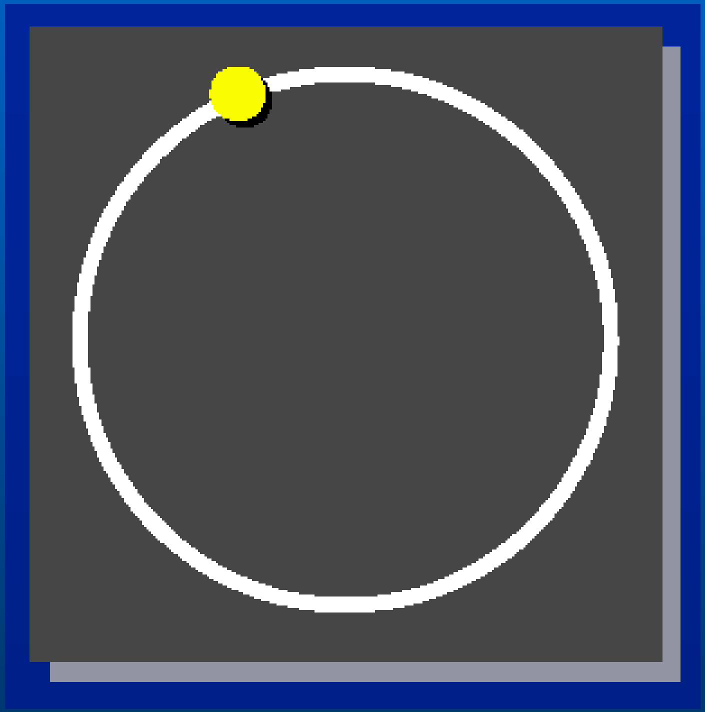
Announcements

- Homework #1 is due on Tuesday, 2/14/06
- Reading: SIGGRAPH 2001 Course Notes on Physically-based Modeling

Disclaimer

- The following slides reuse materials from SIGGRAPH 2001 Course Notes on Physically-based Modeling (copyright © 2001 by Andrew Witkin at Pixar).

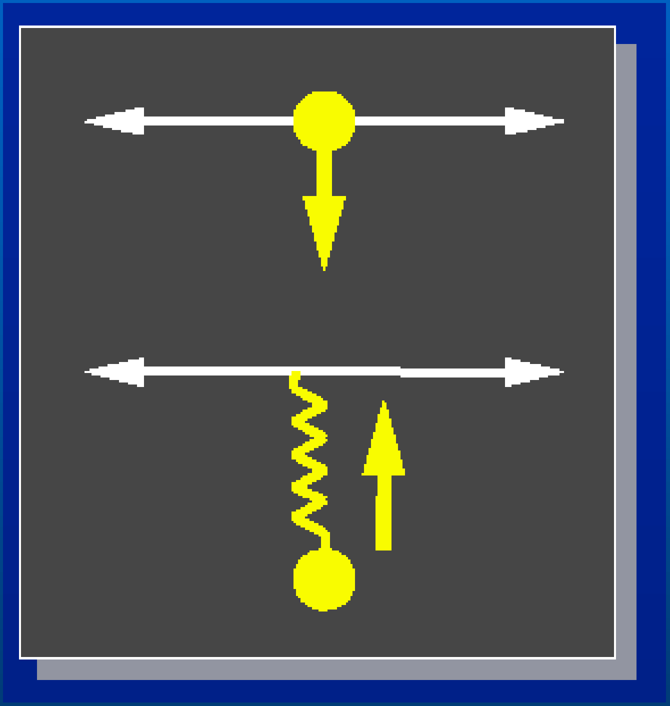
A Bead on a Wire



- Desired Behavior
  - The bead can slide freely along the circle.
  - It can never come off no matter how hard we pull.

So, how do we make this happen?

Penalty Constraints

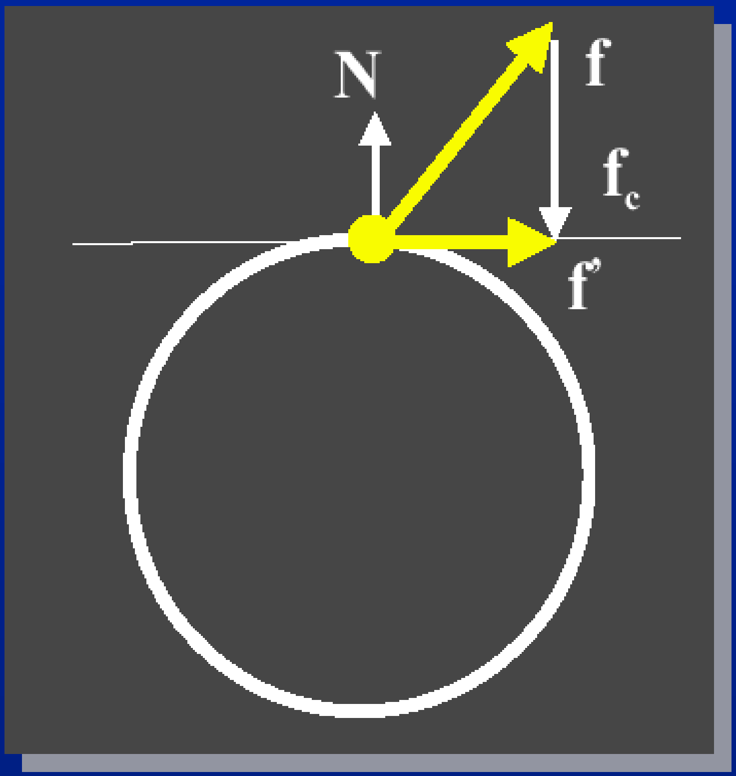


- How about using a spring to hold the bead to the wire?
- Problem
  - Weak springs ⇒ sloppy constraints
  - Strong springs ⇒ instability

