1.7, we obtain a sequence (x_n) in $[a, b]$ such that $f(x_n) \to \sup\big(f(A)\big)$. Since the sequence (x_n) is bounded, it has a convergent subsequence (x_{n_k}); let's call its limit $x_M \in [a, b]$. Since f is continuous, it follows that $f(x_{n_k}) \to f(x_M)$, but we know that $f(x_{n_k}) \to \sup\big(f(A)\big)$, hence $f(x_M) = \sup\big(f(A)\big)$ and f reaches its upper bound. $\square$

Theorem 4.17 (Intermediate value property)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f has the intermediate value property, i.e. if $y \in \mathbb{R}$ is in between $f(a)$ and $f(b)$, there exists $c \in (a, b)$ such that $f(c) = y$.

Proof. Assume that $f(a) < y < f(b)$ and consider the set $S := \{x \in [a, b] \mid f(x) \leq y\}$. Take

$$c := \sup(S)$$

Let $\varepsilon > 0$, then $\exists \delta > 0$ such that $|f(x) - f(c)| < \varepsilon$, whenever $|x - c| < \delta$. Since $c = \sup(S)$, we have from theorem 1.7 that there exists $x_1 \in S$ such that $c - \delta < x_1 \leq c$. From continuity we have that $f(c) < f(x_1) + \varepsilon \leq y + \varepsilon$. Also, for $x_2 \in (c, c + \delta)$, we have from continuity that $f(c) > f(x_2) - \varepsilon$. From the definition of the supremum, $x_2 \notin S$ hence $f(x_2) > y$ and $f(c) > y - \varepsilon$. We conclude that $y - \varepsilon < f(c) < y + \varepsilon$, for any $\varepsilon > 0$. Hence $f(c) = y$. □

Definition 4.18. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$. The *derivative* of f at x_0 is

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \overline{\mathbb{R}}$$

If $f'(x_0) \in \mathbb{R}$ (finite) we say that f is *differentiable* at x_0.

Remark 4.19. $f'(x_0)$ represents the gradient of the tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$. The equation of the tangent is $f(x) - f(x_0) = f'(x_0)(x - x_0)$.

Theorem 4.20. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$. If f is differentiable at x_0, then f is continuous at x_0.

Proof. Since $f(x) = f(x_0) + \frac{f(x) - f(x_0)}{x - x_0}(x - x_0)$, we have that $\lim\limits_{x \to x_0} f(x) = f(x_0) + 0 = f(x_0)$. □

Example 4.21. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is not differentiable in 0 since $\nexists \lim\limits_{x \to 0} \dfrac{|x|}{x}$.

Theorem 4.22 (Calculus Rules)**.**

- $(cf)'(x) = cf'(x)$, for any constant $c \in \mathbb{R}$.
- $(f + g)'(x) = f'(x) + g'(x)$.
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$. (Product Rule)
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$. (Quotient Rule)
- $(f \circ g)'(x) = f'\big(g(x)\big)g'(x)$. (Chain Rule)

Proposition 4.23 (l'Hôpital's rule)**.** Let I be an open interval, $x_0 \in \overline{\mathbb{R}}$ and $f, g : I \setminus \{x_0\} \longrightarrow \mathbb{R}$ differentiable. If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$ or $\pm\infty$, and $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \in \overline{\mathbb{R}}$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Definition 4.24. $f : A \longrightarrow \mathbb{R}$ has a local extremum (minimum or maximum) at $x_0 \in A$ if

$$\exists V \in \mathcal{V}(x_0) \text{ s.t. } f(x_0) \leq f(x) \text{ or } f(x_0) \geq f(x),\ \forall x \in V \cap A.$$

Theorem 4.25 (Fermat)**.** Let $f : (a, b) \longrightarrow \mathbb{R}$ and $x_0 \in (a, b)$. If f is differentiable at x_0 and x_0 is a local extremum, then $f'(x_0) = 0$.

Proof. The lateral derivatives at x_0 are equal. Since x_0 is a local extremum, one of them is ≥ 0, the other ≤ 0. Hence $f'(x_0) = 0$. □

Theorem 4.26 (Rolle)**.** Let $f : (a, b) \longrightarrow \mathbb{R}$ with $f(a) = f(b)$. If is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ s.t. $f'(c) = 0$.

Proof. Since f is continuous, it is bounded and it attains its bounds. Denote by x_m and x_M the minimum and maximum points of f on $[a, b]$. If at least one of x_m and x_M belongs to (a, b), then $f'(x_m) = 0$ or $f'(x_M) = 0$. Otherwise, $x_m, x_M \in \{a, b\}$ and $f(x_m) = f(x_M)$, hence the function is constant and its derivative is zero on (a, b). □

Theorem 4.27 (Mean value theorem)**.** Let $f : (a, b) \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Consider the function $g : (a, b) \longrightarrow \mathbb{R}$, $g(x) := f(x) - x\frac{f(b)-f(a)}{b-a}$. Since $g(a) = g(b)$, the conclusion follows from Rolle's theorem. □

Theorem 4.28 (Monotony)**.** Let $f : (a, b) \longrightarrow \mathbb{R}$ be differentiable on (a, b). Then

$$f \text{ is increasing iff } f' \geq 0,$$

$$f \text{ is decreasing iff } f' \leq 0.$$

Proof. $\Longrightarrow$ follows from the definition of the derivative; $\Longleftarrow$ from the mean value theorem. □

※ Taylor series

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$ a point where f is differentiable n times ($n \in \mathbb{N}$). Does there exist a polynomial $P : \mathbb{R} \to \mathbb{R}$ that matches the function f and all its derivatives up to order n at the point x_0? That is

$$\begin{aligned} P(x_0) &= f(x_0) \\ P'(x_0) &= f'(x_0) \\ P''(x_0) &= f''(x_0) \\ &\vdots \\ P^{(n)}(x_0) &= f^{(n)}(x_0). \end{aligned}$$

Let us look for P of degree at most n of the form

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n.$$

By imposing the conditions at x_0 and differentiating P we have that

$$P(x_0) = a_0 = f(x_0),\ P'(x_0) = a_1 = f'(x_0),\ P''(x_0) = 2a_2 = f''(x_0),\ \ldots,\ P^{(n)}(x_0) = n!a_n = f^{(n)}(x_0).$$

We thus see that there exists a unique such polynomial P of degree at most n given by

$$P(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

that matches the function f and all its derivatives up to order n at the point x_0.

Definition 5.1. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$ where f is differentiable n times. The polynomial $T_n : \mathbb{R} \to \mathbb{R}$,

$$T_n(x) := f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the *Taylor polynomial* of degree n centered around x_0. □

The Taylor polynomial T_n gives a good approximation of f around x_0, i.e. when $x \approx x_0$,

$$f(x) \approx T_n(x).$$

The simplest approximations are: the *linear approximation* of f around x_0 given by T_1, i.e.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

and the *quadratic approximation* of f around x_0 given by T_2, i.e.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

The closer x is to x_0 and the higher the degree of T_n is, the better $T_n(x)$ approximates $f(x)$.

