

# Piano Tuning Method

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## CONTENTS

Abstract .....	2
Project Location.....	2
1    Introduction .....	2
2    Technical Knowledge.....	3
2.1    Key Names .....	3
2.2    Key Numbers.....	3
2.3    Functions .....	3
2.4    Tuning Methodology .....	4
3    Piano Tuning Method .....	4
3.1    Traditional Method.....	4
3.1.1    Sampling Piano.....	4
3.1.2    Audio Processing.....	4
3.1.3    Frequency Analysis .....	4
3.1.4    Catchup Overtone.....	5
3.1.5    Inharmonicity Model .....	6
3.1.6    Tuning Curve Optimization Model .....	8
3.1.7    Temperament Model.....	11
3.1.8    Creating Tuning Strategy Table .....	12
3.2    Entropy Tuning Method .....	12
3.2.1    Sampling Piano & Audio Processing .....	12
3.2.2    Construct Spectrum .....	12
3.2.3    Tuning with Entropy Optimizer .....	13
3.2.4    Creating Tuning Strategy Table .....	15
4    Audio Processing & Pure Sound Tuner.....	16
4.1    Tuning .....	16
4.2    Sound Purify .....	17
5    Future Work .....	18
6    Reference.....	18
7    Appendix .....	19

## ABSTRACT

*Since the piano string is considered to be a stick rather than a pure ideal string, it contains stiffness and its overtone will shift in such way that make piano tuning a difficult work. In this work, two optimization algorithms for piano tuning method are presented. The traditional tuning algorithm is divided into several models that use various fitting techniques to model the target piano, and then convert to linear regression problem for optimization. The entropy tuning method is a trial method to tune the piano to minimize the entropy value when all keys are pressed – to achieve simpler spectrum in pitch domain. In addition, a pure tuner method is invented to get rid of all inharmonic effect of piano sound.*

**Keyword:** *piano tuning, inharmonicity, entropy, audio processing*

## PROJECT LOCATION

*Reference [2]*

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## 1 INTRODUCTION

Piano tuning is a difficult work since the harmonics shift that make the piano hard to tune. The tuning process will be a task to highly reduce the audible cacophonous. There are several factors we need to consider, which the rule of harmony is.

- The cacophonous created by its base frequency and audible harmonics; a good tuning will largely reduce the inharmonic effects for harmonics (the frequency domain should be simple, which the frequency peaks should merged or coincide).
- The inner music scales related pitch; the odd pitch tuning will result in the weird effect when playing music scales.

Other famous related works are:

- Tunelab (closed source; has trial version)
- Reyburn CyberTuner (closed source; no trial version)
- Entropy Piano Tuner (open source) [1]

The first two are similar, which represent the old tuning techniques, and my work mostly focus on this algorithm.

As for Entropy Piano Tuner, it represents the new way of piano tuning. It can also achieve very good result for tuning a piano, however this temperament is not regular 12-equal temperament, but a piano approximation temperament starting from 12-equal temperament, in order to largely eliminate the non-harmonious effect.

- Since the pitch in the piano does not have relatively same pitch interval, some inner scales sound weird.
- Since the piano optimizes all 88 keys harmony, it values overall harmonious – some simpler chord might not sound harmonious.
- It only considers the sound which at the certain striking level of piano keys, which result in the optimization of keys are based only on sampling striking level. However, it values the average case for piano performance, thus it covers the majority situation of harmony cases.
- The accuracy cannot be too high due to large amount of calculation, it does not achieve an ideal result.

In my work, I will talk about two piano tuning methods, and one audio processing method.

- As for traditional tuning method, since it is closed source, I guessed their tuning method and create a similar solution, and will be shown in this article. Besides, I used more accurate model for inharmonicity coefficients.
- I will reproduce the result for Entropy Piano Tuning method.

- The tuning for audio and a pure sound tuner is introduced.

In this article, the first part is to introduce the technical knowledge for high level modeling algorithms. The second part is to introduce my piano modeling and tuning optimization method. Then, followed an audio processing technique. Finally, the future work will be introduced.

## 2 TECHNICAL KNOWLEDGE

### 2.1 KEY NAMES

The left most key name is defined as “A0”, where “A” is the note name, 0 is the scale number. “C” is the starting point of one scale. It only allowed sharp in the note, flat is not allowed in this naming format.

*A0, A#0, B0, C1, C#1, ..., B1, C2, ..., B7, C8*

There are 88 keys for standard piano.

### 2.2 KEY NUMBERS

In the real world, the piano key will be labeled with numbers when the piano is open and machine part is shown off.

A0 key is labeled to be 1, and “C8” is 88.

However, in my program, “A0” key is labeled as 0 for easier calculation, which is defined as  $k$ .

### 2.3 FUNCTIONS

Frequency ratio to cents function:

$$\text{Fr}_{\rightarrow c}(\gamma) = 1200 \log_2(\gamma) \quad (2.1)$$

The inverse process is:

$$\text{C}_{\rightarrow \text{fr}}(c) = 2^{\left(\frac{c}{1200}\right)} \quad (2.2)$$

Where cents is from 12 equal temperament, each half note has 100 divisions, named cents.

Frequency add cents (pitch) function:

$$\text{F}_{+c}(f, c) = f \cdot 2^{\left(\frac{c}{1200}\right)} \quad (2.3)$$

This function returns the frequency that added the pitch (cents)  $c$ .

The ideal frequency for the key  $k$  is:

$$\tilde{f}_k = \tilde{f}_{[A4]} \cdot 2^{\left(\frac{k-48}{12}\right)} \quad (2.4)$$

Where  $\tilde{f}_{[A4]}$  is the international standard pitch for “A4”, usually defined as 440Hz. Other tuning standard will replace this number, 48 is the key number for “A4”.

## 2.4 TUNING METHODOLOGY

Since the minor tuning for each string will rarely affect its stiffness, from Equation (3.3), we assume that the  $B_k$  is the constant.

## 3 PIANO TUNING METHOD

### 3.1 TRADITIONAL METHOD

The traditional tuning method is to match the specific frequency peaks that aimed at largely eliminating the “beat” (pitch differences from two notes; for example, “A3’s” second overtone matches its octave “A4”, which is denoted to be 2:1). Then, use a smooth curve to optimize/minimize all the differences to achieve relatively good result.

Since the piano overtone shift (inharmonicicity) has a very nice relation, it enables us to just sample very few keys and guess all the properties for all piano; then, get the tuning strategy.

#### 3.1.1 Sampling Piano

Before tuning a piano, we need to sample a piano by recording few piano keys sound audios. This process will roughly or precisely measure the inharmonicity of piano strings (which will talk about later), such that we could model the inharmonicity for the targeted piano.

The sampling is suggested to measure keys “C1”, “C2”, “C3”, “C4”, “C5” (and probably “C6”; user could record more piano keys such as “A1” ~ “A6” for better result). Since the tuning inharmonicity curve is a smooth curve and predictable, thus it is possible to sample fewer notes. The piano key sound should be recorded in a quiet environment, which allows more accuracy for later frequency analysis. In this sampling process, we need to press the key hard in order to get higher harmonic peaks for measurement.

In my program, I use fully or almost fully sampled piano for research purposes.

#### 3.1.2 Audio Processing

Since the real audio may contain the white space at the start or the end, and the sound length varies. I use this method to process my sampled audio:

- Normalize ( $N(x) = x / \max(x)$ ) the audio file into 1, then, find the peak volume of audio, and start from here.
- Slice these audio pieces into tiny partitions, say 0.1 second is one partition. The maximum number of each partition will be its assumed volume at this time point.
- Trim the audio at the volume start from some large number to small number – since piano sound is loud from its beginning and decay by the time. Say from 90% to 2% of the sampled sound audio.

#### 3.1.3 Frequency Analysis

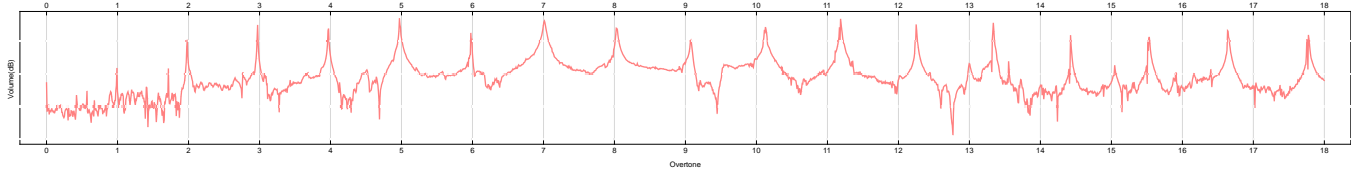


Figure 3-1 “A#0” Key (at Upright Piano Samples) Overtone Plot; Volume at Logarithm Scale

Then, put this audio samples into fourier analysis (FFT algorithm). Then we get the function  $G_k(f) = \|\text{FFT}(S_k(t))\|_2$  where  $S_k(t)$  is the audio function, and  $G_k(f)$  is the frequency domain function,  $k$  is

piano key number,  $f$  is the frequency variable,  $\|\cdot\|_2$  is the 2-norm of complex numbers. In our work, the frequency domain is converted to the ratio to its ideal fundamental frequency, thus we can see the Figure 3-1, the peaks will always almost lies in the grid by dividing its ideal frequency.

From Figure 3-1, we can see that the higher overtone (right hand side peaks with larger numbers) shifts higher.

It is a problem to capture all these peaks numbers, since some are not clear: the fundamental frequency (at 1), and some has multiple peaks: at 15 ~ 16.

In my work, I use the frequency *Catchup Method* to get octave values for all these peaks.

### 3.1.4 Catchup Overtone

From the charactors of these peaks, there are several charactors will be considered:

- From left to right, the gap between two peaks are increasing gradually.
- The largest value of this plot is probably some peak of overtone
- The valid peak should be nearly larger than fundamental frequency position: at 1.
- The peak may be broken into several peaks, we need centralize the targeted position.

From this characteristics, the *Catchup Method* could be built:

- Analyze the frequency samples which roughly larger than 1 (my program is starting from 0.8), get the peak frequency  $f_{k,peak}$  at key number  $k$ , and overtone number  $peak$ .
- Comparing with ideal frequency  $\tilde{f}_k$ . We can then assume that it is  $n = \text{round}(f_{k,peak} / \tilde{f}_k)$  harmonics.

Then, we can know its guessed fundamental frequency is  $\hat{f}_k = f_{k,peak} / n$ . Then, this should be the step size for catchup method.

