Seminar 7

1. Compute the following limits using Riemann integrals:

   (a) $\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{2n}\right)$.

   (b) ★ $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{e}+2\sqrt[n]{e^2}+\cdots+n\sqrt[n]{e^n}}{n^2}$.

   (c) $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n}$.

   (d) ★ $\lim\limits_{n\to\infty}\sqrt[n]{\sin\dfrac{\pi}{2n}\sin\dfrac{2\pi}{2n}\ldots\sin\dfrac{(n-1)\pi}{2n}}$.

2. Study the Riemann integrability of the function $f:[0,1]\to\mathbb{R}$,

$$f(x)=\begin{cases}1, & \text{if } x\in\mathbb{Q}\\ 0, & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}.\end{cases}$$

3. Compute the following improper integrals:

   (a) $\displaystyle\int_1^2\frac{1}{x(x-2)}\,\mathrm{d}x$.

   (b) $\displaystyle\int_0^\infty xe^{-x^2}\,\mathrm{d}x$.

   (c) $\displaystyle\int_0^1\frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$.

   (d) ★ $\displaystyle\int_0^\infty e^{-x}\sin x\,\mathrm{d}x$.

4. Study the convergence of the following improper integrals:

   (a) $\displaystyle\int_1^\infty\frac{1}{x\sqrt{1+x^2}}\,\mathrm{d}x$.

   (b) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{1}{\cos x}\,\mathrm{d}x$.

   (c) $\displaystyle\int_1^\infty\frac{\ln x}{x\sqrt{x^2-1}}\,\mathrm{d}x$.

5. Using the integral test, study the convergence of the following series:

   (a) $\displaystyle\sum_{n\geq1}\frac{1}{n^p}$, $p>0$.

   (b) $\displaystyle\sum_{n\geq2}\frac{1}{n(\ln n)^2}$.

   (c) $\displaystyle\sum_{n\geq2}\frac{\ln n}{n^2}$.

6. ★ [Python] The integral $\displaystyle\int_{-\infty}^{\infty}e^{-x^2}\,\mathrm{d}x$ represents the area under the bell curve $y=e^{-x^2}$ and it is related to the normal (Gaussian) probability distribution. It is essential in probability theory and has a wide range of applications. Considering intervals of the form $[-a,a]$, for increasing $a>0$, show numerically (e.g. trapezium rule) that $\displaystyle\int_{-\infty}^{\infty}e^{-x^2}\,\mathrm{d}x=\sqrt{\pi}$.

Homework questions are marked with ★.