Example 5.2. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ and $x_0 = 0$. Then $f(0) = f'(0) = \ldots = f^{(n)}(0) = 1$ and

$$T_n(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!}.$$

Definition 5.3. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$ where f is differentiable n times. We define $R_n : \mathbb{R} \to \mathbb{R}$ to be the remainder when approximating f by T_n,

$$R_n(x) := f(x) - T_n(x).$$

Theorem 5.4 (Taylor-Lagrange)**.** Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ differentiable $n+1$ times. Then for any $x, x_0 \in I$, there exists $c \in (x_0, x)$ or $c \in (x, x_0)$ such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$

called *the remainder in Lagrange's form*. Taylor's formula with Lagrange remainder is

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Remark 5.5. There exist other forms of the remainder, but we will only use this one. Its main advantage is that assuming that all the derivatives of f are bounded by $M > 0$,

$$|f(x) - T_n(x)| = |R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1} \to 0 \text{ as } n \to \infty.$$

Definition 5.6. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be infinitely differentiable. For $x_0 \in I$ and $x \in \mathbb{R}$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the *Taylor series* of f around x_0. If the series converges and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

we say that f can be expanded in a Taylor series around x_0 (also called Taylor expansion).

Remark 5.7. The partial sum of a Taylor series is the Taylor polynomial $T_n(x)$. A Taylor series converges to $f(x)$ if and only if the remainder $f(x) - T_n(x) = R_n(x) \to 0$ as $n \to \infty$.

Remark 5.8. The Taylor series around 0 is called the *MacLaurin series*,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Example 5.9. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ and $x_0 = 0$. Then

$$T_n(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!}.$$

Consider Taylor's formula $f(x) = T_n(x) + R_n(x)$ with the Lagrange remainder, for which there exists c in between 0 and x such that

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1} \to 0$$

since $\frac{|x|^n}{n!} \to 0$ as $n \to \infty$. It follows that e^x can be expanded as a Taylor series around 0:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x}{2} + \ldots + \frac{x^n}{n!} + \ldots, \forall x \in \mathbb{R}.$$

Example 5.10. The functions sin and cos can be expanded in a Taylor series around 0.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots.$$

Example 5.11. The function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is infinitely differentiable at 0, but f is not expandable in a Taylor series around 0.

Example 5.12 (Convex/concave)**.** Let $f : I \to R$ be two times differentiable, with a critical point at x_0, i.e. $f'(x_0) = 0$. Then from Taylor's formula we have that

$$f(x) = f(x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + R_2(x).$$

When x is very close to x_0, the quadratic approximation is very accurate and the remainder $R_2(x)$ is very small. Thus the behaviour of $f(x)$ around x_0 is dictated by the quadratic term $f''(x_0)(x - x_0)^2$ and we see that:

- If $f''(x_0) > 0$ (convexity), then $f(x) > f(x_0)$ and x_0 is a local minimum.
- If $f''(x_0) < 0$ (concavity), then $f(x) < f(x_0)$ and x_0 is a local maximum.

Theorem 5.13 (Local optimality conditions)**.** Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ and $x_0 \in I$ a point where f is differentiable n times and

$$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0 \text{ and } f^{(n)}(x_0) \neq 0.$$

1. If n is even and $f^{(n)}(x_0) > 0$, then x_0 is a *local minimum* of f.
2. If n is even and $f^{(n)}(x_0) < 0$, then x_0 is a *local maximum* of f.
3. If n is odd, then x_0 is not a local extremum point of f.

Proof. It follows from the Taylor approximation $f(x) = f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$. □

※ Riemann integrals. Improper integrals

Let $[a, b]$ be a compact interval and let $f : [a, b] \to \mathbb{R}$. The points $a = x_0 < x_1 < \ldots < x_n = b$ define a partition of the interval $[a, b]$

$$\mathcal{P} = \{[x_{k-1}, x_k] \mid k = \overline{1, n}\},$$

whose norm is given by

$$\|\mathcal{P}\| = \max_{k=\overline{1,n}}\{x_k - x_{k-1}\}.$$

Consider also a set of intermediate points $c_k \in [x_k, x_{k-1}]$ attached to the partition $\mathcal{P}$.

Definition 6.1. For $f : [a, b] \to \mathbb{R}$ and a partition $\mathcal{P}$ of $[a, b]$, the Riemann sum is given by

$$\sigma(f, \mathcal{P}) := \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}).$$

Remark 6.2. The Riemann sum collects the areas of the rectangles defined by the partition $\mathcal{P}$ (and the intermediate points). In the limit one obtains the area below the curve $y = f(x)$.

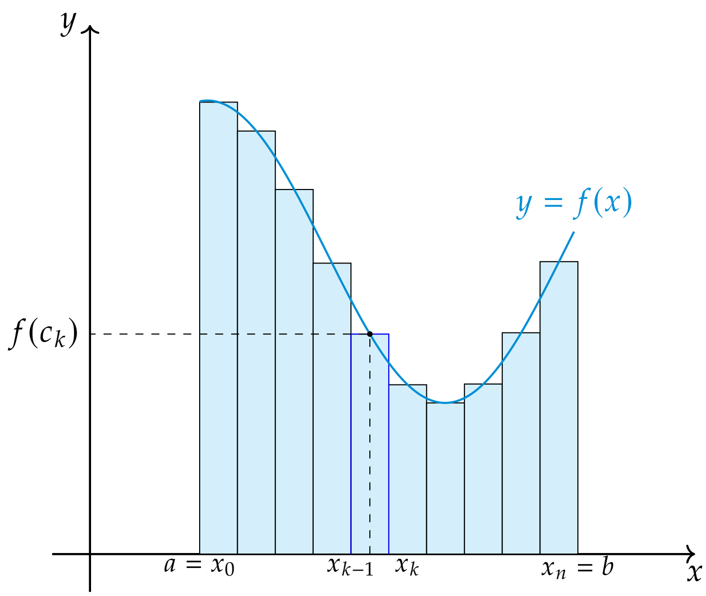


Figure 1: Area under a curve approximated through rectangles. Riemann sum.

Definition 6.3. We say that $f : [a, b] \to \mathbb{R}$ is *Riemann integrable* if there exists $I \in \mathbb{R}$ s.t. for any partition $\mathcal{P}$ of $[a, b]$ the Riemann sum $\sigma(f, \mathcal{P})$ converges to I as $\|\mathcal{P}\| \to 0$, i.e.

$$\lim_{\|\mathcal{P}\|\to 0} \sigma(f, \mathcal{P}) = I =: \int_a^b f(x)\,\mathrm{d}x.$$

Proposition 6.4. Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable and $\alpha \in \mathbb{R}$. Then

- $\displaystyle\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x.$
- $f + g$ is Riemann integrable and $\displaystyle\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x.$
- If $f \leq g$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$

Proposition 6.5. Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $c \in (a,b)$. Then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

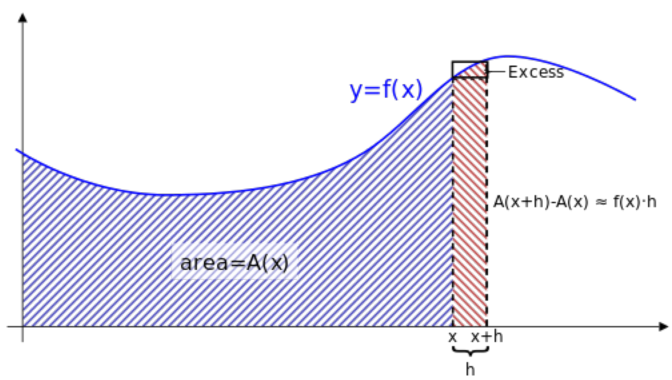


Figure 2: The derivative of the area function A is f. Source: wikipedia.

Theorem 6.6. Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. Then the function $A : [a,b] \to \mathbb{R}$, $A(x) := \displaystyle\int_a^x f(t)\,\mathrm{d}t$ is continuous. Furthermore, if f is continuous, then A is differentiable and $A'(x) = f(x)$.