- The catchup method is forward (goes to the right), and the backward (goes to the left). If we are in the forward operation, the next guessed target frequency is  $\hat{f}_{k,peak+1} = f_{k,peak} + f'_k$ , where  $f'_k$  is the assumed gap between two peak at this position. In the first try, we set this number to  $f'_k = \hat{f}_k$ , and this number will be increasing for more right harmonics. Then, we get the around data (in a relatively small area) for guessed target frequency  $\hat{f}_{k,peak+1} \pm \delta$ . We can find its maximum number these data to be the frequency candidate  $\hat{f}_{k,peak+1}^{candidate}$ , then we get the data of smaller surround area  $\hat{f}_{k,peak+1}^{candidate} \pm \delta'$  where  $\delta' \ll \delta$ . Then, we calculate the weighted average for this smaller area, and the result is the actual frequency of this peak  $f_{k,peak+1} = \int_{\hat{f}-\delta'}^{\hat{f}+\delta'} \omega \cdot G(\omega) d\omega$ , where  $\omega$  is proportional to frequency. Then, the assumed gap between two peak at this step is updated to be  $f'_k = f_{k,peak+1} - f_{k,peak}$ .
- Iterate this method for forward catchup to get all higher frequencies.
- If the highest peak is not fundamental frequency, we will perform the backward catchup. Since there are less peaks and the overtone shift will be far less than the right, the assumed targeted gap between two peaks is set to be the assumed fundamental frequency  $\hat{f}_k$ .

From this method, we can get a overtone (frequency) list for the key  $k$ . Which is:

$$k \rightarrow \{f_{k,1}, f_{k,2}, \dots\} \quad (3.1)$$

### 3.1.5 Inharmonicity Model

From Figure 3-1, we can see that the overtone will shift higher and higher as the frequency goes higher. This effect is caused by the stiffness of an object, its natural frequency will follow a certain pattern.

From reference [1], we assume that the piano string is a bar with two fixed ends, which approximately follows the partial differential equation:

$$\ddot{y} \propto -y'' - \varepsilon y'''' \quad (3.2)$$

Where  $y$  is the special position of piano string (bar model). The prime is the derivative to spatial domain, and dots is the derivative to time domain.

Then, use the modal analysis and solved the natural frequencies for this string are:

$$f_{k,n} \propto n \cdot f_{k,1} \sqrt{1 + B_k \cdot n^2} \Rightarrow f_{k,n} = A_k \cdot n \cdot f_{k,1} \sqrt{1 + B_k \cdot n^2} \quad (3.3)$$

Here we have two unknown variables  $A_k$  and  $B_k$ .

Then, we use this function to fit all frequency results at Equation (3.1). The parameter  $A_k$  is set since not all fundamental frequency is guessing perfectly. We can ignore this number by making sure the fundamental frequency always target on 1, and focus only on  $B_k$ .

Then, we can get inharmonicity parameter list  $\{\{k, B_k\}\}$ .

From my observation, the logarithm of this number has some beautiful properties with the data  $\{\{k, \ln(s \cdot B_k)\}\}$ , where  $s$  is a scaling parameter (I set to 10000).

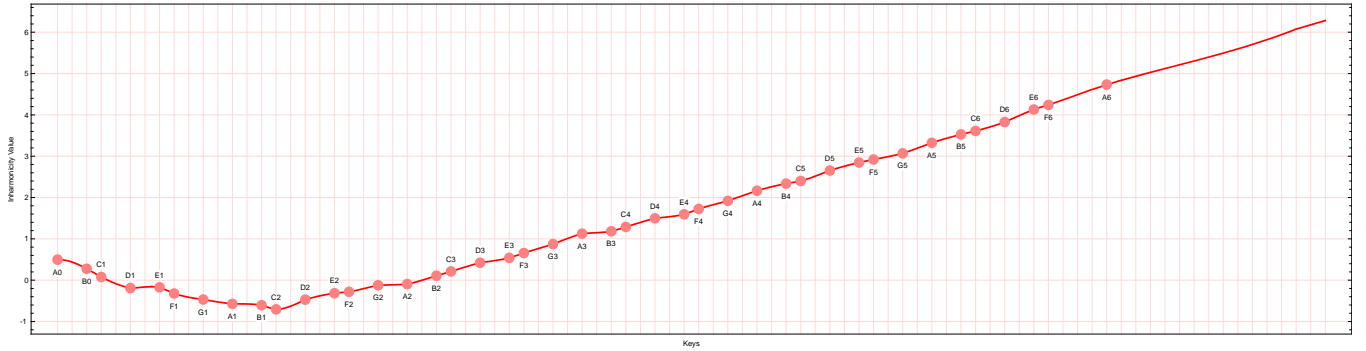
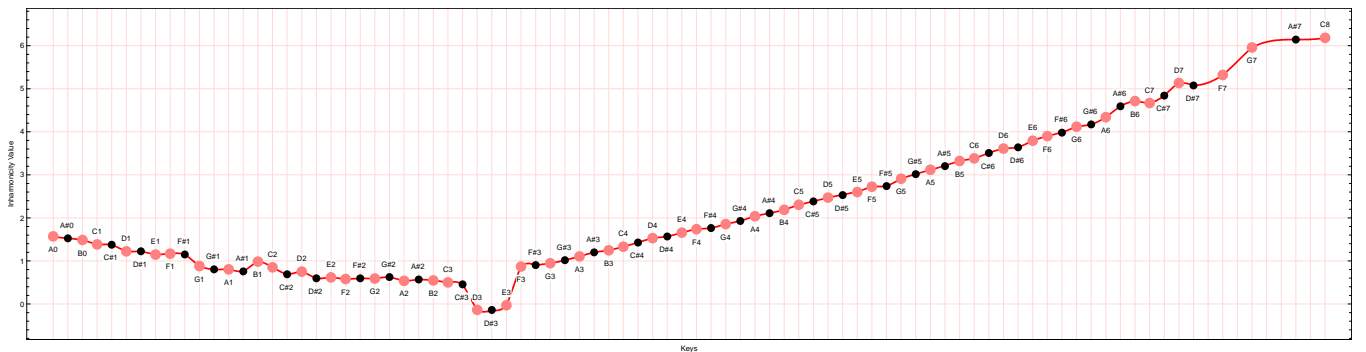


Figure 3-2 Inharmonicity Plot of Grand Piano  $IH(k)$



**Figure 3-3 Inharmonicity Plot of Upright Piano  $IH(k)$**

From Figure 3-2 and Figure 3-3, we can clearly see the line is divided into 2 parts.



**Figure 3-4 Grand Piano String Arrangement**



**Figure 3-5 Upright Piano String Arrangement**

From Figure 3-4 and Figure 3-5, we can clearly see that the string is divided into two parts, the steel string and copper string (may be covered by silver for highly expensive pianos). The upright piano has more copper strings since the steel string cannot go longer, and the string will become thicker to make the string vibrate slower. From spring vibration formula:

$$\omega = \sqrt{\frac{K}{m}} \quad (3.4)$$

Where  $\omega$  is proportional to frequency,  $m$  is the mass of spring,  $K$  is the stiffness of spring.

When  $m$  increases,  $K$  increase a little bit,  $\omega$  decreases, then frequency decrease.

Since the piano cannot growing longer, it become thick and more like a stick rather than an ideal string. For higher notes strings, it is too short, and the thickness become relatively larger comparing to its length, thus it is more likely to be a bar.

Thus, from the plot, we can see the inharmonicity increases at two ends, and break at the position of separation of two kinds of strings.

Since grand concert piano is longer, and can have more steel strings, less copper strings, thus the break will become more left side.

The figure of inharmonicity plot also tell us that two separate line are almost linear. In my model, I used the valid sampled points are modeled with interpolation function, and two edges are modeled with linear function, and it is method is shown below.

- We get several samples from one line, and fit in a linear form.
- Get its slope, and build a line which pass the right end point (since I will not wish to have a break for the interpolation function), and add some samples for edges situation to sample pool.
- Similar to the left hand side.
- We use interpolation for these samples of sample pool – “left hand side + samples + right hand side”, which is our final model for inharmonicity model function  $\text{IH}(k)$ .

$$\text{IH}(k) = \ln(s \cdot B_k) \quad (3.5)$$

Thus, we can have the modeled parameter  $B_k$  with:

$$B_k = \frac{e^{\text{IH}(k)}}{s} \quad (3.6)$$

Then, the frequencies  $\tau(k, n)$  will be:

$$\tau(k, n) = f_{k,1} \cdot n \cdot \sqrt{1 + B_k \cdot n^2} \quad (3.7)$$

Where  $f_{k,1}$  is currently unknown but it will be eliminated, since it is in frequency ratio form.

### 3.1.6 Tuning Curve Optimization Model

Similar to Tunelab ®, I set the tuning optimization method to separate the lower tones (bass) and higher tones (tenor) into two tuning target optimization method, the separation point  $k_0$  is “C#4/D4”. And the default tuning method for bass is to set 6:3. Since  $6/3=2$  ( $a/b$ ), this frequency ratio is  $\gamma = a/b$ , and its corresponding pitch range is  $\text{Fr}_{\rightarrow c}(\gamma)$  which is 1200, and 1200 is an octave, it means the tone say “A0”’s 6<sup>th</sup> harmonics will largely match its octave’s “A1”’s 3<sup>rd</sup> harmonics.

Here pitch is defined by cents.

The error function  $\mathcal{E}_k$  is defined as:



$$\begin{aligned}
\varepsilon_k &= \text{Fr}_{\rightarrow c} \left( \frac{\tau(k, a)}{\tau(k + \text{Fr}_{\rightarrow c}(a/b), b)} \right) \\
&= \text{Fr}_{\rightarrow c} \left( \sqrt{\frac{1 + B_k \cdot a^2}{1 + B_{k + \text{Fr}_{\rightarrow c}(a/b)} \cdot b^2}} \cdot \frac{a}{b} \cdot \left( \frac{f_{k,1}}{f_{k + \text{Fr}_{\rightarrow c}(a/b),1}} \right) \right) \\
&= \text{Fr}_{\rightarrow c} \left( \sqrt{\frac{1 + B_k \cdot a^2}{1 + B_{k + \text{Fr}_{\rightarrow c}(a/b)} \cdot b^2}} \right)
\end{aligned} \tag{3.8}$$

We can do this for all bass strings.

