Disclaimer

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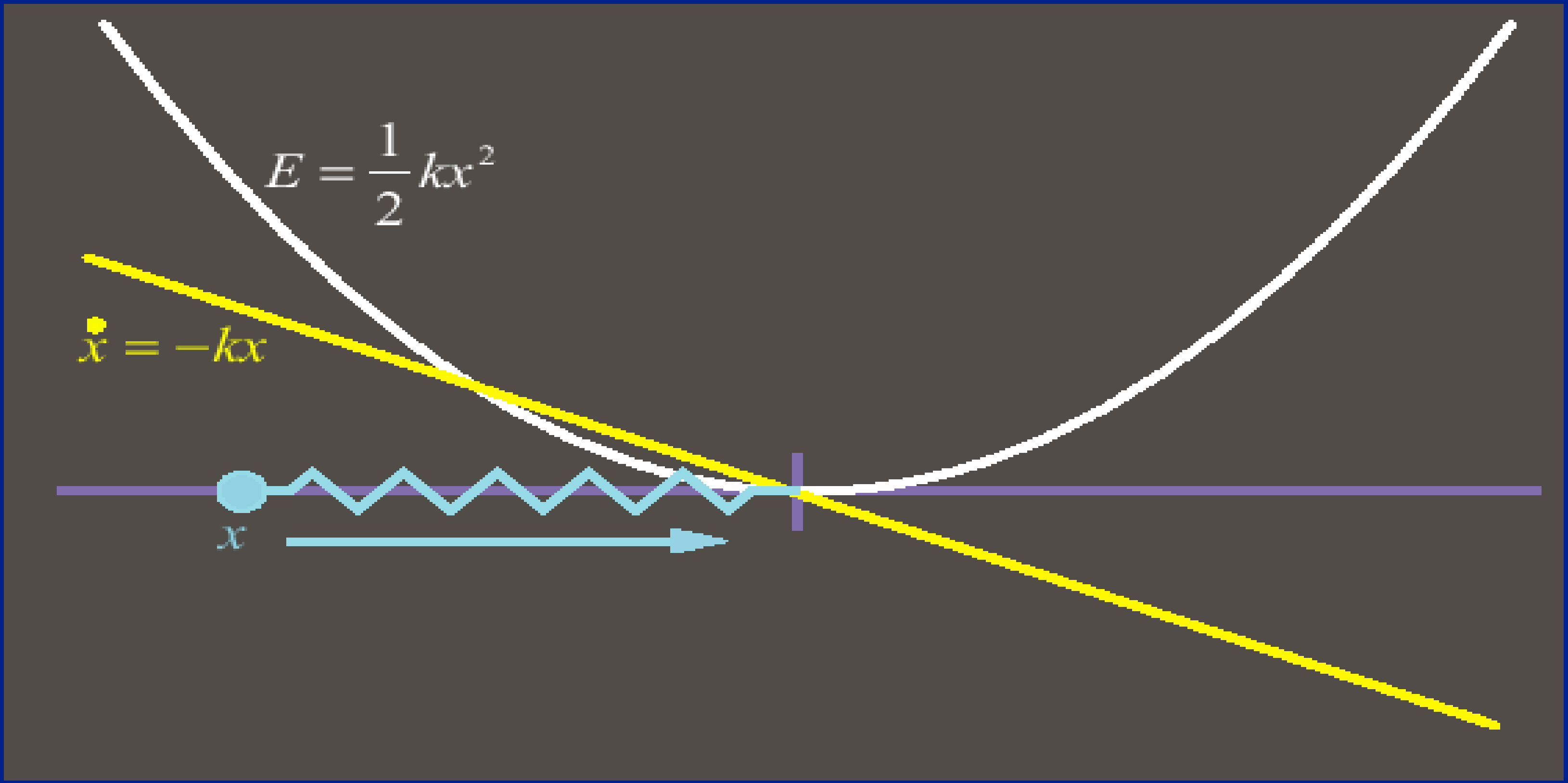
Determining Step Size

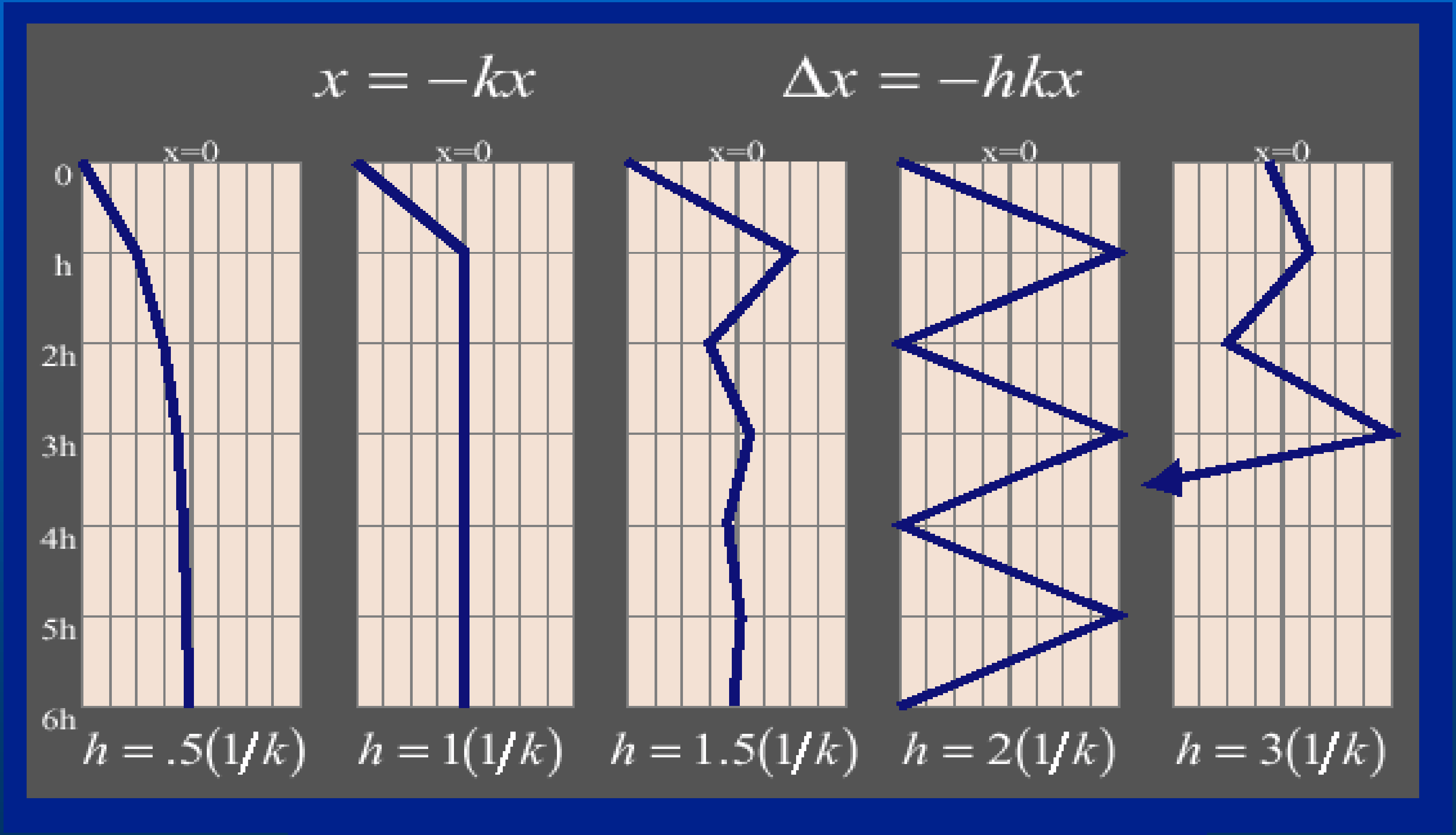
- Explicit Integration
  - Too big, unstable!
  - Too small, too slow
  - Adaptive, maybe
  - Ultimately the constants decide!
- Implicit Methods
  - Taking large steps when possible

