Determining Frictional Inconsistency
for Rigid Bodies is *NP*-Complete

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Abstract

The computational complexity of computing the forces between bodies in contact is presented. The bodies are restricted to be perfectly rigid bodies that contact at finitely many points. It has been known for some time that under the Coulomb model of friction, some configurations of bodies are inconsistent; that is, no contact forces satisfying the constraints of the Coulomb friction model exist for the configuration. The main result of this paper is a proof that determining if a configuration is inconsistent is an *NP*-complete problem. An immediate corollary of this proof is that computing the contact forces for a configuration of bodies is *NP*-hard. Computing contact forces remains *NP*-hard even if configurations are restricted to be consistent.

1. Introduction

The computational problem which motivates this research is the following: given a collection of perfectly rigid bodies that contact at finitely many points at time t_0, compute a valid set of contact forces for the contact points at t_0. A set of contact forces is *valid* if the following three conditions hold. First, the forces between the bodies are directed outwards; that is, the forces between bodies can "push" but not "pull". Second, the forces must be such that bodies do not inter-penetrate immediately after time t_0. Third, the non-frictional components of the contact forces must not perform any net work on the bodies. We will assume that the relative velocity of bodies at contacts points is such that no impulsive forces arise.

For systems without friction, a valid set of contact forces exists for any configuration of bodies[8]. Although a valid set of contact forces is guaranteed to exist, it is not necessarily unique. However, the accelerations of the bodies due to the contact forces are unique[3]. A valid set of contact forces can be found by solving a convex quadratic program (QP). The solution, if it exists, of a convex QP can be found in polynomial time[6]. In practice, convex QP's are solved by efficient algorithms whose worst case behavior is exponential but whose expected running time is polynomial[10]. Since a valid set of contact force is guaranteed to exist, the problem of determining contact forces is well in hand for systems without friction.

For systems with Coulomb friction, it has been known for almost a century that the accelerations due to the contact forces are not necessarily unique. Two valid sets of contact forces may give rise to two different accelerations of the bodies. Furthermore, there are configurations of bodies for which no set of contact forces is valid. We will call such a configuration an *inconsistent* configuration. As in the frictionless case, a quadratic program, not necessarily convex, can be formulated for a given configuration of bodies. For inconsistent configurations, there is no solution to the QP; otherwise, the solution of the QP translates to a valid set of contact forces for the configuration.

A recent paper by Mason and Wang[9] discusses inconsistent systems and advocates a resolution of the situation that involves introducing impulses into the system. However, it is first necessary to identify a given configuration as inconsistent. Clearly, the consistency of a configuration can be determined by checking the associated QP for a solution. However, determining whether or not a non-convex QP has a solution is an *NP*-complete problem. If every non-convex QP was associated with some configuration of bodies, this would immediately establish determining inconsistency as an *NP*-hard problem. Unfortunately, it is not clear that every QP problem is associated with some configuration of bodies.

We shall show that determining inconsistency for a configuration of bodies is an *NP*-complete problem. The proof will be by reducing the *NP*-complete problem "subset sum"[5] to the problem of determining inconsistency for a configuration of bodies. A corollary of this result will be that computing contact forces, even for configurations known to be consistent, is hard; that is, no polynomial time algorithm exists to compute contact forces unless $P = NP$.

2. Contact Force Model

In this section we present a configuration involving two bodies and one contact point. By choosing different values for the angular velocities and external forces acting on the bodies, we can produce a configuration with two distinct valid contact forces, and an inconsistent configuration. These particular configurations may be found in a number of papers; for example, Lötstedt[7], or Mason and Wang[9]. In the next section, these configurations will form the basic building blocks of a more complex configuration.

Let two bodies A and B be in contact as shown in figure 1. Assignments of various physical and geometric constants are summarized in this figure. Body A is a thin rod of mass m with a length of 2. Body B (the "base") is fixed in place. We will assume that the linear and angular velocities of A are such that the point p_a on A has a non-zero velocity tangent to B. The unit vector $\hat{n}$ is normal to the surface of B. The unit vector $\hat{t}$ is tangent to the surface of B, and is directed opposite to the motion of the point p_a; $\hat{n}$ and $\hat{t}$ are perpendicular. The vector $\vec{r}$ is the displacement of p_a from the center of mass of A. If A had uniform density, its moment of inertia I would be

