Probability & Statistics for Data Science

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Random Variable and Distribution

- A ***random variable X*** is a numerical outcome of a random experiment
- The ***distribution*** of a random variable is the collection of possible outcomes along with their probabilities:
  - Discrete case: $\Pr(X = x) = p_\theta(x)$
  - Continuous case: $\Pr(a \le X \le b) = \int_a^b p_\theta(x)dx$

- Distribution simulation
  - https://www.youtube.com/watch?v=6YDHBFVIvIs

Random Variables Distributions

- Cumulative Probability Distribution (CDF):

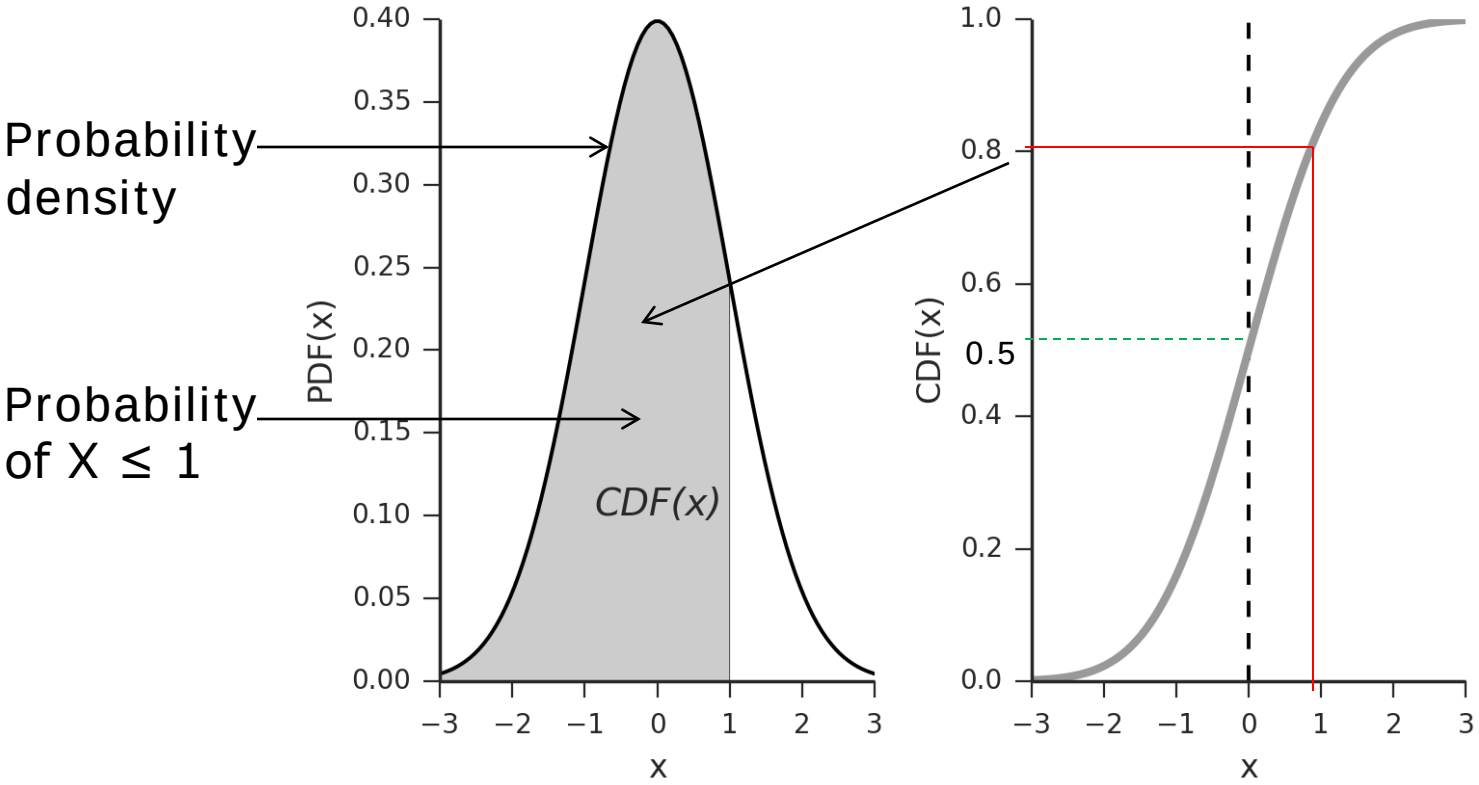
$$F_X(x) = P(X \leq x)$$

- Probability Density Function (PDF):

$$p_X(x) = \frac{dF_X(x)}{dx}$$

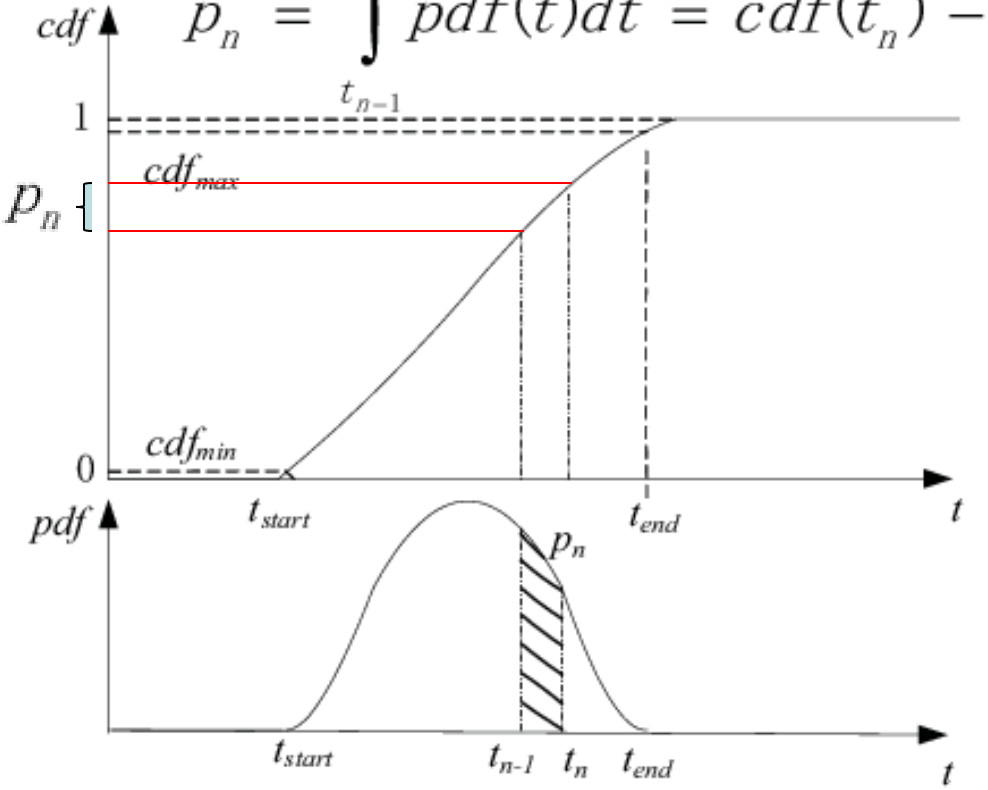
$$\int_{-\infty}^{\infty} p_X(x)dx = 1.0$$

Random Variables Distributions



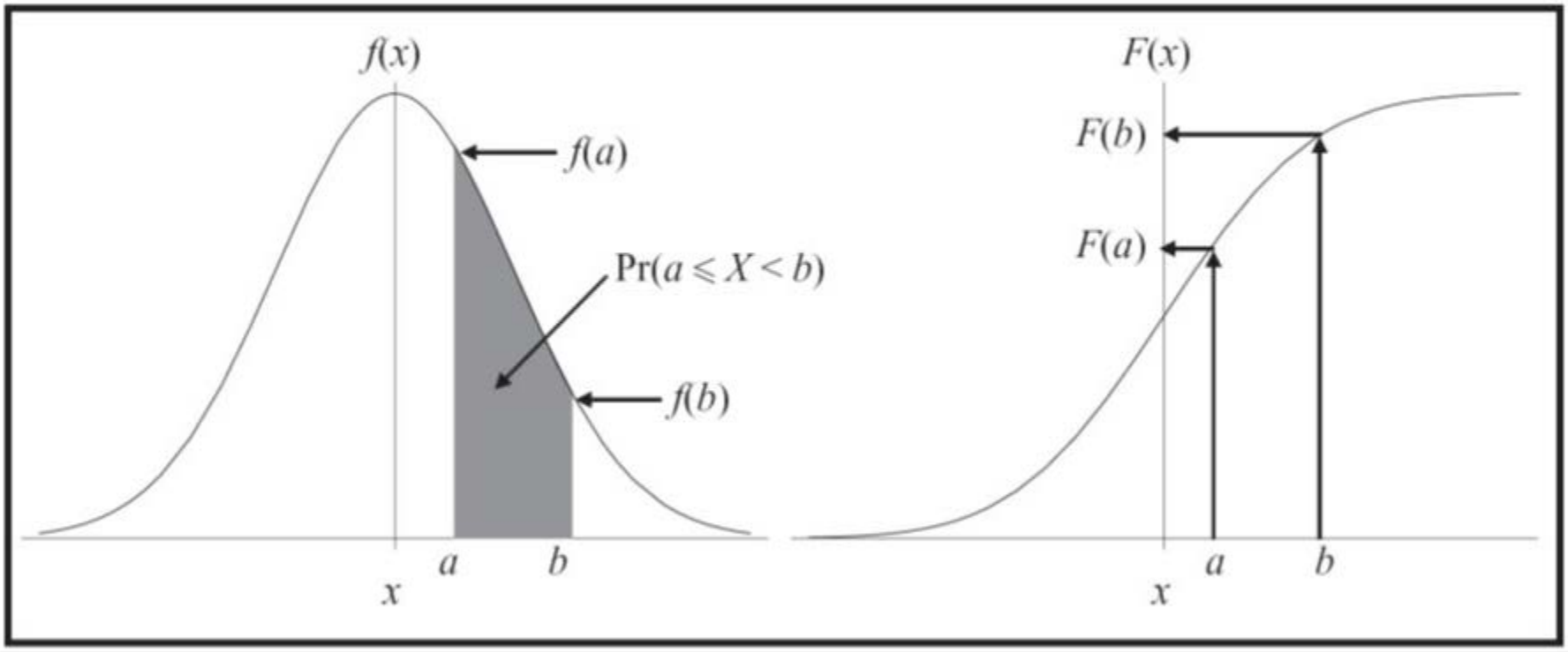
Random Variables Distributions

$$p_n = \int_{t_{n-1}}^{t_n} pdf(t)dt = cdf(t_n) - cdf(t_{n-1})$$



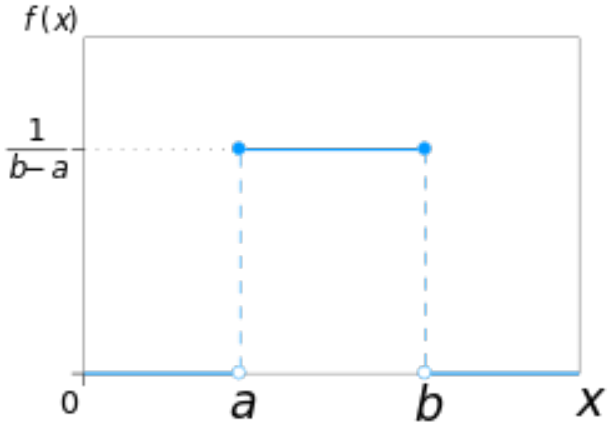
Random Variables Distributions

Probability density function **Cumulative distribution function**



A frequently asked question:

- Can a probability density function (pdf) be greater than 1?
- Let’ s take an example of the easiest PDF — the uniform distribution defined on the domain [a, b]. The PDF of the uniform distribution is 1/(b-a) within the domain, which is constantly 2 throughout when b=1 and a=0.5, and 0 elsewhere



- The total probability is the total area under the graph f(x), which is 2 * 0.5 = 1.