Theorem 6.7 (Fundamental theorem of calculus)**.** Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $F : [a,b] \to \mathbb{R}$ an antiderivative (primitive) of f, i.e. $F' = f$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Definition 6.8 (Trapezium rule)**.** Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and consider $a = x_0 < x_1 < \ldots < x_n = b$. The area below the curve $y = f(x)$ can be approximated by

$$\sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2}(x_k - x_{k-1}).$$

Note that $\frac{f(x_{k-1})+f(x_k)}{2}(x_k - x_{k-1})$ is the area of the trapezium determined by $x_{k-1}, x_k, f(x_{k-1}), f(x_k)$. In the case of a uniform partition with $x_k - x_{k-1} = \frac{b-a}{n}$, $\forall k \in \overline{1,n}$, we have that

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{n}\left(\tfrac{1}{2}f(a) + \sum_{i=1}^{n-1} f(x_k) + \tfrac{1}{2}f(b)\right).$$

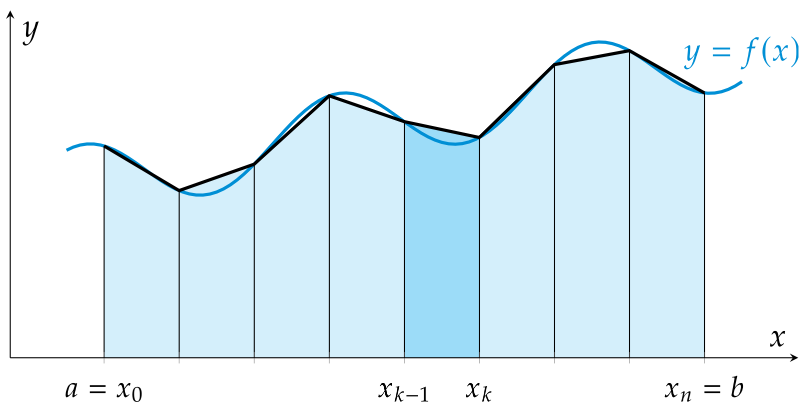


Figure 3: Trapezium rule.

Definition 6.9. Let $a, b \in \mathbb{R}$. If the following limits exist, we define the *improper integrals*

- If $f : [a, \infty) \to \mathbb{R}$ is Riemann integrable on any compact interval in the domain,

$$\int_a^\infty f(x)\,\mathrm{d}x := \lim_{t\to\infty} \int_a^t f(x)\,\mathrm{d}x.$$

- If $f : [a, b) \to \mathbb{R}$ is Riemann integrable on any compact interval included in the domain,

$$\int_a^{b-0} f(x)\,\mathrm{d}x := \lim_{t\nearrow b} \int_a^t f(x)\,\mathrm{d}x.$$

- If $f : (a, b] \to \mathbb{R}$ is Riemann integrable on any compact interval included in the domain,

$$\int_{a+0}^{b} f(x)\,\mathrm{d}x := \lim_{t\searrow a} \int_t^b f(x)\,\mathrm{d}x.$$

The notation $\int_a^{b-0} \ldots$, $\int_{a+0}^{b} \ldots$ emphasizes that the integrals are improper, but we can also simply write $\int_a^b \ldots$ even when dealing with an improper integral.

Definition 6.10. We say that an improper integral is convergent if it is finite (finite limit).

Note that an improper integral represents the area of an infinite region.

Example 6.11. Let $a > 0$ and $p \in \mathbb{R}$. The improper integral

$$\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x$$

converges when $p > 1$ and diverges when $p \le 1$. Indeed, for $p = 1$ the integral diverges ($\ln(\infty)$) and for $p \neq 1$,

$$\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x = \lim_{t\to\infty}\frac{t^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1},$$

which converges when $-p+1 < 0$, i.e. $p > 1$, and diverges when $p < 1$.

Example 6.12. Let $0 < a < b$ and $p \in \mathbb{R}$. The improper integrals

$$\int_a^b \frac{1}{(b-x)^p}\,\mathrm{d}x, \int_a^b \frac{1}{(x-a)^p}\,\mathrm{d}x$$

converge when $p < 1$ and diverge when $p \ge 1$. Indeed, for $p = 1$ the integrals diverge ($\ln(0)$) and for $p \neq 1$ the first integral, for example, is

$$\int_a^b \frac{1}{(b-x)^p}\,\mathrm{d}x = -\lim_{t\nearrow b}\frac{(b-t)^{-p+1}}{-p+1} + \frac{(b-a)^{-p+1}}{-p+1},$$

which converges when $-p+1 > 0$, i.e. $p < 1$, and diverges when $p > 1$.

Theorem 6.13. Let $a < b \le \infty$ and $f, g : [a,b) \to [0,\infty)$. If there exists $c \in (a,b)$ s.t. $f(x) \le g(x), \forall x \ge c$, then

- If $\int_a^b g(x)\,\mathrm{d}x$ converges, then $\int_a^b f(x)\,\mathrm{d}x$ converges.
- If $\int_a^b f(x)\,\mathrm{d}x$ diverges, then $\int_a^b g(x)\,\mathrm{d}x$ diverges.
- If $\lim\limits_{x\nearrow b}\dfrac{f(x)}{g(x)} \in (0,\infty)$, then $\int_a^b f(x)\,\mathrm{d}x$ and $\int_a^b g(x)\,\mathrm{d}x$ have the same nature.

Theorem 6.14 (Integral test for series)**.** Let $f : [1,\infty) \to [0,\infty)$ be decreasing, then $\int_1^\infty f(x)\,\mathrm{d}x$ and $\sum\limits_{n=1}^{\infty} f(n)$ have the same nature.

Proof. Let $N \in \mathbb{N}$ and write $\int_1^N f(x)\,\mathrm{d}x = \sum\limits_{n=1}^{N-1}\int_n^{n+1} f(x)\,\mathrm{d}x$. Since f is decreasing we have that

$$\sum_{n=1}^{N-1} f(n+1) \le \int_1^N f(x)\,\mathrm{d}x \le \sum_{n=1}^{N-1} f(n).$$

The conclusion follows by letting $N \to \infty$ and using the comparison test. □

※ The Euclidean space $\mathbb{R}^n$

Elements in R^n are vectors with n components. We will write $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ most of the time, apart from situations where matrices will also be involved – in this case we will adopt the linear algebra notation of writing $x \in \mathbb{R}^n$ as a column vector $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$, since this allows to multiply matrices $[\quad]_{m\times n}$ with vectors $[\quad]_{n\times 1}$ to get vectors $[\quad]_{m\times 1}$.

As you've seen in your Algebra course, $\mathbb{R}^n$ is a vector space: two vectors $x, y \in \mathbb{R}^n$ can be added component wise $x + y := (x_1 + y_1, \ldots, x_n + y_n)$, and a vector can be multiplied by a scalar $\alpha \in \mathbb{R}$ to get $\alpha x := (\alpha x_1, \ldots, \alpha x_n)$. We will denote by e_i the canonical basis vector with a 1 in the ith component and 0's everywhere else, giving $x = x_1 e_1 + \ldots x_n e_n$.

Definition 7.1. A map $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is called a *scalar product (or inner product)* if

(a) $\langle x, y\rangle = \langle y, x\rangle$, for any $x, y \in \mathbb{R}^n$.

(b) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$, for any $x, y, z \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.

(c) $\langle x, x\rangle > 0$, for any $x \in \mathbb{R}^n \setminus \{0\}$.

Example 7.2. A symmetric positive definite matrix $M \in \mathbb{R}^{n\times n}$ defines a scalar product $\langle x, y\rangle = x^T M y$.