$$\int_{-1}^{1} \frac{1}{2} r^2 m dr = \frac{1}{3} m \qquad (1)$$

We are therefore justified in assuming a (symmetric) mass distribution such that $I = \frac{1}{16} m$. The assignments $\theta = 72°$, $I = \frac{1}{16} m$ and $\mu \approx \frac{3}{4}$ are somewhat arbitrary; these values are chosen to simplify later computations.
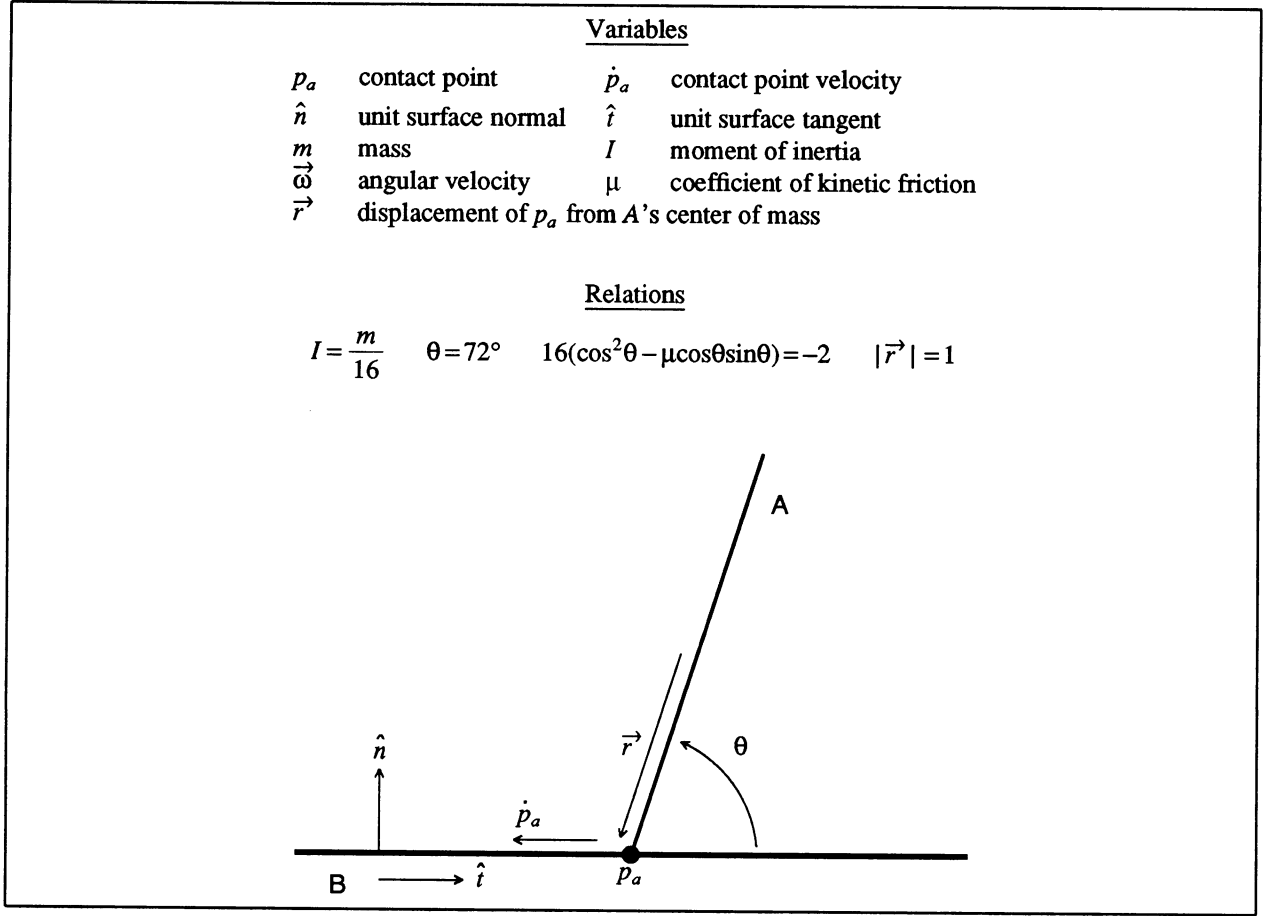
Variables

p_a	contact point	$\dot{p}_a$	contact point velocity
$\hat{n}$	unit surface normal	$\hat{t}$	unit surface tangent
m	mass	I	moment of inertia
$\vec{\omega}$	angular velocity	μ	coefficient of kinetic friction
$\vec{r}$	displacement of p_a from A's center of mass		

Relations

$$I = \frac{m}{16} \qquad \theta = 72° \qquad 16(\cos^2\theta - \mu\cos\theta\sin\theta) = -2 \qquad |\vec{r}| = 1$$

Figure 1. A one contact point configuration between a thin rod A and a fixed base B.

What forces arise from the contact of A and B? The contact force on A may be decomposed into two components (figure 2). The first component, a normal force parallel to $\hat{n}$, acts to prevent inter-penetration between A and B. The second component, a friction force parallel to $\hat{t}$, acts to oppose the sliding motion between A and B. If the normal force has magnitude f, then the Coulomb friction model states that the friction force has a magnitude μf, where μ is the coefficient of sliding friction between A and B. The net contact force acting on A is therefore

$$f\hat{n} + \mu f \hat{t} = f(\hat{n} + \mu\hat{t}). \tag{2}$$

An equal and opposite force acts on B without effect, since B is fixed.