For tenor strings, the default tuning method is set to 4:1 ( $c/d$ ). But this time we count the higher note as the target to calculate.

$$\varepsilon_k = \text{Fr}_{\rightarrow c} \left( \sqrt{\frac{1 + B_{k - \text{Fr}_{\rightarrow c}(c/d)} \cdot c^2}{1 + B_k \cdot d^2}} \right) \tag{3.9}$$

The combined expression is:

$$E(k) = \begin{cases} \text{Fr}_{\rightarrow c} \left( \sqrt{\frac{1 + B_k \cdot a^2}{1 + B_{k + \text{Fr}_{\rightarrow c}(a/b)} \cdot b^2}} \right) & k \leq k_0 \\ \text{Fr}_{\rightarrow c} \left( \sqrt{\frac{1 + B_{k - \text{Fr}_{\rightarrow c}(c/d)} \cdot c^2}{1 + B_k \cdot d^2}} \right) & k > k_0 \end{cases} \tag{3.10}$$

From this equation, we can see  $E(k)$  is only a value for calculation at given  $k$ .

From this point, we need a function to largely eliminate these errors. The piano tuning curve  $C(k)$  is introduced, it represent the deviation of the actual tuning pitch to the ideal 12-equal temperament pitch.

The optimizer deviation function  $D(k)$  is:

$$D(k) = C(k) - E(k) \tag{3.11}$$

The cost function  $J(k)$  for optimization is:

$$J(k) = \sum_k (D(k))^2 \tag{3.12}$$

Which minimize the square error of these functions.

Here I use polynomial for easier calculation:

$$C(x) = \sum_{i=1}^n \chi_i \cdot x^i \quad (3.13)$$

Since  $C(x)$  will pass the fix point, which is “A4” pitch at 440Hz frequency at pitch deviation of 0, thus  $i$  is from 1 and  $x = k - k_{[A4]}$ , where  $k_{[A4]}$  is the key number (index) at “A4”, which is 48.

Thus,  $J(k)$  is the second order multi-variable polynomial function, which is very easy to minimize by linear regression method to calculate the fitting parameter  $\{\chi_i\}$ , and rebuild the functions.

Then, we can bring  $\{\chi_i\}$  to the  $D(k)$  function to calculate its deviations.

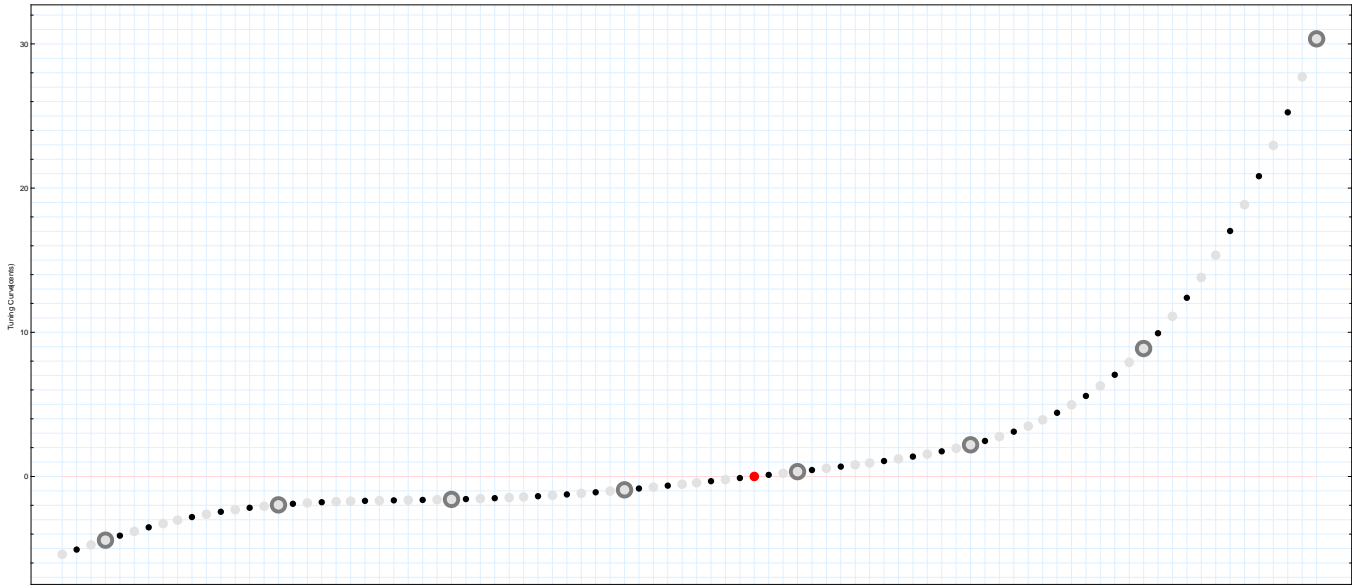


Figure 3-6  $C(k)$  for Grand Piano

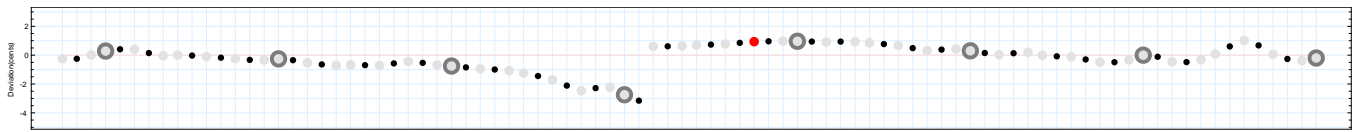


Figure 3-7  $D(k)$  for Grand Piano

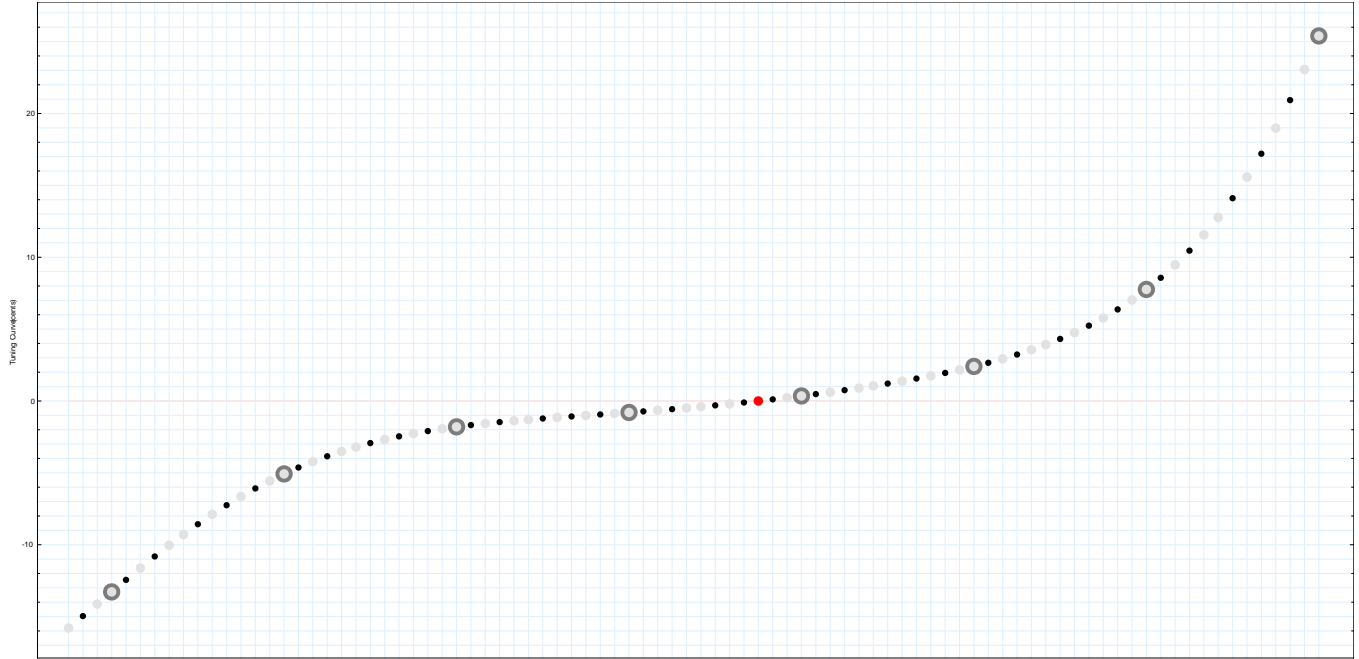


Figure 3-8  $C(k)$  for Upright Piano

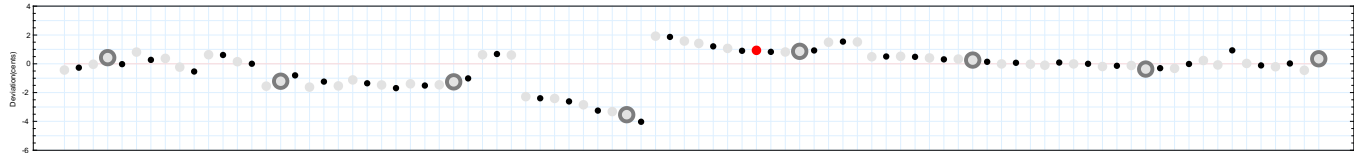


Figure 3-9  $D(k)$  for Upright Piano

The result of two piano is shown above. Horizontal axis is the key number, and the vertical axis the pitch interval with its ideal frequencies represented by cents.

From this tuning method, we can see that the bass tuning will consider the deviations from the tenor part, and vice versa. The effect is inner related. Thus this tuning method is theoretically to optimize almost the whole piano keys tuning.

### 3.1.7 Temperament Model

With the development of music, various temperament appears and create unique flavor of music. The temperament model is using the pitch deviation tables of different temperament (the unit is cent). We can then create the non-12 equal temperament tuning strategy. The temperament function is defined to be  $T(k)$ .

The tuning table such as “Bach - Bradley Lehman” is:

C	C#	D	D#	E	F	F#	G	G#	A	A#	B
5.87	3.91	1.96	3.91	-1.96	7.82	1.96	3.91	3.81	0	3.91	0

Table 3-1 Table for “Bach - Bradley Lehman” Temperament

Where A note will always be 0 since A is the reference frequency and will always keep to 440 Hz (if is standard situation).

This table shows the situation of “C” major.

The other major tuning will follow the rotation of table. For example: if tuning “D” major, the “D” will rotate to current “D”  $\rightarrow$  “C” place, which is rotating left 2 times. However, we will make sure “A” note will always be 0, then, we can subtract the number at “B”  $\rightarrow$  “A” to make it possible.

