Mock Exam (2h)

1. Draw the interior and the boundary of the unit ball in $\mathbb{R}^2$ for the norm $\|(x,y)\|_1 = |x| + |y|$.

2. Study the continuity and the differentiability (partial and total) of the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

3. (a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and let $v \in \mathbb{R}^n$. Write the definition for the directional derivative $D_v f(x)$ and prove that it equals $\nabla f(x) \cdot v$.
   (b) Let $A \in \mathbb{R}^{2\times 2}$ be a symmetric matrix and let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x^T Ax$. In which direction does f decrease the most at the point $(1,1)$?

4. Let $f(x,y) = x^2 e^{-xy}$. Find the second order Taylor expansion of f around $(1,1)$.

5. Find and classify all the critical points of $f(x,y) = (x^2 + 3y^2)e^{1-x^2-y^2}$.

6. Find the extrema of $3x + 2y$ subject to $2x^2 + 3y^2 = 1$.

7. Compute the following integrals:

   (a) $\displaystyle\int_0^\infty e^{-2x^2}\,\mathrm{d}x.$ (b) $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,\mathrm{d}x\,\mathrm{d}y.$

8. Let D be the triangle with vertices $(0,0), (1,0)$ and $(0,1)$. Compute $\displaystyle\iint_D (x^2 - y^2)\,dx\,dy.$

This is a mock exam for January 2024 intended to give you an idea about what to expect.