In order for equation (2) to represent a valid contact force, the following conditions must hold. First, since contact forces may "push" but not "pull", the linear inequality

$$f \geq 0 \tag{3}$$

must be satisfied. Second, f must be sufficiently strong to prevent inter-penetration. For the simple geometry and initial conditions of figure 1, this is equivalent to stating that the point p_a have no acceleration opposite $\hat{n}$.[1] This may be expressed as the inequality

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[1] This constraint becomes more complicated when more general contact geometries and initial conditions are allowed. See Baraff[1, 2] for details.
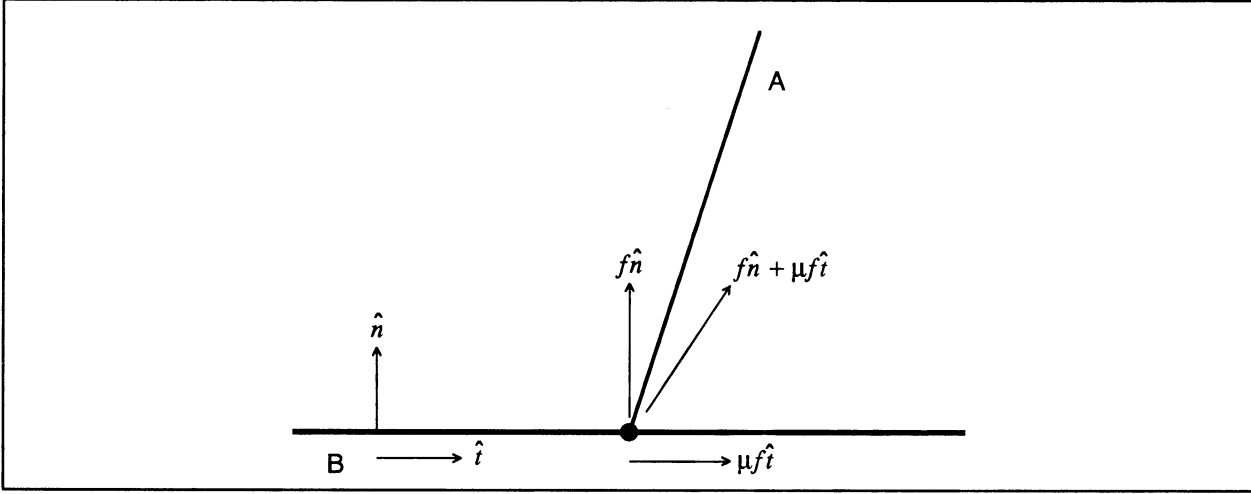
Figure 2. Normal and friction forces acting on A.

$$\hat{n} \cdot \ddot{p}_a \geq 0. \tag{4}$$

Third, the normal force must not perform any work. By D'Alembert's principle, this may be expressed as

$$f(\hat{n} \cdot \ddot{p}_a) = 0. \tag{5}$$

An intuitive explanation of equation (5) is that as long as f is positive, A must remain in contact with B, or net work will be done by the normal force $f\hat{n}$. Otherwise, if $\hat{n} \cdot \ddot{p}_a > 0$, A and B are separating and the contact force must be zero. In either case, at least one of f and $\hat{n} \cdot \ddot{p}_a$ must be zero.

We now compute equations (4) and (5) in terms of f for the configuration of figure 1. Define the following quantities: $\vec{a}$, the linear acceleration of A, $\vec{\alpha}$, the angular acceleration of A, and $\vec{\omega}$, the angular velocity of A. Since the coordinate system is two-dimensional, the angular quantities $\vec{\omega}$ and $\vec{\alpha}$ are perpendicular to both $\hat{n}$ and $\hat{t}$. The quantity $\ddot{p}_a$ may be decomposed into three terms: linear acceleration, tangential acceleration, and centripetal acceleration.

The linear acceleration, $\vec{a}$, is given by

$$\vec{a} = \frac{f\hat{n} + \mu f\hat{t} + (-mg)\hat{n}}{m} = \frac{f\hat{n} + \mu f\hat{t}}{m} - g\hat{n} \tag{6}$$

where $(-mg)\hat{n}$ is an external gravity force in the $-\hat{n}$ direction that acts (only) on A. The torque on A due to the normal and friction forces is

$$r \times (f\hat{n} + \mu f\hat{t}). \tag{7}$$

This yields an angular acceleration of

$$\vec{\alpha} = \frac{r \times (f\hat{n} + \mu f\hat{t})}{I}. \tag{8}$$

The tangential acceleration of p_a, $\vec{\alpha} \times \vec{r}$, is then

$$\frac{r \times (f\hat{n} + \mu f\hat{t})}{I} \times \vec{r}. \tag{9}$$

The centripetal acceleration of p_a is

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}). \tag{10}$$

Thus,

$$\ddot{p}_a = \vec{a} + \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}). \tag{11}$$

Substituting equations (6), (9) and (10),

$$\ddot{p}_a = \frac{f\hat{n} + \mu f\hat{t}}{m} - g\hat{n} + \frac{r \times (f\hat{n} + \mu f\hat{t})}{I} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}). \tag{12}$$

Taking the dot product of equation (6) with $\hat{n}$,

$$\hat{n} \cdot \vec{a} = \frac{f\hat{n} \cdot \hat{n} + \mu f\hat{n} \cdot \hat{t}}{m} - g\hat{n} \cdot \hat{n} = \frac{f}{m} - g. \tag{13}$$

