Seminar 11

1. Find the second-order Taylor polynomial for the following functions at the given points:

   (a) $f(x, y) = \sin(x + 2y)$ at $(0, 0)$.

   (b) $f(x, y) = e^{x+y}$ at $(0, 0)$ and $(1, -1)$.

   (c) $f(x, y) = \sin(x)\sin(y)$ at $(\pi/2, \pi/2)$.

   (d) $f(x, y) = e^{-(x^2+y^2)}$ at $(0, 0)$.

2. Compute the Hessian matrix and its eigenvalues for the following:

   (a) $f(x, y) = (y - 1)e^x + (x - 1)e^y$ at $(0, 0)$.

   (b) $f(x, y) = \sin(x)\cos(y)$ at $(\pi/2, 0)$.

3. Find and classify the critical points for each of the following functions:

   (a) $f(x, y) = x^3 - 3x + y^2$.

   (b) $f(x, y) = x^3 + y^3 - 6xy$.

   (c) $f(x, y) = x^4 + y^4 - 4(x - y)^2$.

   (d) $f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z$.

4. Let A be a symmetric $n \times n$ matrix and the quadratic function $f : \mathbb{R}^n \to R$, $f(x) = \frac{1}{2}x^T Ax$. Prove that $\nabla f(x) = Ax$ and $H(x) = A$. *Hint: use the Taylor expansion.*

5. Let A be an $m \times n$ matrix, b a vector in R^m and the least squares minimization problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|^2.$$

   Prove that the solution x^* of this problem satisfies (the so-called normal equations)

$$A^T Ax^* = A^T b.$$

6. ★[Python] Let A be a 2×2 matrix and let the quadratic function $f : \mathbb{R}^2 \to R$, $f(x) = \frac{1}{2}x^T Ax$.

   (a) Give a matrix A such that f has a unique minimum.

   (b) Give a matrix A such that f has a unique maximum.

   (c) Give a matrix A such that f has a unique saddle point.

   In each case plot the 3d surface, three contour lines and the gradient at three different points.

Homework questions are marked with ★.