Then, add these pitch errors to all the notes of tuning, the modified tuning curve is:

$$C'(k) = C(k) + T(k) \quad (3.14)$$

### 3.1.8 Creating Tuning Strategy Table

The final tuning strategy  $\tau(k, n)$  (unit: Hz) is:

$$f_{k,1} = F_{+c}(\tilde{f}_k, C'(k)) \quad (3.15)$$

$$\begin{aligned} \tau(k, n) &= f_{k,1} \cdot n \cdot \sqrt{1 + B_k \cdot n^2} f \\ &= F_{+c}(\tilde{f}_k, C'(k)) \cdot n \cdot \sqrt{1 + B_k \cdot n^2} \\ &= F_{+c}(\tilde{f}_k, C'(k)) \cdot n \cdot \sqrt{1 + \frac{e^{IH(k)}}{s} \cdot n^2} \end{aligned} \quad (3.16)$$

From Equation (3.16), we can see only  $C(\cdot)$  and  $IH(\cdot)$  function is modeled function, other function are basic mathematics functions.

From the modeling, we can get a strategy of piano tuning, then we can convert this strategy into a tuning table, which shows all the frequency of fundamental and its harmonics frequencies, and corresponding deviation to ideal frequencies represented by cents.

The grand and upright piano tuning strategy is shown in Figure 7-1 and Figure 7-2.

The red font is the frequencies recommended for the devices to tune.

## 3.2 ENTROPY TUNING METHOD

Entropy tuning method is not to model the exact value of frequencies or pitches, it simulates the condition that simultaneously press down all piano keys, and uses entropy method as cost function to largely merge the peaks at pitch domain to create more sharp and simple sound for piano, which optimize the piano sound. The method is extremely simple, however, it is really computational intensive.

### 3.2.1 Sampling Piano & Audio Processing

In entropy piano tuning method, sampling every piano key is necessary. Other requirement is similar to traditional method. The audio processing is also similar to traditional method.

### 3.2.2 Construct Spectrum

Since human ear is sensitive to the pitch (“pitch” is equivalent to the logarithm of frequency component for approximation: ignore non-linear effect of ear structures) within the hearing range (20Hz ~ 10000Hz is reasonable for optimizing algorithm). Thus, the model should be built by putting equal significance to the pitch scale. Traditionally, the pitch is represented as music note. If we evaluate the “pitch” content/data by equally sampling from the pitch scale of spectrum, it put the equal importance to the pitch scale – logarithm scale of frequencies. In my experiment, I put 0.1 cent as the precision.

Then, we have the converted the spectrum into pitch domain  $I(\kappa)$ , to resample the data with the key number:

$$I(\kappa) = \left\| G(f_\kappa) \right\|^\beta \Big|_{\kappa \rightarrow 12 \cdot \log_2 \left( \frac{f_\kappa}{f_{[A0]}} \right), \beta \rightarrow 2} \quad (3.17)$$

Where for each key  $k$  we will have 1000 samples in total, each sample pitch denote as  $\kappa$ . Namely, each sample will represent 0.1 cent. Since the audio is also the limited samples, I use the interpolation function to resample the data.

In this model, I use the square of spectrum  $\beta = 2$ . The reason is that: although human ear sensitive to the sound pressure level is based on logarithm of magnitude of sound, unit could be decibel (dB), however human ear also has the auditory mask, which mask small peaks around it, thus we should value more on major peaks, and ignore minor one. From the paper [1], and my trial and error, the square is actually achieve very ideal result. I also tried other numbers for  $\beta$ , when  $\beta = 1$ , the sound is messy at all;  $\beta = 2$  is perfect;  $\beta$  is larger, the simpler sound will hear more harmonious, however the complicated chord may not hear well since the algorithm may value more on merging major peaks of spectrum and ignore the little ones. If people need to play more simple chord songs, they may try larger numbers of  $\beta$ , if need to play more messy types songs like Impressionist or Jazz, I suggest they will use smaller  $\beta$ . On average, 2 is great number for  $\beta$ .

Since for each key sound, the first peak of spectrum should start from its fundamental frequency, thus, we will set it 0 to ignore these noise.

### 3.2.3 Tuning with Entropy Optimizer

The tuning process from programming point of view is to move left or right of array  $I(\cdot)$  as minor tuning process with  $+c$  cent shift.

$$I_k(\kappa - c) = \left\| G(f_{\kappa - c}) \right\|^\beta \quad (3.18)$$

The entropy function is defined as:

$$\text{Entropy}(x) = -x \cdot \log(x) \quad (3.19)$$

Entropy for a function is defined as:

$$\begin{aligned} \text{Entropy}(\phi(x)) &= \int_{-\infty}^{+\infty} (-\phi(x) \cdot \log(\phi(x))) dx \\ &= \sum_x (-\phi(x) \cdot \log(\phi(x))) \end{aligned} \quad (3.20)$$

Where  $\phi(\cdot)$  is the density function:

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} \phi(x) dx \\ &= \sum_x \phi(x) \end{aligned} \quad (3.21)$$

### 3.2.3.1 How to calculate entropy value for tuning strategies.

Since the algorithm optimize the case that all sound volume is equal, however the sampling time are different, we will make a standard case to simulate all keys are pressed in an equal strength. In my program, I use density function  $\bar{I}_k(\kappa)$  to simulate the equal strength for each piano key sound in pitch domain:

$$\bar{I}_k(\kappa) = \frac{I_k(\kappa)}{\sum_{\kappa} (I_k(\kappa))} \quad (3.22)$$

When press all piano keys, the total volume  $V(\kappa)$  for each key pitch shift  $+c_k$  cents for tuning is:

$$V(\kappa) = \sum_k (\bar{I}_k(\kappa - c_k)) \quad (3.23)$$

The density function for this function is:

$$\bar{V}(\kappa) = \frac{V(\kappa)}{\sum_{\kappa} (V(\kappa))} \quad (3.24)$$

Then, the cost function value  $J$  (entropy value for function  $\bar{V}(\kappa)$ ) is:

$$J = \sum_{\kappa} (-\bar{V}(\kappa) \cdot \log(\bar{V}(\kappa))) \quad (3.25)$$

### 3.2.3.2 Steps to calculate tuning strategy

In my program, there are several steps to dig out the good strategy for tuning.

- Step 1: Calculate the traditional tuning strategy which is simpler version of Traditional Tuning strategy, to be the initial starting point for entropy minimizer to begin. In this algorithm, no inharmonicity model is built, but just use the captured frequency to optimize.
- Step 2: Randomly change tuning for one key for  $c_k$  cents, and check its entropy value. If entropy value is smaller than last time, we keep this tuning strategy, otherwise, drop. Where the changing pitch is defined as a random number between 0 to some small number  $p$ . We will try both side of tuning by adding and subtracting the pitches. The “A4” key never change since it is standard pitch.
- Step 3: We do “step 2” experiment for all keys and all directions as one round of experiment. Each time we count the times of successfully tuning, until we cannot find a round with no improvement.
- Step 4: We stop the algorithm with the test for  $p$  precision. Then we shrink the  $p$  and more accurate spectrum data (more data), and calculate “Step 2” and “Step 3”
- Step 5: Calculate tuning strategy and get report.

In this process, “Step 1” is because the algorithm has many local minimums; although some local minimum can achieve similar simple and sharp harmony, it perform badly in simpler harmonies, such as an octave. A traditional tuning method can roughly optimize major overtones, the best result for entropy minimizer should be around the traditional tuning strategy.

In “Step 2”, although there should be more improvement during this step, however from probability point of view, when it stops, the result is good enough for this precision. It could also use the parallel algorithm. In my program, I modeled several CPUs (not GPU program this time: GPU should calculate array sum much faster) with one shared

memory to modify the result altogether. Although all CPUs will affect the overall result, however, if we can understand it will stop at the point that several CPUs could not find improvement, the effect are the same.

In “Step 4”, my program uses 3 round with 1, 0.5 and 0.2 cent boundaries as step size for entropy minimizers. Since there are many local minimums, and we need to achieve a smooth tuning strategy for not creating weird music scale sound, we cannot set the step size to be really large. Thus, 1 cent boundary is a good point to start. The, next two round is accurate tuning, the accuracy will be increased to 0.1 cent, which is desirable.

In “Step 5”, the frequency peaks frequencies  $f_{k,n}$  are captured also by “catchup method”, but without weighted average.

### 3.2.4 Creating Tuning Strategy Table

The method to get the frequencies components for each key sound is simple:

$$\tau'(k, n) = f_{k,n} \cdot C_{\rightarrow \text{fr}}(c_k) \quad (3.26)$$

However, this process is problematic. Since the whole process is based on pitch shift with certain precision, the “A4” standard frequency will not be the fix number. Here we need to eliminate this tuning error by introducing a correction factor  $\varepsilon_{[A4]}$ :

$$\varepsilon_{[A4]} = \frac{\tau'([A4], 1)}{\tilde{f}_{[A4]}} \quad (3.27)$$

Thus, the tuning strategy  $\tau(k, n)$  is modified to be:

$$\tau(k, n) = f_{k,n} \cdot C_{\rightarrow \text{fr}}(c_k) \cdot \varepsilon_{[A4]} \quad (3.28)$$

To build the tuning curve, the pitch deviation to the ideal frequency function  $C(k)$  is shown:

$$C(k) = \text{Fr}_{\rightarrow c} \left( \frac{\tau(k, n)}{\tilde{f}_k} \right) \quad (3.29)$$

The tuning strategy is shown in Figure 7-3.