Random Variable: Example

- Let S be the set of all sequences of three rolls of a dice. Let X be the sum of the number of dots on the three rolls.
- What are the possible values for X?
- Pr(X = 5) = ?, Pr(X = 10) = ?
- **Solution**

  5=1+1+3=1+2+2

  (3+3)/(6*6*6) = 1/36

  10=1+3+6=2+2+6=2+3+5=2+4+4

  (6+3+6+3)/(6*6*6)=18/(6*6*6)=1/12

Expectation

- Definition: the expectation of a random variable is

$$E[X] = \sum_x x \Pr(X = x)$$ discrete case

What is "average"?

$$E[X] = \int_{-\infty}^{\infty} x p_\theta(x) dx$$ continuous case

- Properties
  - Summation: For any n≥1, X_1, X_2, …, X_n may be dependent and any constants $k_1,\ldots,k_n$

$$E[\sum_{i=1}^{n} k_i X_i] = \sum_{i=1}^{n} k_i E(X_i)$$

  - Product: If X_1, X_2, …, X_n are independent

$$E[\prod_{i=1}^{n} X_i] = \prod_{i=1}^{n} E(X_i)$$

Expectation

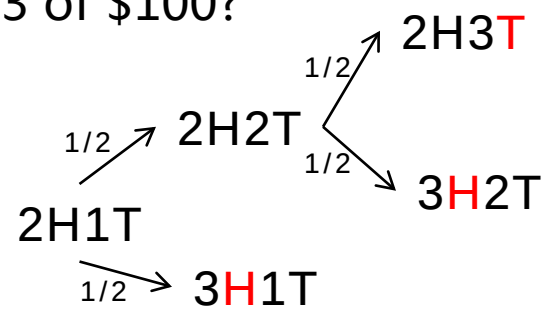
- Division of the stakes problem

  Henry and Tony play a game. They toss a fair coin, if get a Head, Henry wins; Tail, Tony wins. They contribute equally to a prize pot of $100, and agree in advance that the first player who has won 3 rounds will collect the entire prize. However, the game is interrupted for some reason after 3 rounds. They got 2 H and 1 T. How should they divide the pot fairly?

a) It seems unfair to divide the pot equally Since Henry has won 2 out of 3 rounds. Then, how about Henry gets 2/3 of $100?

b) Other thoughts?

X	0	100
P	0.25	0.75



X is what Henry will win if the game not interrupted

Expectation: Example

- Let S be the set of all sequence of three rolls of a dice. Let X be the sum of the number of dots on the three rolls.
- What is E(X)?
  - *E[X1+X2+X3] = E[X1]+E[X2]+E[X3]=3.5*3=10.5*
- Let S be the set of all sequence of three rolls of a dice. Let X be the product of the number of dots on the three rolls.
- What is E(X)?
  - *E[X1*X2*X3] = E[X1]*E[X2]*E[X3]=3.5^3*

Variance

- Informally, variance measures how far a set of (random) numbers are spread out from their average value.
- Mathematically, the variance of a random variable X is the expectation of $(X-E[x])^2$:

$$\begin{aligned}\operatorname{Var}(X) &= \operatorname{E}\left[(X - \operatorname{E}[X])^2\right] \\ &= \operatorname{E}\left[X^2 - 2X\operatorname{E}[X] + \operatorname{E}[X]^2\right] \\ &= \operatorname{E}\left[X^2\right] - 2\operatorname{E}[X]\operatorname{E}[X] + \operatorname{E}[X]^2 \\ &= \operatorname{E}\left[X^2\right] - \operatorname{E}[X]^2\end{aligned}$$

- Properties
  - For any constant C, $Var(CX)=C^2Var(X)$
  - If $X_1, X_2, \ldots, X_n$ are independent

$$Var(X_1 + X_2 + \ldots + X_n) = Var(X_1) + Var(X_2) + \ldots + Var(X_n)$$

Standard Deviation

- Definition: the **square root** of the **Variance**:

The "**Population** Standard Deviation":

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}$$

The "**Sample** Standard Deviation":

$$s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}$$

Looks complicated, but the important change is to divide by **N-1** (instead of **N**) when calculating a Sample Variance.

Covariance and Correlation

$$COV(x,y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n-1}$$

x = the independent variable
y = the dependent variable
n = number of data points in the sample
$\bar{x}$ = the mean of the independent variable x

$\bar{y}$ = the mean of the dependent variable y

From Wikipedia:
Covariance is a measure of the joint variability of two random variables. If the greater values of one variable mainly correspond with the greater values of the other variable, and the same holds for the lesser values, the covariance is positive

$$r_{(x,y)} = \frac{COV(x,y)}{s_x s_y}$$

Normalize covariance to the range [-1, 1] to make it comparable

r(x,y) = correlation of the variables x and y
COV(x, y) = covariance of the variables x and y
sx = sample standard deviation of the random variable x
sy = sample standard deviation of the random variable y

Standard Deviation of two correlated variables

$$Cor(R_i, R_j) = \frac{Cov(R_i, R_j)}{\sigma_i \sigma_j}$$

$$\sigma_p = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho_{1,2} \sigma_1 \sigma_2}$$

$$\sigma_p = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 Cov_{1,2}}$$

Example: stock portfolio

- Suppose you want to invest your wealth in the stocks of ABC, Inc. and XYZ, Inc. The expected returns and standard deviations of the returns for the two corporations are

$$E(r_{ABC}) = 20\% \qquad \sigma_{ABC} = .26$$
$$E(r_{XYZ}) = 10\% \qquad \sigma_{XYZ} = .16$$

- Now assume that the covariance of the returns from the two companies is zero

- What are the expected return and standard deviation of your portfolio if you invested 50 percent in ABC, Inc. and 50 percent in XYZ, Inc.

Example: stock portfolio answers

$$E(r_{Port}) = .5(20\%) + .5(10\%)$$
$$= 15\%$$

$$\rho_{i,j} = \frac{COV(r_i, r_j)}{\sigma_i \sigma_j}$$
$$= \frac{0}{.26 \times .16}$$
$$= 0$$

$$\sigma_{Port} = \left(\sigma_i^2 w_i^2 + \sigma_j^2 w_j^2 + 2 r_{i,j} \sigma_i \sigma_j w_i w_j\right)^{\frac{1}{2}}$$
$$= \begin{pmatrix} (.26)^2 (.5)^2 + (.16)^2 (.5)^2 \\ +2(0)(.26)(.16)(.5)(.5) \end{pmatrix}^{\frac{1}{2}}$$
$$= .1526$$

Bernoulli Distribution

- The outcome of an experiment can either be success (i.e., 1) and failure (i.e., 0).
- Pr(X=1) = p, Pr(X=0) = 1-p, or

$$p_\theta(x) = p^x(1-p)^{1-x}$$

- E[X] = p, Var(X) = p(1-p)
- Bernoulli random variable can take either 0 or 1 using certain probability as a parameter.
- To generate 10000 Bernoulli random numbers with success probability p=1/4 , we will use sample()

Bernoulli Distribution

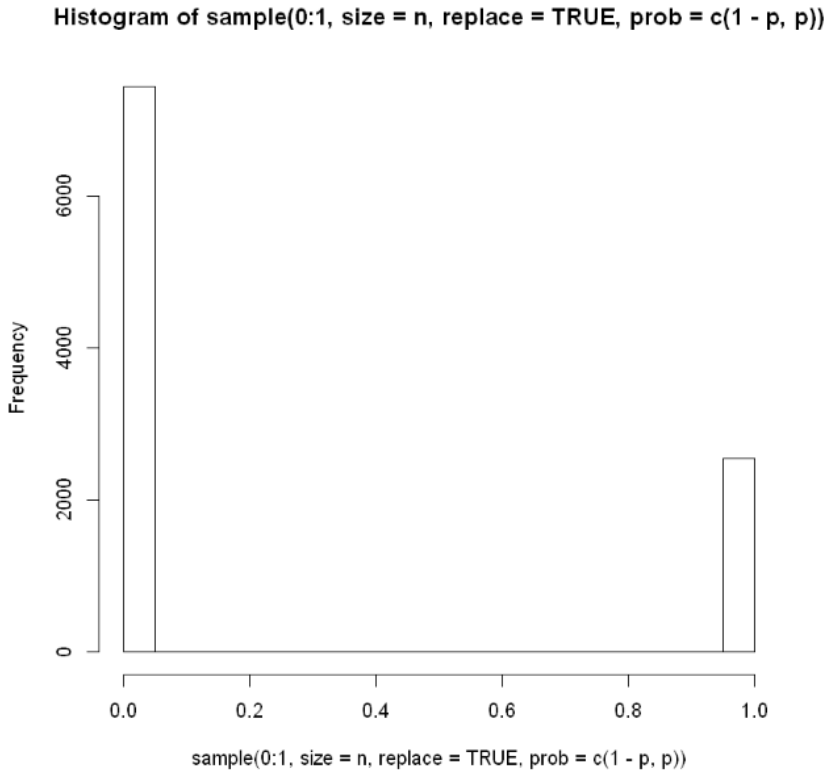
```
n = 10; p = 1/4;
sample(0:1, size=n, replace=TRUE, prob=c(1-p, p))
```

```
0 0 0 0 1 1 0 0 0 0
```

```
n = 10000; p = 1/4;
hist(sample(0:1, size=n, replace=TRUE, prob=c(1-p, p)))
```

```
n = 10000; p = 1/4;
table(sample(0:1, size=n, replace=TRUE, prob=c(1-p, p)))
```

```
   0    1
7496 2504
```



Binomial Distribution

- n draws of a Bernoulli distribution
  - $X_i \sim \text{Bernoulli}(p)$, $X=\sum_{i=1}^{n} X_i$, $X \sim \text{Bin}(p, n)$
- Random variable X stands for the number of times that experiments are successful.

$$\Pr(X = x) = p_\theta(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0,1,2,\ldots,n \\ 0 & \text{otherwise} \end{cases}$$

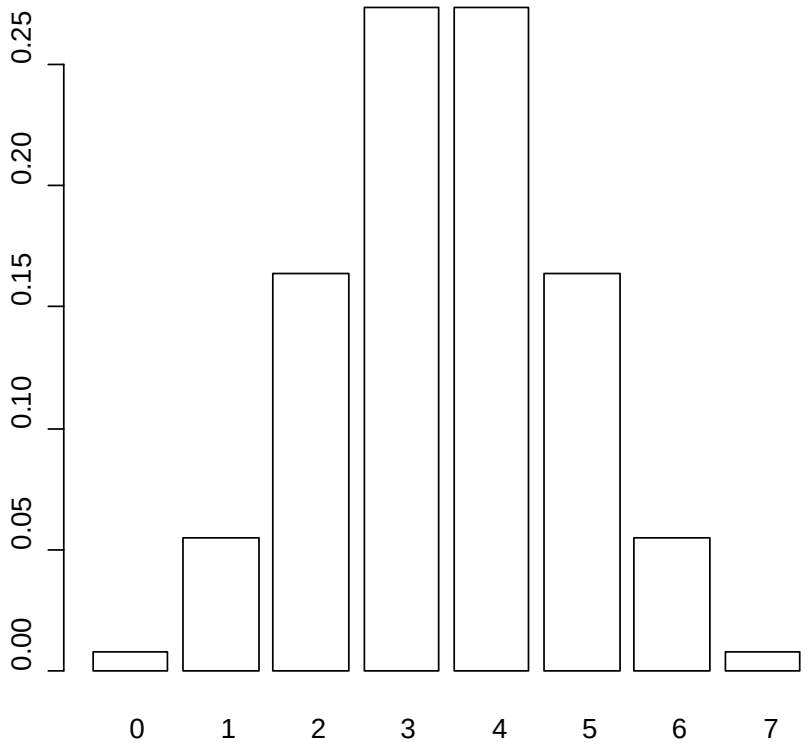
  - n = the number of trials
  - x = the number of successes
  - p = the probability of success
- E[X] = np, Var(X) = np(1-p)
  - Intuitively binomial just repeats bernoulli n times

the binomial distribution in R

- dbinom(x, size, prob) where "d" means density
- Try 7 times, equally likely succeed or fail. The prob of succeeding 3 times in the 7 tries.

```
> dbinom(3,7,0.5)
[1] 0.2734375
```