An Example

A 1-D particle governed by $\dot{x} = -kx$ where k is a stiffness constant.



Speed Limitation of Euler's Method

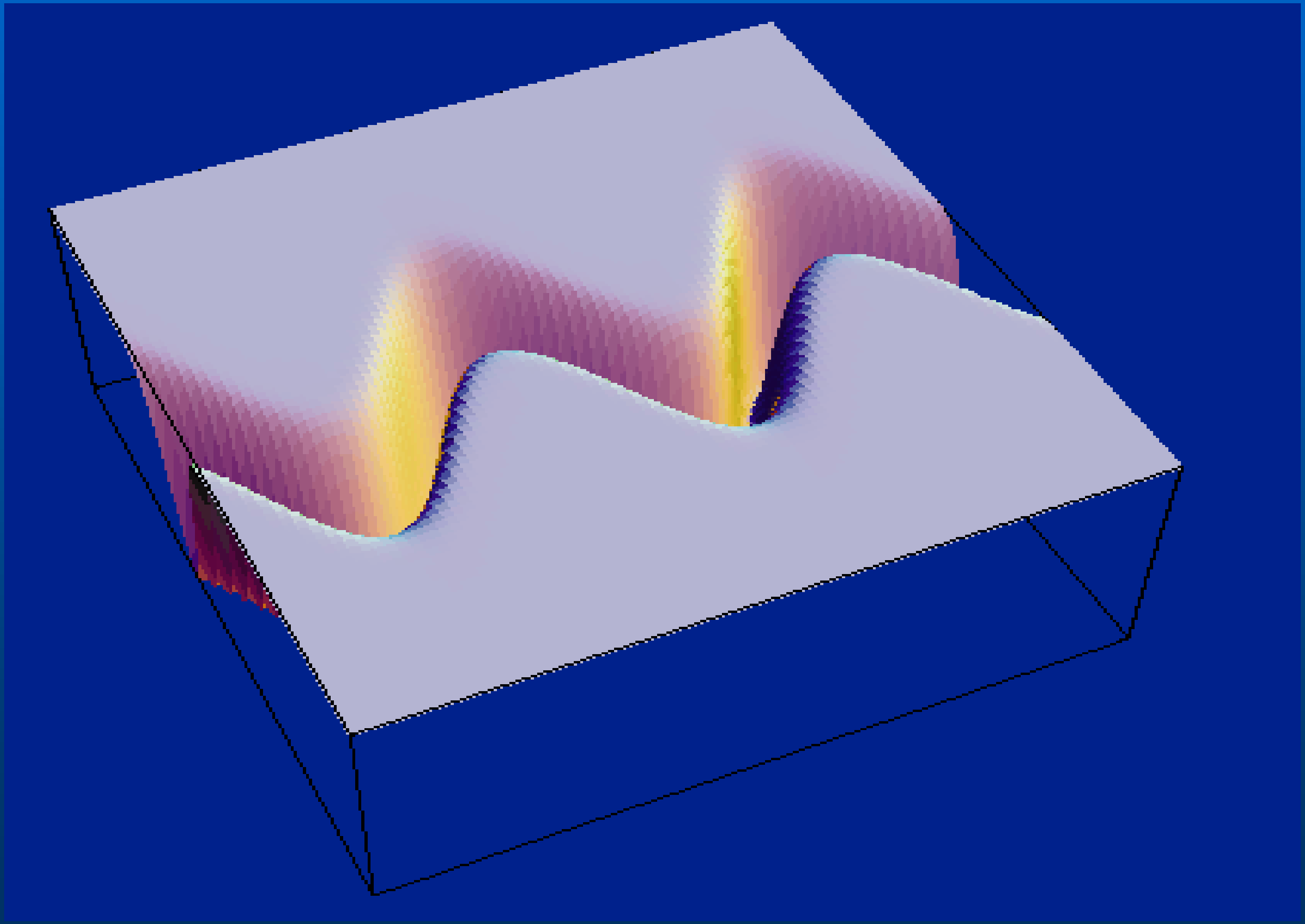


$h > 1/k$: oscillate. $h > 2/k$: explode!

