Mathematical Analysis

1st Year Computer Science

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Last updated: October 5, 2023

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※ Real numbers

Let us start with some standard notation: $\emptyset$ is the empty set; $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers; $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\} = \{m - n \mid m, n \in \mathbb{N}\}$ the set of integers; $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ the set of rational numbers; $\mathbb{R}$ the set of real numbers.

You are very much used with the real numbers. However, it is not trivial to define them in a rigorous way, see for example [1] or [2] for different methods of constructing $\mathbb{R}$ from $\mathbb{Q}$. We will straightforwardly start working with the real numbers – for this reason some of their properties, e.g. definition 1.5, will simply be given as definitions.

Definition 1.1. Let A be a subset of $\mathbb{R}$, denoted as $A \subseteq \mathbb{R}$. We define $x \in \mathbb{R}$ to be

a *lower bound* for A if $x \leq a$, $\forall a \in A$; an *upper bound* for A if $x \geq a$, $\forall a \in A$.

We define

$$lb(A) := \{x \in \mathbb{R} \mid x \leq a, \ \forall a \in A\} \text{ the set of lower bounds of } A,$$

$$ub(A) := \{x \in \mathbb{R} \mid x \geq a, \ \forall a \in A\} \text{ the set of upper bounds of } A.$$

We define $x \in \mathbb{R}$ to be

the minimum of A if $x \in lb(A) \cap A$; the maximum of A if $x \in ub(A) \cap A$,

denoted by $\min(A)$, respectively $\max(A)$. In other words, we have that

$$\min(A) \in A \text{ and } \min(A) \leq a, \ \forall a \in A; \quad \max(A) \in A \text{ and } \max(A) \geq a, \ \forall a \in A.$$

Note that there are sets which do no have minimum or maximum, e.g. $(0, 1)$.

Definition 1.2. A set $A \subseteq \mathbb{R}$ is defined to be

- bounded (from) below if $lb(A) \neq \emptyset$;
- bounded (from) above if $ub(A) \neq \emptyset$;
- bounded if it is both bounded below and above;
- unbounded if it is not bounded.

Definition 1.3. We say that $x \in \mathbb{R}$ is the *supremum* of $A \subseteq \mathbb{R}$, $x := \sup(A)$, if and only if:

1. $x \geq a$, $\forall a \in A$, that is $x \in ub(A)$.
2. if u is an upper bound for A, then $x \leq u$.

The supremum is *the least upper bound*, i.e. $\sup(A) := \min(ub(A))$.

Definition 1.4. We say that $x \in \mathbb{R}$ is the *infimum* of $A \subseteq \mathbb{R}$, $x := \inf(A)$, if and only if:

1. $x \leq a$, $\forall a \in A$, that is $x \in lb(A)$.
2. if u is a lower bound for A, then $x \geq u$.

The infimum is *the greatest lower bound*, i.e. $\inf(A) := \max(lb(A))$.

Definition 1.5 (Completeness Axiom)**.** Every set $A \subseteq \mathbb{R}$ that is bounded above has a supremum. Similarly, every set $A \subseteq \mathbb{R}$ that is bounded below has an infimum.

Note that if A has a maximum, then $\sup(A) = \max(A)$. Similarly, if A has a minimum, then $\inf(A) = \min(A)$. Also, if $\sup(A) \in A$, then $\max(A) = \sup(A)$.

Example 1.6. (a) $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, $\sup(A) = 1 = \max(A)$, $\inf(A) = 0$, $\nexists \min(A)$.

(b) $A = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$, $\sup(A) = \sqrt{2}$, $\nexists \max(A)$, $\inf(A) = -\sqrt{2}$, $\nexists \min(A)$.

Theorem 1.7. Let $A \subseteq \mathbb{R}$ be a bounded set. For $\sup(A)$ and $\inf(A)$ the following are true:

$$\forall \varepsilon > 0, \exists x \in A \text{ such that } \sup(A) - \varepsilon < x,$$

$$\forall \varepsilon > 0, \exists x \in A \text{ such that } x < \inf(A) + \varepsilon.$$

Proof. By definition, for any $y < \sup(A)$, say $y = \sup(A) - \varepsilon$ with $\varepsilon > 0$, we have that $y \notin ub(A)$. Hence there exists $x \in A$ such that $y < x$. Similar proof for $\inf(A)$. □

Proposition 1.8. Let $A \subseteq B \subseteq \mathbb{R}$ be (nonempty) bounded sets. Then

$$\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$$

and

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\},$$
$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}.$$

Proof. It follows directly from the definitions. □

Definition 1.9. Define the *extended real line* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, where ∞ and $-\infty$ are such that

$$\forall x \in \mathbb{R}, -\infty < x < \infty.$$

If a set A is not bounded above, we define $\sup(A) := \infty$.
If a set A is not bounded below, we define $\inf(A) := -\infty$.