Basic Ideas

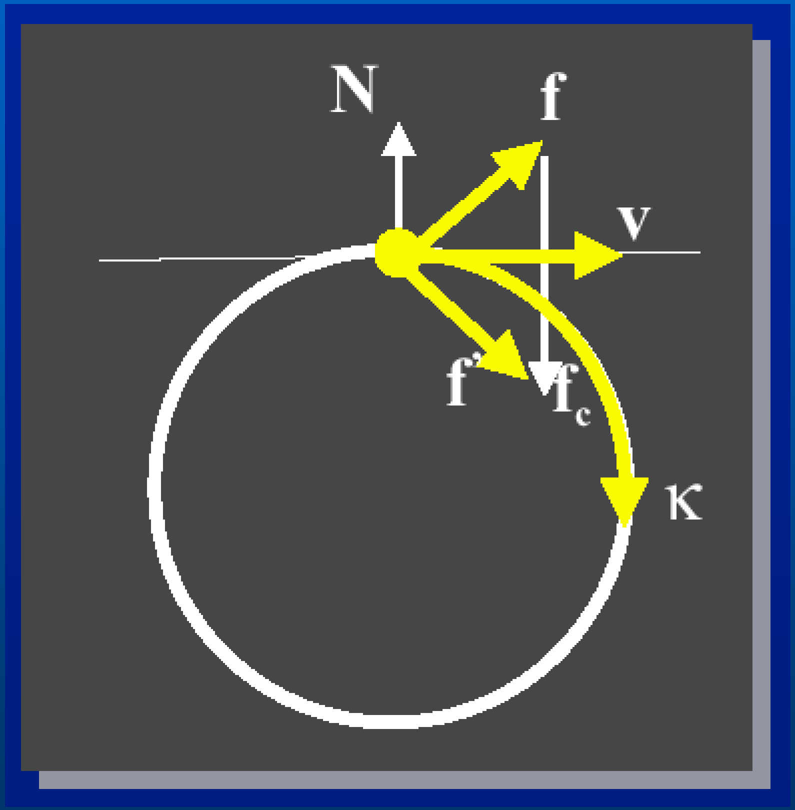
- Convert each constraint into a force imposed on a particle (system)
- Use principle of virtual work – constraint forces do not add or remove energy
- Solve the constraints using Lagrange multipliers, λ's
  - For particle systems, need to use the derivative matrix, J, or the Jacobian Matrix.

Geometric Interpretation



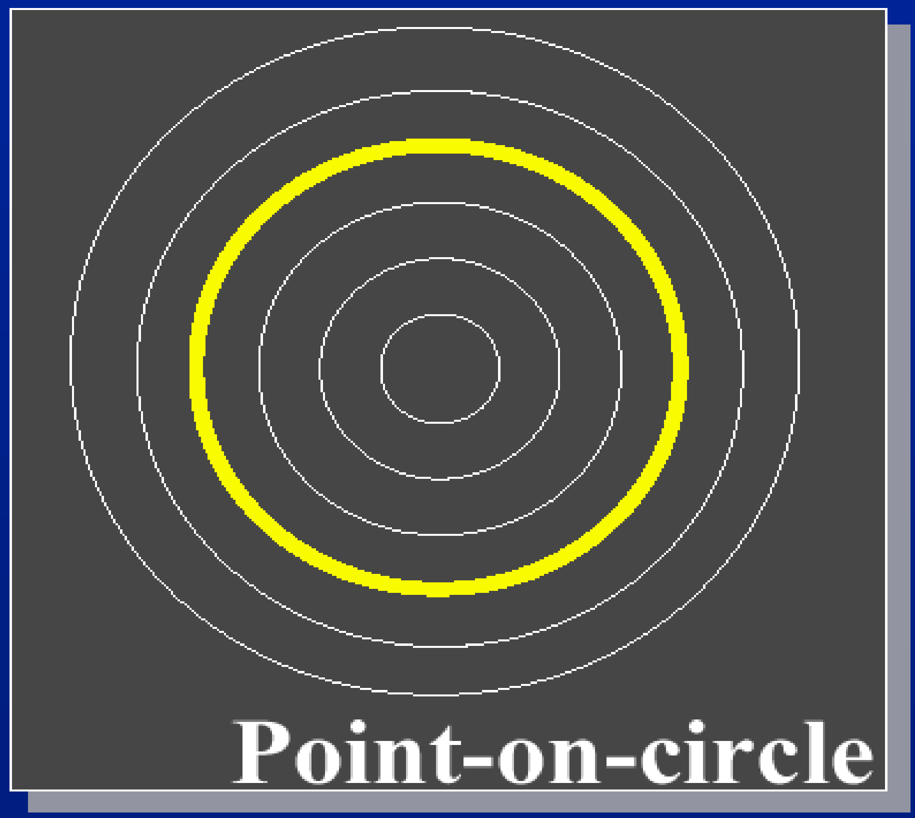
- 1st order world.
- *Legal velocity:* tangent to circle ($N \cdot v = 0$).
- *Project* applied force f onto tangent: $f' = f + f_c$ $\quad f_c = -\frac{f \cdot N}{N \cdot N} N$
- Added normal-direction force f_c: *constraint force.*
- No tug-of-war, no stiffness.

Basic Formulation (F = ma)



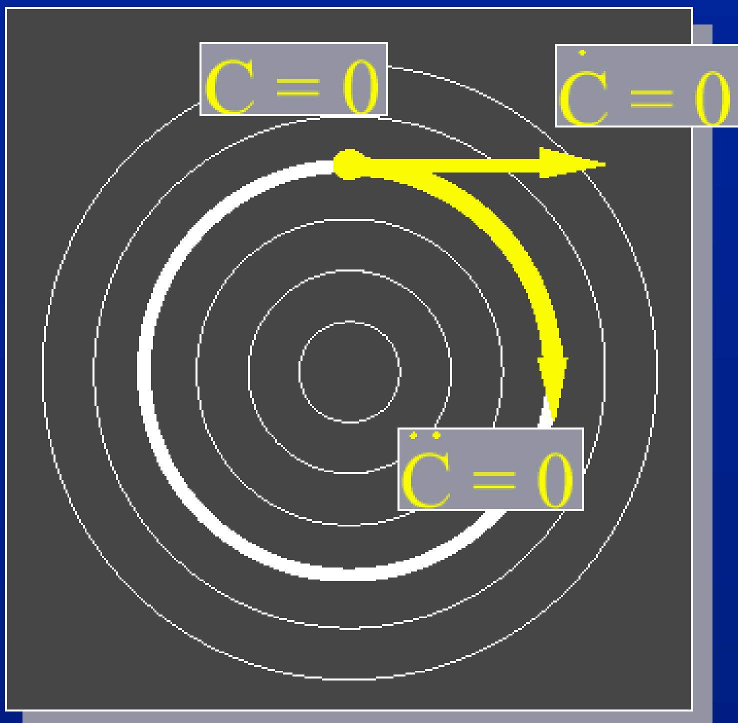
- Curvature(*k*) has to match
- *k* depends on both *a* & *v*:
  - The faster the bead is going, the faster it has to turn
- Calculate f_c to yield a legal combination of *a* & *v*