The tuning curve is shown in Figure 3-10, the spectrum of optimized result is shown in Figure 3-11:

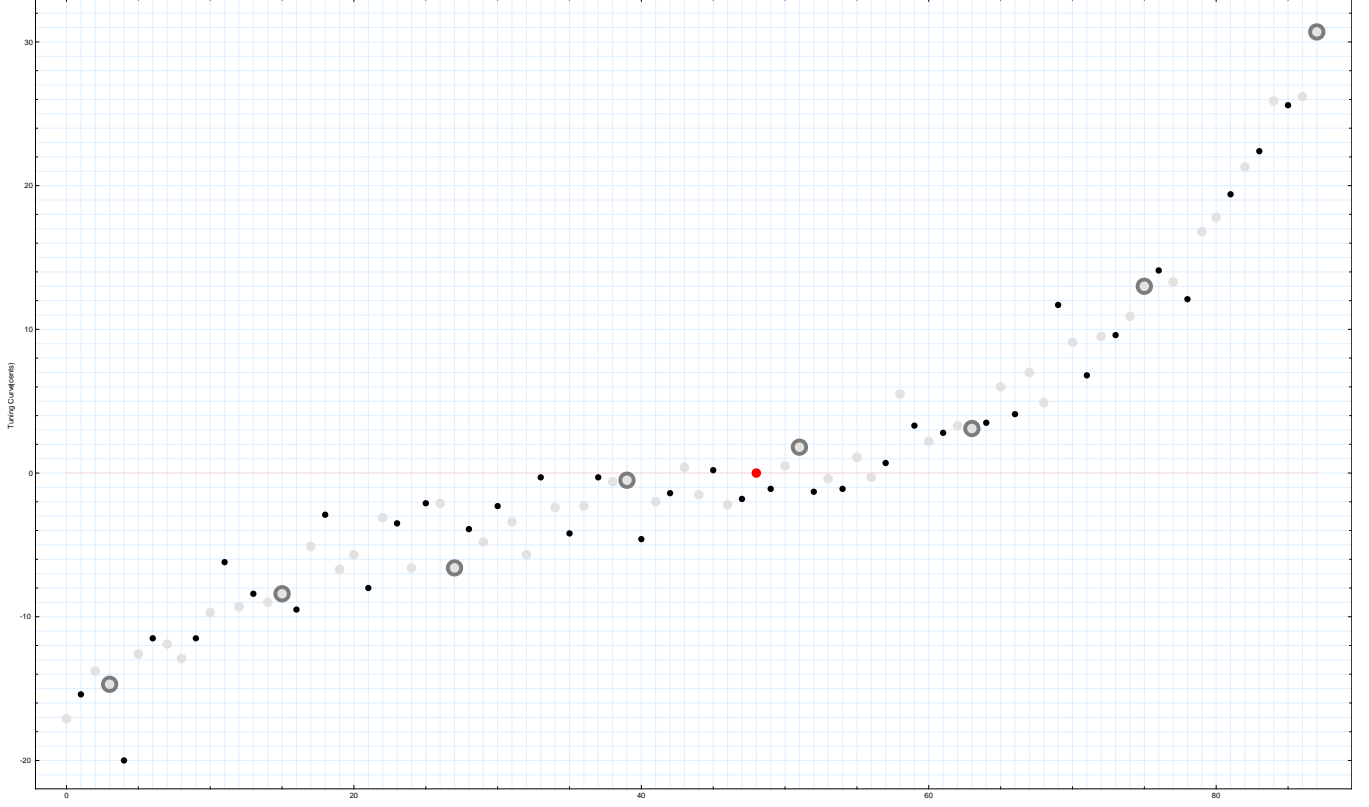


Figure 3-10 Tuning Curve for Upright Piano Optimized by Entropy Minimizer

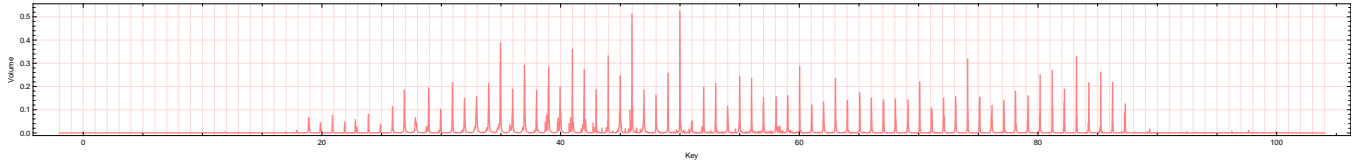


Figure 3-11 Spectrum for Optimized Result

From Figure 3-11, we could see the spectrum are largely merged. From sound quality point of view, the harmony will sound sharp and clear.

## 4 AUDIO PROCESSING & PURE SOUND TUNER

### 4.1 TUNING

Tuning process for an audio is to create samples for virtual instrument so that we can hear the tuning result before tuning process to make a decision whether to adopt or drop this tuning strategy.

The sound function  $S(t)$  tunes in order to add pitch  $c$  cents:

$$S_{+c}(t) = S\left(t \cdot 2^{\left(\frac{c}{1200}\right)}\right) \quad (4.1)$$

The  $S(t)$  function is modeled as interpolation function.



## 4.2 SOUND PURIFY

This audio processing technique is invented by myself. It removes the inharmonic effect of piano sound.

Since the inharmonicity model has been built, it is possible to use audio processing technique to shrink the harmonics in order to remove the inharmonicity.

If the key  $k$  sound with the inharmonicity coefficient  $\text{IH}(k)$  and tuned to the fundamental frequency to be the frequency (ideal frequency)  $\tilde{f}_k$ ; the  $f_k$  is the fundamental frequency.

We firstly get the FFT of the audio sample with  $\Gamma_k(f)$  of complex number samples:

$$\Gamma_k(f) = \text{FFT}(\mathbf{S}_k(t)) \quad (4.2)$$

Since the FFT is creating an almost symmetry data from the middle, we can extract this data into 4 parts: the real head data  $\Gamma_k^{(0)}(f)$ , the imaginary head data  $\Gamma_k^{(1)}(f)$ , the real tail reverse data  $\Gamma_k^{(2)}(f)$  and the tail imaginary reverse data  $\Gamma_k^{(3)}(f)$ . Four of them looks similar, however it contains all the details of the sound. Since it samples the piano keys, the spectrum is pretty obvious. At its high frequencies, it is almost 0, and it is almost out of hearing range, thus if we need to compress the frequency domain, as for higher frequencies, we could regard it to be 0. For each component we write it as  $\Gamma_k^{(m)}(f)$ , where  $m$  is from 0 to 3 (4 cases),  $i$  is the unit imaginary number.

$$\Gamma_k(f) = \left\{ \Gamma_k^{(0)}(f), \text{rev}(\Gamma_k^{(2)}(f)) \right\} + \left\{ \Gamma_k^{(1)}(f), \text{rev}(\Gamma_k^{(3)}(f)) \right\} \cdot i \quad (4.3)$$

From Equation (3.6) and Equation (3.7), we could get the compression functions, which is  $\tau(k, n)$ . Here the overtone is continuous, which is  $f / f_k$ , rather than  $n$ . Thus, we have the compressed frequency scaler  $\tilde{f}_k$  and its pitch component  $\tilde{\Gamma}_k^{(m)}(f)$ :

$$\tilde{f}_k = \tilde{f}_k \cdot \tau\left(k, \frac{f}{f_k}\right) \quad (4.4)$$

$$\tilde{\Gamma}_k^{(m)}(f) = \begin{cases} \Gamma_k^{(m)}(\tilde{f}_k) & \tilde{f}_k \in \text{defined} \\ 0 & \tilde{f}_k \notin \text{defined} \end{cases} \quad (4.5)$$

Where  $\Gamma_k^{(m)}(f)$  and  $\tilde{\Gamma}_k^{(m)}(f)$  will be same size of samples.

Use the interpolation function to stretch, and do this for four functions; then, combine them in original way, and use inverse Fourier function to restore the audio  $\tilde{\mathbf{S}}_k(t)$ .

$$\tilde{\Gamma}_k(f) = \left\{ \tilde{\Gamma}_k^{(0)}(f), \text{rev}(\tilde{\Gamma}_k^{(2)}(f)) \right\} + \left\{ \tilde{\Gamma}_k^{(1)}(f), \text{rev}(\tilde{\Gamma}_k^{(3)}(f)) \right\} \cdot i \quad (4.6)$$

$$\tilde{\mathbf{S}}_k(t) = \text{Re}(\text{invFFT}(\tilde{\Gamma}_k(f))) \quad (4.7)$$

Where  $i$  is imaginary number,  $\text{invFFT}(\cdot)$  is the inverse FFT,  $\text{Re}(\cdot)$  is to get the real part of a number or array,  $\text{rev}(\cdot)$  is the reverse of an array.

Then, do this for 2 channels and create the audio as Pure Sound Tuner result.

From this function, it needs 3 data: the audio data  $S_k(t)$ , the inharmonicity coefficient  $\text{IH}(k)$ , and its fundamental frequency  $f_k$  (which could be captured by audio data).

## 5 FUTURE WORK

Over-pull tuning is implemented in some tuning apps, and I do not know its method. Since I am still lack of research on this area, I will leave it as future work to think about. I know this effect is caused by the experimental result of the percentage that the tuning pins will loosen and drop the pitch, it should have the correction coefficient for the tuner will make up the errors of this effect by over pull to tune the frequency higher than its actual one.

## 6 REFERENCE

[1] Hinrichsen, Haye. "Entropy-based tuning of musical instruments." *Revista brasileira de Ensino de Física* 34.2 (2012): 1-8.

[2] Github for Piano Tuning Project [[https://github.com/RobertBoganKang/piano\\_tuning](https://github.com/RobertBoganKang/piano_tuning)]