Definition 7.3. The *dot product* of two vectors $x, y \in \mathbb{R}^n$ is given by

$$x \cdot y := x_1 y_1 + \ldots + x_n y_n.$$

The dot product is the most important scalar product. In matrix notation, it is written as

$$x \cdot y = x^T y = [x_1 \ldots x_n]_{1\times n} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1} = x_1 y_1 + \ldots + x_n y_n.$$

Definition 7.4. Two vectors $x, y \in \mathbb{R}^n$ are perpendicular (or orthogonal) iff $x \cdot y = 0$.

Definition 7.5. A function $\| \cdot \| : \mathbb{R}^n \longrightarrow [0, \infty)$ is called a *norm* if

(a) $\|x\| = 0$ if and only if $x = 0$.

(b) $\|\alpha x\| = |\alpha| \|x\|$, for any $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

(c) $\|x + y\| \leq \|x\| + \|y\|$, for any $x, y \in \mathbb{R}^n$ (triangle inequality).

Proposition 7.6. Any scalar product generates a norm on $\mathbb{R}^n$ given by $\|x\| = \sqrt{\langle x, x\rangle}$.

Theorem 7.7 (Cauchy-Schwarz inequality)**.** For any $x, y \in \mathbb{R}^n$ it holds that

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Here the norm $\|\cdot\|$ is generated by the scalar product $\langle\cdot,\cdot\rangle$.

Proof. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(t) = \langle tx + y, tx + y\rangle = \|tx + y\|^2 \geq 0$. Since $f(t) = t^2\|x\|^2 + 2t\langle x, y\rangle + \|y\|^2$ is a quadratic in t, we must have that $\Delta = 4\langle x, y\rangle^2 - 4\|x\|^2\|y\|^2 \leq 0$ □

Definition 7.8. The Euclidean norm is generated by the dot product and it is given by

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \ldots + x_n^2}.$$

This represents the length of the vector $x \in \mathbb{R}^n$ measured using the Euclidean norm.

Theorem 7.9. For $n \in \{2, 3\}$ the dot product of $x, y \in \mathbb{R}^n$ is

$$x \cdot y = \|x\|\|y\| \cos \angle(x, y).$$

Proof. Consider the triangle with sides determined by the vectors x, y and $x - y$. From the cosine rule we have that

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos \angle(x, y).$$

Since $\|x-y\|^2 = (x-y)\cdot(x-y) = x\cdot x+y\cdot y-2x\cdot y$, we obtain that $x\cdot y = \|x\|\|y\| \cos \angle(x, y)$. □

Example 7.10. (a) $\|x\|_1 := |x_1| + \ldots + |x_n|$ is a norm (so-called Manhattan norm).

(b) $\|x\|_p := \left(|x_1|^p + \ldots + |x_n|^p\right)^{\frac{1}{p}}$, $p > 1$, is a norm.

(c) $\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$ is a norm.

Definition 7.11. A function $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is called a *distance (or metric)* if

(a) $d(x, y) = 0$ if and only if $x = y$.

(b) $d(x, y) = d(y, x)$, for any $x, y \in \mathbb{R}^n$.

(c) $d(x, z) \leq d(x, y) + d(y, z)$, for any $x, y, z \in \mathbb{R}^n$ (triangle inequality).

Proposition 7.12. Any norm generates a distance on $\mathbb{R}^n$ given by $d(x, y) = \|x - y\|$.

Definition 7.13. The Euclidean distance is generated by the Euclidean norm and it is given by

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$

We will be using the Euclidean norm and distance, unless we specify otherwise.

Neighborhoods. Interior. Closure. Boundary.

Definition 7.14. A set $A \subseteq \mathbb{R}^n$ is called *bounded* if there exists $r > 0$ such that

$$\|x\| \leq r, \ \forall x \in A.$$

Definition 7.15. Let $x_0 \in \mathbb{R}^n$ and $r > 0$. The open ball of centre x_0 and radius r is given by

$$B(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\},$$

and the closed ball of centre x_0 and radius r is given by

$$\overline{B}(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}.$$

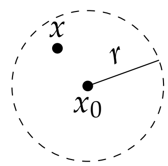


Figure 4: Open ball $B(x_0, r)$.

Definition 7.16. A set $V \subseteq \mathbb{R}^n$ is a *neighborhood (vecinity)* of $x \in \mathbb{R}^n$ if

$$\exists r > 0 \text{ such that } B(x, r) \subseteq V.$$

We denote all the neighborhoods of x by $\mathcal{V}(x) := \{V \subseteq \mathbb{R}^n \mid V \text{ is a neighborhood of } x\}$.

Definition 7.17. Let $A \subseteq \mathbb{R}^n$. The following set is called the *interior* of A

$$\operatorname{int}(A) := \{x \in \mathbb{R}^n \mid \exists V \in \mathcal{V}(x) \text{ such that } V \subseteq A\},$$

the following set is called the *closure* of A

$$\operatorname{cl}(A) := \{x \in \mathbb{R}^n \mid \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\},$$

and the following set is called the *boundary* of A

$$\operatorname{bd}(A) := \{x \in \mathbb{R}^n \mid \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset \text{ and } V \cap A^c \neq \emptyset\}.$$

Example 7.18. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Then

$$\begin{aligned}
\operatorname{int}(A) &= A, \\
\operatorname{cl}(A) &= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}, \\
\operatorname{bd}(A) &= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.
\end{aligned}$$

Proposition 7.19. For any $A \subseteq \mathbb{R}^n$, it holds that $\mathrm{cl}(A) = A \cup \mathrm{bd}(A)$.

Definition 7.20. If $A = \mathrm{int}(A)$, then A is called *open*. If $A = \mathrm{cl}(A)$, then A is called *closed*.

Proposition 7.21. For any $A \subseteq \mathbb{R}^n$, it holds that $\mathrm{int}(A) \subseteq A \subseteq \mathrm{cl}(A)$.

Proof. Similar to proposition 1.12. □

Remark 7.22. To prove that a set A is open, it is sufficient to prove that $A \subseteq \mathrm{int}(A)$.
To prove that a set A is closed, it is sufficient to prove that $\mathrm{cl}(A) \subseteq A$.

Proposition 7.23. The following statements are true:

- The complement of an open set is closed.
- The complement of a closed set is open.

Proof. Similar to proposition 1.15. □

Sequences.

A sequence (x^k) in $\mathbb{R}^n$ indexed by $k \in \mathbb{N}$ has vector elements $x^1, x^2, \ldots, x^k, \ldots$. Notice that the index k appears as superscript (in order to avoid confusion with the coordinates of the vectors).

Definition 7.24. A sequence (x^k) converges to $x \in \mathbb{R}^n$ if $\lim\limits_{k\to\infty} \|x^k - x\| = 0$. We write $\lim\limits_{k\to\infty} x^k = x$.

Example 7.25. Let $x^k = (\frac{1}{k}, \frac{k}{k+1})$, then $\lim\limits_{k\to\infty} x^k = (0,1)$.

Theorem 7.26. A sequence (x^k) converges to $x \in \mathbb{R}^n$ if and only if $\lim\limits_{k\to\infty} x_i^k = x_i$, $\forall i = \overline{1,n}$.

Proof. Consider first $i \in \{1, \ldots, n\}$. We have that

$$|x_i^k - x_i| = \sqrt{(x_i^k - x_i)^2} \le \sqrt{(x_1^k - x_1)^2 + \ldots + (x_n^k - x_n)^2} = \|x^k - x\|,$$

hence if (x^k) converges to $x \in \mathbb{R}^n$, i.e. $\|x^k - x\| \to 0$, then $|x_i^k - x_i| \to 0$ and $x_i^k \to x_i$.