Stiff Equations

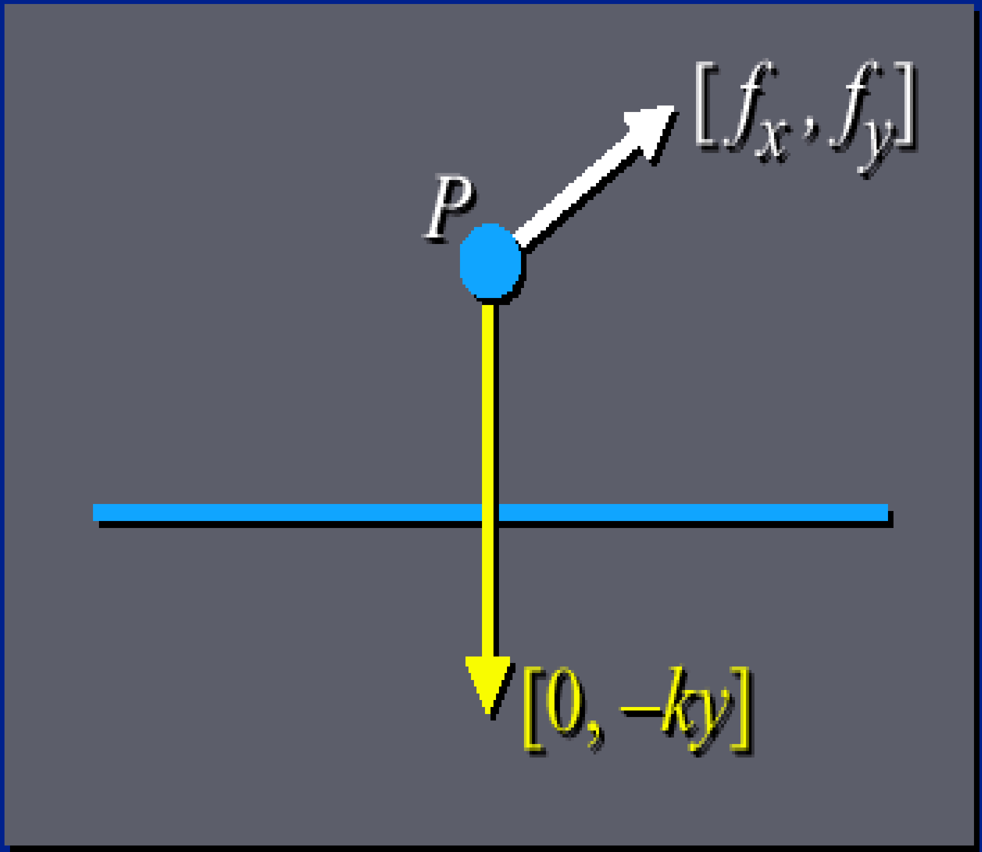
- In more complex systems, step size is limited by the largest k. One stiff spring can screw it up for everyone else.
- Systems that have some big k's mixed in are called stiff systems.

A Stiff Energy Landscape



Example: Particle-on-line

- A particle P in the plane.
- Interactive "dragging" force $[f_x, f_y]$.
- A penalty force $[0, -ky]$ tries to keep P on the x-axis.



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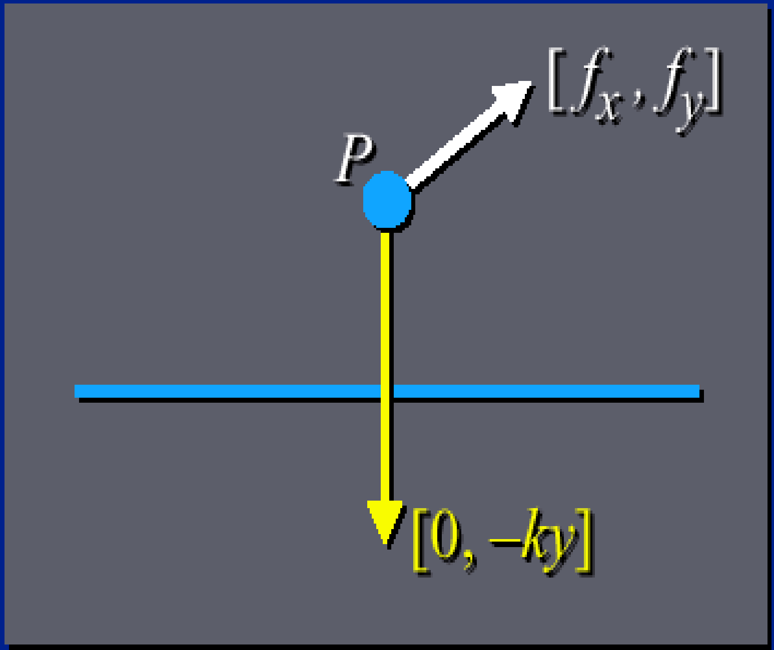
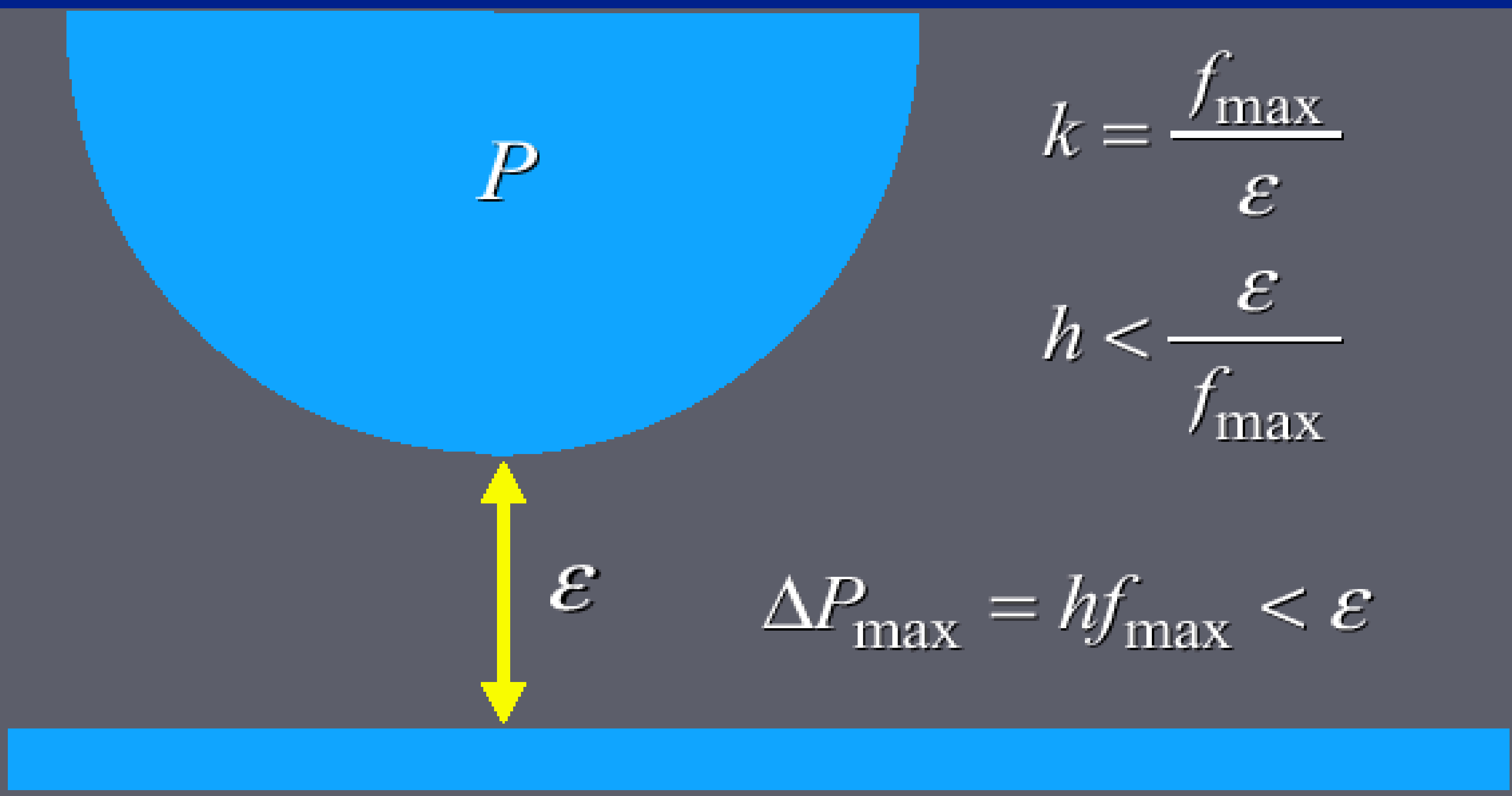


