Detecting Shifts of Parameter in Gamma Sequences with Applications to Stock Price and Air Traffic Flow Analysis

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In this article a technique for detecting shift of scale parameter in a sequence of independent gamma random variables is discussed. Distribution theories and related properties of the test statistic are investigated. Numerical critical points and test powers are tabulated for two specific variables. Other useful techniques are also summarized. The methods are then applied to the analysis of stock-market returns and air traffic flows. These two examples are studied in detail to illustrate the use of the proposed method compared to other available techniques. The empirical examples also illuminate the importance of the treatment of stochastic instability in statistical applications.

KEY WORDS: Shift in scale parameter; Variance heterogeneity; Gamma random variables; Locally most powerful test; Dirichlet variables; Monte Carlo methods.

1. INTRODUCTION

Phenomena in real life can often be modeled as if they were the results of sequences of independent gamma random variables, whose probability density function is known to be

$$f(x;\xi,\theta) = [\theta^{\xi}\Gamma(\xi)]^{-1}x^{\xi-1}e^{-x/\theta} ,$$
$$\theta > 0 \; ; \quad \xi > 0 \; ; \quad x > 0 \; , \qquad (1.1)$$

where x represents the observed value of the random variable; $\Gamma(\cdot)$ is the gamma function; ξ is a shape parameter whose value determines the shape of the distribution function; and θ represents the unit scale for the random variable. (For expository convenience, we call θ a *scale parameter* throughout this article, and note that the mean $(=\xi\theta)$ and the standard deviation $(=\theta\sqrt{\xi})$ of the gamma distribution cannot be functionally separated.)

For instance, in a sequence of events generated by a Poisson process, the time intervals between consecutive events are distributed as exponential variables (a special case of gamma variables with $\xi = 1$; cf. Cox and Lewis 1966, Ch. 2). The value of the associated scale parameter, θ, of the exponential variables is the reciprocal of the occurrence rate. In another case, squares of a sequence of independent normal random variables after subtracting off the known mean form a series of *scaled* χ^2 variables with one degree of freedom (abbreviated "scaled $\chi^2(1)$," a special case of gamma variables with $\xi = \frac{1}{2}$). The scale parameter in this case equals twice the variance of the underlying normal variables. Cases in which the value of ξ is other than $\frac{1}{2}$ and 1 can also arise in many practical situations (cf. Johnson and Kotz 1970, Ch. 17, Sec. 2).

The value of the scale parameter, θ, may be subject to changes (more specifically, step-type changes) for reasons of practical concern. To investigate the problems of shift in the scale parameter and thus to illuminate their importance in application, we first describe a simple test relevant for the purpose of detection, and review some useful techniques. Two examples of general interest are then provided: (1) a study of U.S. stock-market price volatility during 1971–74; and (2) a test of the regularity of aircraft arrivals at an air traffic control sector in the New York area during a busy afternoon. Detailed investigations of these two examples are furnished to illuminate the relative merits of the test methods considered.

2. THE TEST FOR SHIFT OF θ IN A GAMMA SEQUENCE

In some applications, a shift in the scale parameter of a gamma sequence can be related to an apparent causal event at a time that can be precisely specified. However, in many other cases, potential causal events cannot be identified with reasonable confidence, and the timing of the possible shift-point is uncertain. For this reason, we formulate a test problem which is relevant for the latter cases, and investigate a suggested detection technique. Insights into the adequacy of this test formulation are given in Sections 4 and 5 with real-world examples.

2.1 Formulation of the Test Problem

Let $X_1, X_2, X_3, \ldots, X_M$ be a sequence of independent gamma random variables as defined in (1.1). The value of ξ is assumed known; θ is assumed *unknown* and is sub-

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ject to a single step change at an *unknown* point. In formal terms, our problem here is to test

$$H_0: \theta_1 = \theta_2 = \ldots = \theta_M = \theta_0 \ (\theta_0 \text{ unknown})$$

against

$$H_1: \theta_1 = \theta_2 = \ldots = \theta_k = \theta_0 \ ; \quad \theta_{k+1} = \ldots = \theta_M = \theta_0 + \delta \ , \quad |\delta| > 0 \ ,$$

where k is unknown ($k = 1, 2, \ldots, M - 1$) and δ is unknown $(\theta_0 + \delta) \in (0, \infty)$.

2.2 Previous Work

Studies concerning testing a shift in parameter of a distribution model, occurring at an unknown time, have been reported in the literature, usually for the mean of a normal sequence. Little has been done for problems formulated in terms of gamma variables. The only direct reference is Kander and Zacks (1966). In their work, a test problem was formulated as just stated (Section 2.1), except that the initial level of the scale parameter, θ_0, was assumed known. In practice, however, the "known" or "nominal" value of θ_0 is usually unavailable, so we relax this assumption and let the value of θ_0 be estimated from the data.