Let us now prove the converse statement and assume that $\lim\limits_{k\to\infty} x_i^k = x_i$, $\forall i = \overline{1,n}$. Then

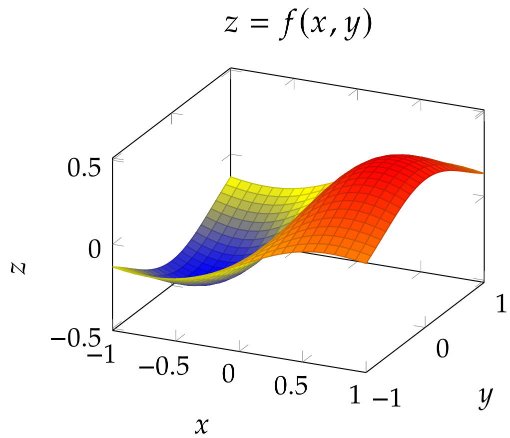
$$\|x^k - x\| = \sqrt{(x_1^k - x_1)^2 + \ldots + (x_n^k - x_n)^2} \to 0,$$

hence $x^k \to x$. □

Note that this is telling us that a sequence of vectors converges if and only if the components of the vectors converge, respectively.

※ Functions of several variables. Limits and continuity

We will now introduce functions of several variables, focusing on those having real (scalar) values. This means we will mostly consider functions $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ mapping vectors in $\mathbb{R}^n$ into real numbers. As you already know, when $n = 1$ the graph of a function is a curve in $\mathbb{R}^2$. When $n = 2$, the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by points with coordinates $(x, y, f(x, y))$ – this represents a surface in $\mathbb{R}^3$ (an example is shown in the figure below).



What about when $n \geq 3$? The graph of the function, $\{(x, f(x) \in \mathbb{R}^{n+1}) \mid x \in A \subseteq \mathbb{R}^n\}$, would be a set in $\mathbb{R}^{n+1}$ and we are able to visualize only its projections in lower dimensional spaces ($\mathbb{R}^3$ or $\mathbb{R}^2$). Apart from the graph, another way of visualizing a function is through

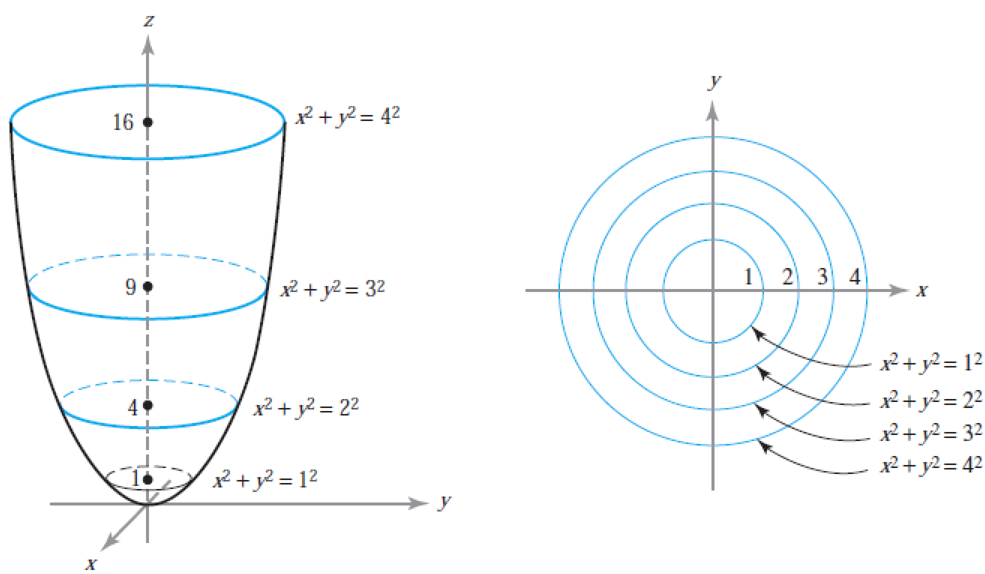


Figure 5: Graph and level curves for $f(x, y) = x^2 + y^2$. Source: [3, page 80].

its level sets, which are given by

$$L_c := \{x \in A \subseteq \mathbb{R}^n \mid f(x) = c\},$$

for a constant $c \in \mathbb{R}$. If $n = 2$, the set $L_c = \{(x, y) \in A \mid f(x, y) = c\}$ describes a *level curve* (see the figure above). If $n = 3$, the set $L_c = \{(x, y, z) \in A \mid f(x, y, z) = c\}$ describes a *level surface*.

Limits of functions of several variables. Continuity.

Using neighbourhoods in $\mathbb{R}^n$, we can define limits and continuity exactly as in section 4.

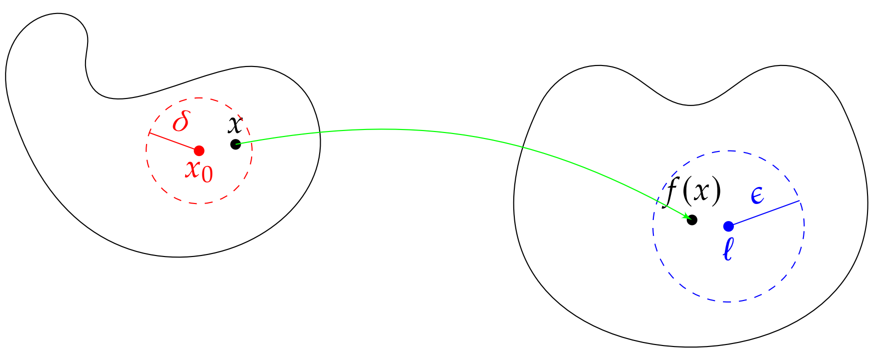
Definition 8.1. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in A'$. We say that $\lim_{x \to x_0} f(x) = \ell \in \overline{\mathbb{R}}$ if

$$\forall V \in \mathcal{V}(\ell),\ \exists U \in \mathcal{V}(x_0) \text{ s.t. } f(x) \in V,\ \forall x \in U \cap \big(A \setminus \{x_0\}\big).$$

Remark 8.2. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in A'$. We say that $\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - \ell| < \varepsilon,\ \forall x \in A \text{ with } \|x - x_0\| < \delta.$$

A similar definition can be given when $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, illustrated below. □



Theorem 8.3. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in A'$. Then $\lim_{x \to x_0} f(x) = \ell \in \overline{\mathbb{R}}$ if and only if

$$\text{for any sequence } (x^k) \text{ in } A \setminus \{x_0\} \text{ with } \lim_{k \to \infty} x^k = x_0, \text{ we have that } \lim_{k \to \infty} f(x^k) = \ell \in \overline{\mathbb{R}}.$$

□

Let us now consider some limits in $\mathbb{R}^2$ and explore some methods of computing them.

Example 8.4. (a) $\displaystyle\lim_{(x,y) \to (0,0)} \frac{x^2}{\sqrt{x^2 + y^2}} = 0.$

We will try a simple strategy: to bound the function and use the squeeze theorem. Since $0 \leq \frac{x^2}{\sqrt{x^2+y^2}} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} \leq \sqrt{x^2 + y^2}$ and $\sqrt{x^2 + y^2} \to 0$, as $(x, y) \to (0, 0)$, we have that the limit is zero.

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = 1.$

Here we can use a remarkable limit since $t := x^2 + y^2 \to 0$ as $(x,y) \to (0,0)$, hence the limit equals $\lim\limits_{t\to 0} \dfrac{\sin t}{t} = 1$.