- Suppose you want P to stay within a miniscule ε of the x-axis when you try to pull it off with a huge force f_{max}.
- How big does k have to be? How *small* must h be?

Example: Particle-on-line

Really big k. Really small h.

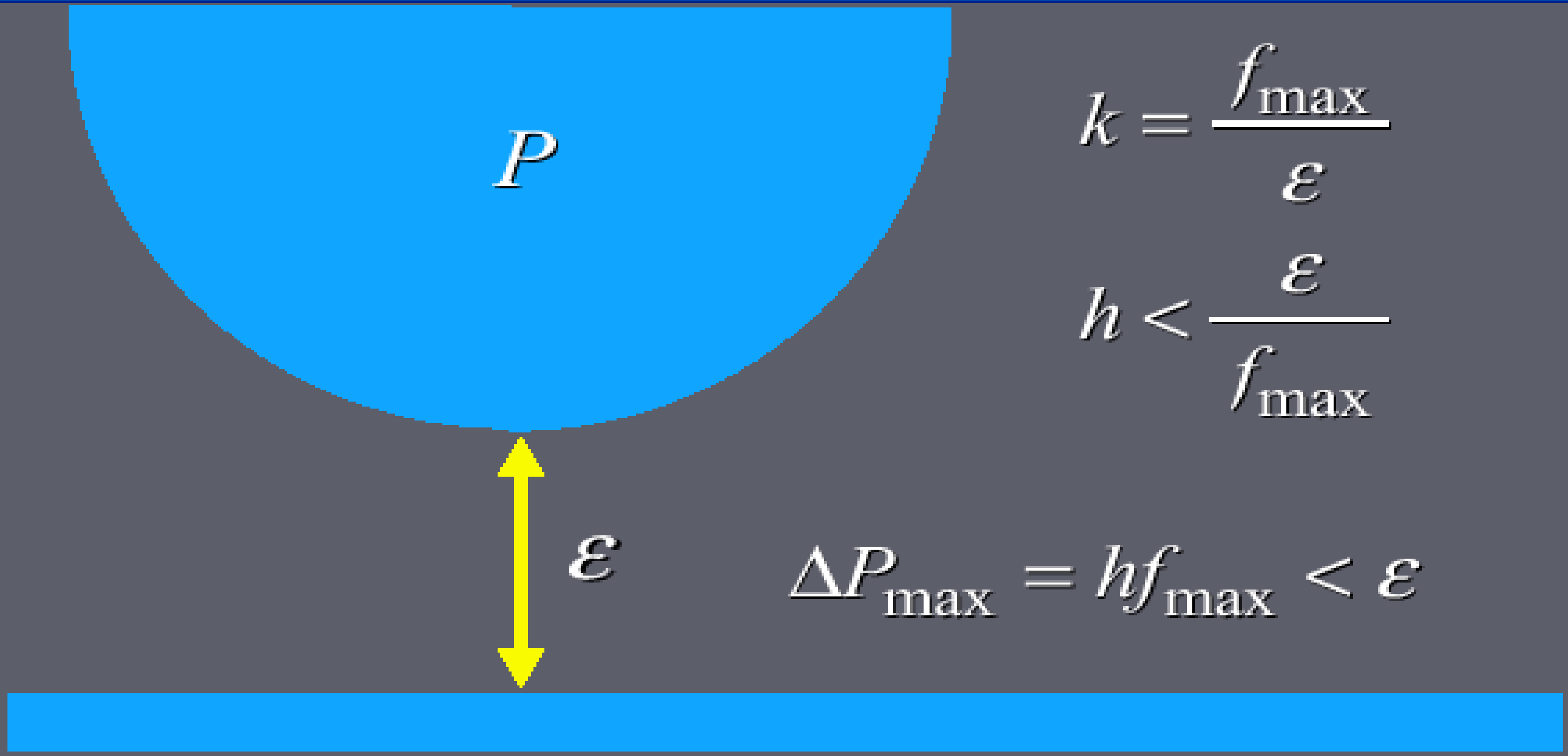


$$k = \frac{f_{\max}}{\varepsilon}$$

$$h < \frac{\varepsilon}{f_{\max}}$$