# 7 APPENDIX

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A0	27.414	54.832	82.270	109.739	137.252	164.824	192.467	220.196	248.023	275.962	304.026	332.227	360.579	389.094	417.785	446.665
A#0	29.05	58.104	87.177	116.281	145.437	174.637	203.916	233.28	262.742	292.316	322.015	351.851	381.837	411.987	442.313	472.827
B0	30.7833	61.5706	92.3741	123.206	154.079	185.004	215.994	247.061	278.216	309.473	340.842	372.335	403.965	435.742	467.679	499.787
C1	32.6198	65.2432	97.8806	130.543	163.24	195.983	228.782	261.648	294.592	327.622	360.751	393.988	427.344	460.828	494.451	528.223
C#1	34.5657	69.1346	103.716	138.32	172.966	207.633	242.36	277.148	312.005	346.942	381.966	417.089	452.318	487.664	523.135	558.741
D1	36.6274	73.2579	109.9	146.564	183.258	219.991	256.773	293.612	330.517	367.497	404.562	441.72	478.979	516.35	553.84	591.458
D#1	38.8117	77.6267	116.455	155.306	194.19	233.117	272.096	311.138	350.252	389.447	428.734	468.123	507.622	547.242	586.992	626.882
E1	41.1257	82.2549	123.398	166.565	205.767	247.014	288.315	329.683	371.126	412.655	454.281	496.013	537.861	579.836	621.948	664.206
F1	43.5772	87.1575	130.75	174.366	218.012	261.7	305.438	349.236	393.103	437.048	481.082	525.213	569.45	613.803	658.281	702.894
F#1	46.1742	92.3514	138.541	184.752	230.993	277.274	323.604	369.952	416.447	462.979	509.595	556.306	603.12	650.046	697.093	744.271
G1	48.9253	97.8537	146.794	195.756	244.749	293.782	342.863	392.003	441.21	490.494	539.863	589.326	638.893	688.573	738.374	788.306
G#1	51.8397	103.682	155.537	207.414	259.321	311.268	363.263	415.317	467.438	519.636	571.918	624.295	676.776	729.369	782.084	834.929
A1	54.9269	109.857	164.799	219.764	274.759	329.794	384.879	440.023	495.235	550.524	605.899	661.371	716.947	772.637	828.45	884.395
A#1	58.1974	116.398	174.612	232.849	291.118	349.43	407.794	466.22	524.719	583.298	641.969	699.72	759.624	818.627	877.76	937.033
B1	61.6619	123.327	185.006	246.708	308.444	370.224	432.057	493.954	555.926	617.981	680.129	742.382	804.749	867.239	929.863	992.631
C2	65.3319	130.667	196.015	261.396	326.789	392.233	457.73	523.287	589.915	654.624	720.422	786.32	852.327	918.453	985.322	1051.1
C#2	69.2198	138.443	207.682	276.946	346.247	415.596	485.004	554.482	624.041	693.692	763.446	833.315	903.308	973.437	1043.71	1114.15
D2	73.3384	146.681	220.043	293.436	366.875	440.374	513.946	587.605	661.364	735.238	809.239	883.382	957.679	1032.14	1106.79	1181.63
D#2	77.7015	155.408	233.136	310.902	388.72	466.608	544.58	622.653	700.843	779.164	857.635	936.268	1015.08	1094.49	1173.31	1252.75
E2	82.3238	164.654	247.007	329.403	411.859	494.393	577.023	659.766	742.682	825.667	908.859	992.366	1075.82	1159.62	1243.65	1327.95
F2	87.2205	174.448	261.701	349.001	436.366	523.817	613.373	699.054	786.879	874.868	963.041	1051.42	1140.31	1228.85	1317.95	1407.33
F#2	92.4082	184.824	277.274	369.768	462.342	555.014	647.805	740.874	833.84	927.127	1020.62	1114.35	1208.33	1302.59	1397.15	1492.03
G2	97.9041	195.817	293.764	391.772	489.867	588.074	686.419	784.928	883.627	982.541	1081.7	1181.12	1280.83	1380.96	1481.17	1582.03
G#2	103.727	207.463	311.236	415.074	519.005	623.056	727.256	831.631	936.299	1041.02	1146.09	1251.44	1357.1	1463.11	1569.48	1676.25
A2	109.896	219.801	329.747	436.763	549.878	660.123	770.529	881.123	991.936	1103.14	1214.34	1325.99	1437.97	1550.32	1663.07	1776.24
A#2	116.431	232.874	349.366	469.935	582.621	699.459	816.481	933.724	1051.22	1169.01	1287.12	1405.58	1524.44	1643.72	1763.45	1883.68
B2	123.356	246.726	370.15	493.061	617.329	741.166	865.221	989.537	1114.15	1239.11	1364.45	1490.21	1616.43	1743.16	1870.42	1998.27
C3	130.693	261.401	392.175	523.061	654.109	785.367	916.882	1048.7	1180.88	1313.46	1446.48	1580.1	1714.07	1848.22	1984.01	2122.97
C#3	138.466	276.951	415.512	550.707	693.093	832.226	971.665	1109.37	1258.66	1392.37	1533.59	1675.4	1817.85	1960.98	2104.88	2249.57
D3	146.702	293.426	440.24	587.21	734.403	881.887	1029.73	1177.99	1321.74	1476.05	1625.98	1776.59	1927.95	2080.12	2233.16	2387.15
D#3	155.428	310.881	466.434	622.162	778.139	934.439	1091.14	1248.31	1406.02	1564.36	1723.38	1883.17	2043.8	2205.33	2367.83	2531.39
E3	164.674	329.377	494.192	659.205	824.5	990.162	1156.27	1322.92	1490.18	1658.15	1826.89	1996.5	2167.05	2338.63	2511.31	2685.17
F3	174.471	348.975	523.614	698.489	873.7	1049.35	1225.53	1402.35	1579.91	1758.29	1937.61	2117.96	2299.43	2482.11	2666.11	2851.51
F#3	184.851	369.741	554.791	740.118	925.842	1112.08	1298.95	1486.57	1675.06	1864.53	2055.1	2246.87	2439.97	2634.49	2830.57	3028.29
G3	195.849	391.745	582.898	784.24	981.119	1178.61	1376.84	1575.95	1775.09	1977.38	2179.97	2383.97	2589.53	2796.78	2995.34	3216.84
G#3	207.503	415.063	622.85	831.036	1039.79	1249.28	1459.68	1671.14	1883.85	2097.96	2313.63	2531.02	2750.3	2971.62	3195.13	3420.98
A3	219.851	439.769	659.951	880.622	1101.96	1324.18	1547.47	1772.04	1997.89	2225.8	2455.36	2686.96	2920.83	3157.1	3395.96	3637.6
A#3	232.934	465.942	699.243	933.059	1167.61	1403.11	1639.78	1877.83	2117.48	2358.93	2602.4	2848.09	3096.2	3346.93	3600.47	3857.25
B3	246.797	493.674	740.431	988.637	1237.2	1486.81	1737.77	1990.11	2244.26	2500.41	2758.74	3019.51	3282.94	3549.27	3818.67	4091.26
C4	261.485	523.066	785.025	1047.65	1311.21	1576.01	1842.31	2110.39	2380.52	2652.98	2928.03	3205.93	3486.94	3771.31	4059.28	4351.11
C#4	277.049	554.21	831.819	1110.21	1389.72	1670.68	1953.42	2238.27	2525.53	2815.55	3108.61	3405.04	3705.13	4009.17	4317.45	4630.26
D4	293.539	587.21	881.405	1176.52	1472.93	1771.05	2071.24	2373.89	2679.37	2988.05	3300.29	3616.45	3936.88	4261.91	4591.87	4927.09
E4	311.012	622.169	933.903	1246.65	1560.83	1876.88	2195.21	2516.25	2840.41	3168.09	3499.69	3835.59	4176.19	4521.84	4872.92	5229.77
F4	329.525	659.213	989.55	1321.02	1654.11	1989.3	2327.06	2667.86	3012.15	3360.41	3713.06	4070.55	4433.29	4801.74	5176.2	5557.15
F#4	349.141	698.478	1048.6	1400.09	1753.52	2109.48	2468.54	2831.26	3198.18	3569.85	3946.8	4329.55	4718.6	5114.43	5517.52	5928.33
G4	369.925	730.079	1111.5	1483.83	1858.79	2236.71	2618.25	3004.07	3394.72	3791.12	4193.58	4602.81	5019.38	5443.85	5876.77	6318.66
G#4	391.946	784.159	1177.44	1572.59	1970.39	2371.63	2777.09	3187.52	3603.65	4026.23	4455.96	4893.52	5339.57	5794.76	6259.71	6735.13
A4	415.278	830.877	1247.76	1666.87	2089.17	2515.59	2947.06	3384.46	3828.7	4280.62	4741.06	5210.83	5690.71	6181.43	6683.73	7198.27
A#4	440.14	880.383	1340.19	1769.89	2215.28	2668.59	3127.93	3594.35	4068.9	4552.6	5046.43	5551.32	6068.2	6597.91	7141.29	7699.13
B4	466.193	932.833	1401.26	1872.8	2348.76	2830.46	3319.15	3816.08	4322.45	4839.44	5368.15	5909.68	6465.05	7035.25	7621.21	8223.81
C5	493.946	988.044	1484.91	1984.97	2490.11	3001.8	3521.49	4050.6	4590.49	5142.48	5707.85	6287.8	6883.5	7496.05	8126.49	8775.8
C#5	522.552	1047.28	1573.51	2106.78	2639.72	3183.07	3735.42	4298.35	4873.42	5462.08	6065.75	6685.78	7323.47	7980.01	8656.57	9354.21
D5	549.805	1108.7	1627.64	2230.37	2799.89	3378.19	3967.18	4568.75	5184.68	5816.3	6466.45	7135.5	7825.33	8527.32	9222.78	9934.25
D#5	587.52	1175.88	1767.56	2365.06	2970.82	3587.2	4216.54	4861.07	5522.94	6204.18	6906.74	7632.44	8382.99	9160.01	9964.98	10799.3
E5	622.499	1245.98	1873.37	2507.59	3151.49	3807.86	4479.39	5168.69	5878.23	6610.36	7367.29	8151.12	8963.77	9807.06	10682.6	11592.1
F5	659.562	1320.26	1985.49	2658.61	3342.92	4041.63	4757.84	5494.53	6254.51	7040.46	7854.87	8700.08	9578.27	10491.4	11441.4	12429.8
F#5	698.833	1398.96	2104.26	2818.56	3545.62	4289.09	5052.47	5839.13	6652.24	7494.8	8369.59	9279.22	10226.1	11212.3	12240.1	13311.1
G5	740.447	1482.36	2230.15	2968.16	3760.64	4551.73	5365.37	6205.35	7075.21	7978.31	8917.74	9896.37	10916.8	11981.5	13092.7	14252.3
G#5	784.542	1570.77	2363.74	3168.43	3989.72	4832.3	5700.69	6599.17	7531.74	8502.16	9513.9	10570.1	11673.8	12827.4	14033.6	15294.3
A5	831.27	1664.56	2505.93	3361.33	4236.58	5137.26	6068.69	7035.89	8043.54	9095.97	10197.1	11350.6	12559.8	13827.5	15156.4	16500.7
A#5	880.789	1764.02	2656.99	3566.88	4500.65	5464.98										