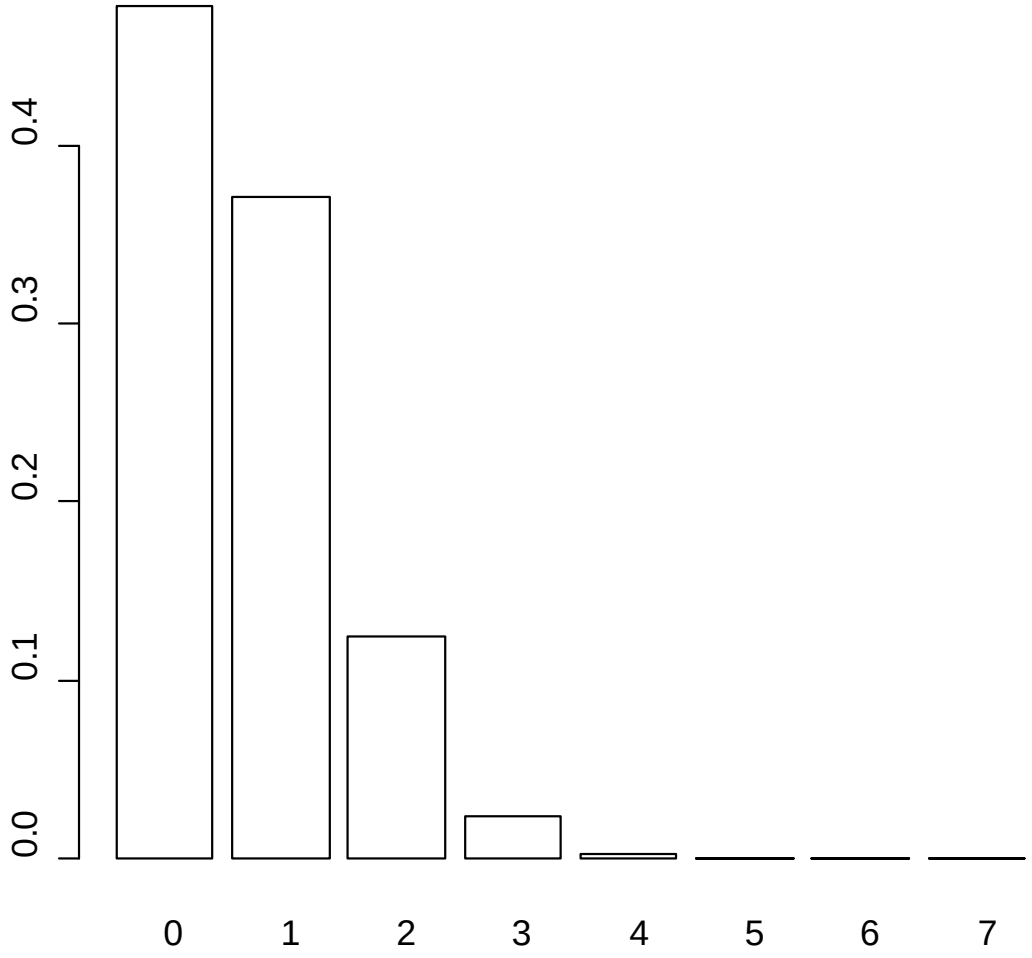
```
>barplot(dbinom(0:7,7,0.5),
   names.arg=0:7)
```



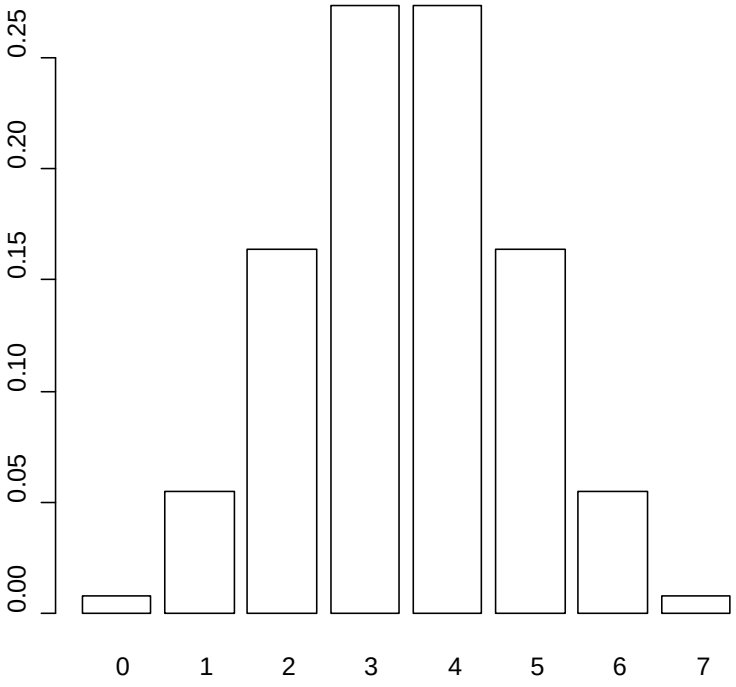
Why is the plot symmetric?

what if p ≠ 0.5?

- > barplot(dbinom(0:7,7,0.1),names.arg=0:7)

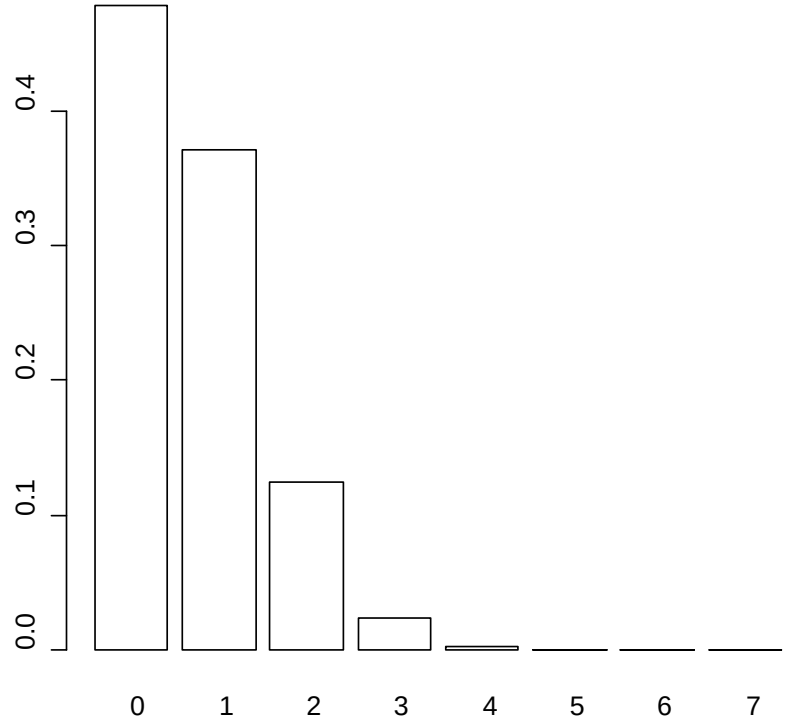


Which distribution has greater variance?