$$\Delta P_{\max} = h f_{\max} < \varepsilon$$

Example: Particle-on-line



$$k = \frac{f_{\max}}{\varepsilon}$$

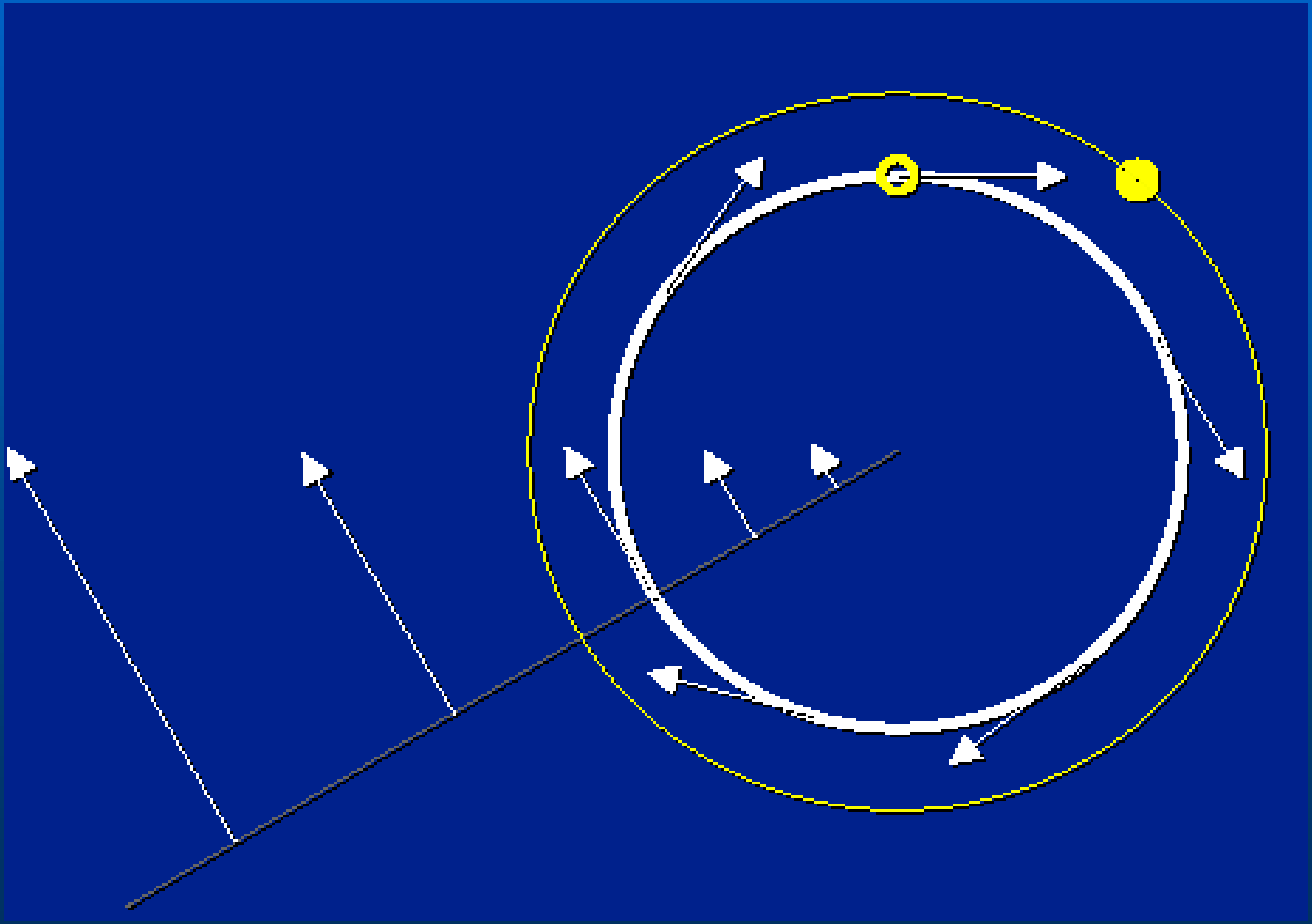
$$h < \frac{\varepsilon}{f_{\max}}$$

$$\Delta P_{\max} = h f_{\max} < \varepsilon$$

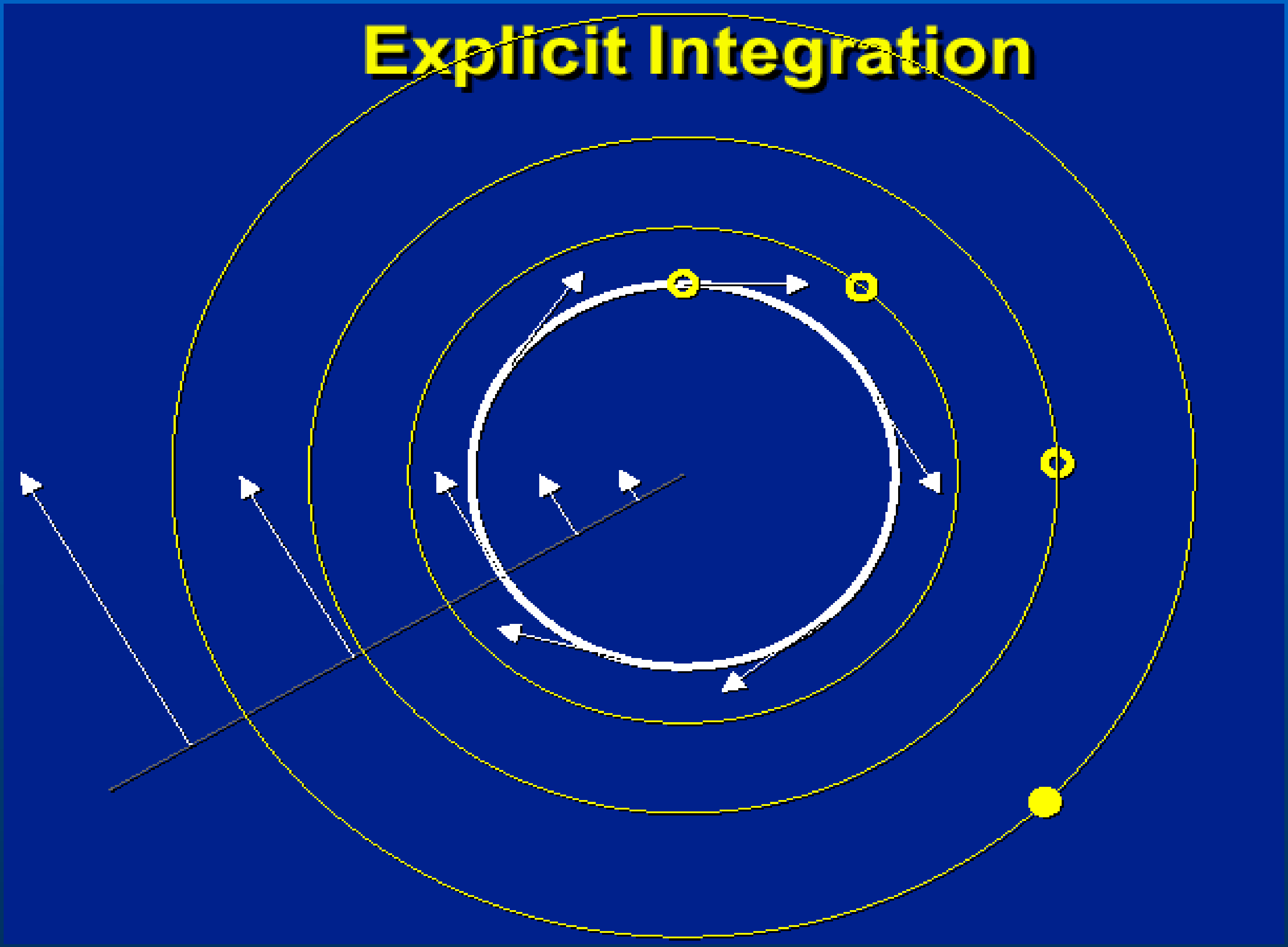
Answer: h has to be so small that P will never move more than ε per step.