Implicit Representation of Constraints



$$C(\mathbf{x}) = |\mathbf{x}| - r = 0$$

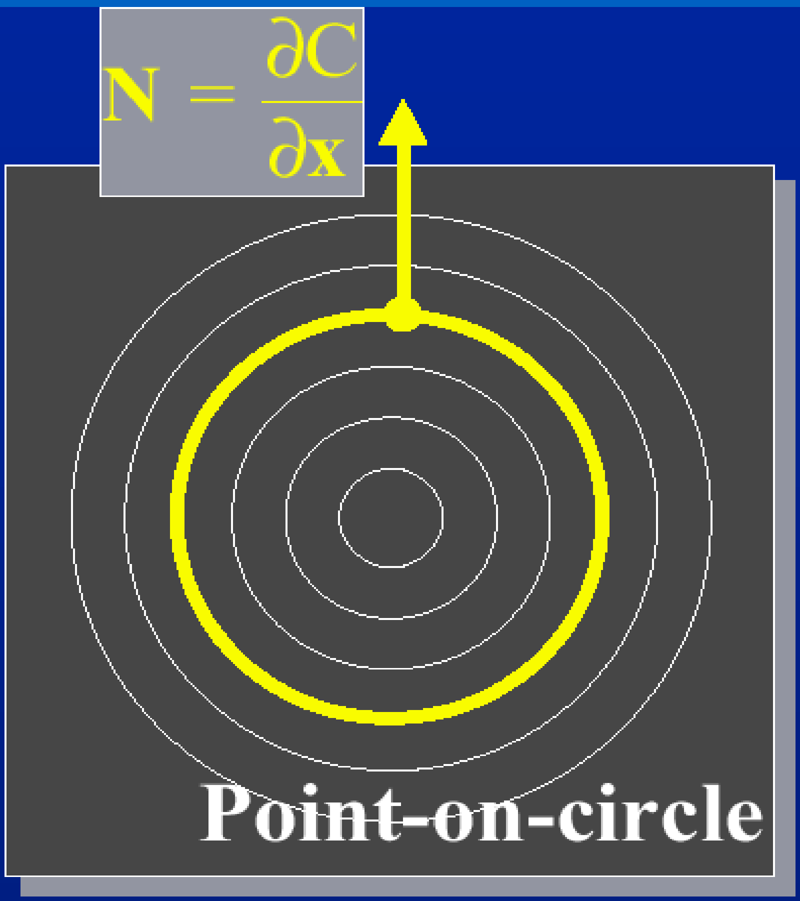
Maintaining Constraints



- Start with legal position and velocity.
- Use constraint forces to ensure legal curvature.

$C = 0$	*legal position*
$\dot{C} = 0$	*legal velocity*
$\ddot{C} = 0$	*legal curvature*

Constraint Gradient



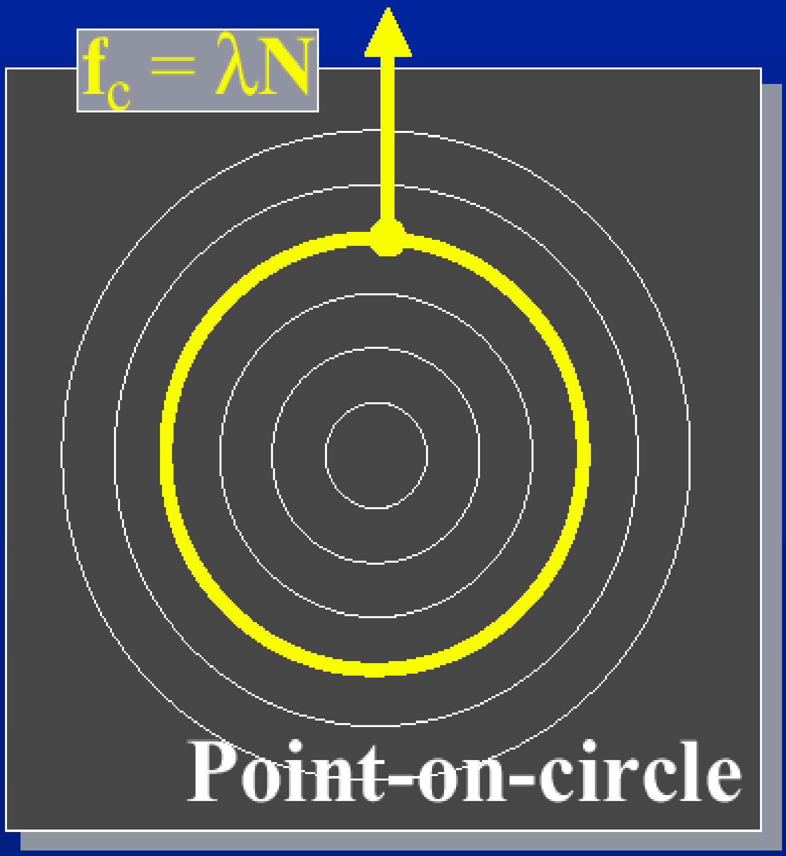
Implicit:

$$C(\mathbf{x}) = |\mathbf{x}| - r = 0$$

Differentiating C gives a normal vector.

This is the direction our constraint force will point in.

Constraint Forces



Constraint force: gradient vector times a scalar, λ.

Just one unknown to solve for.

Assumption: constraint is passive—no energy gain or loss.

