APM466 A2

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Questions - 100 points

1. *(40 points)* Suppose that company X has four states of solvency: good, bad, crisis, and default. Suppose also that the one year transition (between solvency states) probability matrix is given by:

$$P = \begin{array}{c|cccc} \text{state} & \text{good} & \text{bad} & \text{crisis} & \text{default} \\ \hline \text{good} & 8/10 & 1/10 & 1/10 & 0 \\ \text{bad} & 1/10 & 5/10 & 2/10 & 2/10 \\ \text{crisis} & 1/10 & 3/10 & 3/10 & 3/10 \\ \text{default} & 0 & 0 & 0 & 1 \end{array}$$

For the following questions, feel free to use a computer to aid your calculations. For part a)&b), you must state your final answer with a small explanation (explicit calculations discouraged in your report). For parts c)& d), a formal proof is not needed, just a 1 or 2 sentence explanation.

(a) *(10 points)* What is the two year transition probability matrix?

Answer: The two year transition probability matrix is:

$$P^2 = \begin{pmatrix} 0.66 & 0.16 & 0.13 & 0.05 \\ 0.15 & 0.32 & 0.17 & 0.36 \\ 0.14 & 0.25 & 0.16 & 0.45 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

because: According to the long term Markov Chain transition probability matrix

$$p_{ij}^{(2)} = \sum_{k=1}^{4} p_{ik}p_{kj}$$

Then we are able to calculate each entry in two year transition probability matrix.
Python used for calculation. See Appendix.

(b) *(10 points)* What is the probability that if company X is currently in a "crisis" solvency state, they will default within the next month?

Answer: If company X is currently in a "crisis" solvency state, the probability that they will default within the next month is 0.04, because:

$$P = P^{\frac{1}{12}} = \begin{pmatrix} 0.98 & 0.01 & 0.01 & 0 \\ 0.01 & 0.93 & 0.04 & 0.02 \\ 0.01 & 0.07 & 0.88 & 0.04 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

P is transition probability matrix in one year. $P^{\frac{1}{12}}$ is the transition probability matrix in one month. And we are looking at $P_{\text{crisis}\to\text{default}}$, which is 0.04.

Calculation:(Python used for calculation. See Appendix.)

$$X_{k+1} = X_k + \frac{1}{n}(P - X_k^n), X_0 = 0$$

For positive-definite P, $0 \leq X_k \leq X_{k+1}, \forall k, X_k \to P^{\frac{1}{n}}$

(c) *(10 points)* What is $\lim_{t\to\infty} P^t$?

Answer: $\lim_{t\to\infty} P^t = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, because for good or bad or crisis, their probability is smaller than 0 and it is not possible to transfer from default to other state, they will converge to 0 as time goes to infinity. In addition, since the probability have to be summed up to one, all the company shift to default state as time goes to infinity.

(d) *(10 points)* If $t \in \mathbb{N}$, $(t < \infty)$, given that the company X has not yet defaulted, is it guaranteed (/with probability 1) that company X will default within t years?
(Hint: Either use induction or show that $\exists t < \infty$ for which $P_{ij}^t = 0\ \forall j \neq 4, P_{ij}^t = 1$ if $j = 4$.)

Answer: No, because: If the company has not yet defaulted, the transition probability of a company to become not-default can not be 0 in each period, since it is within finite number of years. Unless it starts with 0, which guarantees the company will default within 1 year. However, this does not guarantee the company to be defaulted for the later period.

Proof Sketch:
Suppose a not-yet defaulted company is being defaulted after finite number of years t. $P_{ij}^t = 1$ if $j = 4$. This means that a company is not in not defaulted zone after finite number of years: $P_{ij}^t = 0\ \forall j \neq 4$. Therefore, $p_{ij}^{(t)} = \sum_{k=1}^{4} p_{ik}^{t-1} p_{kj}^{t-1} = 0$. if all the not defaulted state is not start with 0, then neither p_{ik}^{t-1} nor p_{kj}^{t-1} can be 0, that means, $p_{ij}^{(t)}$ is not 0, contradiction aroused!
Thus, with finite time t, it is not guaranteed (/with probability 1) that company X will default within t years.