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A0	27.2503	54.5137	81.8296	109.237	136.775	164.482	192.397	220.557	249.109	277.763	306.88	336.388	366.321	396.713	427.595	459.757
A#0	28.8845	56.7822	84.9673	113.777	143.092	174.3	203.857	233.663	263.756	294.172	324.984	356.121	387.724	419.793	452.36	485.459
B0	30.6169	61.2473	91.9319	122.711	153.628	184.715	216.019	247.577	279.427	311.609	344.158	377.112	410.507	444.378	478.759	513.683
C1	32.4532	64.9194	97.4374	130.046	162.783	195.688	228.798	262.152	295.785	329.735	364.038	398.729	433.843	469.414	505.475	542.059
C#1	34.3997	68.8219	103.281	137.844	172.543	207.418	242.509	277.856	313.497	349.473	385.819	422.575	459.775	497.457	535.655	574.403
D1	36.4625	72.9374	109.462	146.073	182.807	219.702	256.793	294.118	331.717	369.607	407.843	446.452	485.467	524.923	564.851	605.284
D#1	38.6486	77.3104	116.025	154.831	193.768	232.875	272.191	311.753	351.601	391.771	432.301	473.226	514.583	556.406	598.731	641.591
E1	40.9652	81.9434	122.973	164.093	205.342	246.758	288.379	330.243	372.388	414.851	457.666	500.871	544.502	588.594	633.179	678.293
F1	43.4199	86.8538	130.344	173.931	217.658	261.565	305.695	350.087	394.782	439.821	485.242	531.084	577.386	624.186	671.52	719.425
F#1	46.0208	92.0562	138.15	184.345	230.685	277.213	323.972	371.005	418.353	466.059	514.162	562.705	611.726	661.265	711.361	762.051
G1	48.7765	97.5648	146.4	195.318	244.353	293.54	342.915	392.511	442.364	492.507	542.974	593.798	645.013	696.65	748.743	801.322
G#1	51.6961	103.404	155.158	206.992	258.942	311.042	363.328	415.828	468.581	521.62	574.978	628.687	682.78	737.289	792.246	847.682
A1	54.7892	109.591	164.441	219.377	274.436	329.653	385.065	440.709	496.62	552.833	609.384	666.308	723.639	781.411	839.659	898.414
A#1	58.0661	116.145	174.272	232.487	290.825	349.323	408.019	466.948	526.147	585.651	645.497	705.72	766.355	827.435	888.997	951.072
B1	61.5375	123.091	184.711	246.446	308.346	370.458	432.833	495.518	558.562	622.011	685.914	750.315	815.262	880.79	946.971	1013.82
C2	65.2149	130.445	195.737	261.125	326.688	392.435	458.427	524.707	591.321	658.112	725.724	793.601	861.985	930.918	1000.45	1070.6
C#2	69.1106	138.235	207.414	276.69	346.104	415.695	485.507	555.778	625.95	696.662	767.755	839.268	911.239	983.708	1056.71	1130.29
D2	73.2374	146.493	219.805	293.229	366.808	440.588	514.615	589.335	663.593	738.636	814.106	890.05	966.51	1043.3	1121.15	1199.42
D#2	77.6991	155.232	232.912	310.69	388.608	466.71	545.036	623.628	702.527	781.775	861.413	941.481	1022.02	1103.07	1184.66	1266.85
E2	82.24	164.495	246.811	329.234	411.81	494.583	577.599	660.903	744.541	828.556	912.992	1000.14	1087.89	1169.27	1255.83	1343.02
F2	87.1456	174.307	261.563	348.862	436.35	524.039	611.976	700.207	788.777	877.733	967.119	1056.98	1147.36	1238.91	1326.85	1416.08
F#2	92.3422	184.701	277.217	369.67	462.381	555.309	648.504	742.016	835.894	930.187	1024.94	1120.21	1216.04	1312.47	1409.56	1507.35
G2	97.8471	195.712	293.647	391.706	488.406	586.972	687.153	788.335	889.675	995.602	1085.99	1186.92	1288.45	1390.63	1493.46	1597.04
G#2	103.679	207.376	311.152	415.063	519.167	623.522	728.186	833.216	938.668	1044.6	1151.07	1258.12	1365.83	1474.23	1583.39	1693.36
A2	109.675	219.731	329.681	439.762	550.022	660.545	771.357	882.525	994.104	1106.15	1218.71	1331.85	1445.62	1560.07	1675.26	1791.22
A#2	116.4	232.821	349.324	465.971	582.822	699.941	817.387	935.222	1053.5	1172.3	1291.66	1411.65	1532.32	1653.74	1775.96	1899.04
B2	123.333	246.687	370.126	493.714	617.516	741.593	866.01	990.833	1116.12	1241.93	1368.33	1495.39	1623.15	1751.69	1881.06	2011.32
C3	130.677	261.375	392.126	523.096	654.246	785.677	917.451	1049.631	1182.29	1315.47	1449.26	1583.7	1718.87	1854.81	1991.6	2129.73
C#3	138.457	276.936	415.003	554.222	693.159	832.38	971.95	1113.9	1252.39	1393.39	1535.1	1677.28	1820.28	1964.37	2108.74	2254.31
D3	146.7	293.412	440.175	587.029	734.01	881.158	1028.51	1176.11	1324.33	1472.18	1620.73	1769.67	1919.05	2068.9	2219.25	2370.15
D#3	155.432	310.877	466.376	621.97	777.699	933.603	1089.72	1246.1	1402.77	1559.78	1717.17	1874.97	2033.23	2191.98	2351.27	2511.13
E3	164.683	329.381	494.144	659.018	824.053	989.294	1154.79	1320.59	1486.74	1653.29	1820.28	1987.77	2155.79	2324.4	2493.64	2663.56
F3	174.483	349.009	523.7	698.683	874.081	1050.02	1226.62	1404.877	1582.19	1761.61	1942.07	2123.8	2306.91	2491.52	2677.33	2865.67
F#3	184.867	369.78	554.875	740.289	926.158	1112.62	1299.81	1487.85	1676.89	1867.06	2058.48	2251.29	2445.61	2641.56	2839.29	3038.89
G3	195.868	391.786	597.907	804.38	1013.56	1228.99	1445.82	1676.82	1913.91	2147.22	2386.77	2627.53	2873.08	3119.45	3372.44	3633.33
G#3	207.523	415.104	622.914	831.126	1039.91	1249.44	1459.87	1671.39	1884.16	2098.34	2314.1	2531.59	2750.98	2972.43	3196.08	3422.09
A3	219.872	439.811	660.014	880.682	1102.01	1324.2	1547.45	1771.94	1997.87	2225.44	2454.83	2686.22	2919.88	3155.76	3394.25	3635.47
A#3	232.956	465.988	699.333	933.213	1167.87	1403.52	1640.4	1878.73	2118.71	2360.65	2604.68	2851.03	3099.92	3351.57	3608.76	3863.91
B3	246.818	493.721	740.968	988.816	1237.52	1487.34	1738.52	1991.31	2245.97	2502.73	2761.84	3023.54	3288.06	3555.63	3826.47	4100.81
C4	261.505	523.109	785.106	1047.8	1311.48	1576.43	1842.96	2111.34	2381.77	2654.82	2930.46	3209.08	3490.94	3776.29	4058.45	4340.82
C#4	277.067	554.249	831.892	1110.34	1389.94	1671.04	1953.96	2239.05	2526.64	2817.04	3100.59	3382.62	3671.07	3960.37	4250.11	4540.25
D4	293.555	587.245	874.187	1176.65	1463.82	1751.46	2041.82	2334.84	2630.72	2929.91	3230.77	3532.77	3836.97	4143.5	4451.59	4760.25
D#4	311.025	622.199	933.969	1246.78	1561.08	1877.31	2195.9	2517.28	2841.88	3170.11	3502.39	3839.11	4180.67	4527.45	4879.82	5238.13
E4	329.535	659.243	989.642	1321.25	1654.57	1990.13	2328.41	2669.93	3015.16	3364.6	3718.7	4077.93	4442.73	4813.55	5190.88	5574.89
F4	349.148	698.495	1048.63	1400.16	1753.66	2109.72	2468.91	2831.81	3198.97	3570.95	3948.26	4331.45	4721.01	5117.45	5521.23	5932.82
F#4	369.929	737.074	1111.08	1483.6	1858.26	2235.71	2616.56	2997.01	3380.44	3765.7	4156.26	4553.19	4955.04	5362.36	5780.45	6209.42
G4	391.948	784.147	1177.35	1572.3	1969.75	2370.42	2775.05	3184.36	3599.03	4019.76	4442.21	4882.04	5344.87	5819.67	6306.96	6797.38
G#4	415.279	830.844	1247.55	1666.25	2087.8	2513.02	2942.74	3377.78	3818.93	4266.96	4722.63	5186.67	5659.79	6142.68	6635.99	7140.35
A4	440.194	880.338	1322.03	1766.07	2213.48	2665.23	3122.29	3585.63	4056.17	4534.81	5022.45	5519.92	6028.05	6547.63	7079.41	7624.12
A#4	466.194	932.772	1400.88	1871.68	2346.29	2825.83	3311.41	3804.1	4304.97	4815.02	5335.25	5866.61	6410.03	6966.39	7536.52	8121.24
B4	493.948	988.337	1484.49	1983.71	2487.3	2996.54	3512.7	4037.25	4570.63	5114.76	5670.51	6238.97	6821.15	7418.07	8030.65	8659.79
C5	523.357	1047.24	1572.22	2102.85	2637.69	3179.25	3729.03	4288.47	4858.98	5441.93	6038.63	6650.33	7278.23	7923.46	8587.12	9270.22
C#5	554.518	1106.64	1667.15	2228.86	2796.51	3371.85	3956.57	4552.32	5140.26	5740.49	6342.9	6947.88	7576.77	8230.93	8894.37	9584.34
D5	587.537	1175.77	1766.78	2367.7	2965.37	3577.77	4199.47	4834.68	5481.66	6150.57	6834.7	7538.43	8263.3	9010.73	9782.06	10578.6
D#5	622.524	1245.83	1872.25	2504.12	3143.71	3793.28	4455.01	5131.02	5823.36	6533.97	7264.73	8007.41	8793.67	9595.1	10423.2	11279.2
E5	659.596	1320.08	1984.11	2654.33	3333.33	4023.67	4727.8	5448.14	6186.97	6946.5	7728.81	8540.29	9358.81	10203.5	11097.2	12047.2
F5	698.88	1398.82	2103.02	2814.64	3536.77	4272.46	5024.65	5796.16	6589.7	7407.85	8253.03	9127.54	10033.5	10972.9	11943.7	12959.2
F#5	740.506	1482.16	2228.38	2982.56	3748.05	4528.1	5325.87	6144.38	6986.55	7855.13	8752.72	9681.76	10644.5	11643.2	12679.6	13755.6
G5	784.616	1570.67	2362.47	3164.27	3980.25	4814.46	5670.81	6553.03	7464.66	8409.03	9389.26	10408.2	11468.6	12540.3	13723.3	14921.9
G#5	831.357	1664.41	2504.23	3355.83	4224.11	5113.82	6029.51	6975.51	7955.9	8974.51	10034.9	11140.3	12293.7	13497.9	14755.3	16084.7
A5	880.899	1763.77	2654.98	3559.2												