[Seminar] The empty set $\emptyset$ is bounded by any number. In $\overline{\mathbb{R}}$, $\sup(\emptyset) = -\infty$, $\inf(\emptyset) = \infty$.

Definition 1.10. A set $V \subseteq \mathbb{R}$ is a *neighborhood (vecinity)* of $x \in \mathbb{R}$ if

$$\exists \varepsilon > 0 \text{ such that } (x - \varepsilon, x + \varepsilon) \subseteq V.$$

A set $V \subseteq \mathbb{R}$ is a *neighborhood* of ∞ if $\exists a \in \mathbb{R}$ such that $(a, \infty) \subseteq V$.

A set $V \subseteq \mathbb{R}$ is a *neighborhood* of $-\infty$ if $\exists a \in \mathbb{R}$ such that $(-\infty, a) \subseteq V$.

We denote all the neighborhoods of x by $\mathcal{V}(x) := \{V \subseteq \mathbb{R} \mid V \text{ is a neighborhood of } x\}$.

Definition 1.11. Let $A \subseteq \mathbb{R}$. The following set is called the *interior* of A

$$\operatorname{int}(A) := \{x \in \mathbb{R} \mid \exists V \in \mathcal{V}(x) \text{ such that } V \subseteq A\},$$

and the following set is called the *closure* of A

$$\operatorname{cl}(A) := \{x \in \mathbb{R} \mid \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}.$$

Proposition 1.12. For any $A \subseteq \mathbb{R}$, it holds that $\operatorname{int}(A) \subseteq A \subseteq \operatorname{cl}(A)$.

Proof. To prove that $\operatorname{int}(A) \subseteq A$ we prove that if $x \in \operatorname{int}(A)$, then $x \in A$. Let $x \in \operatorname{int}(A)$, then $\exists \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq A$. Since $x \in (x - \varepsilon, x + \varepsilon)$, we have that $x \in A$.
To prove that $A \subseteq \operatorname{cl}(A)$ we show that if $x \in A$, then $x \in \operatorname{cl}(A)$. Let $x \in A$. Then for any $V \in \mathcal{V}(x)$ it holds that $x \in V$, giving that $x \in V \cap A$. Hence $x \in \operatorname{cl}(A)$ since $V \cap A \neq \emptyset$. □

Definition 1.13. If $A = \operatorname{int}(A)$, then A is called *open*. If $A = \operatorname{cl}(A)$, then A is called *closed*.

Remark 1.14. To prove that a set A is open, it is sufficient to prove that $A \subseteq \operatorname{int}(A)$.
To prove that a set A is closed, it is sufficient to prove that $\operatorname{cl}(A) \subseteq A$.

Proposition 1.15. The following statements are true:

- The complement of an open set is closed.
- The complement of a closed set is open.

Proof. Let us prove the first statement, the other one being similar. Consider A an open set, i.e. $A = \operatorname{int}(A)$, and denote by $A^c = \{x \in \mathbb{R} \mid x \notin A\}$ its complement. To prove that A^c is closed, we prove that $\operatorname{cl}(A^c) \subseteq A^c$. Consider $x \in \operatorname{cl}(A^c)$ and let's assume that $x \notin A^c$, i.e. $x \in A$, aiming to obtain a contradiction. Since A is open, there exists $V \in \mathcal{V}(x)$ such that $V \subseteq A$, giving that $V \cap A^c = \emptyset$: contradiction with $x \in \operatorname{cl}(A^c)$. Hence the assumption $x \notin A^c$ is false, and we have that if $x \in \operatorname{cl}(A^c)$, then $x \in A^c$. In other words, $\operatorname{cl}(A^c) \subseteq A^c$. □

Proposition 1.16. Any union of open sets is open. Any finite intersection of closed sets is closed.

Proof. (Optional) Left as extra homework. □

References

[1] W. Rudin, *Principles of Mathematical Analysis 3rd ed*, McGraw-Hill, 1976.

[2] T. Tao, *Analysis I*, Springer, 2016.

[3] J.E. Marsden, A. Tromba, *Vector Calculus 6th ed*, W.H. Freeman and Company, 2012

[4] M. Oberguggenberger, A. Ostermann, *Analysis for Computer Scientists*, Springer, 2018

[5] G. Strang, *Linear Algebra and Learning from Data*, Wellesley Cambridge Press, 2019