Derivation

$$\boxed{\mathbf{C}(\mathbf{x}(t))}$$

$$\mathbf{f_c} = \lambda \mathbf{N}$$

$$\dot{\mathbf{C}} = \mathbf{N} \cdot \dot{\mathbf{x}}$$

$$\ddot{\mathbf{x}} = \frac{\mathbf{f} + \mathbf{f}_c}{m}$$

$$\ddot{\mathbf{C}} = \frac{\partial}{\partial t}[\mathbf{N} \cdot \dot{\mathbf{x}}]$$

$$= \dot{\mathbf{N}} \cdot \dot{\mathbf{x}} + \mathbf{N} \cdot \ddot{\mathbf{x}}$$

Set $\ddot{\mathbf{C}} = 0$, solve for λ:

$$\lambda = -m \frac{\dot{\mathbf{N}} \cdot \dot{\mathbf{x}}}{\mathbf{N} \cdot \mathbf{N}} - \frac{\mathbf{N} \cdot \mathbf{f}}{\mathbf{N} \cdot \mathbf{N}}$$

Constraint force is λN.

Notation: $\mathbf{N} = \frac{\partial \mathbf{C}}{\partial x}, \; \dot{\mathbf{N}} = \frac{\partial^2 \mathbf{C}}{\partial \mathbf{x} \partial t}$

Example: A Bead on a Wire

$$\mathbf{C} = |\mathbf{x}| - r$$

$$\mathbf{N} = \frac{\partial \mathbf{C}}{\partial \mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$$

$$\dot{\mathbf{N}} = \frac{\partial^2 \mathbf{C}}{\partial \mathbf{x} \partial t} = \frac{1}{\mathbf{x}} \left[\dot{\mathbf{x}} - \frac{\mathbf{x} \cdot \dot{\mathbf{x}}}{\mathbf{x} \cdot \mathbf{x}} \right]$$

$$\lambda = -m \frac{\dot{\mathbf{N}} \cdot \dot{\mathbf{x}}}{\mathbf{N} \cdot \mathbf{N}} - \frac{\mathbf{N} \cdot \mathbf{f}}{\mathbf{N} \cdot \mathbf{N}} = \left[m \frac{(\mathbf{x} \cdot \dot{\mathbf{x}})^2}{\mathbf{x} \cdot \mathbf{x}} - m(\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}) - \mathbf{x} \cdot \mathbf{f} \right] \frac{1}{|\mathbf{x}|}$$

Write down the constraint equation.

Take the derivatives.

Substitute into generic template, simplify.

Drift and Feedback

- In principle, clamping $\ddot{C}$ at zero is enough.
- Two problems:
  - Constraints might not be met initially.
  - Numerical errors can accumulate.
- A feedback term handles both problems:

$$\ddot{C} = -\alpha C - \beta \dot{C}, \text{ instead of}$$
$$\ddot{C} = 0$$

α and β are magic constants.

Constrained Particle Systems

- Multiple constraints:
  - each is a function $C_i(x_1,x_2,\ldots)$
  - *Legal state:* $C_i = 0, \forall\ i.$
  - *Simultaneous* projection.
  - Constraint force: *linear combination* of constraint gradients.
- Matrix equation.

Compact Notation

$$\ddot{q} = WQ$$

q: *3n*-long *state vector.*

Q: *3n*-long *force vector.*

M: *3n* x *3n* diagonal *mass matrix.*

W: M-inverse (element-wise reciprocal)

$$q = \left[x_1, x_2, \quad , x_n\right]$$

$$Q = \left[f_1, f_2, \quad , f_n\right]$$

$$M = \begin{bmatrix} m_1 & & & & & \\ & m_1 & & & & \\ & & m_1 & & & \\ & & & m_n & & \\ & & & & m_n & \\ & & & & & m_n \end{bmatrix}$$

$$W = M^{-1}$$

Particle System Constraint Equations

Matrix equation for λ

$$\left[\mathbf{JWJ}^{T}\right]\lambda = -\dot{\mathbf{J}}\dot{\mathbf{q}} - \left[\mathbf{JW}\right]\mathbf{Q}$$

Constrained Acceleration

$$\ddot{\mathbf{q}} = \mathbf{W}\left[\mathbf{Q} + \mathbf{J}^{T}\lambda\right]$$

More Notation

$$\mathbf{C} = \left[C_1, C_2, \quad ,C_m\right]$$

$$\boldsymbol{\lambda} = \left[\lambda_1, \lambda_2, \quad ,\lambda_m\right]$$

$$\mathbf{J} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}$$

$$\dot{\mathbf{J}} = \frac{\partial^2 \mathbf{C}}{\partial \mathbf{q}\partial t}$$

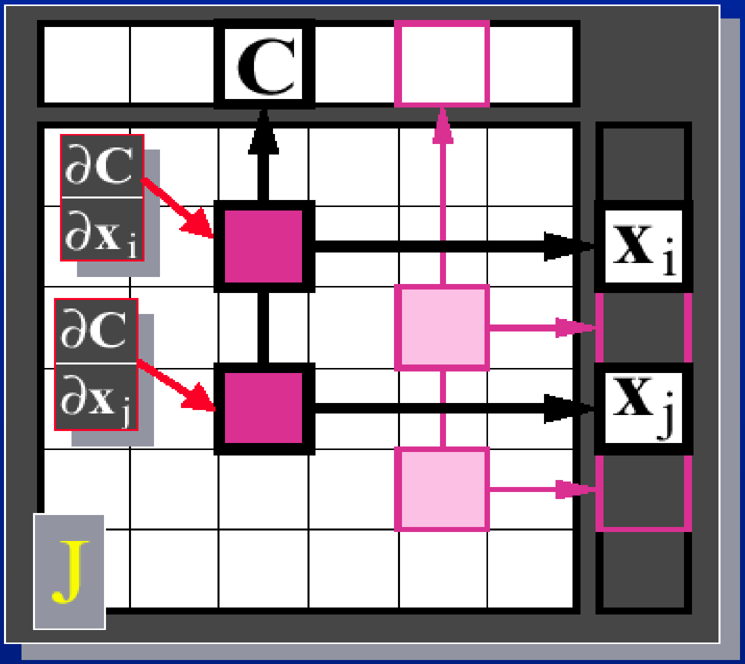
Derivation: just like bead-on-wire.