p = 0.5

var = n*p*(1-p) = 7*0.5*0.5 = 7*0.25

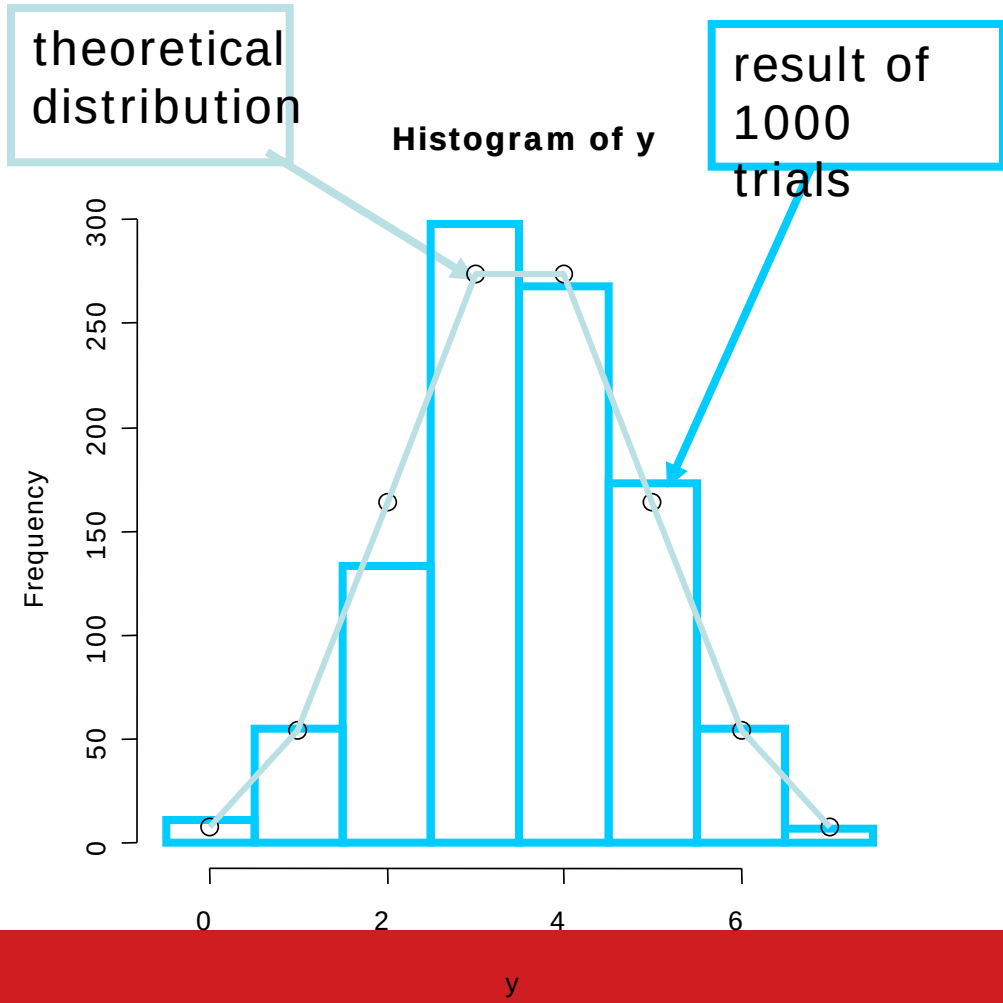


p = 0.1

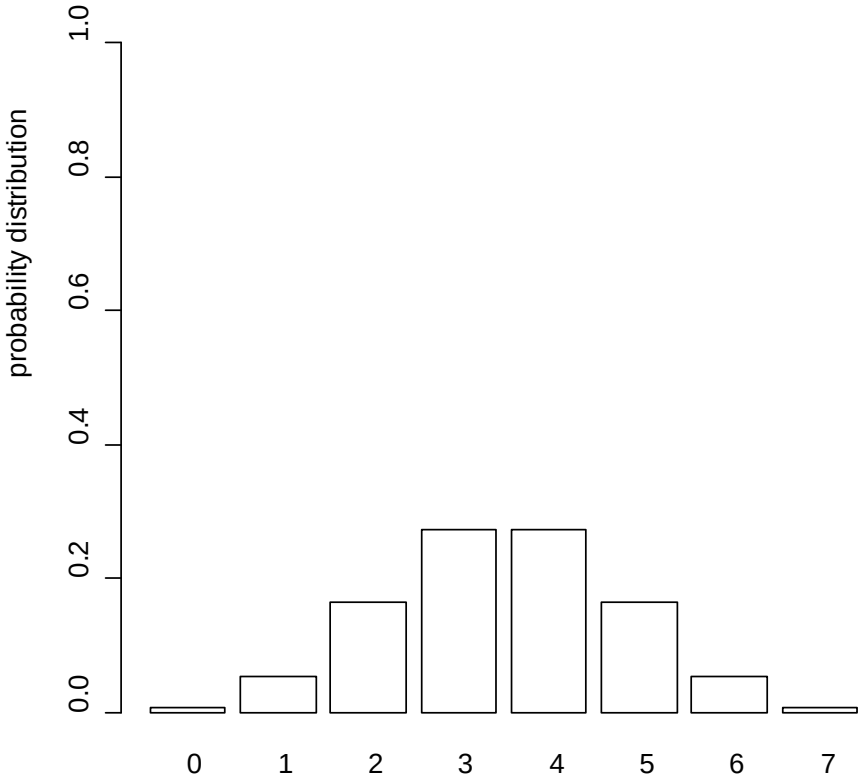
var = n*p*(1-p) = 7*0.1*0.9=7*0.09

briefly comparing an experiment to a distribution

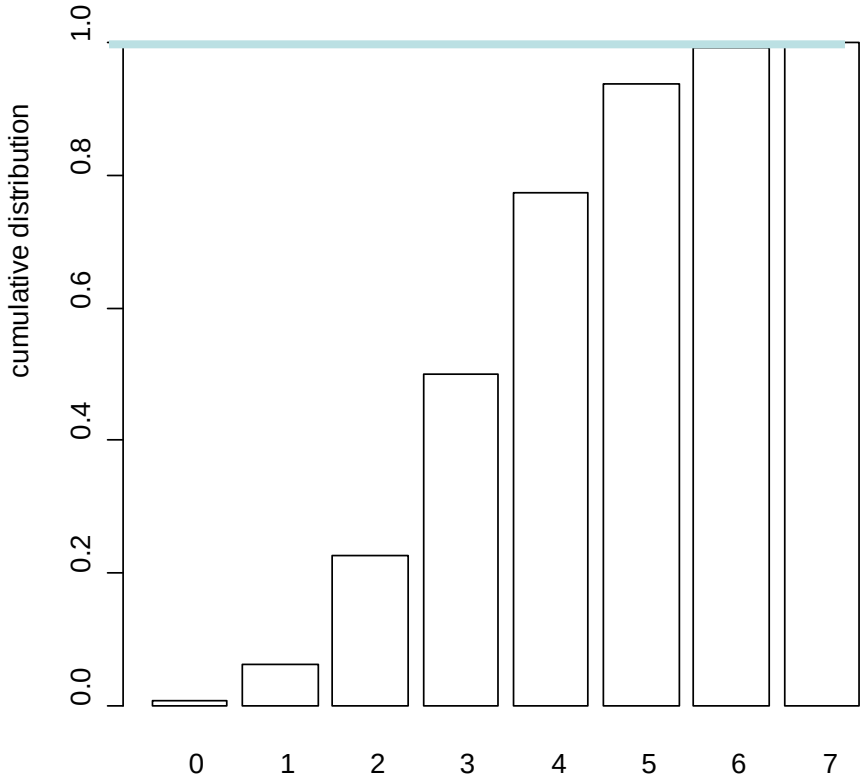
```
nExpr = 1000
tosses = 7; y=rep(0,nExpr);
for (i in 1:nExpr) {
    x = sample(c("H","T"),
    tosses, replace = T)
    y[i] = sum(x=="H")
}
hist(y,breaks=-0.5:7.5)
lines(0:7,dbinom(0:7,7,0.5)*nEx
    pr)
points(0:7,dbinom(0:7,7,0.5)*n
    Expr)
```



Cumulative distribution



P(X=x)

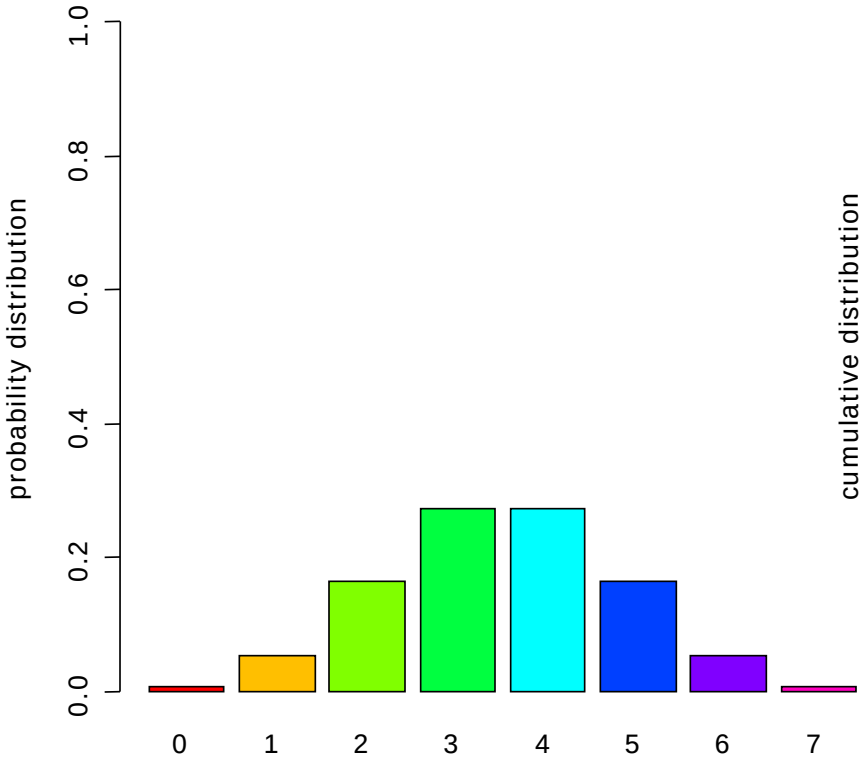


P(X≤x)

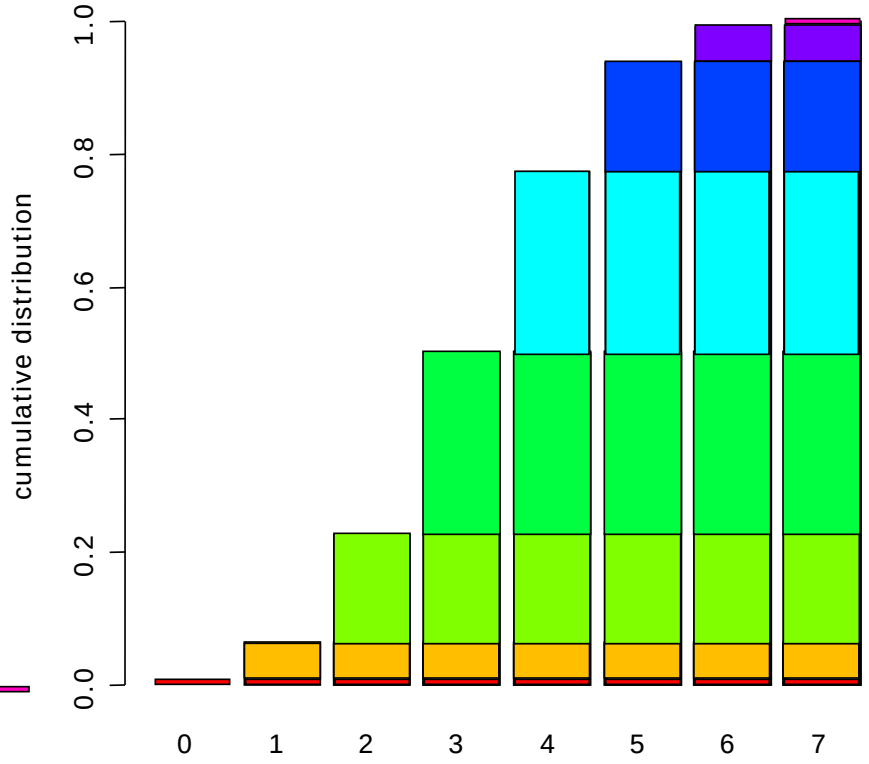
```
> barplot(dbinom(0:7,7,0.5),names.arg=0:7)
```

```
> barplot(pbinom(0:7,7,0.5),names.arg=0:7)
```

cumulative distribution



P(X=x)



P(X≤x)

example: surfers on a website

- Your site has a lot of visitors 45% of whom are female
- Out of the first 100 visitors, what' s the probability that 55 or more are female.

- Hint:
  - probability that 55 or more are female is same as (1 - probability that 54 or less are female)
  - Or, probability that 55 or more are female is same as probability of 45 or less are male and 55% visitors are male
  - Use cumulative density function

Solutions

- probability that 55 or more are female is same as (1 - probability that 54 or less are female)
- $P(X \geq 55) = 1 - P(X \leq 54)$
  $= 1\text{-pbinom}(54,100,0.45)$
  $= 0.02839342$

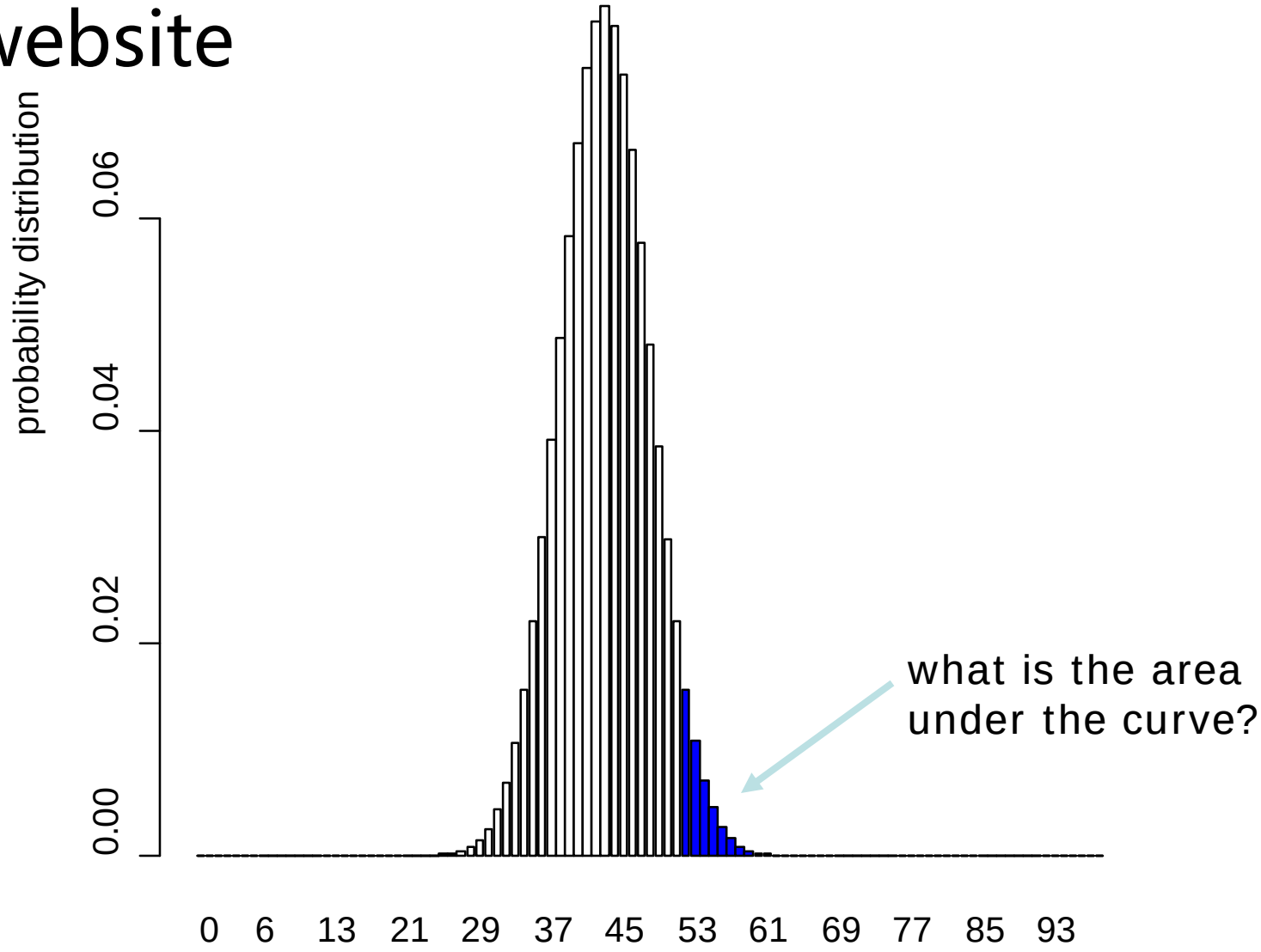
- Or, probability that 55 or more are female is same as probability of 45 or less are male and 55% visitors are male
- $P(X \geq 55) = \text{pbinom}(100\text{-}55, 100, 1\text{-}0.45)$
  $= \text{pbinom}(45, 100, 0.55)$
  $= 0.02839342$