Taking the dot product of equation (9) with $\hat{n}$ and using the geometry of figure 1,

$$\hat{n} \cdot (\vec{\alpha} \times r) = \hat{n} \cdot \left[\frac{r \times (f\hat{n} + \mu f\hat{t})}{I} \times \vec{r} \right] \tag{14}$$

which simplifies to

$$\hat{n} \cdot (\vec{\alpha} \times r) = \frac{f(\cos^2\theta - \mu\cos\theta\sin\theta)}{I}. \tag{15}$$

Last, taking the dot product of equation (10) with $\hat{n}$,

$$\hat{n} \cdot \vec{\omega} \times (\vec{\omega} \times \vec{r}) = |\vec{\omega}|^2\cos\theta. \tag{16}$$

Thus,

$$\hat{n} \cdot \ddot{p}_a = \frac{f}{m} - g + \frac{f(\cos^2\theta - \mu\cos\theta\sin\theta)}{I} + |\vec{\omega}|^2\cos\theta. \tag{17}$$

Factoring out f, and using $I = \frac{1}{16}m$,

$$\hat{n} \cdot \ddot{p}_a = \frac{f}{m}(1 + 16(\cos^2\theta - \mu\cos\theta\sin\theta)) + |\vec{\omega}|^2\cos\theta - g. \tag{18}$$

Since μ and θ were chosen such that

$$16(\cos^2\theta - \mu\cos\theta\sin\theta) = -2, \tag{19}$$

we have simply

$$\hat{n} \cdot \ddot{p}_a = \frac{f}{m}(1 - 2) + |\vec{\omega}|^2\cos\theta - g = -\frac{f}{m} + (|\vec{\omega}|^2\cos\theta - g). \tag{20}$$

The configuration of figure 1 can be made to have two distinct contact forces as follows. Let g and $\vec{\omega}$ be chosen[2] such that

$$|\vec{\omega}|^2\cos\theta - g = 1. \tag{21}$$

Then

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[2]Given a particular $\vec{\omega}$, the linear velocity of A must be chosen such that p_a's velocity is tangent to B. The linear velocity of A does not appear in any of the computations of this paper.

$$\hat{n} \cdot \ddot{p}_a = -\frac{f}{m} + 1. \tag{22}$$

From equations (3), (4), and (5), f must therefore satisfy the QP

$$f \geq 0, \quad -\frac{f}{m} + 1 \geq 0 \quad \text{and} \quad f(-\frac{f}{m} + 1) = 0. \tag{23}$$

Equation (23) has two solutions for f, and thus two valid contact forces. The two solutions of equation (23) are

$$f = 0 \quad \text{and} \quad f = m. \tag{24}$$

For the $f = 0$ solution, $\hat{n} \cdot \ddot{p}_a = 1$. In this solution, the centripetal acceleration of $\ddot{p}_a$ is stronger than the force of gravity pulling A down; thus, A merely continues its rotation and the point p_a moves off of B (figure 3a), although the center of mass of A accelerates downwards.

In the second solution, $f = m$ and $\hat{n} \cdot \ddot{p}_a = 0$. A normal force of $m\hat{n}$ and a friction force of $\mu m \hat{t}$ act on A. The friction force induces a torque opposite to $\vec{\omega}$, pushing p_a right and down; the gravity force $(-mg)\hat{n}$ also pulls p_a downwards. The normal force induces a torque that pushes p_a left and up. Additionally, p_a has an upwards component of centripetal acceleration. The net result is that p_a remains in contact with B. A continues to rotate, although its center of mass does accelerate downwards (figure 3b). Note that the *only* valid values of f for this configuration are $f = 0$ or $f = m$.
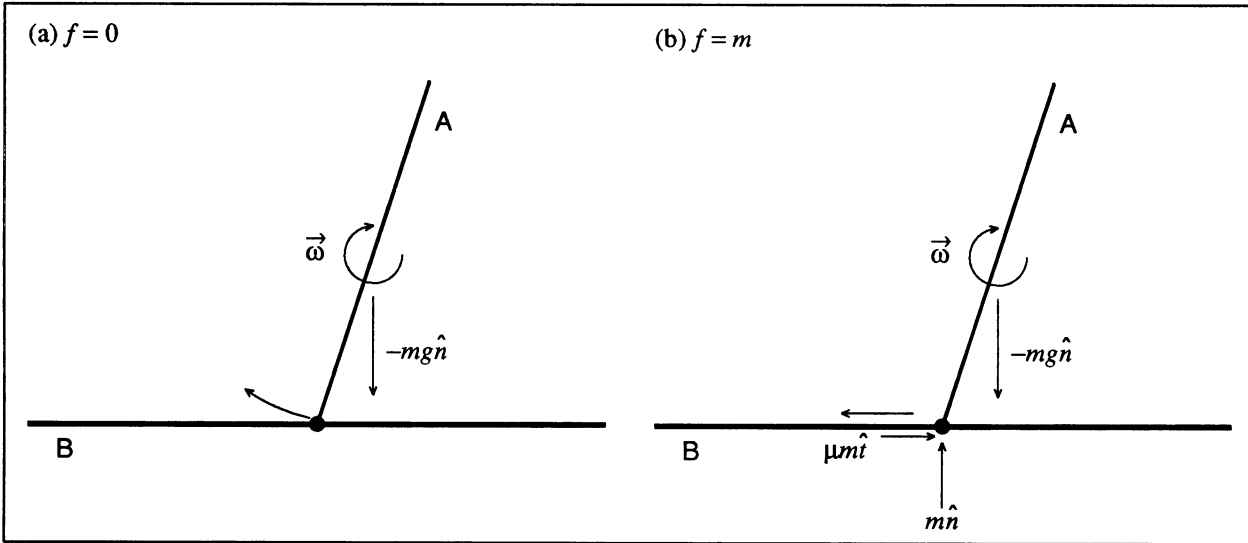