Implementations

- A global matrix equation
- Matrix block structure with sparsity
- Each constraint adds its own piece to the equation

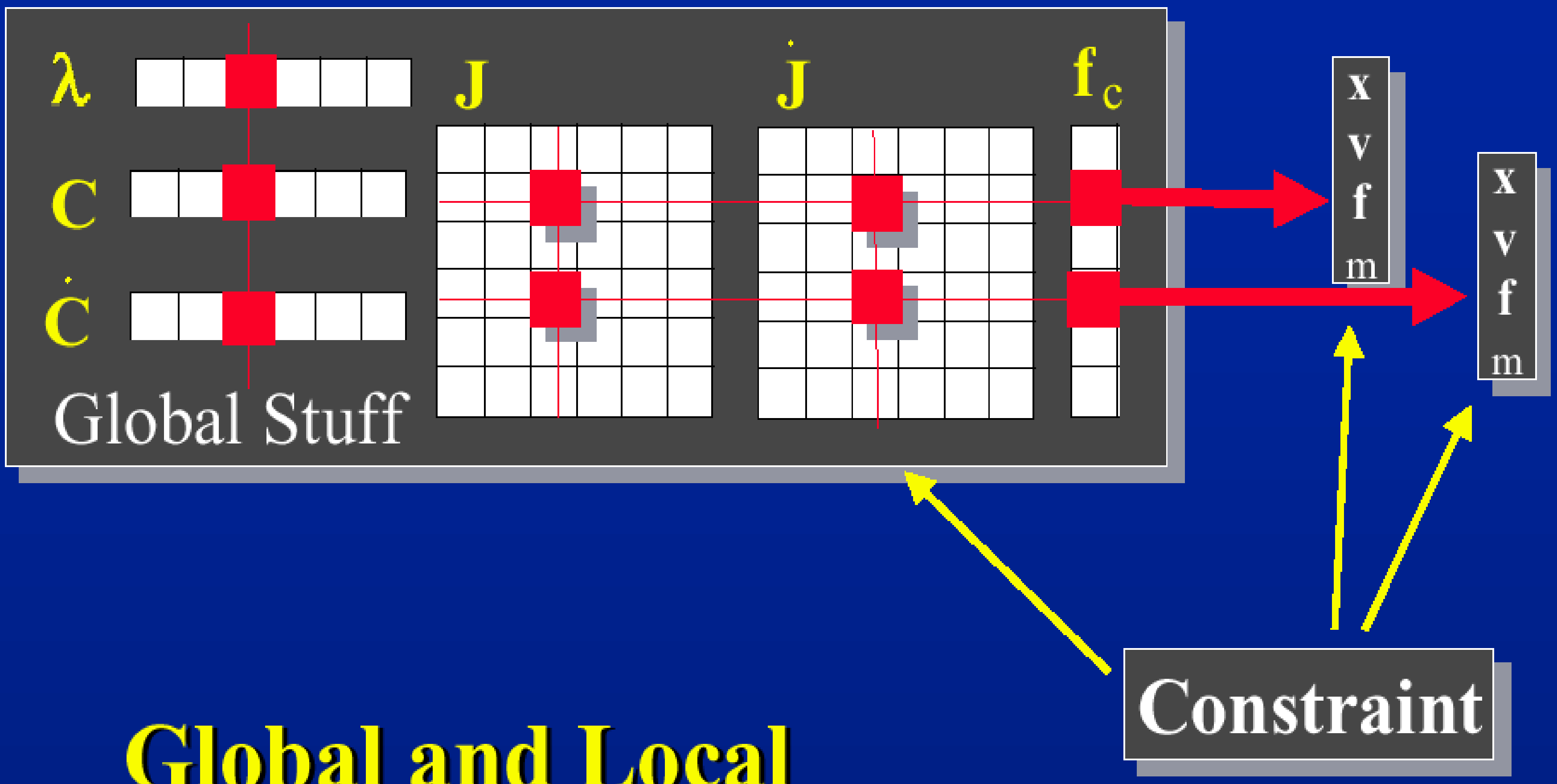
Mathematical Formulation

Matrix Block Structure



- Each constraint contributes one or more *blocks* to the matrix.
- Sparsity: many empty blocks.
- Modularity: let each constraint compute its own blocks.
- Constraint and particle indices determine block locations.