Result: Your simulation grinds to a halt.

Explicit Integration



Problems



Explicit vs. Implicit Euler Method

$$x(t_0 + h) = x(t_0) + h\,\dot{x}(t_0)$$

vs.

$$x(t_0 + h) = x(t_0) + h\,\dot{x}(t_0 + \Delta t)$$

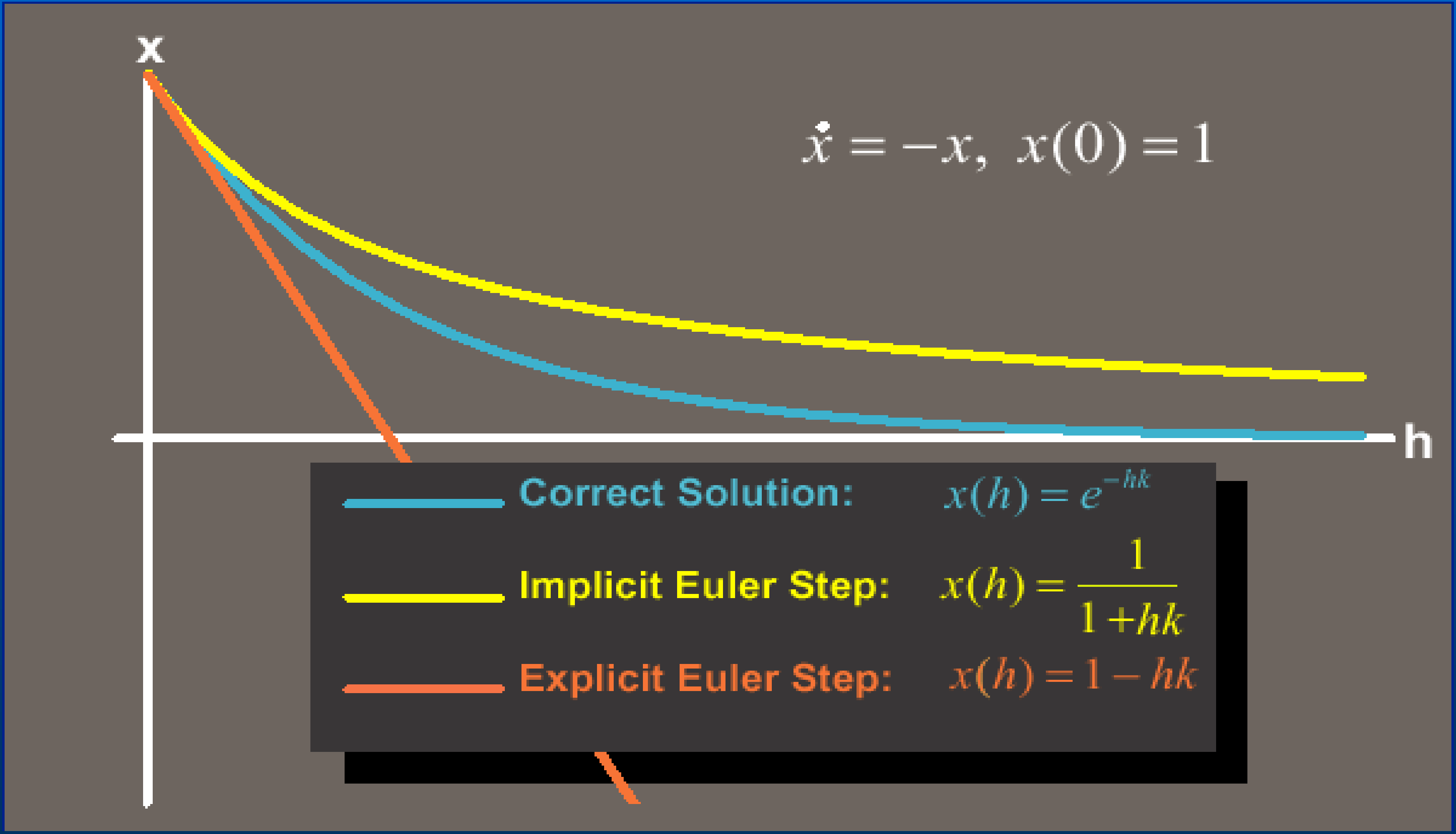
Implicit Euler for $\dot{x} = -kx$