(c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

As $(x,y) \to (0,0)$ we are having points in $\mathbb{R}^2$ that converge towards the origin. The points can approach the origin along any path – if the limit exists, we will always get the same thing. One important strategy is thus to approach the origin along different paths: if the function converges to different values, then the limit doesn't exist! The simplest paths we could consider are lines that pass through the origin, i.e. points (x, mx). For our example, $\lim\limits_{(x,mx)\to(0,0)} \dfrac{xy}{x^2+y^2} = \lim\limits_{(x,mx)\to(0,0)} \dfrac{mx^2}{(1+m^2)x^2} = \dfrac{m}{1+m^2}$. The limit value depends on the gradient m, e.g. for $m = 0$ we get 0 and for $m = 1$ we get $1/2$, so the limit does not exist.

Remark 8.5. If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, then $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y) = \lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$.

Definition 8.6. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in A$. We say that f is continuous at x_0 if

$$\forall V \in \mathcal{V}(f(x_0)),\ \exists U \in \mathcal{V}(x_0) \text{ s.t. } f(x) \in V,\ \forall x \in U \cap A.$$

Remark 8.7. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in A \cap A'$ an accumulation point. Then f is continuous at x_0 if and only if

$$\lim_{x\to x_0} f(x) = f(x_0).$$

Remark 8.8. If $x_0 \in A$ is an isolated point, then f is continuous at x_0.

Theorem 8.9. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in A \cap A'$. The following are equivalent:

(a) f is continuous at x_0.

(b) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$, $\forall x \in A$ with $\|x - x_0\| < \delta$.

(c) for any sequence (x^k) in A with $\lim\limits_{n\to\infty} x^k = x_0$, we have that $\lim\limits_{k\to\infty} f(x^k) = f(x_0)$.

Proposition 8.10 (Any norm is continuous)**.** $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|x\|$ is continuous on $\mathbb{R}^n$.

Proof. Let $x_0 \in \mathbb{R}^n$ and (x^k) with $x^k \to x_0$. We have that $\|x^k - x^0\| \to 0$. By the triangle inequality $\big|\|x^k\| - \|x^0\|\big| \le \|x^k - x^0\| \to 0$, hence $\|x^k\| - \|x^0\| \to 0$, i.e $\|x^k\| \to \|x^0\|$. □

Theorem 8.11 (Weierstrass)**.** Let $A \subseteq \mathbb{R}^n$ be closed and bounded, and $f : A \to \mathbb{R}$ a continuous function. Then f is bounded and it attains its bounds, i.e. there exist $\min\big(f(A)\big), \max\big(f(A)\big)$. □

※ Partial derivatives and differentiability in $\mathbb{R}^n$. Gradient descent

Definition 9.1. Let $A \subseteq \mathbb{R}^n$ be an open set and $f : A \to \mathbb{R}$. The *partial derivative* of f with respect to x_i at the point $x = (x_1, \dots, x_n) \in A$ is given by

$$\begin{aligned}\frac{\partial f}{\partial x_i}(x) = \partial_i f(x) &:= \lim_{h\to 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h} \\ &= \lim_{h\to 0} \frac{f(x + he_i) - f(x)}{h}.\end{aligned}$$

Note that $\frac{\partial f}{\partial x_i}$ is the derivative of f with respect to x_i, with the other variables held fixed.

Definition 9.2. For a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ that has partial derivatives at $x \in A$ with respect to all its variables, the *gradient* at x is given by the vector $\nabla f(x) \in \mathbb{R}^n$,

$$\nabla f(x) := \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right).$$

Example 9.3. For $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2y + y^2$ we have that $\nabla f(x, y) = (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)) = (2xy, x^2 + 2y)$.

Example 9.4 (With partial derivatives but discontinuous)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

Since $f(x, 0) - f(0, 0) = 0$, $\dfrac{\partial f}{x}(0, 0) = \lim\limits_{x\to 0} \dfrac{f(x, 0) - f(0, 0)}{x} = 0 = \dfrac{\partial f}{y}(0, 0)$, so f has partial derivatives zero at $(0, 0)$. But $\lim\limits_{(x,mx)\to(0,0)} \dfrac{mx^2}{(m^2 + 1)x^2} = \dfrac{m}{(m^2 + 1)}$ depends on m, so f doesn't have a limit at $(0, 0)$, which means that f is discontinuous at $(0, 0)$. □

As the above example shows, a function that has partial derivatives at a point doesn't have to be continuous. This means that if we want to have good properties for differentiable functions, we have to find a better way of defining differentiability.

Let us recall an important idea for differentiable functions in $\mathbb{R}$: $f(x_0) + f'(x_0)(x - x_0)$ is the linear approximation to $f(x)$. This comes from the definition of the derivative

$$\lim_{x\to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0,$$

which can also be written as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R(x - x_0), \text{ with } \frac{R(x - x_0)}{x - x_0} \to 0,$$

where $R(x - x_0)$ is the remainder of the linear approximation.

Definition 9.5. Let $A \subseteq \mathbb{R}^n$ be an open set and $f : A \to \mathbb{R}$. We say that f is *differentiable* at $x_0 \in A$ if there exists a vector $Df(x_0) \in \mathbb{R}^n$, called the differential/derivative of f at x_0, s.t.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - Df(x_0) \cdot (x - x_0)}{\|x - x_0\|} = 0.$$

Here $Df(x_0) \cdot (x - x_0)$ is the dot product of two vectors. This can also be written as

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - Df(x_0) \cdot h}{\|h\|} = 0.$$

Note that differentiability is equivalent to

$$f(x) = f(x_0) + Df(x_0) \cdot (x - x_0) + R(x - x_0), \text{ with } \frac{R(x - x_0)}{\|x - x_0\|} \to 0,$$

where $R(x - x_0)$ is the remainder of the linear approximation.

Definition 9.6. Let $A \subseteq \mathbb{R}^n$ be an open set. If $f : A \to \mathbb{R}^m$, $f = (f_1, \ldots, f_m)$, then f is differentiable at x_0 if there exists a matrix $Df(x_0) \in \mathbb{R}^{m \times n}$, called the differential/derivative of f at x_0, s.t.

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - Df(x_0)(x - x_0)\|_m}{\|x - x_0\|_n} = 0.$$

Here $Df(x_0)(x - x_0)$ is a matrix–vector product: $[\quad]_{m \times n}[\quad]_{n \times 1} = [\quad]_{m \times 1}$.

Example 9.7. Constant functions have zero derivative and linear functions have a constant derivative.

- If $f : \mathbb{R}^n \to \mathbb{R}^m$ is constant, then $Df(x) = 0$ since $f(x) = f(x_0)$ for any $x, x_0 \in \mathbb{R}^n$.
- If $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = a \cdot x$ with $a \in \mathbb{R}^n$, then $Df(x) = a$ since $f(x) - f(x_0) - a \cdot (x - x_0) = 0$.
- If $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = Ax$ with $A \in \mathbb{R}^{m \times n}$, then $Df(x) = A$; $f(x) - f(x_0) - A(x - x_0) = 0$.

Theorem 9.8. Let $A \subseteq \mathbb{R}^n$ be an open set. If $f : A \to \mathbb{R}^m$ is differentiable at x_0, then f is continuous at x_0.