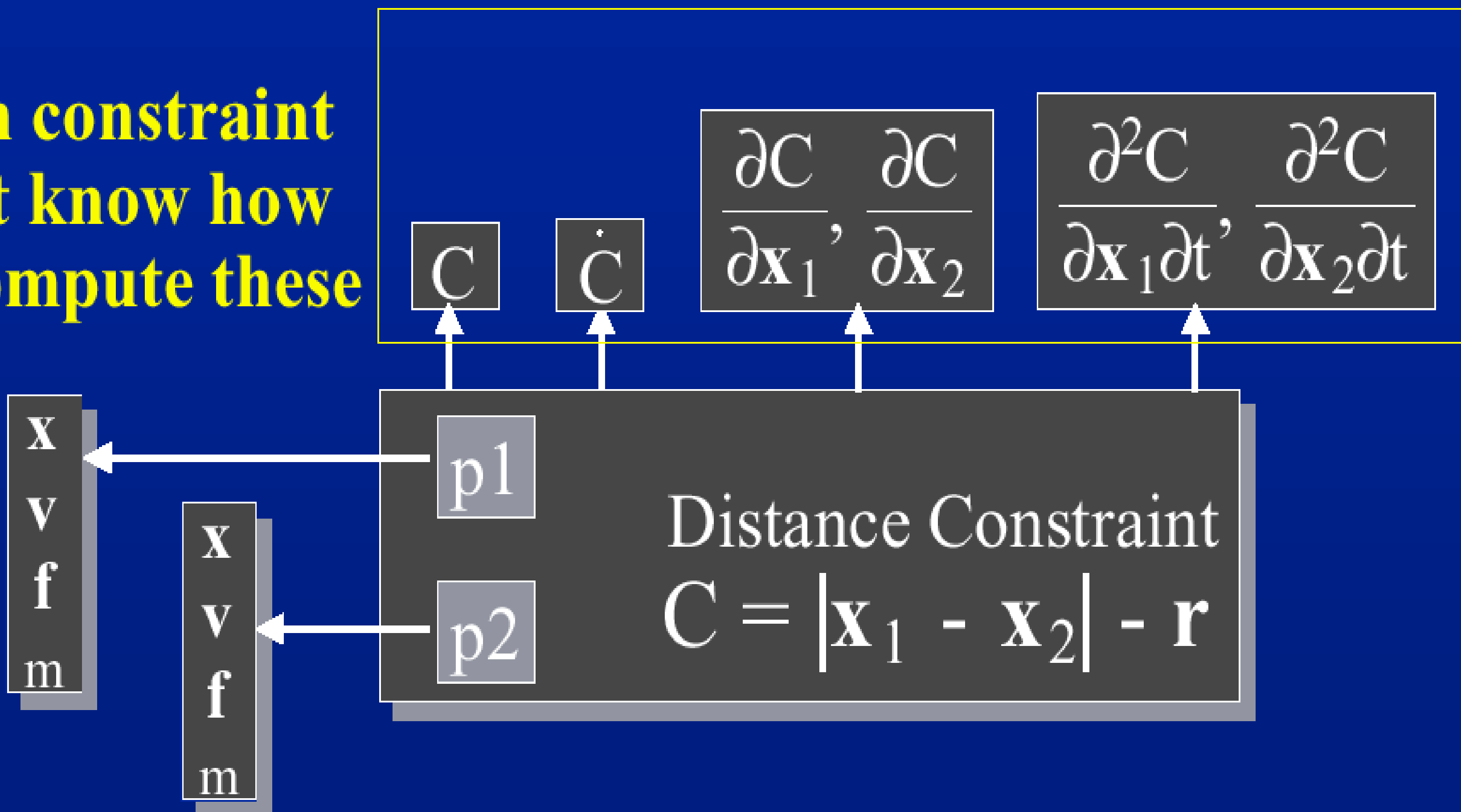
Take a Closer Look



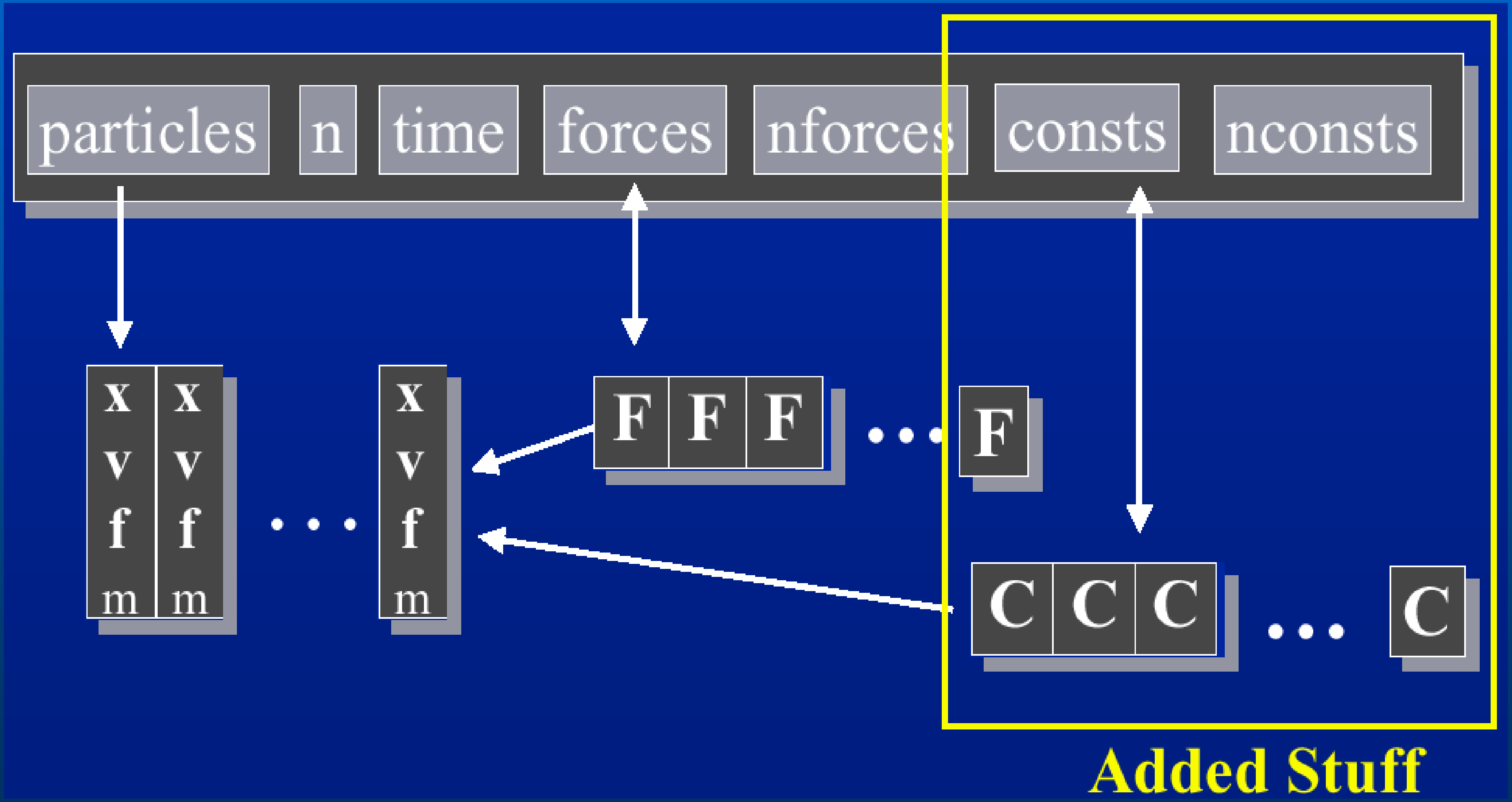
Global and Local

Constraint Structure

Each constraint must know how to compute these

