Q&A

Friday, July 12, 2024 4:58 PM

$$\begin{cases} x'' + t x' = 1 \\ x(0) = x'(0) = 0 \end{cases}$$

$x'' + t x' = 1$ second order linear non-homogeneous with non-constant coeff

$x = x_h + x_p$ $x_h \to$ the gen sol. of $x'' + t x' = 0$

$x_p \to$ a partic. sol. of the given LN-HDE

$x'' + t x' = 0$ LHDE

$x'' + t x' = 1$ $y = x'$ $y(t) = x'(t) \Rightarrow y'(t) = x''(t)$

$y' + t y = 1$ first order linear non-hom with non-const coeff.

$y = y_h + y_p$ y_h : $y' + t y = 0$ $\frac{dy}{dt} = -ty$ $y = 0$ sol.

$y \neq 0$ $\frac{dy}{y} = -t\,dt$ $\ln|y| = -\frac{t^2}{2} + c$ $|y| = \underbrace{e^{-\frac{t^2}{2} + c}}_{e^c \cdot e^{-t^2/2}}$

$$\boxed{y_h = c\,\underbrace{e^{-\frac{t^2}{2}}}, \quad c \in \mathbb{R}}$$

$\left.\begin{array}{l} y = \pm e^c \cdot e^{-t^2/2} \\ y = 0 \end{array}\right| \Rightarrow$

$y_p = \varphi(t) e^{-\frac{t^2}{2}}$ $y' + t y = 1$

$$\varphi' e^{-\frac{t^2}{2}} + \cancel{\varphi \cdot e^{-\frac{t^2}{2}} \cdot (-t)} + \cancel{t \varphi e^{-\frac{t^2}{2}}} = 1$$

$\varphi'(t) = e^{\frac{t^2}{2}}$ $\varphi(t) = \int_0^t e^{\frac{s^2}{2}} ds$

$y_p = e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} ds$ $y = c \cdot e^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} ds$, $c \in \mathbb{R}$

$\left.\begin{array}{l} x'(0) = 0 \\ y = x' \end{array}\right\} \Rightarrow \underline{y(0) = 0}$ $\left.\begin{array}{l} \\ y(0) = c \end{array}\right\} \Rightarrow c = 0$

$\left. y = e^{-\frac{t^2}{2}} \int^t e^{+\frac{s^2}{2}} ds \right\} \Rightarrow$ $x' = e^{-\frac{t^2}{2}} \int^t e^{\frac{s^2}{2}} ds \Rightarrow$

$$y = e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} ds \Bigg\} \Rightarrow \quad x' = e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} ds \Rightarrow$$

$y = x'$

$$\Rightarrow \quad x(t) = \int_0^t e^{-\frac{\tau^2}{2}} \int_0^\tau e^{\frac{s^2}{2}} ds \, d\tau + c \,, \quad c \in \mathbb{R}$$

$x(0) = 0$

$\Rightarrow \quad c = 0$

The unique sol. of the IVP is $\boxed{x(t) = \int_0^t e^{-\frac{\tau^2}{2}} \int_0^\tau e^{\frac{s^2}{2}} ds \, d\tau}$

<u>Lecture 8.</u> $\rightarrow \quad \ddot{\theta} + \omega^2 \sin\theta = 0 \qquad \omega > 0$

$x = \theta \,, \; y = \dot{\theta}$

we write it as the following planar system:

$$(2) \begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 \sin x . \end{cases} \qquad \text{the equil. points are } (k\pi, 0) : k \in \mathbb{Z} .$$

$\theta(t) = 0, \quad \forall t \in \mathbb{R}$ is the unique sol. of the IVP $\begin{cases} \ddot{\theta} + \omega^2 \sin\theta = 0 \\ \theta(0) = 0 \\ \dot{\theta}(0) = 0 \end{cases}$

Thus, it corresponds to the equilibrium $(0,0)$ of the planar system (2).
We have to justify that $(0,0)$ is stable but is not an attractor of the planar system (2).

if we linearize (2) around the equil. point $(0,0)$ we get

$$f(x,y) = \begin{pmatrix} y \\ -\omega^2 \sin x \end{pmatrix} \qquad Jf(x,y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & 0 \end{pmatrix} \quad A := Jf(0,0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$$

$f(x,y) = \begin{pmatrix} \cdots \\ -\omega^2 \sin x \end{pmatrix}$ $\quad Df(x,y) = \begin{pmatrix} \cdots \\ -\omega^2 \cos x & 0 \end{pmatrix}$ $\quad \cdots \begin{pmatrix} \cdots \\ -\omega^2 & 0 \end{pmatrix}$

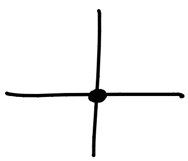
the eigenvalues of A are $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$.

$Re(\lambda_1) = 0$ and $Re(\lambda_2) = 0$ $\Rightarrow$ the equil $(0,0)$ is not hyperbolic

$\Rightarrow$ the LM (linearization method) fails

$\rightsquigarrow$ look for a first integral

we find $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = \frac{y^2}{2} + \omega^2(1 - \cos x)$



In particular, this first integral is well-defined in a neighborhood of the equil point $(0,0)$.

$\Rightarrow$ $(0,0)$ is stable, but it is not an attractor.

the 2 theorems from Lecture 8