$$x(t+h) = x(t) + h\,\dot{x}(t+h)$$

$$= x(t) - hkx(t+h)$$

$$= \frac{x(t)}{1+hk}$$

One Step: Implicit vs. Explicit

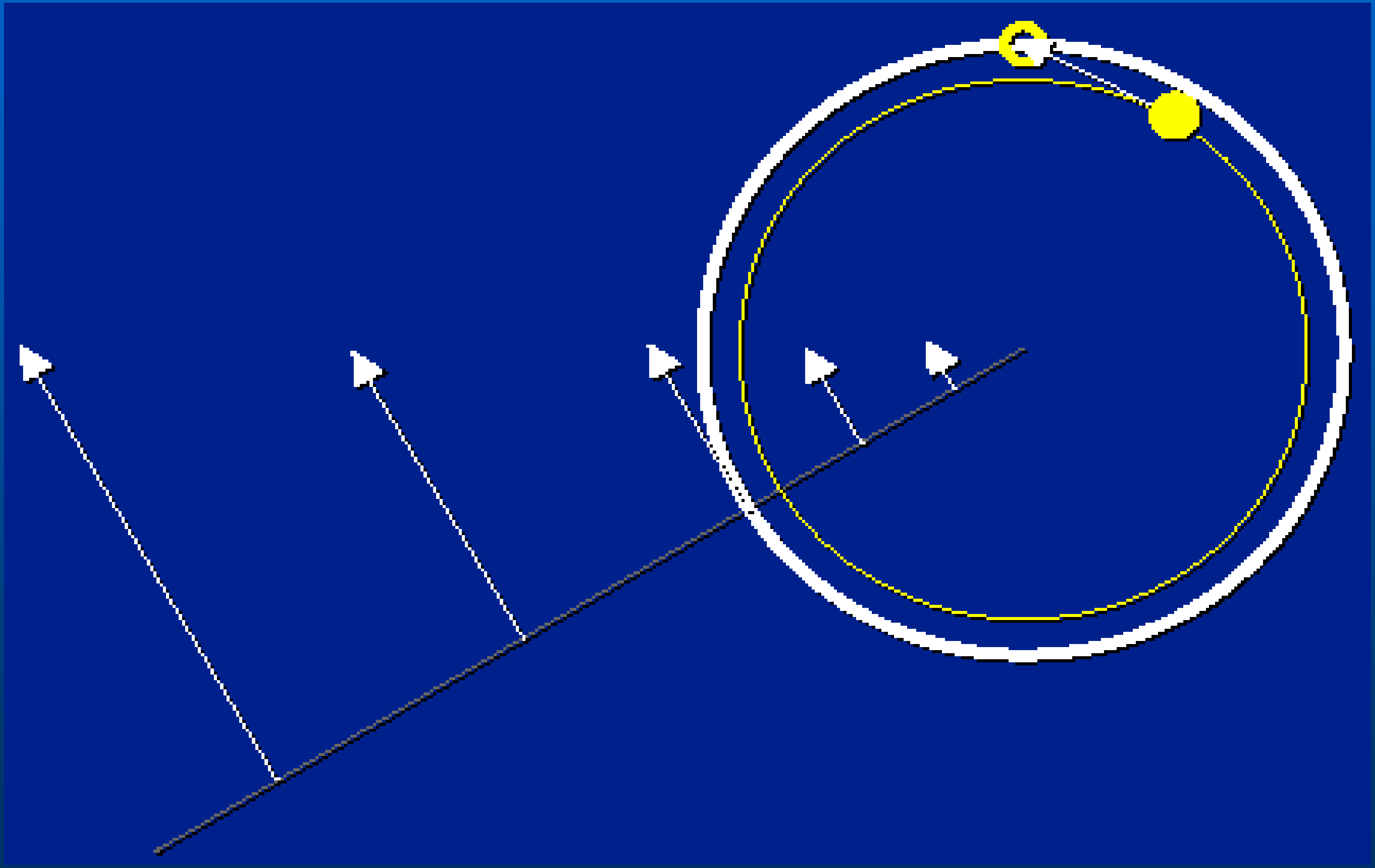


Large Systems

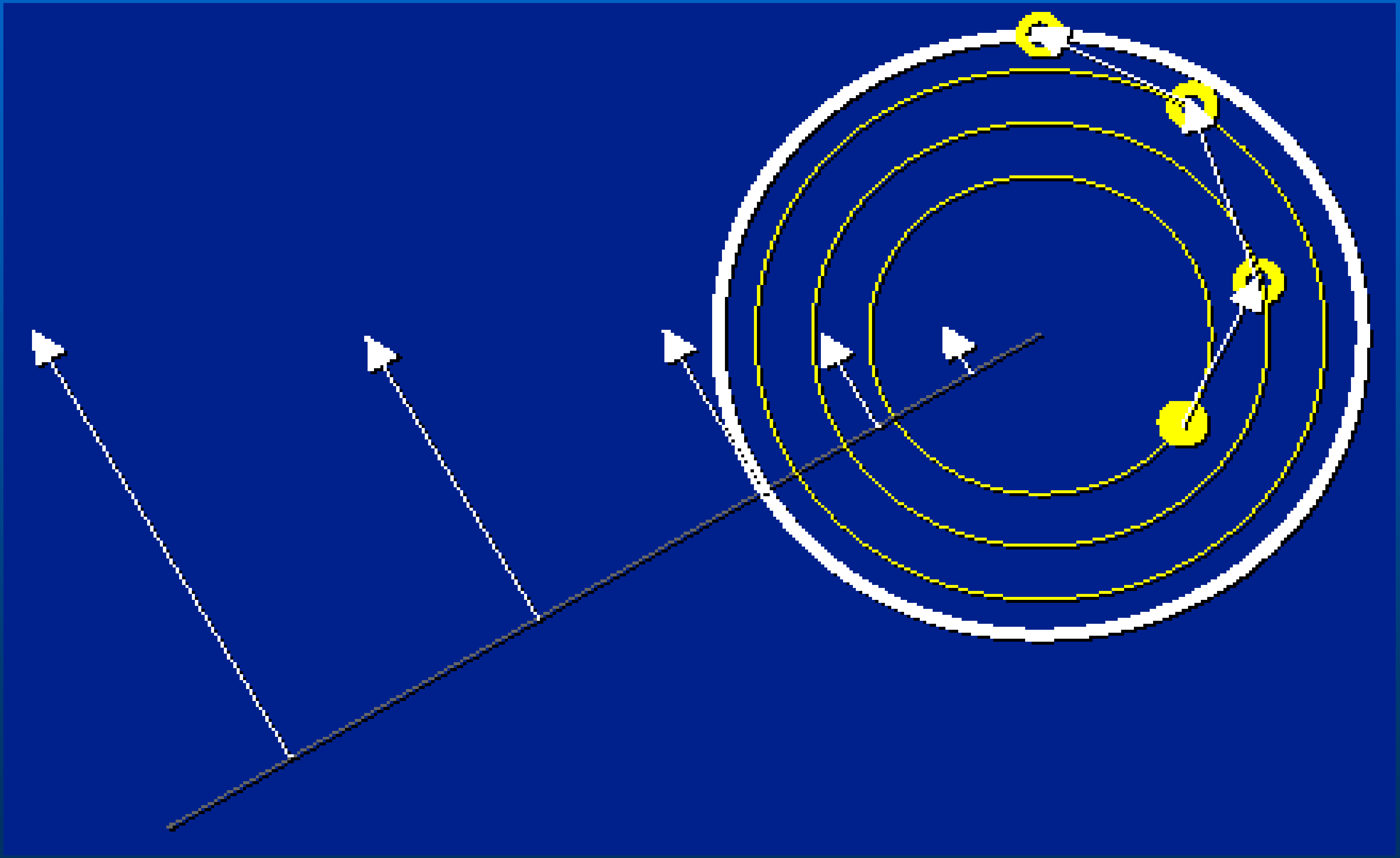
$$\frac{d}{dt}\mathbf{X}(t) = \dot{\mathbf{X}}(t) = f\left(\mathbf{X}(t)\right)$$