Proof. Since f is differentiable, $f(x) = f(x_0) + Df(x_0)(x - x_0) + \frac{R(x - x_0)}{\|x - x_0\|}\|x - x_0\|$. Letting $x \to x_0$ we use that $\frac{\|R(x - x_0)\|}{\|x - x_0\|} \to 0$ to obtain that $f(x) \to f(x_0)$. □

Theorem 9.9. Let $A \subseteq \mathbb{R}^n$ be an open set and $x \in A$. If $f : A \to \mathbb{R}$ is differentiable at x, then the partial derivatives exist at x and

$$Df(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right).$$

Proof. Differentiability at x gives

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - Df(x)\cdot h}{\|h\|} = 0.$$

Let us take the vector h in the direction of e_i, with a non-zero value on the ith component only, i.e. $h = (0, \dots, 0, h_i, 0, \dots, 0) = h_i e_i$. Then we have that

$$\lim_{h_i\to 0} \frac{f(x+h_ie_i) - f(x) - Df(x)\cdot h_ie_i}{|h_i|} = 0 = \lim_{h_i\to 0} \frac{f(x+h_ie_i) - f(x) - Df(x)\cdot h_ie_i}{h_i},$$

which gives that

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h_i\to 0} \frac{f(x+h_ie_i) - f(x)}{h_i} = Df(x)\cdot e_i = Df(x)_i,$$

hence the ith component of $Df(x)$ is $\frac{\partial f}{\partial x_i}(x)$, which means that $Df(x) = \nabla f(x)$. □

Theorem 9.10. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and $x \in A$. If all the partial derivatives exist and are continuous at x, then f is differentiable at x.

Proof. (Optional) See [1][Theorem 9.21]. □

It is possible for a function to have partial derivatives, but not be differentiable if the partial derivatives are not continuous. The function in example 9.4 has partial derivatives at $(0,0)$, but it is discontinuous there, so it is not differentiable at that point.

Theorem 9.11. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $f = (f_1, \dots, f_m)$ be differentiable at $x \in A$, then

$$Df(x) = J = \begin{bmatrix} \nabla f_1(x) \\ \vdots \\ \nabla f_m(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}_{m\times n}.$$

This matrix is called the *Jacobian matrix* and is typically denoted by J.

Theorem 9.12 (Calculus rules)**.** Let $A \subseteq \mathbb{R}^n$ and $f, g : A \to \mathbb{R}$ differentiable at $x \in A$. Then

1. $\nabla(f+g)(x) = \nabla f(x) + \nabla g(x)$.
2. $\nabla(fg)(x) = g(x)\nabla f(x) + f(x)\nabla g(x)$.
3. $\nabla\left(\frac{f}{g}\right)(x) = \frac{g(x)\nabla f(x) - f(x)\nabla g(x)}{g^2(x)}$.

Theorem 9.13 (Chain rule)**.** Let $g : \mathbb{R}^n \to \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}^p$ differentiable at x and $g(x)$, respectively. Then

$$D(f \circ g)(x) = Df\big(g(x)\big)Dg(x).$$

In terms of matrix dimensions: $[\quad]_{p\times n} = [\quad]_{p\times m}[\quad]_{m\times n}$.

Proof. (Optional) Considering $E(h) := \|f\big(g(x+h)\big) - f\big(g(x)\big) - Df\big(g(x)\big)Dg(x)h\|$ we aim to prove that $\lim_{h\to 0} \frac{E(h)}{\|h\|} = 0$. Since

$$\begin{aligned} E(h) = \|&f\big(g(x+h)\big) - f\big(g(x)\big) - Df\big(g(x)\big)\big(g(x+h) - g(x)\big) \\ &+ Df\big(g(x)\big)\big(g(x+h) - g(x)\big) - Df\big(g(x)\big)Dg(x)h\|, \end{aligned}$$

using the triangle inequality we have that

$$\begin{aligned} E(h) \leq &\|f\big(g(x+h)\big) - f\big(g(x)\big) - Df\big(g(x)\big)\big(g(x+h) - g(x)\big)\| \\ &+ \|Df\big(g(x)\big)\|\|g(x+h) - g(x) - Dg(x)h\|. \end{aligned}$$

At this point, after a few intermediate steps, one can now divide by $\|h\|$, take $h \to 0$ and use the differentiability of f at $g(x)$, and of g at x, together with the fact that $\|Df\big(g(x)\big)\|$ is bounded and independent of h. ◻

Example 9.14 (Chain rule)**.** Let $g : \mathbb{R} \to \mathbb{R}^n$, $g = (g_1, \dots, g_n)$, $f : \mathbb{R}^n \to \mathbb{R}$, $f \circ g : \mathbb{R} \to \mathbb{R}$. With $g'(t) = (g_1'(t), \dots, g_n'(t))$, we have that

$$\begin{aligned} (f \circ g)'(t) &= \nabla f\big(g(t)\big) \cdot g'(t) \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big(g(t)\big) \cdot g_i'(t) \end{aligned}$$

If $n = 2$ and $g(t) = (x(t), y(t))$, then $(f \circ g)(t) = f(x(t), y(t)) = h(t)$ and

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Definition 9.15. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and a vector $v \in \mathbb{R}^n$. The derivative of f in the direction of v at $x \in A$ (*directional derivative*) is given by

$$Df_v(x) := \lim_{h\to 0} \frac{f(x+hv) - f(x)}{h}.$$

Note that here h is a scalar. The directional derivative $Df_v(x)$ is also denoted by $\partial_v f(x)$.

Theorem 9.16. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and $v \in \mathbb{R}^n$. If f is differentiable at $x \in A$, then

$$Df_v(x) = \nabla f(x) \cdot v.$$

Proof. From differentiability we have that

$$\lim_{h\to 0} \frac{f(x+hv) - f(x) - \nabla f(x)\cdot hv}{\|hv\|} = 0.$$

Since $\|hv\| = |h|\|v\|$, this gives that

$$\lim_{h\to 0} \frac{f(x+hv) - f(x) - \nabla f(x)\cdot hv}{h} = 0,$$

which can be rearranged as

$$Df_v(x) = \lim_{h\to 0} \frac{f(x+hv) - f(x)}{h} = \nabla f(x)\cdot v.$$

□

Theorem 9.17 (Fermat)**.** Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $x \in A$. If x is a local extremum, then it is a critical point, i.e. $\nabla f(x) = 0$.

Proof. x is an extremum in every direction, thus $0 = Df_v(x) = \nabla f(x)\cdot v$ for every $v \in \mathbb{R}^n$ (including the canonical vectors e_i). This gives that $\nabla f(x) = 0$. □

Proposition 9.18 (Direction of steepest ascent/descent)**.** Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $x \in A$ with $\nabla f(x) \neq 0$. Then

- $\nabla f(x)$ gives the direction of fastest increase (steepest ascent).
- $-\nabla f(x)$ gives the direction of fastest decrease (steepest descent).

Proof. Since $Df_v(x) = \nabla f(x)\cdot v$, by the Cauchy-Schwarz inequality we have that

$$-\|\nabla f(x)\|\|v\| \le Df_v(x) \le \|\nabla f(x)\|\|v\|,$$

with the maximum obtained for $v = \alpha\nabla f(x)$, the minimum for $v = -\alpha\nabla f(x)$, $\alpha > 0$. □

Proposition 9.19 (Gradient perpendicular to the level set)**.** Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $x \in A$. Then $\nabla f(x)$ is perpendicular to the level set containing x, i.e. if v is a tangent vector to the level set then $Df_v(x) = \nabla f(x)\cdot v = 0$.

