Seminar 13

1. By changing the order of integration, evaluate the following:

   (a) $\displaystyle\int_0^1 \int_y^1 \sin(x^2)\,\mathrm{d}x\,\mathrm{d}y.$ (b) $\displaystyle\int_0^1 \int_x^1 e^{y^2}\,\mathrm{d}y\,\mathrm{d}x.$ (c) ★ $\displaystyle\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin(y)}{y}\,\mathrm{d}y\,\mathrm{d}x.$

2. Compute the following integrals by doing a change of variables:

   (a) $\displaystyle\iint_D e^{\frac{x-y}{x+y}}\,\mathrm{d}x\,\mathrm{d}y$, where $D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0, x+y \leq 1\}$.

   (b) $\displaystyle\iint_D \left(\frac{x-y}{x+y+2}\right)^2 \mathrm{d}x\,\mathrm{d}y$, where $D = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}$.

   (c) $\displaystyle\iint_D \frac{y^2}{x}\,\mathrm{d}x\,\mathrm{d}y$, where D is the region between the parabolas $x = 1-y^2$ and $x = 3(1-y^2)$.

   (d) ★ $\displaystyle\iint_D xy\,\mathrm{d}x\,\mathrm{d}y$, where D is the parallelogram with vertices $(0,0)$, $(2,2)$, $(1,2)$, $(3,4)$.

3. Compute the following integrals using polar coordinates:

   (a) $\displaystyle\iint_D \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$, where $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4\}$.

   (b) $\displaystyle\iint_D \sin(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$, where $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq a^2,\ x \leq 0\}$.

   (c) $\displaystyle\iint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,\mathrm{d}x\,\mathrm{d}y$, where $D = \{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$.

   (d) ★ $\displaystyle\iint_D \ln(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$, where D is the region in the first quadrant between the circles $x^2+y^2 = a^2$ and $x^2+y^2 = b^2$, with $0 < a < b$.

4. Find the area of the region bounded by the curve defined through the equation:

   (a) $(x^2+y^2)^2 = a^2(x^2-y^2)$, $x > 0$, $a > 0$. (b) $(x^2+y^2)^2 = 2a^2xy$, $a > 0$.

5. Compute $\displaystyle\iiint_D \sqrt{x^2+y^2+z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where $D = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq az\}$.

6. Show that the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is given by $\frac{4\pi}{3}abc$.

Homework questions are marked with ★.