Figure 3. (a) The contact force between A and B is zero. p_a rotates to the left and up, losing contact with B. (b) The normal and friction forces balance gravity and centripetal acceleration; p_a moves horizontally and maintains contact with B.

Now suppose that $|\vec{\omega}| = 0$ and A's linear velocity is opposite $\hat{t}$ (figure 4). Then f must satisfy

$$f \geq 0, \quad -\frac{f}{m} - g \geq 0 \quad \text{and} \quad f(-\frac{f}{m} - g) = 0. \tag{25}$$

However, if $g > 0$ (figure 4a), then

$$f \geq 0 \quad \text{and} \quad \frac{-f}{m} - g \geq 0 \tag{26}$$

cannot be simultaneously satisfied. For $g > 0$, there is no valid contact force for this configuration and the configuration is inconsistent. A physical description of the inconsistency is the following: no matter what value is assigned to f, the direction of the contact force

$$f\hat{n} + \mu f \hat{t} \tag{27}$$

is unchanged. As f is increased, the normal force pushes p_a upwards from B more strongly. However, the torque due to friction, which causes p_a to rotate downwards into B, increases as well. Because of the geometry, the downwards acceleration of p_a from the friction force outweighs the upwards component of the acceleration of p_a from the normal force. Thus, no (non-negative) value of f is sufficient to prevent inter-penetration. The existence of such a configuration may seem counter-intuitive; the reader is referred to Mason and Wang[9] for more discussion on this phenomenon.
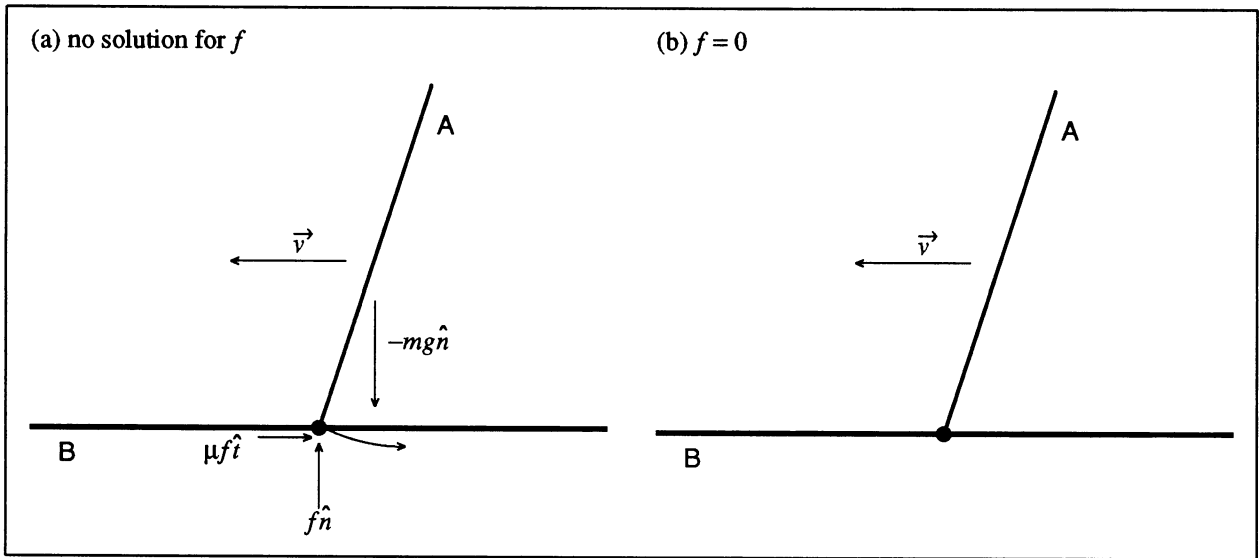

Figure 4. (a) An inconsistent configuration. For any $f \geq 0$, p_a is accelerated downwards into B. (b) The configuration has a unique solution of $f = 0$ when gravity is removed; A skims along the surface of B.

Note that the value of g is crucial. If $g = 0$, so that no external force acts on A, then $f = 0$ becomes the (unique) valid contact force (figure 4b). This corresponds to p_a "skimming" horizontally over B, with neither a normal force nor a friction force. If g becomes even slightly positive however, the configuration is inconsistent.