Another way to calculate cumulative probabilities

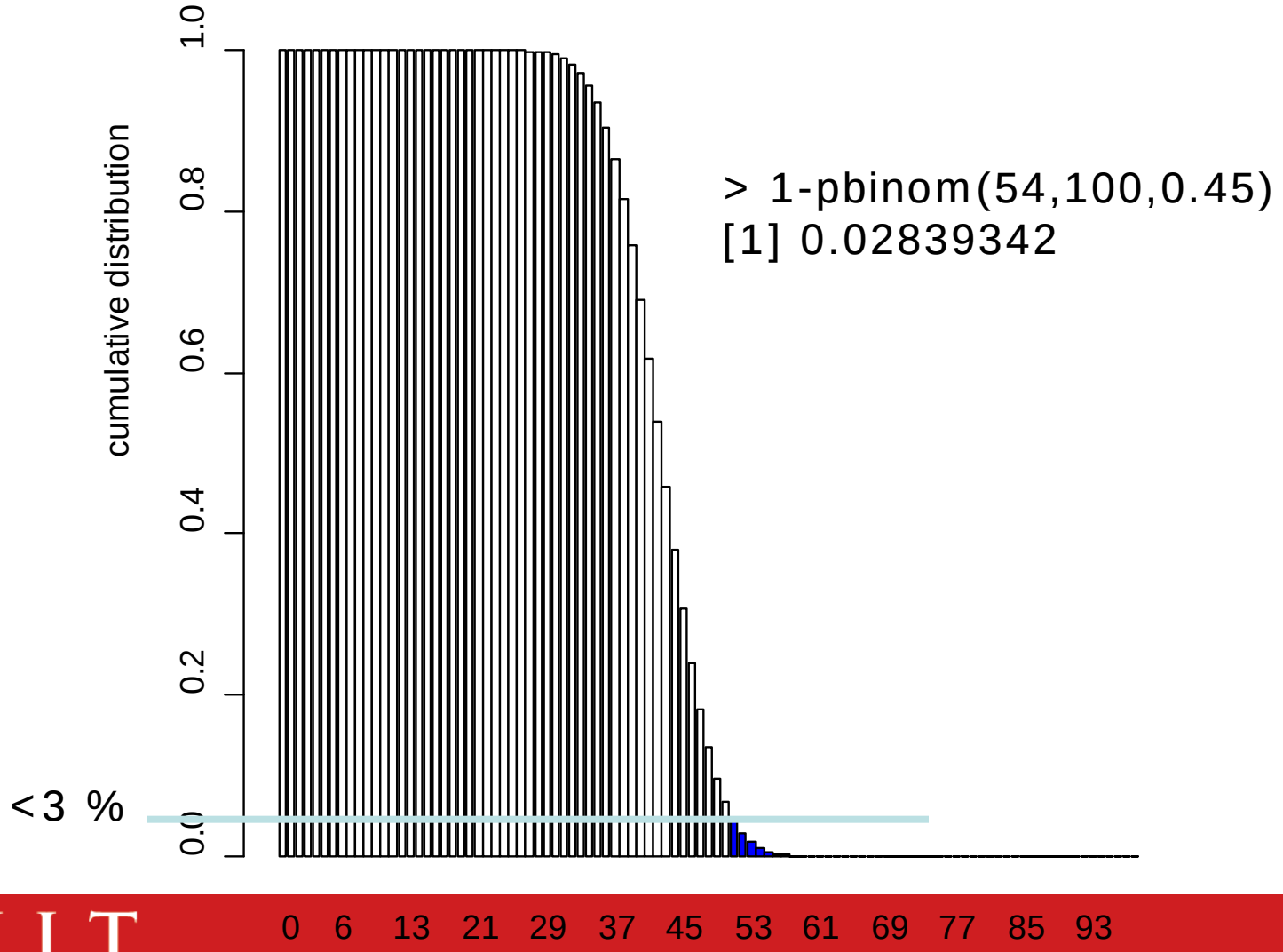
- ?pbinom
- P(X≤x) = pbinom(x, size, prob, lower.tail = T)
- P(X>x) = pbinom(x, size, prob, lower.tail = F)

```
> 1-pbinom(54,100,0.45)
[1] 0.02839342
> pbinom(54,100,0.45,lower.tail=F)
[1] 0.02839342
```

Female surfers visiting a section of a website

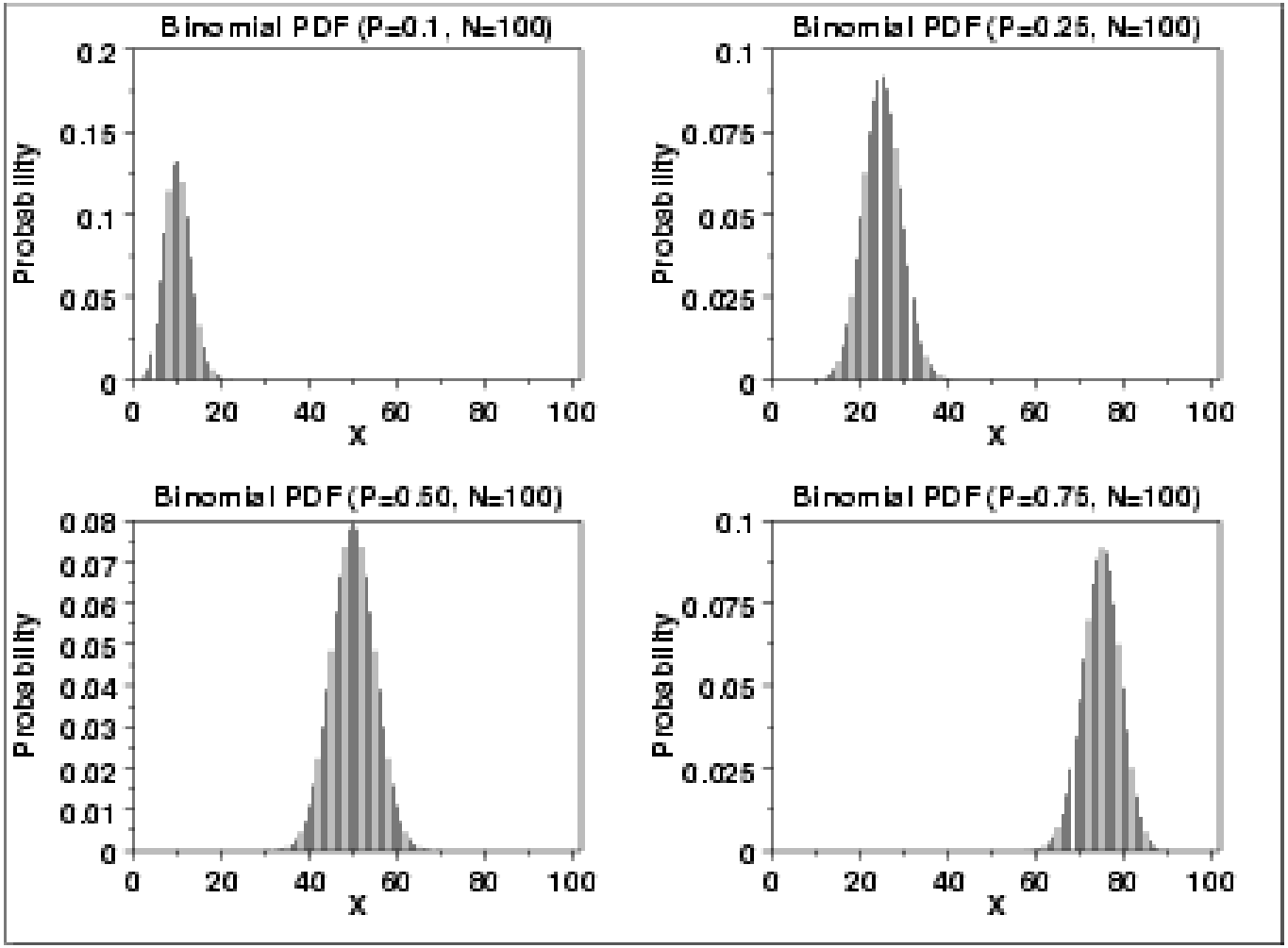


Cumulative distribution




```
> 1-pbinom(54,100,0.45)
[1] 0.02839342
```

Plots of Binomial Distribution



Another discrete distribution: hypergeometric

- Randomly draw n elements without replacement from a set of N elements, r of which are S′ s (successes) and (N-r) of which are F′ s (failures)
- hypergeometric random variable ***x*** is the number of S′ s in the draw of n elements

$$p(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

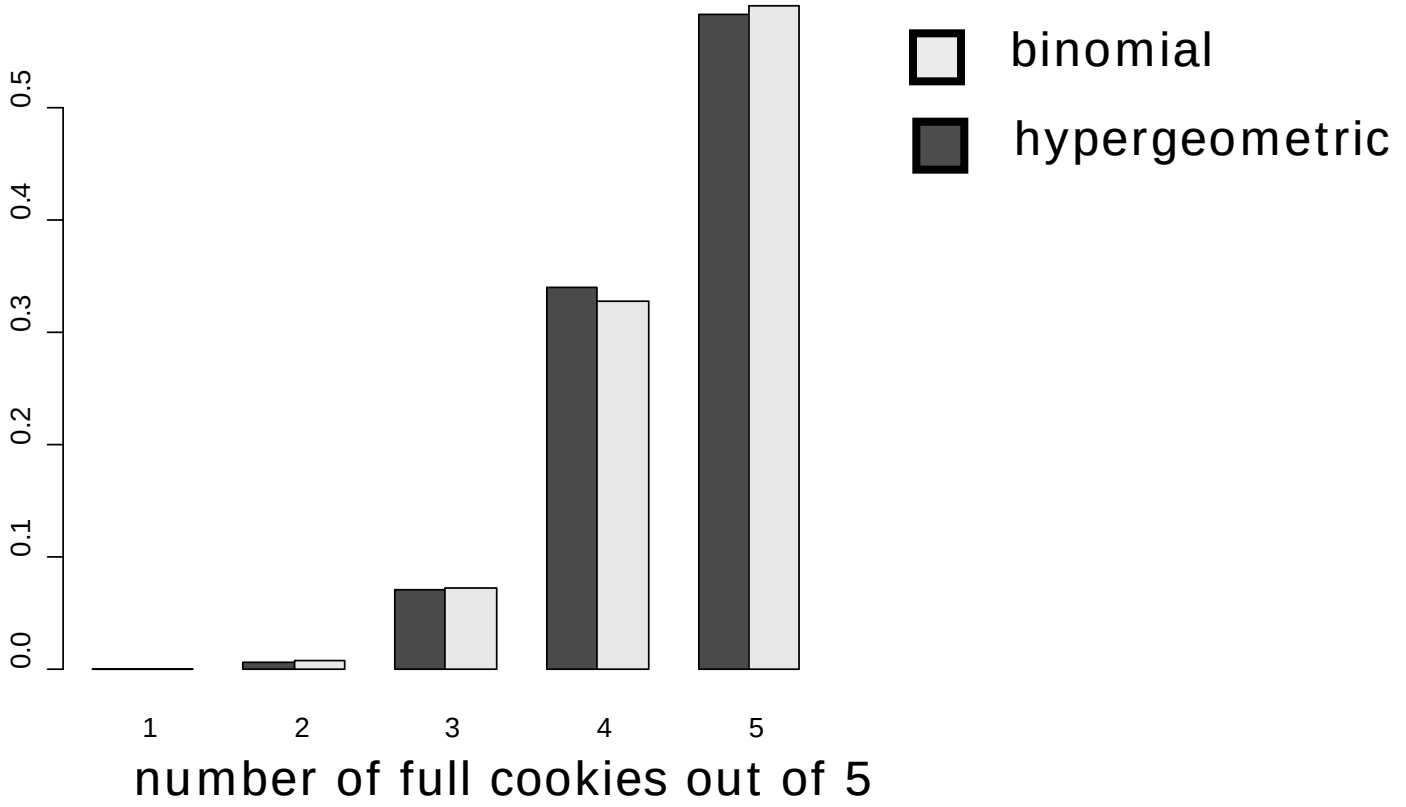
hypergeometric example

- fortune cookies
- there are N = 20 fortune cookies
- r = 18 have a fortune, N-r = 2 are empty
- What is the probability that out of n = 5 cookies, s=5 have a fortune (that is we don’ t notice that some cookies are empty)
- > dhyper(5, 18, 2, 5)
- [1] 0.5526316

- So there is a greater than 50% chance that we won’ t notice.

hypergeometric and binomial

- When the population N is (very) big, whether one samples with or without replacement is pretty much the same
- 100 cookies, 10 of which are empty



code aside

```
> x = 1:5
> y1 = dhyper(1:5,90,10,5)
> y2 = dbinom(1:5,5,0.9)
> tmp = as.matrix(t(cbind(y1,y2)))
> barplot(tmp,beside=T,names.arg=x)
```

hypergeometric probability

binomial probability

Poisson distribution

- the number of times an event happened in a time interval
  - e.g. number of light bulbs burning out in a building in a year
  - number of people arriving in a queue per minute
  - number of users visited your website in an interval