Proof. Let $c(t)$, $t \ge 0$ be a path on the level set, i.e. $f(c(t)) =$ constant, that starts from $x = c(0)$. Let $v = c'(0)$ be the tangent vector to the path at $t = 0$. By the chain rule

$$0 = \frac{d}{dt} f(c(t))\bigg|_{t=0} = \nabla f(c(0))\cdot c'(0) = \nabla f(x)\cdot v.$$

□

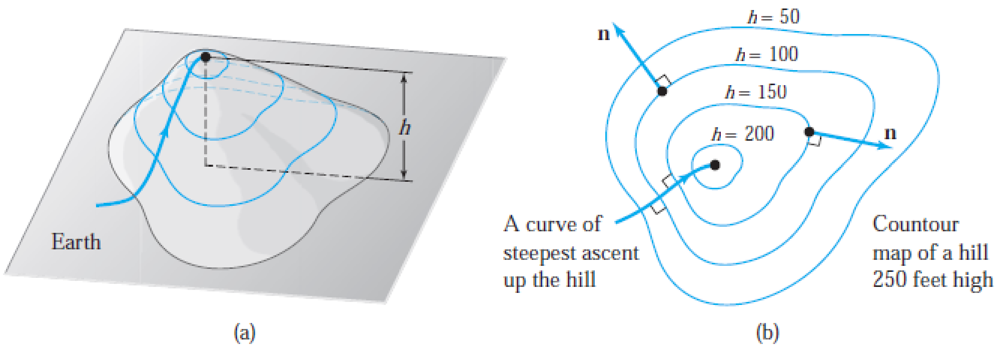


Figure 6: The gradient is the steepest ascent direction and is perpendicular to the level curves. Source: [3, page 140].

Example 9.20 (Tangent line to a level curve)**.** Consider a level curve $L = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$ and a point (x_0, y_0) on it. Take the tangent line at that point. If (x, y) is a point on the tangent line, then the gradient is perpendicular to the vector $(x - x_0, y - y_0)$,

$$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0,$$

hence the equation of the tangent line is given by

$$\frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0) = 0.$$

Example 9.21 (Tangent plane to a level surface)**.** Consider a level curve $L = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = c\}$ and a point (x_0, y_0, z_0) on it. Take the tangent plane at that point. If (x, y, z) is a point on the tangent plane, then

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

hence the equation of the tangent plane is given by

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0) \cdot (y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0) \cdot (z - z_0) = 0.$$

Gradient descent. One of the most important optimization methods is based on the fundamental idea that the direction of steepest descent on the graph of a function f is given by $-\nabla f$ (see proposition 9.18). If we want to minimize a function f, this naturally suggests an iterative method in which from a current position x_k we move in the direction of $-\nabla f(x_k)$ with a step size $s_k > 0$ in order to get to the next position x_{k+1} with $f(x_{k+1}) < f(x_k)$. The gradient descent method starts from an initial value $x_0 \in \mathbb{R}^n$ and then for $k \geq 0$ takes

$$x_{k+1} = x_k - s_k \nabla f(x_k).$$

The step size (learning rate) s_k has to be chosen at every iteration. One way of doing this is called *exact line search*: we look for the optimal step size s_k that minimizes the function $\varphi(s) = f(x_{k+1}) = f(x_k - s\nabla f(x_k))$. By the chain rule we have that

$$\varphi'(s) = \nabla f(x_{k+1}) \cdot \frac{d}{ds} x_{k+1} = \nabla f(x_{k+1}) \cdot \big(-\nabla f(x_k)\big).$$

Since $\varphi'(s_k) = 0$ for the optimal step size s_k, we see that $\nabla f(x_{k+1}) \cdot \nabla f(x_k) = 0$, which means that two consecutive search directions are perpendicular, i.e. $(x_{k+2} - x_{k+1}) \perp (x_{k+1} - x_k)$.

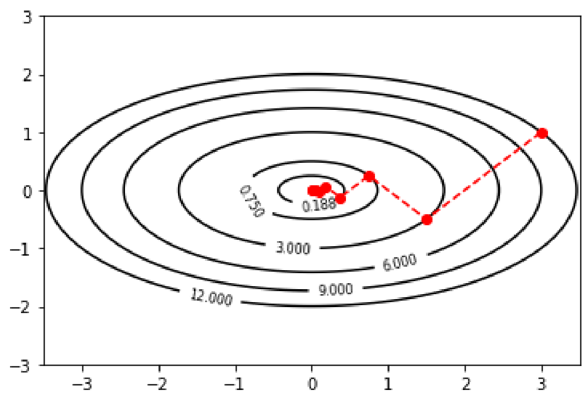


Figure 7: Level curves and gradient descent iterations for example 9.22.

Example 9.22. Consider the quadratic function $f(x, y) = x^2 + 3y^2$ which has a unique global minimum at the origin $(0, 0)$. The gradient is given by $\nabla f(x, y) = (2x, 6y)$.
Gradient descent: starting from an initial value (x_0, y_0) consider the sequence

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - s\nabla f(x_k, y_k),$$

that is

$$x_{k+1} = (1-2s)x_k, \; y_{k+1} = (1-6s)y_k.$$

The step size is determined using exact line search. We look for the optimal step size (learning rate) $s > 0$ by minimizing the function

$$\varphi(s) = f(x_{k+1}, y_{k+1}) = (1-2s)^2 x_k^2 + 3(1-6s)^2 y_k^2 \to \min .$$

We want

$$\varphi'(s) = 0 \text{ with } \varphi'(s) = -4(1-2s)x_k^2 - 36(1-6s)y_k^2,$$

hence we obtain the optimal step size $s = \frac{x_k^2+9y_k^2}{2x_k^2+54y_k^2}$. In fig. 7 we have the gradient descent iterations starting from the initial value $(3, 1)$ converging towards the solution $(0, 0)$. □

※ Higher order derivatives. Local extremum conditions. Applications

The second order partial derivative with respect to x_i is simply

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right) =: \frac{\partial^2 f}{\partial x_i^2} == \partial_i^2 f$$

and the mixed second order partial derivative is

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) =: \frac{\partial^2 f}{\partial x_i \partial x_j} = \partial_{i,j}^2 f.$$

Example 10.1. For $f(x,y) = x^2y + (x+2y)^3$ we have that

$$\frac{\partial f}{\partial x} = 2xy + 3(x+2y)^2, \quad \frac{\partial^2 f}{\partial x^2} = 2y, \quad \frac{\partial^2 f}{\partial y \partial x} = 2x + 12(x+2y)$$
$$\frac{\partial f}{\partial y} = x^2 + 6(x+2y)^2, \quad \frac{\partial^2 f}{\partial y^2} = 24(x+2y), \quad \frac{\partial^2 f}{\partial x \partial y} = 2x + 12(x+2y)$$

Theorem 10.2 (Schwarz)**.** If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ has continuous second order partial derivatives, then if $i \neq j$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Proof. (Optional) See [3][page 151]. The proof is based on the mean value theorem. □

Definition 10.3. For $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ the *Hessian matrix* is defined by

$$H(x) = D^2 f(x) = D(\nabla f)(x) = \begin{bmatrix} \nabla\left(\frac{\partial f}{\partial x_1}\right) \\ \nabla\left(\frac{\partial f}{\partial x_2}\right) \\ \vdots \\ \nabla\left(\frac{\partial f}{\partial x_n}\right) \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{n\times n}.$$

If the second order derivatives are continuous, then the Hessian matrix $H(x)$ is symmetric.

References

[1] W. Rudin, *Principles of Mathematical Analysis 3rd ed*, McGraw-Hill, 1976.

[2] T. Tao, *Analysis I*, Springer, 2016.

[3] J.E. Marsden, A. Tromba, *Vector Calculus 6th ed*, W.H. Freeman and Company, 2012

[4] M. Oberguggenberger, A. Ostermann, *Analysis for Computer Scientists*, Springer, 2018

[5] G. Strang, *Linear Algebra and Learning from Data*, Wellesley Cambridge Press, 2019