In anticipation of developments in the next section, we present a slightly modified version of the configuration of figure 4. Body B will be free to move vertically, either up or down. However, we will still constrain body B from either rotating or moving horizontally. We will let the mass, M, of body B be $M = 100m$ so that B is still massive compared to A. Lastly, we will assume that there is no external gravity force acting on A, but that B is subject to a vertical acceleration of magnitude a_b, from external forces (figure 5)). A positive value for a_b indicates an upwards acceleration of B, towards A.

Since B may now move vertically, equation (25) must be modified. Since there is no external force on A, $g = 0$. Instead of

$$-\frac{f}{m} - g \geq 0, \tag{28}$$

we have

$$-\frac{f}{m} + \frac{f}{M} - a_b \geq 0. \tag{29}$$

The term $\frac{f}{M}$ is the linear acceleration of B away from A from the contact force, and the term a_b is the upwards acceleration of B from external forces. Since $M = 100m$, we have

$$-\frac{f}{m} + \frac{f}{M} - a_b = -\frac{99}{100}\frac{f}{m} - a_b. \tag{30}$$

Equation (25) is then rewritten as

$$f \geq 0, \quad -\frac{99}{100}\frac{f}{m} - a_b \geq 0 \quad \text{and} \quad f(-\frac{99}{100}\frac{f}{m} - a_b) = 0. \tag{31}$$

As before, if $a_b > 0$, an inconsistency occurs. If $a_b \leq 0$, then $f = 0$ is a valid solution and the configuration is consistent.
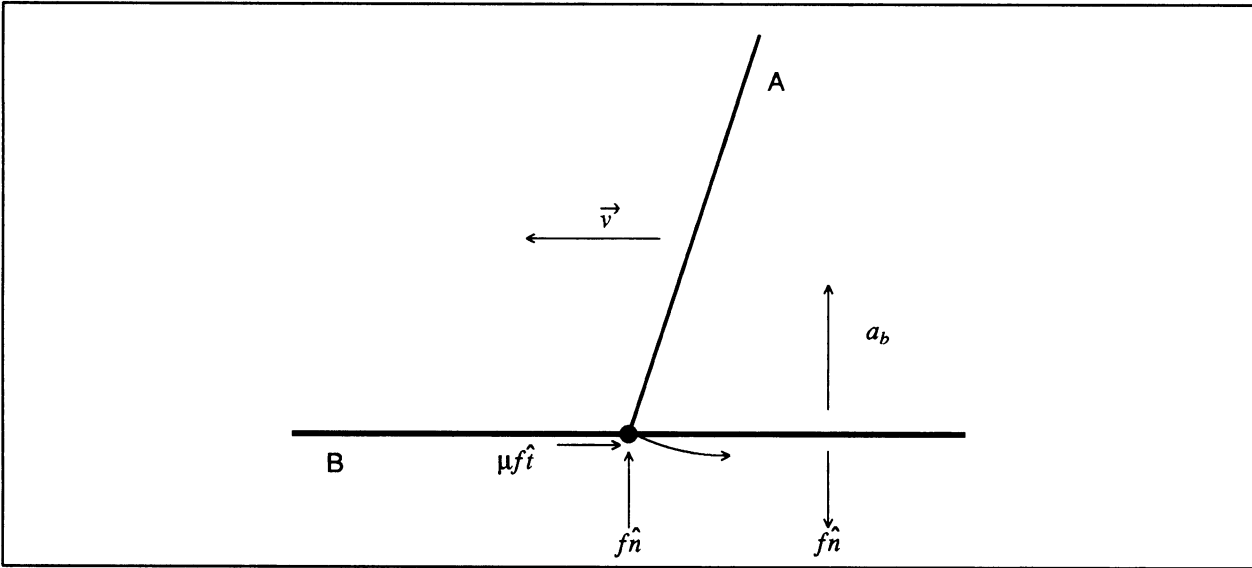

Figure 5. B **may move vertically, but is not allowed to move horizontally or rotate. If the external vertical acceleration a_b of B is positive, the configuration is inconsistent.**

3. *NP*-complete configurations

In this section, we will prove that determining frictional inconsistency is *NP*-complete. We will begin by showing that the problem of determining frictional inconsistency lies in *NP*.

Definition *An instance of the frictional inconsistency problem is a configuration C of bodies that contact at n distinct contact points. The physical properties of each body (mass, moment of inertia, linear and angular velocity, position and orientation, and external forces) are described by rational numbers. The specifics of a contact point (position, coefficient of friction, surface normal) are also described by rational numbers. The notation $|C| = k$ means that configuration C is describable in k bits. Clearly $n < k$.*

Theorem 1 *Determining whether or a not an instance C of the frictional inconsistency problem is consistent lies in NP.*

Proof. Given an instance of C, a QP of size n with the following properties exists. If C is consistent, then an n-vector $\vec{x}$ that is a solution to the QP exists. The set of contact forces such that the magnitude of the normal force at the ith contact point is x_i is a valid set of contact forces for the configuration C. Otherwise, if C is inconsistent, the QP has no solution. The specifics of constructing the QP may be found in Lötstedt[8] or Featherstone[4]. The numerical quantities in the QP are computed from the rational entries of C in a total of $O(n^3)$ arithmetical operations. The QP can therefore be constructed in time polynomial to k. Vavasis[11] has recently shown that quadratic programming lies in NP. It follows from this that determining frictional inconsistency is also in NP. ∎

In order to show that determining frictional inconsistency is NP-hard, we will reduce the NP-complete problem "subset sum" to the frictional inconsistency problem.