$$\Delta\mathbf{X}(t_0) = h\,\dot{\mathbf{X}}(t_0 + \Delta t) = h\,f\left(\mathbf{X}(t_0 + \Delta t)\right)$$
$$= h\,f\left(\mathbf{X}(t_0) + \Delta\mathbf{X}(t_0)\right)$$

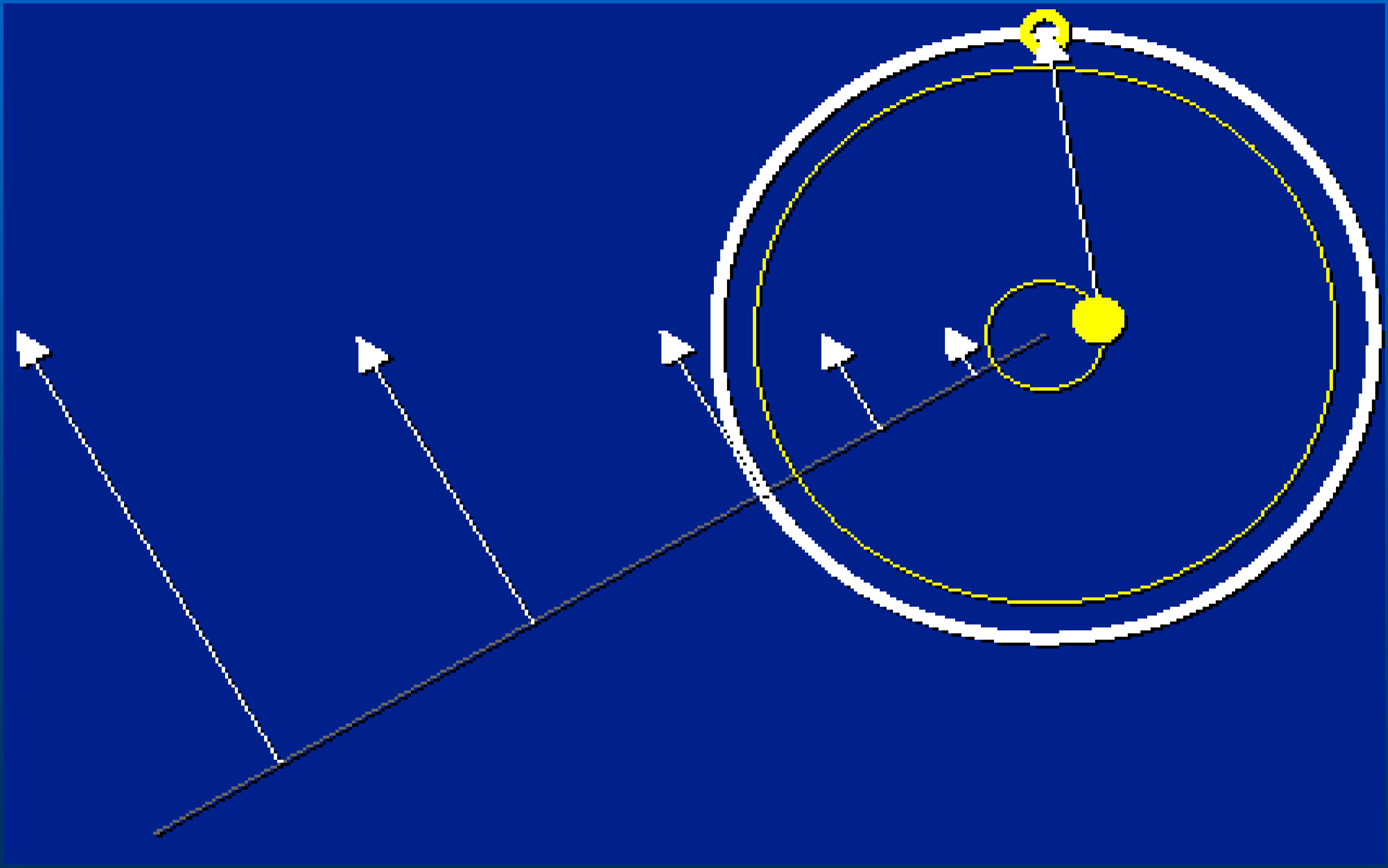
Implicit Integration



Implicit Integration



Implicit Integration



Linearized Implicit Integration

$$\dot{\mathbf{X}}(t) = f\left(\mathbf{X}(t)\right)$$

$$\Delta\mathbf{X} = h\, f\left(\mathbf{X}_0 + \Delta\mathbf{X}\right)$$

$$\Delta\mathbf{X} = h\left(f\left(\mathbf{X}_0\right) + \left(\frac{\partial f}{\partial \mathbf{X}}\right)\Delta\mathbf{X} \right)$$

Single-Step Implicit Euler Method

$$\Delta\mathbf{X} = h\left(f(\mathbf{X}_0) + \left(\frac{\partial f}{\partial \mathbf{X}}\right)\Delta\mathbf{X} \right)$$

$$\underbrace{\left(\mathbf{I} - h\frac{\partial}{\partial \mathbf{X}}\left(\dot{\mathbf{X}}(t_0) \right)\right)}_{n\times n \text{ sparse matrix}}\Delta\mathbf{X} = h\,\dot{\mathbf{X}}(t_0)$$

Solving Large Systems

- **Matrix structure reflects force-coupling:**
- **(i , j)th entry exists iff f_i depends on X_j**
- **Conjugate gradient a good first choice**