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

  - λ = mean # of events in a given interval

- E[X] = λ, Var(X) = λ

Example: Poisson distribution

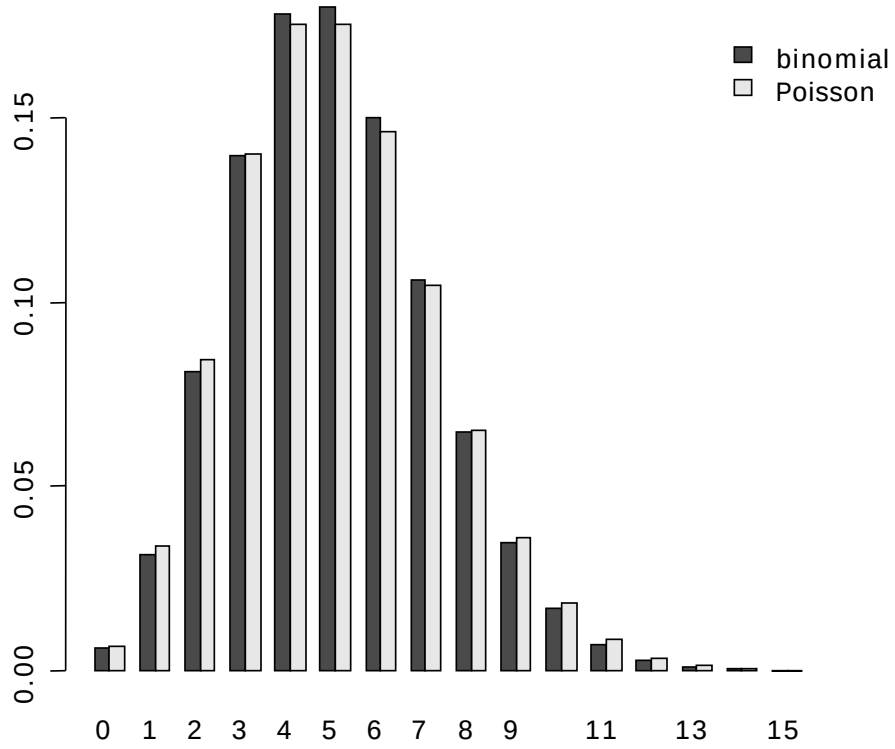
- You got a box of 1,000 widgets.
- The manufacturer says that the failure rate is 5 per box on average.
- Your box contains 10 defective widgets. What are the odds?

```
> ppois(9,5,lower.tail=F)  #the CDF of X > 9
[1] 0.03182806
> sum(dpois(10:1000, 5)) #sum PDF to get CDF
[1] 0.03182806
> dpois(10, 5) #PDF at 10
[1] 0.01813279
```

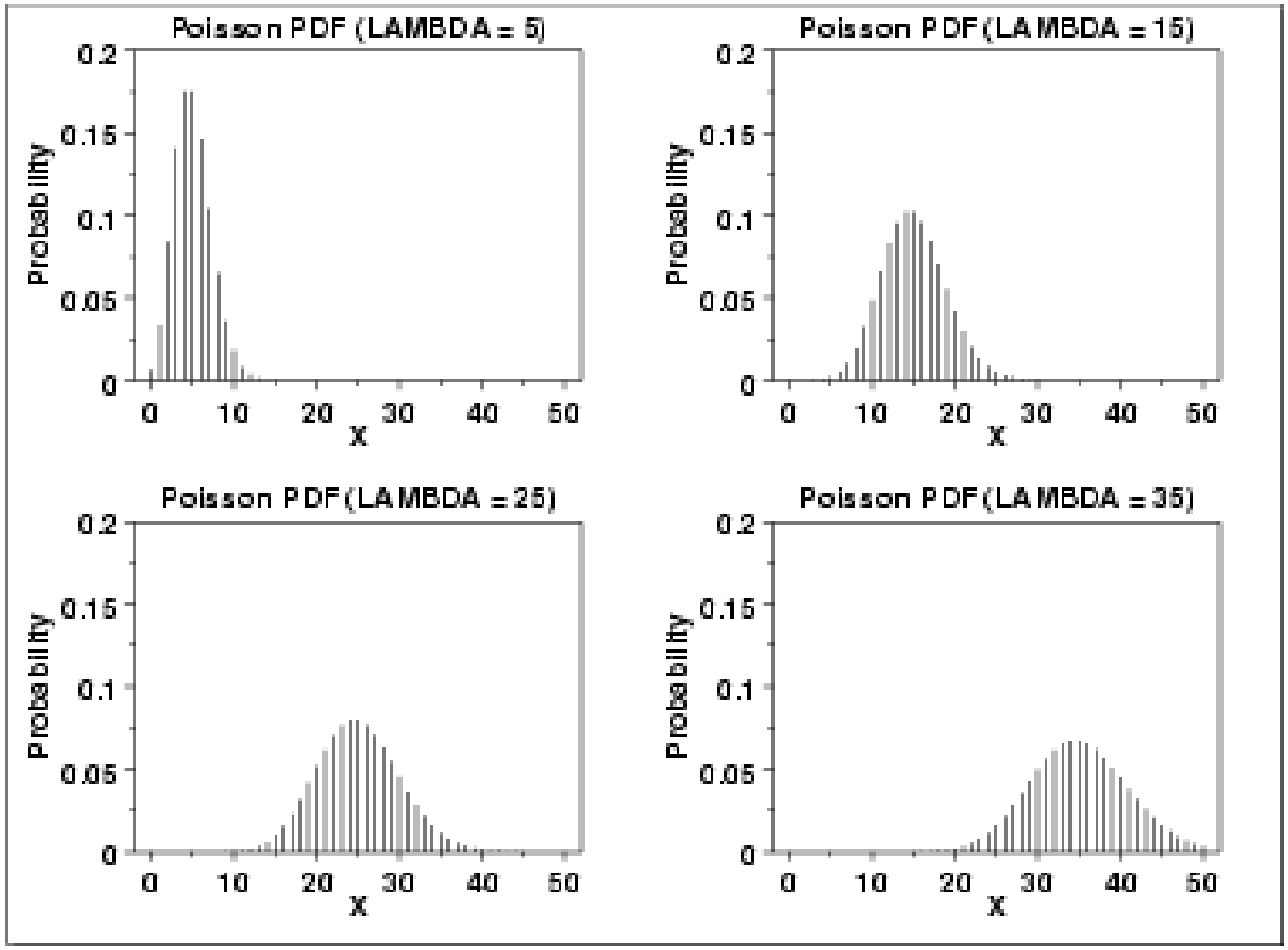
- The chance is 1.8%, maybe the manufacturer is not quite honest.
- Or the distribution is not Poisson?

Poisson approximation to binomial

- If ***n*** is large (e.g. > 100) and ***n*p*** is moderate (p should be small) (e.g. < 10), the Poisson is a good approximation to the binomial with $\lambda = n^*p$



Plots of Poisson Distribution



Normal (Gaussian) Distribution

- Normal distribution (aka “bell curve”)
- fits many biological data well
  - e.g. height, weight
- serves as an approximation to binomial, hypergeometric, Poisson because of the Central Limit Theorem
- Well studied

Normal (Gaussian) Distribution

- $X \sim N(\mu, \sigma^2)$

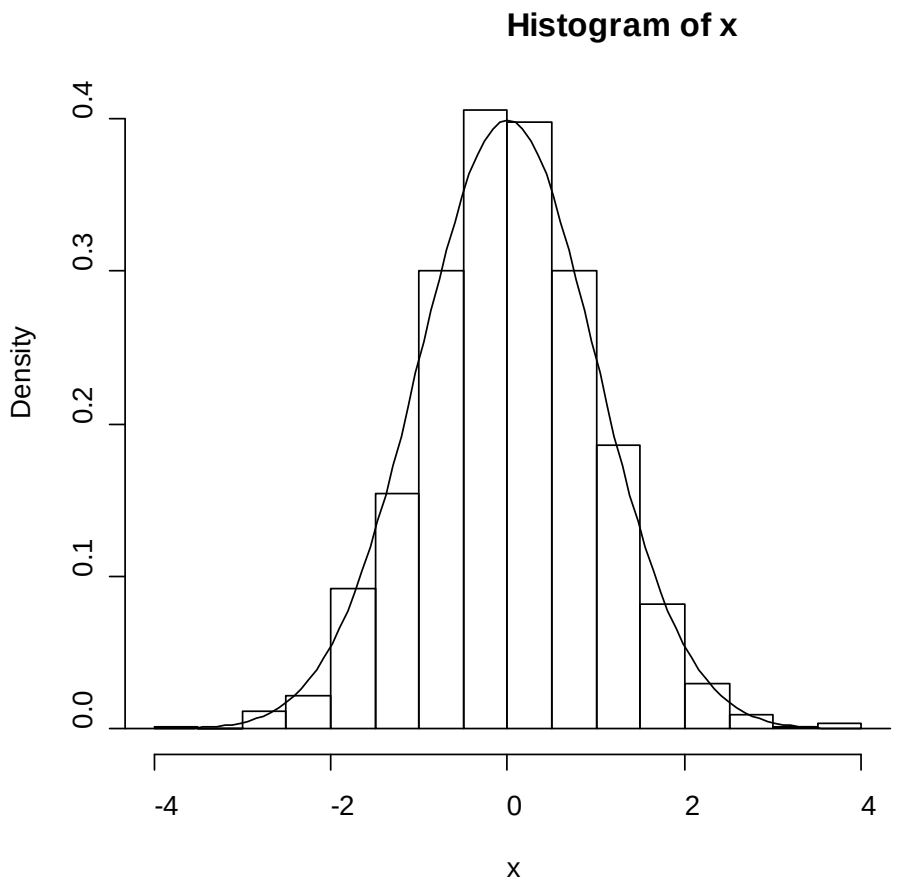
$$p_\theta(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

$$\Pr(a \le X \le b) = \int_a^b p_\theta(x)dx = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

- $E[X] = \mu$, $Var(X) = \sigma^2$
- If $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$, and X_1, X_2 are independent
  - $X = X_1 + X_2$? $X \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$,
  - $X = X_1 - X_2$? $X \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$,

sampling from a normal distribution

```
x <- rnorm(1000)
h <- hist(x, plot=F)
ylim <-
    range(0,h$density,dnorm(
    0))
hist(x,freq=F,ylim=ylim)
curve(dnorm(x),add=T)
```



Normal Approximation based on Central Limit Theorem

- Central Limit Theorem
  - If x_i ~i.i.d with (μ, σ^2) and when n is large, then

    $(x_1+\ldots+x_n)/n \sim N(\mu, \sigma^2/n)$

    Or $(x_1+\ldots+x_n) \sim N(n\mu, n\sigma^2)$
- Example
  - A population is evenly divided on an issue (p=0.5). For a random sample of size 1000, what is the probability of having ≥550 in favor of it?

n=1000, x_i~Bernoulli (p=0.5), i.e. $E(x_i)=p$; $V(x_i)=p(1-p)$

$(x_1+\ldots+x_n) \sim$ Binomial(n=1000, p=0.5)

$Pr((x_1+\ldots+x_n)>=550)$ =1-pbinom(549,1000,0.5) = 0.000865

Normal Approximation:

$(x_1+\ldots+x_n) \sim N(np, np(1-p))=N(500, 250)$

$Pr((x_1+\ldots+x_n)>=550)$=1-pnorm(549, mean=500, sd=sqrt(250)) = 0.000971

Why is the probability so small?

d, p, q, and r functions in R

- In R, a set of functions have been implemented for each of almost all known distributions.
- r<distname>(n,<parameters>)
  - Possible distributions: binom, pois, hyper, norm, beta, chisq, f, gamma, t, unif, etc
- You find other characteristics of distributions as well
  - d<dist>(x,<parameters>): density at x
  - p<dist>(x,<parameters>): cumulative distribution function to x
  - q<dist>(p,<parameters>): inverse cdf

d, p, q, and r functions in R

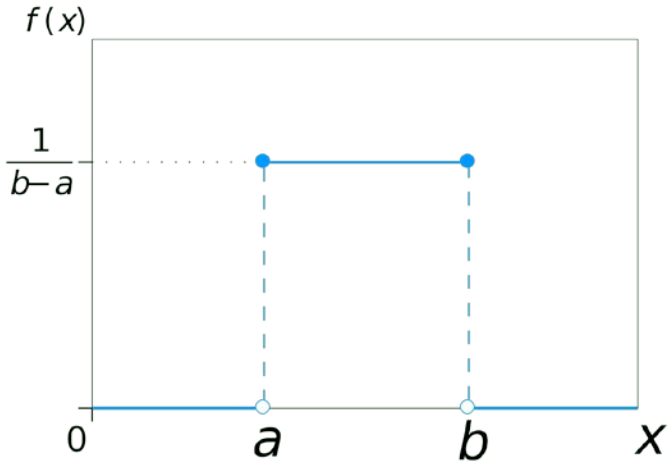
- p for "probability", the cumulative distribution function (c. d. f.)
- q for "quantile", the inverse c. d. f.
- d for "density", the density function (p. f. or p. d. f.)
- r for "random", a random variable having the specified distribution