	1	2	3	4	5	6	7	8	9	10	11	12
A0	27.4413	54.6261	82.0674	100.765	137.463	165.417	193.884	222.095	251.075	280.055	309.804	339.297
A#0	28.8409	57.6818	86.5226	115.579	144.635	173.906	204.038	233.525	264.088	294.65	325.428	356.422
B0	30.6368	61.2735	91.9103	122.766	153.98	185.213	216.845	248.675	280.903	313.33	345.758	378.981
C1	32.4352	64.8703	97.4898	130.109	163.097	196.454	229.626	263.72	297.445	331.539	365.633	399.648
C#1	34.1767	68.502	102.679	137.301	171.924	207.14	242.209	277.723	313.385	349.494	385.751	422.305
D1	36.5828	73.0243	109.748	146.614	183.338	220.768	258.339	296.052	334.189	372.184	410.744	449.304
D#1	38.6613	77.3226	115.984	155.044	194.303	233.761	273.419	313.276	353.532	393.787	434.441	475.295
E1	40.8872	81.9168	122.946	164.404	205.719	247.46	289.06	331.371	373.826	416.85	459.304	502.613
F1	43.2485	86.6554	130.062	173.786	216.668	261.867	306.225	350.899	395.415	440.723	486.031	531.972
F#1	45.9597	91.9195	138.012	184.372	230.731	277.623	324.516	371.808	419.366	467.191	515.282	564.039
G1	48.8379	97.5044	146.514	195.694	244.703	293.713	343.236	392.931	442.968	493.005	543.557	594.451
G#1	51.5976	103.379	154.976	206.941	258.722	310.687	363.019	415.351	465.664	520.199	574.367	627.985
A1	54.5567	109.325	164.093	218.861	273.841	329.032	384.435	440.049	495.874	551.7	608.371	665.254
A#1	57.9676	115.935	174.057	232.487	290.609	349.039	407.623	466.515	525.562	585.38	645.352	705.632
B1	61.2533	122.864	184.474	246.442	308.41	370.556	432.524	495.027	558.423	621.641	685.573	749.862
C2	65.2473	130.223	195.742	261.261	326.78	392.571	458.906	524.969	591.576	658.454	726.148	794.386
C#2	69.1197	137.766	207.043	276.321	345.441	415.034	484.785	554.693	625.233	695.931	766.629	838.589
D2	73.2158	146.432	220.086	293.301	367.394	440.829	515.36	589.453	664.641	739.83	815.457	890.865
D#2	77.3853	155.26	232.523	310.642	387.905	465.779	544.876	622.995	702.336	780.822	860.407	940.822
E2	82.0508	164.102	246.583	328.921	411.402	493.739	576.794	660.136	744.052	827.537	912.313	997.09
F2	86.8265	173.957	261.544	348.827	436.261	523.088	610.979	699.326	785.088	876.78	966.04	1056.21
F#2	91.9479	184.156	276.623	368.831	461.298	554.025	647.012	740.518	834.284	928.57	1022.86	1116.26
G2	97.5987	195.746	293.482	391.629	489.227	587.511	686.481	785.862	885.379	985.171	1085.57	1186.66
G#2	103.234	207.236	311.046	414.856	518.283	622.86	727.821	832.207	938.128	1043.66	1150.16	1257.04
A2	109.361	219.307	329.643	439.591	549.928	659.677	770.403	882.298	993.024	1104.92	1218.18	1330.47
A#2	115.706	232.675	348.855	465.351	582.478	698.658	816.258	933.659	1051.93	1170.95	1289.19	1409.63
B2	123.074	246.49	369.563	493.321	617.078	741.007	865.448	989.89	1115.19	1240.99	1367.32	1493.81
C3	130.589	261.693	392.692	523.081	654.384	785.992	917.6	1050.43	1182.64	1316.08	1449.82	1587.65
C#3	137.462	276.483	414.88	553.278	691.363	830.072	969.405	1109.67	1249.32	1389.9	1531.1	1672.92
D3	146.502	293.308	440.42	587.531	734.642	881.753	1029.17	1176.28	1324.3	1472.94	1621.46	1770.51
D#3	155.145	310.646	466.039	621.57	777.239	932.908	1088.85	1245.08	1401.58	1558.49	1714.99	1873.02
E3	164.034	329.759	493.475	658.528	823.244	988.297	1153.69	1319.08	1484.47	1650.53	1811.54	1985.02
F3	173.983	348.892	523.8	699.711	874.542	1050.38	1227.6	1404.82	1583.43	1762.5	1942.18	2124.35
F#3	183.896	369.662	554.181	739.323	925.713	1112.1	1299.11	1486.75	1675.63	1865.76	2057.14	2249.14
G3	195.777	391.55	587.727	784.308	981.694	1179.48	1378.08	1577.48	1778.09	1979.9	2183.33	2387.57
G#3	206.993	413.985	621.475	829.643	1038.94	1247.43	1457.9	1669.38	1881.34	2094.8	2311.25	2527.2
A3	219.729	439.795	660.198	880.602	1102.69	1325.12	1548.56	1773.35	1999.16	2226.65	2456.17	2687.37
A#3	232.696	465.911	699.127	933.122	1167.38	1403.71	1640.57	1878.72	2119.48	2360.75	2604.63	2850.59
B3	246.266	493.394	740.951	988.94	1237.79	1487.93	1738.5	1992.52	2246.1	2503.13	2761.88	3023.22
C4	261.469	523.505	786.111	1049.28	1313.6	1579.04	1846.19	2115.05	2386.18	2660.16	2935.84	3213.79
C#4	276.399	553.749	831.104	1109.42	1389.65	1670.85	1953.96	2238.99	2528.82	2816.73	3110.4	3405.99
D4	293.461	587.402	881.823	1177.21	1474.03	1773.25	2073.92	2376.5	2683.89	2992.24	3304.92	3619.03
D#4	310.756	621.512	933.128	1245.17	1559.37	1875.71	2193.35	2515.28	2841.94	3168.6	3500.84	3834.38
E4	329.696	656.116	989.137	1321.47	1655.65	1991.21	2329.09	2671.13	3016.87	3364.91	3718.51	4077.18
F4	348.817	698.29	1048.42	1400.52	1754.92	2110.97	2470.62	2832.58	3203.07	3571.92	3947.34	4330.65
F#4	369.811	739.622	1110.68	1483.61	1859.03	2237.58	2617.37	3002.77	3396.28	3786.67	4187.04	4590.52
G4	391.691	784.275	1178.05	1574.2	1972.14	2373.64	2779.31	3187.66	3606.71	4026.36	4454.64	4885.58
G#4	414.798	829.597	1246.18	1664.53	2086.45	2511.93	2942.75	3371.79	3822.2	4267.26	4719.44	5178.75
A4	440.4	881.137	1324.55	1769.1	2219.33	2671.83	3131.16	3595.04	4069.15	4547.8	5035.56	5525.58
A#4	465.705	932.19	1401.01	1872.18	2348.03	2828.55	3316.88	3803.65	4312.26	4819.3	5340.39	5863.04
B4	493.727	987.974	1484.82	1985.31	2490.49	3000.86	3518.52	4039.82	4578.81	5122.48	5678.64	6244.69
C5	524.062	1048.12	1577.04	2108.74	2645.3	3189.49	3739.93	4301.47	4872.73	5456.49	6045.1	6660.09
C#5	553.989	1110.06	1668.2	2231.55	2801.13	3376.94	3963.15	4556.64	5169.87	5793.5	6428.56	7072.98
D5	587.407	1176.73	1769.9	2367.86	2973.51	3588.75	4212.63	4849.95	5500.7	6164.89	6844.44	7544.15
D#5	621.43	1247.02	1875.39	2507.91	3151.54	3804.87	4467.92	5142.06	5838.39	6543.05	7274.06	
E5	659.347	1318.69	1986.36	2660.27	3342.5	4035.12	4738.15	5466.13	6200.36	6963.71	7735.37	
F5	698.14	1400.02	2104.39	2821.23	3545.56	4282.34	5036.58	5817	6607.4	7418.99	8262.99	
F#5	739.78	1481.48	2230.84	2990.74	3762.14	4549.84	5351.9	6177.93	7025.03	7900.89	8755.66	
G5	786.154	1576.47	2373.74	3179.34	4003	4840.54	5701.7	6590.64	7505.97	8437.96	9419.96	
G#5	831.792	1666.36	2510.62	3367.37	4239.37	5133.54	6049.9	6996.76	7975.5	8984.74		
A5	881.357	1766.12	2663.38	3573.13	4496.51	5451.69	6427.32	7442.7	8485.33	9539.33		
A#5	933.078	1873.96	2824.2	3790.04	4774.61	5793.51	6831.13	7920.24	9038.99			
B5	989.254	1988.5	2995.24	4024.46	5081.17	6165.35	7284.5	8453.62	9652.72			
C6	1048.61	2108.33	3180.55	4272.21	5391.65	6559.7	7761.09	8983.3				
C#6	1110.51	2230.72	3367.56	4530.75	5735.54	6966.66	8253.24	9599.43				
D6	1178.43	2368.09	3577.73	4819.83	6096.88	7421.36	8797.03					
D#6	1246.25	2502.88	3784.44	5093	6451.41	7845.13	9319.85					
E6	1323.47	2656.92	4022.84	5421.22	6872.03	8382.78						
F6	1399.17	2817.04	434.393	4266.11	5756.75	7305.6	8920.98					
F#6	1489.74	2994.47	458.281	4536.7	6128.92	7796.12	9528.32					
G6	1576.77	3171.02	485.261	4815.26	6509.47	8273.65						
G#6	1665.85	3361.63	5094.81	5209.89	6892.84	8788.12						
A6	1769.53	3544.05	5420.89	6110.85	7345.16	9376.75						
A#6	1872.73	3780.37	5750.35	6471.38	7800.13							
B6	1986.02	3972.05	6110.85	7345.16								
C7	2108.65	4259.78	6498.35	8834.36								
C#7	2234.48	4461.48	6870.33	9381.31								
D7	2366.71	4798.33	7344.79									
D#7	2504.12	5072.96	7763.77									
E7	2659.56	5309.16	8130.43									
F7	2816.41	5724.95	8805.32									
F#7	2992.84	5988.17	9061.02									
G7	3175.77	6471.38										
G#7	3360.55	6726.09										
A7	3571.5	7138.01										
A#7	3782.77	7735.22										
B7	4010.53	8026.05										
C8	4259.89	8731.77										

Figure 7-3 Entropy Tuning for Upright Piano