Definition *An instance of the subset sum problem is a pair (A,S) where $A = \{a_1, \cdots, a_n\}$ is a set of positive integers and S is a single positive integer. We say the subset sum instance (A,S) is satisfiable if there exists a subset $A' \subset A$ such that*

$$\sum_{a \in A'} a = S. \tag{32}$$

Determining if an instance of the subset sum problem is satisfiable is an NP-complete problem.

To show that determining frictional inconsistency is NP-hard we will take an instance (A,S) of the subset sum problem and construct a configuration of bodies C. The configuration C will have the property that C will be consistent if and only if (A,S) is satisfiable.

Theorem 2 *Determining frictional inconsistency is NP-hard.*

Proof. Consider the configuration of figure 6. Body B of figure 6 is initially at rest and is positioned by four fixed triangular wedges that contact B without friction. Body B is therefore free to move horizontally, but can neither rotate nor move vertically. On either side of body B are thin rods E_1 and E_2. E_1 and E_2 have no angular velocity and have a linear velocity as indicated. E_1 and E_2 contact B in the same manner as the configuration of figure 5 (although the frames of reference for E_1 and E_2 are rotated by 90° with respect to figure 4). As in figure 5, the only normal force between E_1 and B which does not cause inter-penetration is a normal force of magnitude zero. The same holds for E_2 and B. The configuration of figure 6 is therefore consistent if and only if a normal force of zero between E_1 and B is valid, and similarly for E_2 and B.
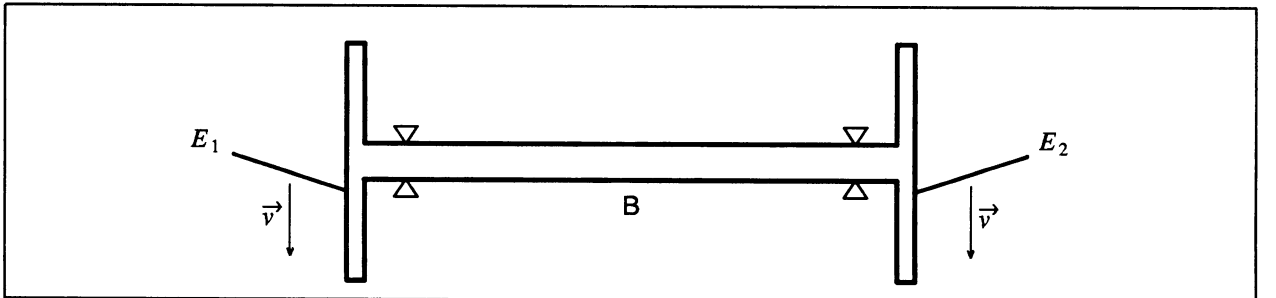

Figure 6. B **is constrained by the fixed wedges and can only move horizontally. However, the configuration is consistent only if** B **is not subject to a net horizontal force.**

Now suppose that B is subject to a net horizontal acceleration to the right from other forces. Then a normal force of zero between E_2 and B is an invalid solution. Since any non-zero normal force between E_2 and B is also invalid, the configuration is inconsistent if B is subject to a net horizontal force to the right. Similarly, the configuration is inconsistent if B is subject to a net horizontal acceleration to the left. Thus, the configuration of figure 6 is consistent only if the net horizontal acceleration of B is zero. In this case, the rods E_1 and E_2 skim along the surface of B as in figure 4b.

Now consider figure 7, where a collection of thin rods $R_1, \cdots, R_n$ have been added. In addition, an external horizontal force with magnitude μS acts on B, trying to accelerate B to the right. Each rod R_i has mass m_i. The configuration between each rod R_i and B is the same as the configuration of figure 3; thus each rod R_i has angular velocity $\vec{\omega}$ and is subject to an external gravity force. (The external gravity force however does not act on E_1 or E_2). Let f_i be the magnitude of the normal force between R_i and B. As in figure 3, the only valid solutions for f_i are $f_i = 0$ and $f_i = m_i$. If $f_i = 0$, then no friction force between R_i and B. Otherwise, $f_i = m_i$ and a friction force of magnitude μm_i acts between R_i and B. The friction force pushes R_i to the right and B to the left, with magnitude μm_i. The friction force on B therefore acts to oppose the external force of magnitude μS.
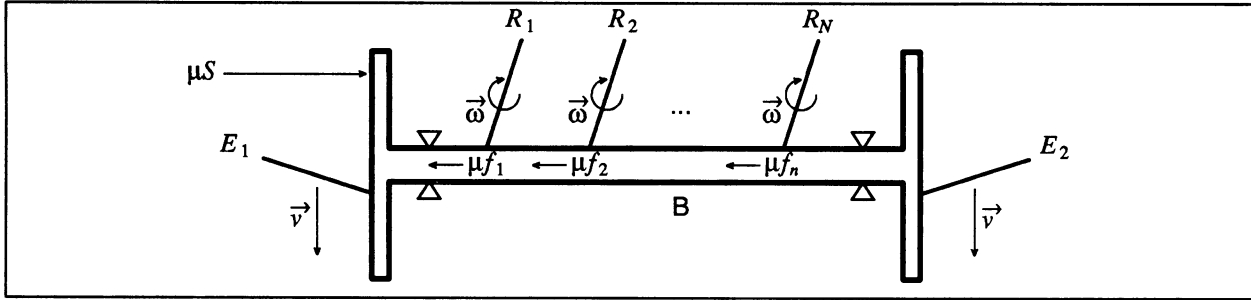

Figure 7. The configuration is consistent if and only if the friction forces on B sum to μS.