Distribution	Functions			
Beta	pbeta	qbeta	dbeta	rbeta
Binomial	pbinom	qbinom	dbinom	rbinom
Cauchy	pcauchy	qcauchy	dcauchy	rcauchy
Chi-Square	pchisq	qchisq	dchisq	rchisq
Exponential	pexp	qexp	dexp	rexp
F	pf	qf	df	rf
Gamma	pgamma	qgamma	dgamma	rgamma
Geometric	pgeom	qgeom	dgeom	rgeom
Hypergeometric	phyper	qhyper	dhyper	rhyper
Logistic	plogis	qlogis	dlogis	rlogis
Log Normal	plnorm	qlnorm	dlnorm	rlnorm
Negative Binomial	pnbinom	qnbinom	dnbinom	rnbinom
Normal	pnorm	qnorm	dnorm	rnorm
Poisson	ppois	qpois	dpois	rpois
Student t	pt	qt	dt	rt
Studentized Range	ptukey	qtukey	dtukey	rtukey
Uniform	punif	qunif	dunif	runif
Weibull	pweibull	qweibull	dweibull	rweibull
Wilcoxon Rank Sum Statistic	pwilcox	qwilcox	dwilcox	rwilcox
Wilcoxon Signed Rank Statistic	psignrank	qsignrank	dsignrank	rsignrank

Uniform Distribution

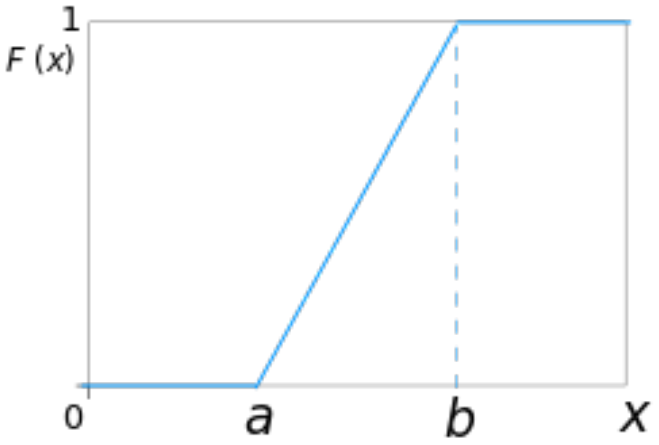
- The probability distribution function of the continuous uniform distribution is:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$



Probability density function

$$\begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a, b] \\ 1 & \text{for } x > b \end{cases}$$

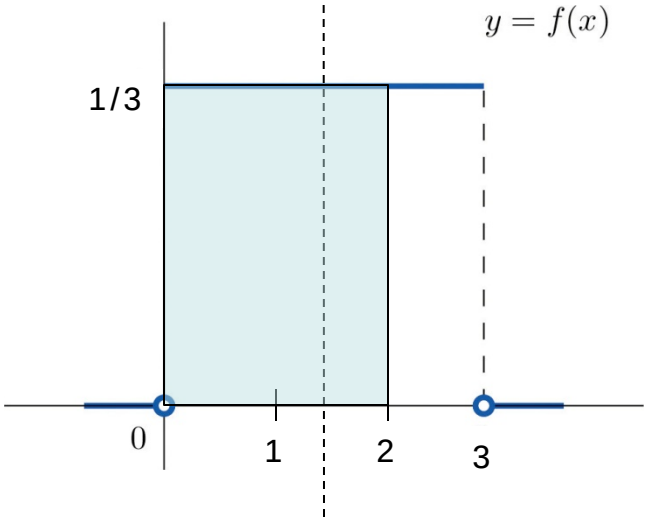


Cumulative distribution function

Example: Uniform Distribution

- The uniform distr. On [a,b] has two parameter. The family name is unif. In R, the parameters are named *min* and *max*
- The uniform distribution has density

  $f(x)=1/(max-min)$ for $min \le x \le max$




```
> dunif(x=1, min=0, max=3)
[1] 0.3333333
> punif(q=2, min=0, max=3)
[1] 0.6666667
> qunif(p=0.5, min=0, max=3)
[1] 1.5
> runif(n=5, min=0, max=3)
[1] 2.7866852 1.9627136 0.9594195 2.6293273 1.6277597
```

Bayes' Rule

- Given two events A and B and suppose that Pr(A) > 0. Then

- Example:

$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\Pr(A \mid B)\Pr(B)}{\Pr(A)}$$

Pr(R) = 0.8

Pr(W|R)	R	¬R
W	0.7	0.4
¬W	0.3	0.6

R: It is a rainy day

W: The grass is wet

Pr(R|W) = ?

Bayes' Rule

$$
\begin{aligned}
P(R|W) &= P(RW)/P(W) \\
&= P(WR)/P(W) \\
&= P(WR)/[P(WR) + P(W\neg R)] \\
&= P(W|R)P(R)/[P(W|R)P(R)+P(W|\neg R)P(\neg R)] \\
&= 0.7*0.8/(0.7*0.8+0.4*0.2)
\end{aligned}
$$

Summation (Integration) out tip

- Suppose that B_1, B_2, … B_k form a partition of Ω:
  $B_1 \cup B_2 \cup \ldots \cup B_k = \Omega$ and B_1, B_2, … B_k are mutually exclusive

Suppose that $\Pr(B_i) > 0$ and $\Pr(A) > 0$. Then

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i)\Pr(B_i)}{\Pr(A)}$$

Summation (Integration) out tip

- Suppose that B_1, B_2, … B_k form a partition of Ω:
  $B_1 \cup B_2 \cup \ldots \cup B_k = \Omega$ and B_1, B_2, … B_k are mutually exclusive

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$$= \frac{\Pr(A \mid B_i)\Pr(B_i)}{\sum_{j=1}^{k} \Pr(AB_j)}$$

Summation (Integration) out tip

- Suppose that B_1, B_2, … B_k form a partition of Ω:
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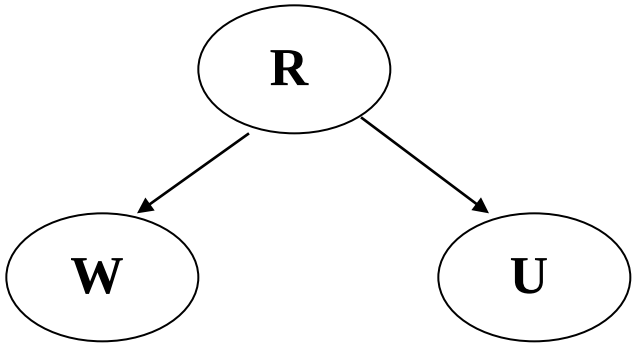
$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i)\Pr(B_i)}{\Pr(A)}$$

$$= \frac{\Pr(A \mid B_i)\Pr(B_i)}{\sum_{j=1}^{k} \Pr(AB_j)}$$

$$= \frac{\Pr(A \mid B_i)\Pr(B_i)}{\sum_{j=1}^{k} \Pr(B_j)\Pr(A \mid B_j)}$$

Key: Joint distribution!

Application of Bayes' Rule

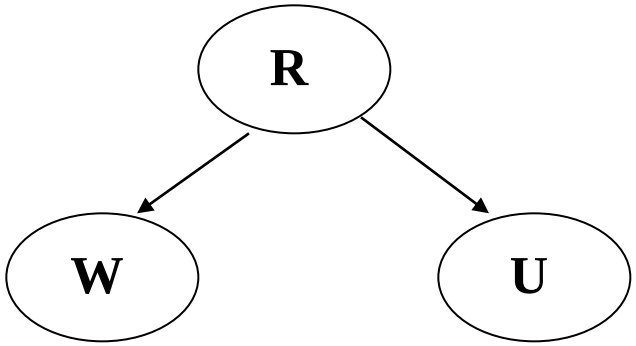


R It rains

W The grass is wet

U People bring umbrella

A More Complicated Example



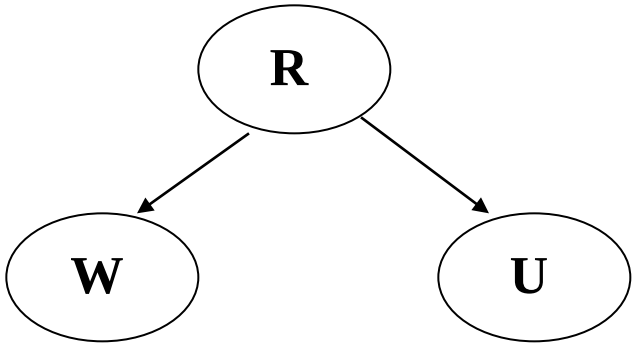
R It rains

W The grass is wet

U People bring umbrella

$$Pr(UW|R) = Pr(U|R)Pr(W|R)$$
$$Pr(UW|\neg R) = Pr(U|\neg R)Pr(W|\neg R)$$

A More Complicated Example



$Pr(R) = 0.8$

R It rains

W The grass is wet

U People bring umbrella

$Pr(UW|R) = Pr(U|R)Pr(W|R)$

$Pr(UW|\neg R) = Pr(U|\neg R)Pr(W|\neg R)$

Pr(W|R)	R	¬R
W	0.7	0.4
¬W	0.3	0.6

Pr(U|R)	R	¬R
U	0.9	0.2
¬U	0.1	0.8

$Pr(U|W) = ?$

Solution:

$$P(UW) = P(UW|R)P(R) + P(UW|\neg R)P(\neg R)$$
$$= P(U|R)P(W|R)P(R) + P(U|\neg R)P(W|\neg R)P(\neg R)$$
$$= 0.9*0.7*0.8 + 0.2*0.4*0.2$$
$$= 0.52$$
$$P(W) = P(W|R)P(R) + P(W|\neg R)P(\neg R)$$
$$= 0.7*0.8 + 0.4*0.2$$
$$= 0.64$$
$$P(U|W) = P(UW)/P(W) = 0.52/0.64 = 0.8125$$

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Introduction to Natural Language Processing

- Bayes Classifier
- Text Similarity

Text/Document Representations

- Document set

$$D = \{d_1, d_2, \ldots, d_n\}$$

- These documents have a "bag-of-words" or the feature set

$$X = \{x_1, x_2, \ldots, x_m\}$$

- The class set is

$$C = \{c_1, c_2, c_k\}.$$

- Assumption: the features in a dataset are mutually independent

$$P(x_1, x_2, \ldots, x_k | C) = \prod_{i=1}^{k} P(x_i | C)$$

Text/Document Representations

```
vocab = ['blue', 'red', 'dog', 'cat', 'biscuit', 'apple']
doc = "the blue dog ate a blue biscuit"

# note that the words that didn't appear in the vocabulary will be discarded
bernoulli = [1 if v in doc else 0 for v in vocab]
multinomial = [doc.count(v) for v in vocab]
print('bernoulli', bernoulli)
print('multinomial', multinomial)
```

```
bernoulli [1, 0, 1, 0, 1, 0]
multinomial [2, 0, 1, 0, 1, 0]
```

Naïve Bayes Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- In NLP, Bayes theorem can be rewritten to

$$p(C = k|D) = \frac{p(C = k)\, p(D|C = k)}{p(D)} \propto p(C = k)\, p(D|C = k)$$

- $\propto$ means is proportional to
- p(C=k) represents the class k's **prior** probabilities.
- p(D|C=k) is the **likelihoods** of the document given the class k.
- p(D) is the **normalizing factor** which we don't have to compute since it does not depend on the class C.

Bernoulli Model

- To calculate the probability of observing features X1 through Xd, given some class C

$$p(x_1, x_2, \ldots, x_d \mid C) = \prod_{i=1}^{d} p(x_i \mid C)$$

- **Bernoulli Model**

$$p(D_i \mid C) = \prod_{t=1}^{d} b_{it} p(w_t \mid C) + (1 - b_{it})(1 - p(w_t \mid C))$$

Where:

- $p(w_t \mid C)$ is the probability of word w_t occurring in a document of class C.
- $1 - p(w_t \mid C)$ is the probability of w_t not occurring in a document of class C.
- b_{it} is either 0 or 1 representing the absence or presence of word w_t in the i_{th} document.