2. *(40 points)* Assume that Germany's bonds are risk-free and Italy's bonds are risk-prone, and that each country issues zero coupon bonds with a face value of 1. We denote a German bond with an outstanding term of i years simply by its current price P_i^G, and an Italian bond with outstanding term of i years also simply by P_i^I. Finally, assume everything henceforth is priced using continuous discounting, and zero recovery under default.

(a) *(10 points)* Given $\{P_1^G, \ldots, P_n^G\}$ and $\{P_1^I, \ldots, P_n^I\}$, derive a closed form formula for the credit spread, h_i, at time $i \in \{1, \ldots, n\}$ for Italy in terms of i, P_i^G, and P_i^I.

Answer: Since each country issues zero coupon bonds with a face value of 1 we see that:

Price of Germany risk-free bond is: $P_i^G = 1 \times e^{-ir_i^G}$
Price of Italy's risk-prone bond is: $P_i^I = 1 \times e^{-ir_i^I}$
Rearrange:

$$r_i^G = \frac{-lnP_i^G}{i}, r_i^I = \frac{-lnP_i^I}{i}$$

$$h_i = r_i^G - r_i^I = \frac{ln(P_i^G / P_i^I)}{i}$$

(b) *(10 points)* Under a two state markov chain model (solvency and default), write Italy's ith-year probability transition matrix, P^i, in terms of just i and h_i.

Answer: Since h_i can be expressed by probability of solvency q_i. Therefore we have: $h_i = -lnq_i/i, q_i = e^{-ih_i}$

$$P^i = \begin{pmatrix} e^{-ih_i} & 1 - e^{-ih_i} \\ 0 & 1 \end{pmatrix}$$

(c) *(10 points)* If the Italian government issues a one-off asset, A, that pays $C_i, i = 1, \ldots, n$, at time i, find the price of this asset in terms of $\{1, \ldots, n\}$, $\{h_1, \ldots, h_n\}$, $\{P_1^G, \ldots, P_n^G\}$, and $\{C_1, \ldots, C_n\}$.

Answer: Since the price of asset is calculated by $A = \sum_{i=1}^{n} C_i e^{-ir_i^I}$,
we see that: $r_i^I = r_i^G + h_i, \rightarrow A = \sum_{i=1}^{n} C_i e^{-i(r_i^G + h_i)} = \sum_{i=1}^{n} C_i e^{-ir_i^G} e^{-ih_i}$.
Since $r_i^G = \frac{-lnP_i^G}{i}$,

$$A = \sum_{i=1}^{n} C_i P_i^G e^{-ih_i}$$

(d) *(10 points)* First find $\partial_{h_i} A$, then use this to say what would happen to the price of A given Italy's probability of default (by any time $i \geq 1$) increases.

Answer: Taking the derivative in h_i, we see that:

$$\partial_{h_i} A = -i \times C_i P_i^G e^{-hi \times i}$$

And thus since exponential function is strictly greater than 0 and $C_i \geq 0, P_i^G \geq 0$, we see that if Italy's probability of default increases, $\partial_{h_i} A < 0$, thus A's price will decrease.

3. *(20 points)* List 4 simplifications (I.e., assumptions that might not be true in real life) that are made under Merton's Credit Risk Model.

Max 1 sentence per assumption.

(a) *Assumption 1:* No taxes or transaction Costs. (commission)

(b) *Assumption 2:* Underlying stock's volatility and Risk-free rates is always constant.

(c) *Assumption 3:* Firm Assets' value follow geometric Brownian motion.

(d) *Assumption 4:* Market movements are unpredictable (efficient markets)

Appendix

GitHub Link:
https://github.com/Robert54/APM466.git