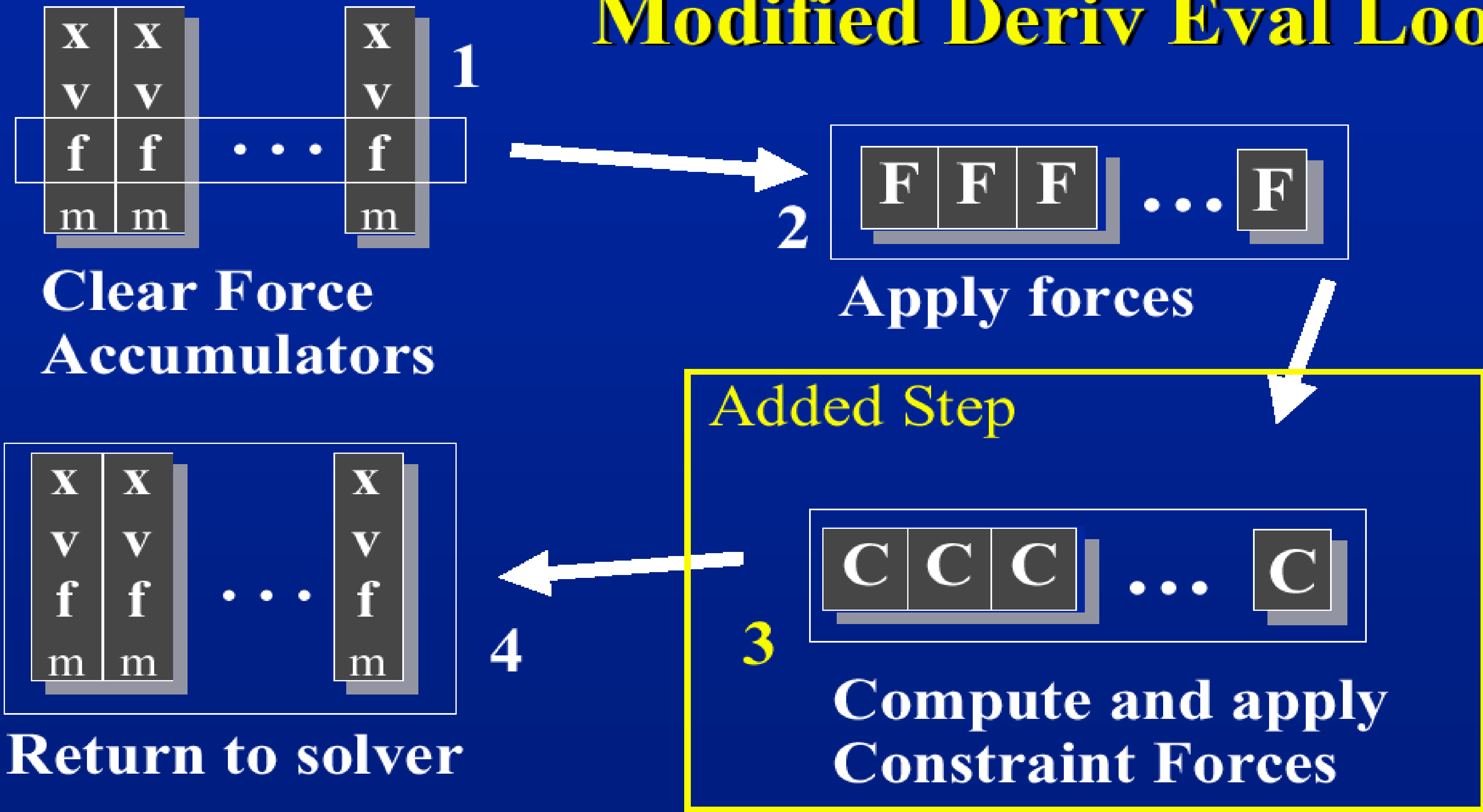


Constrained Particle Systems



Other Modification

Modified Deriv Eval Loop

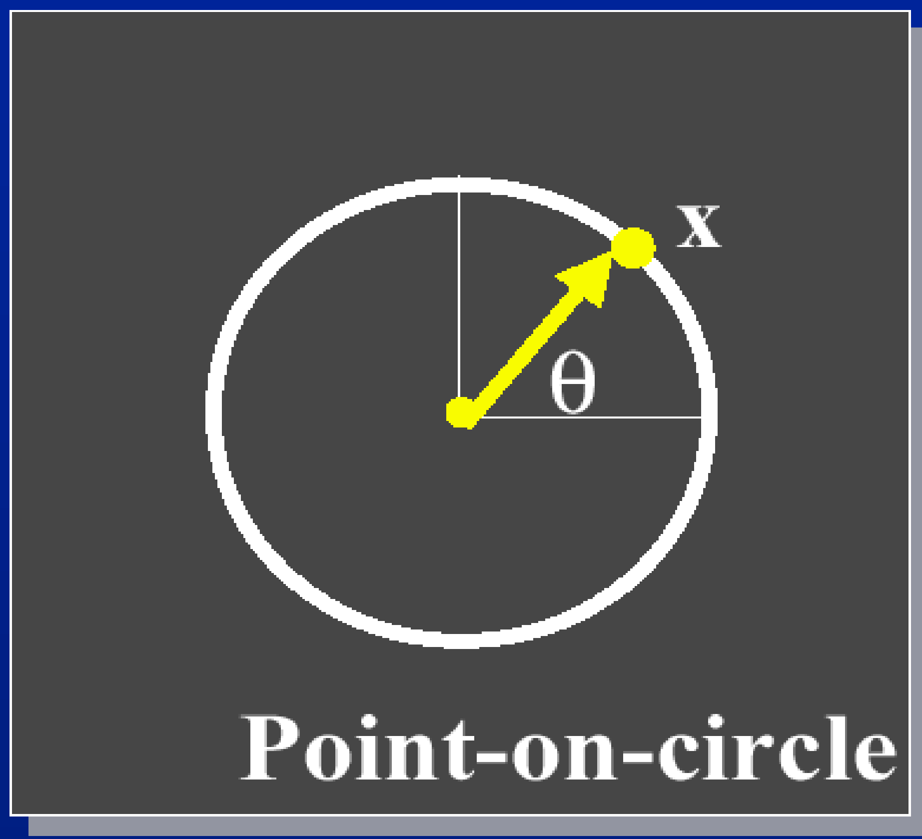


Constraint Force Evaluation

After computing ordinary forces:

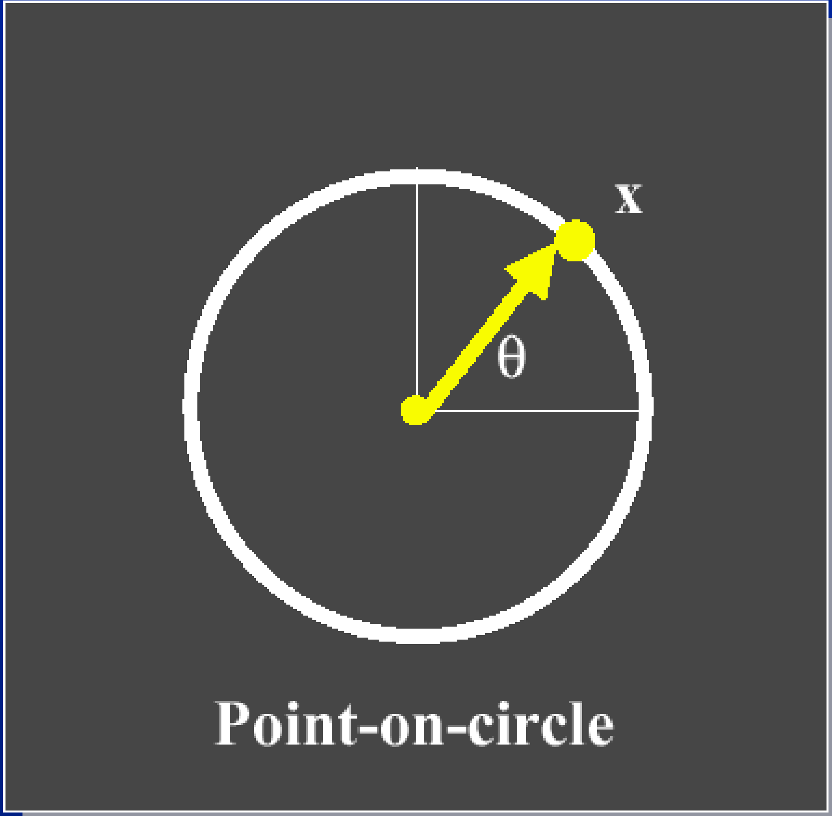
- Loop over constraints, assemble global matrices and vectors.
- Call matrix solver to get λ, multiply by J^T to get constraint force.
- Add constraint force to particle force accumulators.

Parametric Representation of Constraints



$$\mathbf{x} = r\left[\cos\theta, \sin\theta\right]$$

Parametric Constraints

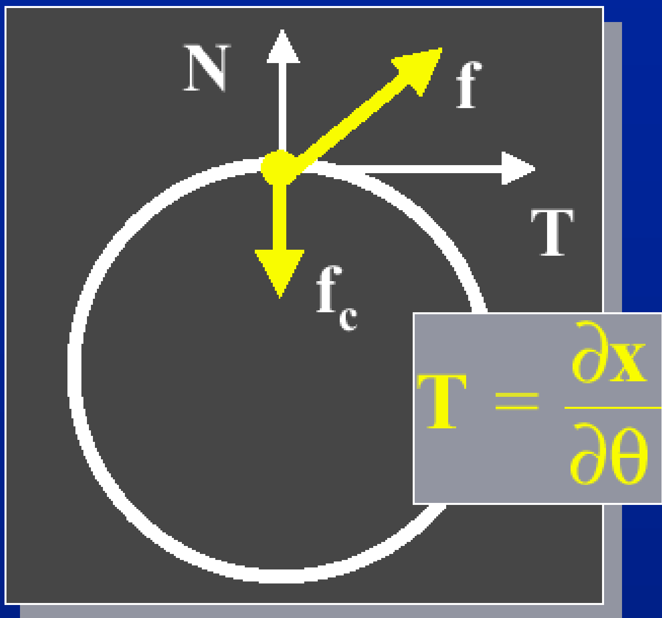


Parametric:

$$\mathbf{x} = r\left[\cos\theta, \sin\theta\right]$$

- **Constraint is always met exactly.**
- **One DOF: θ.**
- **Solve for $\ddot{\theta}$.**

Parametric bead-on-wire (F = mv)



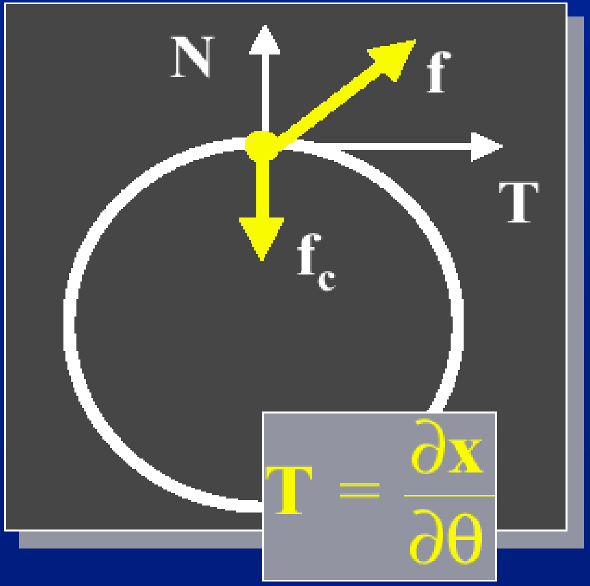
x is not an independent variable.

First step—get rid of it:

$\dot{x} = \frac{f + f_c}{m}$	**f = mv** ***(constrained)***
$\dot{x} = T\dot{\theta}$	**chain rule**
$T\dot{\theta} = \frac{f + f_c}{m}$	**combine**

Some Simplification……

For our next trick…



As before, assume f_c points in the normal direction, so

$$T \cdot f_c = 0$$

We can nuke f_c by dotting T into both sides:

$T\dot{\theta} = \dfrac{f + f_c}{m}$	from last slide
$T \cdot T\dot{\theta} = \dfrac{T \cdot f + \cancel{T \cdot f_c}}{m}$	blam!
$\dot{\theta} = \dfrac{1}{m}\dfrac{T \cdot f}{T \cdot T}$	rearrange.

Lagrangian Dynamics

- Advantages
  - Fewer DOF's
  - Constraints are always met
- Disadvantages
  - Difficult to formulate constraints
  - Hard to combine constraints