Bernoulli Model

- Estimate $p(w_t \mid C)$ and $p(C)$

$$p(w_t \mid C = k) = \frac{n_k(w_t)}{N_k}$$

Where:

- $n_k(w_t)$ is the number of class $C = k$'s document in which w_t is observed.
- N_k is the number of documents that belongs to class k.

$$p(C = k) = \frac{N_k}{N}$$

Where N is the total number of documents in the training set.

Multinomial Model

- In the multinomial case, calculating $p(D|C=k)$ for the i_{th} document becomes

$$p(D_i|C=k) = \frac{x_i!}{\prod_{t=1}^{d} x_{it}!} \prod_{t=1}^{d} p(w_t|C)^{x_{it}} \propto \prod_{t=1}^{d} p(w_t|C)^{x_{it}}$$

Where:

- x_{it}, is the count of the number of times word w_t occurs in document D_i.
- $x_i = \sum_t x_{it}$ is the total number of words in document D_i.
- Often times, we don't need the normalization term $\frac{x_i!}{\prod_{t=1}^{d} x_{it}!}$, because it does not depend on the class, C.
- $p(w_t \mid C)$ is the probability of word w_t occurring in a document of class C. This time estimated using the word frequency information from the document's feature vectors. More specifically, this is: $\text{Number of word } w_t \text{ in class } C / \text{Total number of words in class } C$.
- $\prod_{t=1}^{d} p(w_t|C)^{x_{it}}$ can be interpreted as the product of word likelihoods for each word in the document.

Laplace Smoothing

- What if $p(w_t \mid C)$ is equal to 0? We add a count of one to each word type

$$p(w_t \mid C) = \frac{(\text{Number of word } w_t \text{ in class } C + 1)}{(\text{Total number of words in class } C) + |V|}$$

Log-Transformation

- Our original formula for classifying a document in to a class using Multinomial Naive Bayes was,

$$p(C|D) = p(C) \prod_{t=1}^{d} p(w_t|C)^{x_{it}}$$

- To prevent the small values from being rounded to zero, we can simply apply a log around everything,

$$p(C|D) = log\left(p(C) \prod_{t=1}^{d} p(w_t|C)^{x_{it}}\right)$$

- Which becomes,

$$p(C|D) = log\, p(C) + \sum_{t=1}^{d} x_{it} log\, p(w_t|C)$$

Example

$$P(c) = \frac{N_c}{N}$$

$$P(w \mid c) = \frac{count(w,c)+1}{count(c)+|V|}$$

	Doc	Words	Class
Training	1	Chinese Beijing Chinese	c
	2	Chinese Chinese Shanghai	c
	3	Chinese Macao	c
	4	Tokyo Japan Chinese	j
Test	5	Chinese Chinese Chinese Tokyo Japan	?

Priors:

$P(c) = \frac{3}{4}$

$P(j) = \frac{1}{4}$

Conditional Probabilities:

P(Chinese|*c*) = (5+1) / (8+6) = 6/14 = 3/7

P(Tokyo|*c*) = (0+1) / (8+6) = 1/14

P(Japan|*c*) = (0+1) / (8+6) = 1/14

P(Chinese|*j*) = (1+1) / (3+6) = 2/9

P(Tokyo|*j*) = (1+1) / (3+6) = 2/9

P(Japan|*j*) = (1+1) / (3+6) = 2/9

Choosing a class:

$P(c|d5) \propto 3/4 * (3/7)^3 * 1/14 * 1/14 \approx 0.0003$

$P(j|d5) \propto 1/4 * (2/9)^3 * 2/9 * 2/9 \approx 0.0001$

Applications and Use Cases

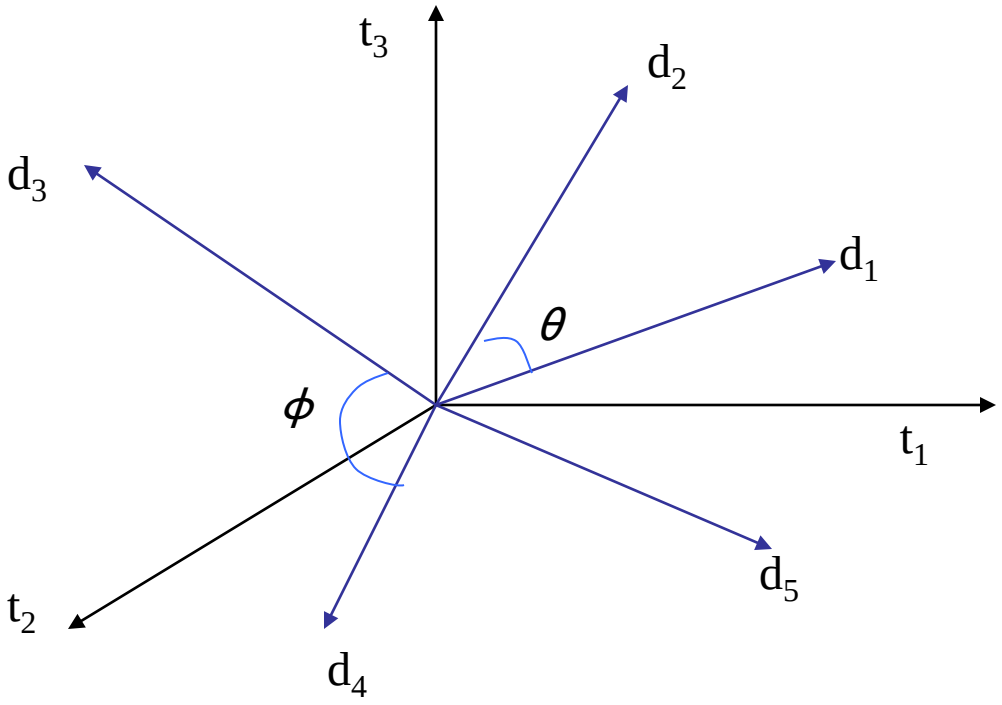
- News and sentiment analysis
- Social media network content analysis
- Marketing
- Customer Services
- Spam email detection
- Advertisement matching by Google AdSense
- Legal documents
- etc

Text Similarity

Question: measure how similar the documents are irrespective of their size

Use case: How to find all job posts that fit my resume?

Intuition



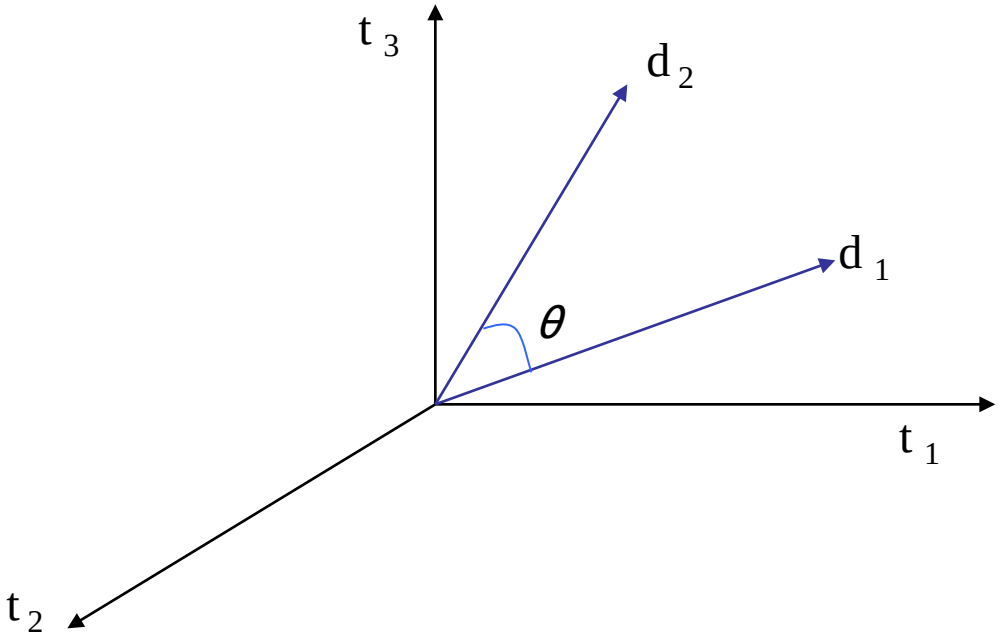
Postulate: Documents that are “close together” in the vector space talk about the same things.

First cut

- Distance between d_1 and d_2 is the length of the vector $|d_1 - d_2|$.
  - Euclidean distance
- Why is this not a great idea?
- We still haven't dealt with the issue of length normalization
  - Long documents would be more similar to each other by virtue of length, not topic
- However, we can implicitly normalize by looking at *angles* instead

Cosine similarity

- Distance between vectors d_1 and d_2 *captured* by the cosine of the angle *x* between them.
- Note – this is *similarity*, not distance
  - No triangle inequality for similarity.



Cosine Similarity

- With cosine similarity we can measure the similarity between two document vectors.

$$\text{similarity} = \cos(\theta) = \frac{A \cdot B}{\|A\|\|B\|} = \frac{\sum_{i=1}^{n} A_i \times B_i}{\sqrt{\sum_{i=1}^{n} (A_i)^2} \times \sqrt{\sum_{i=1}^{n} (B_i)^2}}$$

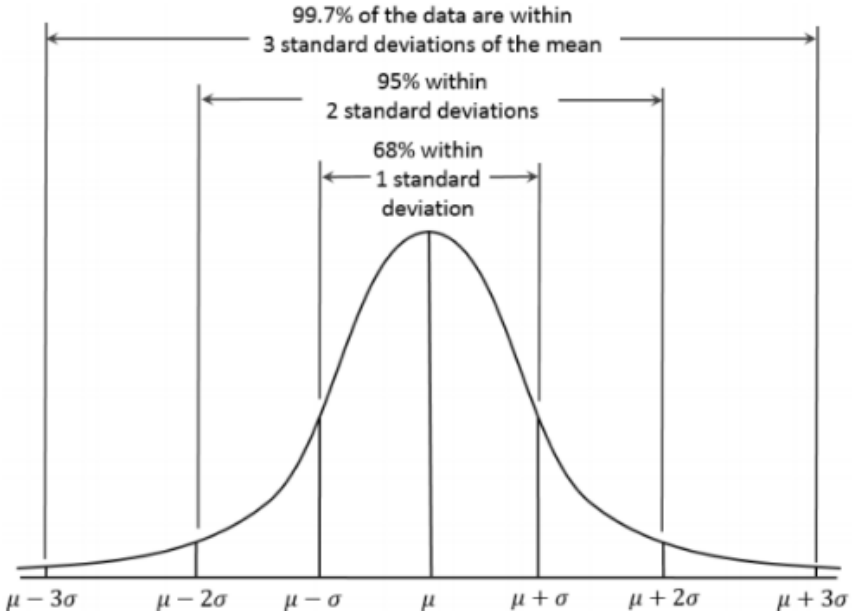
- If the cosine similarity is 1, they are the same document.
- If it is 0, the documents share nothing.
- This is because term frequency cannot be negative so the angle between the two vectors cannot be greater than 90°
- It removes any bias we had towards longer documents.

Homework

Problem 1, Six Sigma

https://en.wikipedia.org/wiki/Six_Sigma

Six Sigma (6σ) is a set of techniques and tools for process improvement. It was introduced by American engineer Bill Smith while working at Motorola in 1980.Jack Welch made it central to his business strategy at General Electric in 1995. A six sigma process is one in which **99.99966% of all opportunities to produce some feature of a part are statistically expected to be free of defects.**



1. Please write R code to verify the probabilities within σ, 2σ and 3σ as shown in the above graph
2. Write R code to find out, within how many σ, the probability is 99.99966%. Is it really 6σ?

Homework

Problem 2, Job Search

You are spamming resumes to apply jobs. Each resume you send out has 1% chance of getting a job offer. You have sent 100 resumes. What's your chance to get at least one job offer?

1. Please solve the problem by math. Use R mathematical expression to get the result.
2. Analytic solution. Use R's probability functions to solve the problem.
3. Answer by simulation. Use sampling function to simulate the process and estimate the answer.
4. How many resumes in total do you have to spam so that you will have 90% chance to get at least one job offer?

Problem 3, President Election Polls

Half of the population supports the president (i.e., p=0.5). For a random sample of size 1000, what is the probability of having ≥600 in support of the president?

1. Use binomial distribution
2. Use normal distribution as approximation.