In order for the configuration of figure 7 to be consistent, B must have no net horizontal acceleration. This means that the friction forces exerted on B from the n rods must sum to μS, balancing the external force applied to B. Thus, the configuration is consistent if and only if

$$\sum_{i=1}^{n} \mu f_i = \mu S. \qquad (33)$$

Since each f_i is either 0 or m_i, the configuration is consistent if and only if there exists an index set $J \subset \{1, 2, \cdots, n\}$ such that

$$\sum_{j \in J} m_j = S. \qquad (34)$$

We can now perform the reduction from subset sum. Given a set $A = \{a_1, \cdots, a_n\}$ and a target sum S, construct the configuration[3] of figure 7. Assign $m_i = a_i$ for $1 \le i \le n$, and let an

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[3]The values chosen in the previous section for μ, θ and $|\vec{\omega}|$ may introduce irrational quantities into the descriptions of the configurations. Since the frictional inconsistency problem is described in terms of rational quantities, we will assume that μ, θ and $|\vec{\omega}|$ are suitably perturbed so that all quantities are rational, without otherwise disturbing the solution properties of the configurations. This can be done since the set of rational numbers is dense with respect to the set of real numbers.

external horizontal force of μS act on B as shown in figure 7.

By the above discussion, the configuration is consistent if and only if there exists $J \subset \{1,2,\cdots,n\}$ such that

$$\sum_{j \in J} m_j = S. \tag{35}$$

Since $a_i = m_i$ for $1 \le i \le n$, the set

$$A' = \{a_j \mid j \in J\} \tag{36}$$

is a set such that $A' \subset A$ and

$$\sum_{a \in A'} a = S. \tag{37}$$

Thus C is consistent if and only if (A,S) is satisfiable. We conclude that determining frictional inconsistency is NP-hard. ∎

Theorem 3 *Determining frictional inconsistency is NP-complete.*

Proof. The result follows immediately from Theorem 1 and Theorem 2. ∎

Corollary 1 *Computing contact forces for configurations is NP-hard.*

Proof. Since deciding if a set of contact forces exists is NP-complete, computing contact forces is NP-hard. ∎

A reasonable response to this corollary is to question its practical applicability to the problem of computing valid contact forces. Suppose that inconsistent configuration are too unlikely to be encountered in practice. Can we then construct an efficient algorithm that computes contact forces just for consistent configurations? The following corollary asserts that a polynomial time algorithm for computing contact forces, even restricted to consistent configurations, does not exist unless $P = NP$.

Corollary 2 *A polynomial time algorithm for computing valid contact forces for consistent configurations exists if and only if $P = NP$.*

Proof. Suppose that $P = NP$. Since quadratic programming is NP-complete, $P = NP$ implies a polynomial time algorithm for finding the solution to a QP. Since valid contact forces for a consistent configuration of bodies can be found by solving the associated QP, valid contact forces are computable in polynomial time if $P = NP$.

Conversely, suppose there exists a polynomial time machine M that, given a consistent configuration C as input, outputs a valid set of contact forces for the configuration. Then a machine M' that determines the consistency of a configuration C in polynomial can be constructed. M' operates as follows. Since M runs in polynomial time, given a consistent configuration C, it must output a valid set of contact forces within time $P(k)$, where P is a polynomial and $k = |C|$ is the size of C. Upon input of a (not necessarily consistent) configuration C, M' runs M on C until $P(k)$ time elapses. If M has not output an answer within time $P(k)$, then C must be inconsistent, and M' reports that C is inconsistent. If M does output an answer within time $P(k)$, M' can check the validity of the answer in polynomial time. If C is consistent, M's output will be valid, and M' reports that C is consistent. Otherwise, M's output is invalid, and C must therefore be inconsistent. M' therefore reports that C is inconsistent. Thus, M' can determine whether or not a configuration C is consistent in polynomial time. Since determining

inconsistency is *NP*-complete, we conclude that the existence of a polynomial time algorithm for computing contact forces on consistent configurations implies $P = NP$. ■

4. Conclusion

We have proven that deciding frictional inconsistency is *NP*-complete. We have also shown that computing contact forces, even when restricted to consistent configurations, is a hard problem. The configurations shown to be *NP*-complete had friction only at contact points that were in relative tangential motion. An interesting open question is the complexity of computing contact forces when friction is restricted to contact points that are not in relative tangential motion.

Acknowledgements

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