Seminar 2

1. Prove using the ε-definition that $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and ★ $\lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$.

2. Study if the sequence (x_n) is bounded, monotone, and convergent, for each of the following:

   (a) $x_n = \sqrt{n+1} - \sqrt{n}$.

   (b) $x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$.

   (c) ★$x_n = \frac{\sin(n)}{n}$.

   (d) $x_n = \frac{2^n}{n!}$.

3. Find the limit for each of the following sequences:

   (a) $\sqrt{n}\big(\sqrt{n+1} - \sqrt{n}\big)$.

   (b) $\big(a_1^n + a_2^n + \ldots + a_k^n\big)^{\frac{1}{n}}$, with $a_i > 0$.

   (c) $\sqrt[n]{n}$.

   (d) ★ $\sqrt[n]{1+2+\ldots+n}$.

   (e) $\frac{2^n + (-1)^n}{3^n}$.

   (f) $\frac{(an+1)^2}{4n^2-2n+1}$, $a \in \mathbb{R}$.

4. Consider the sequence (e_n) given by

$$e_n = \big(1 + \frac{1}{n}\big)^n.$$

   Prove that (e_n) is increasing and bounded – its limit is denoted by e (Euler's number).

5. Find the limit for each of the following sequences:

   (a) $\big(\frac{2n+1}{2n-1}\big)^n$.

   (b) $n\big(\ln(n+2) - \ln(n+1)\big)$.

   (c) ★$\big(\frac{\ln(n+1)}{\ln n}\big)^n$.

6. ★ Prove that the sequence (x_n) given by $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n$ is decreasing and bounded, hence convergent – its limit is denoted by γ (Euler's constant).

7. (Stolz-Cesàro lemma) Let $(a_n), (b_n)$ be two sequences such that (i) $a_n \to 0$ and $b_n \to 0$ with (b_n) decreasing; or (ii) $b_n \to \infty$ with (b_n) increasing.

   If

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell, \quad \text{then } \lim_{n\to\infty} \frac{a_n}{b_n} = \ell.$$

   (a) Let (x_n) be a convergent sequence. What can you say about the sequence (a_n) of averages

$$a_n = \frac{x_1 + x_2 + \ldots + x_n}{n}?$$

   Give an example where the averages converge, even though the sequence does not.

   (b) Compare $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ with n and $\ln n$ by taking the ratio, respectively.

   (c) Let (x_n) be a sequence such that $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell$. Prove that $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = \ell$.

8. Find the limit for each of the following sequences:

   (a) $\frac{n}{\sqrt[n]{n!}}$. (b) ★$\frac{n^n}{1+2^2+3^3+\ldots+n^n}$. (c) $\frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}}$, $p \in \mathbb{N}$.

9. (Banach fixed point theorem) Let $f : [a,b] \to [a,b]$ be a contraction, meaning that there exists $\alpha \in (0,1)$ such that

$$|f(x) - f(y)| \leq \alpha \, |x - y| \,, \quad \forall x, y \in [a,b].$$

   Let an arbitrary $x_1 \in [a,b]$ and consider the sequence (x_n) given by

$$x_{n+1} = f(x_n), \quad \forall n \in \mathbb{N}.$$

   Prove that the sequence (x_n) is Cauchy and that its limit x^* is a fixed point, i.e. $f(x^*) = x^*$.

10. Study the convergence and find the limit of the following sequences:

   (a) $x_{n+1} = \sqrt{2 + x_n}$, $x_1 = 0$. (b) ★$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$, $x_1 = 1$ and $a > 1$.

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.