Seminar 9

1. Study the limits of the following functions when $(x,y) \to (0,0)$:

   (a) $\dfrac{x^2-y^2}{x^2+y^2}$. (b) $\dfrac{x+y}{x^2+y^2}$ (c) $\dfrac{x^3+y^3}{x^2+y^2}$. (d) $\dfrac{\sin x - \sin y}{x-y}$.

2. Compute the partial derivatives (and specify where they exist) for the following functions:

   (a) $f(x,y) = e^{-(x^2+y^2)}$.
   (b) $f(x,y) = \cos x \cos y - \sin x \sin y$.
   (c) $f(x,y) = \|(x,y)\| = \sqrt{x^2+y^2}$.
   (d) $f(x,y,z) = x^2yz + ye^z$.

3. Let $f : \mathbb{R}^2 \to R, f(x,y) = xy$. Using the definition, prove that $Df(x_0,y_0) = (y_0, x_0)$.

4. Prove that

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

   is continuous and has partial derivatives, but it is not differentiable in the origin.

5. Find the gradient of the function f at the point a for the following:

   (a) $f(x,y) = e^{-x}\sin(x+2y)$, $a = (0, \frac{\pi}{4})$.
   (b) $f(x,y) = \arctan(\frac{y}{x})$, $a = (1,1)$.
   (c) $f(x,y,z) = e^{xyz}$, $a = (0,0,0)$.
   (d) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $a = (1,1,1)$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x,y) = f(x^2+y^2),\ \forall\,(x,y) \in \mathbb{R}^2.$$

   Prove that

$$y\frac{\partial g}{\partial x}(x,y) = x\frac{\partial g}{\partial y}(x,y).$$