According to Kander and Zacks (1966, pp. 1197–1198), under the assumption that the location of the change-point (for a step-type change), k, has an equal chance to fall at any of the possible points $i = 1, 2, \ldots, M - 1$, the ratio of the likelihood function under H_1 to that under H_0, for the gamma random variables, as $\delta/\theta_0 \to 0$, can be expressed as

$$R(X_1, X_2, \ldots, X_M; \theta_0, \delta) = 1 + \left(\frac{\delta}{\theta_0}\right)\left[\frac{1}{M-1}\frac{1}{\theta_0}\sum_{i=1}^{M}(i-1)X_i - \frac{\xi M}{2}\right] + 0\left(\frac{\delta}{\theta_0}\right). \tag{2.1}$$

Since the second term in the brackets is a known constant, the likelihood ratio statistic reduces to

$$\tilde{T} = (1/\theta_0)\sum_{i=1}^{M}(i-1)X_i \ . \tag{2.2}$$

2.3 The Test Statistic for Unknown θ_0

When the value of θ_0 is unknown, it is estimated under H_0 by the maximum likelihood method to be:

$$\hat{\theta}_0 = \sum_{i=1}^{M} X_i/(M\xi) \ . \tag{2.3}$$

Substituting the right side of (2.3) into (2.2) for θ_0 and replacing the known constant $M\xi$ by the convenient constant of proportionalities, $1/(M-1)$, we find the test statistic to be:

$$T = \sum_{i=1}^{M}(i-1)X_i/[(M-1)\sum_{i=1}^{M}X_i] \ . \tag{2.4}$$

This statistic, besides a constant, is proportional to the slope of a linear trend, fitted by the least squares method, for the values of X_i, provided that the value of the sum $\sum_{i=1}^{M} X_i$ is known. Moreover, it can be shown that, under H_0, T in (2.4) is a linear function of Dirichlet variables (cf. Johnson and Kotz 1972, Ch. 40, Sec. 5). Based on the expressions given by Johnson and Kotz for the moments of Dirichlet variables, and letting $m = M - 1$, the following moments for the null distribution of T can be obtained:

$$\begin{aligned}
\mu_1(T) &= E(T) = \tfrac{1}{2} \ ; \\
\mu_2(T) &= \operatorname{var}(T) = (m+2)/[12m(M\xi+1)] \ ; \\
\gamma_1(T) &= \mu_3(T)/[\mu_2(T)]^{1.5} = 0 \ ; \\
\gamma_2(T) &= \{\mu_4(T)/[\mu_2(T)]^2\} - 3 \\
&= \frac{3(M\xi+1)[5\xi m(m+1)(m+2) + 6(3m^2+6m-4)]}{5m(m+2)(M\xi+2)(M\xi+3)} - 3.0 \\
&= -(1.2/\xi)m^{-1} + O(m^{-2}) \quad \text{as} \quad m \to \infty \ .
\end{aligned} \tag{2.5}$$

It can be seen from (2.4) that the value of T falls between zero and one, inclusively. For a given set of X_i's, thus a given value of $\sum_{i=1}^{M} X_i$, the value of T, under H_0, is symmetrically distributed about the mean $\frac{1}{2}$ (each permutation of the X_i's has equal chance to be selected, and an exactly reverse ordering of the X_i's will produce its counterpart of symmetry). This symmetry further implies a symmetry of T unconditional on the value of $\sum_{i=1}^{M} X_i$. This property is consistent with the value of the skewness measure; i.e., $\gamma_1(T) = 0$. Moreover, the null distribution of T has a negative kurtosis measure, $\gamma_2(T)$. Further, the distribution of T tends to the normal as M increases, and the rate of approach, as indicated by the last expression of $\gamma_2(T)$ in (2.5), increases with the value of ξ, the shape parameter of gamma variables. The standardized statistic is thus

$$T^* = (T - 0.5)/[\operatorname{var}(T)]^{0.5} \ , \tag{2.6}$$

which, under H_0, is distributed as a standard normal as $M \to \infty$.

Since the test statistic is derived based on the likelihood ratio, approximated for a relatively small magnitude of change in θ, it possesses properties inherent in the tests so obtained. As described in Lehmann (1959, p. 342) and Rao (1965, Ch. 7), a test that rejects H_0 at a specified significance level in favor of the alternative $\delta > 0$, upon observation of a sufficiently large value of T^*, is the locally most powerful one-sided test as $\delta/\theta_0 \to 0+$. And a test that rejects H_0 for a sufficiently large value of $|T^*|$ is the locally most powerful unbiased test against the two-sided alternative at small values of $|\delta/\theta_0|$.

Because of the nature of the limiting approximation in the derivation of the test statistic (where $\delta/\theta_0 \to 0$ is assumed), the T^* test is found to be asymptotically a likelihood ratio test, as the series length, M, tends to infinity, in testing H_0 against $H_2: \theta_i = \theta_0 \exp\{\beta(i-1)\}$, $i = 1, 2, \ldots, M$; i.e., a continuous exponential increase ($\beta > 0$) or decrease ($\beta < 0$) in the value of θ. A test using T^* in this case is an asymptotically most powerful one-sided test for $\beta = 0$ against the alternative of $\beta > 0$ or

$\beta < 0$ and is an asymptotically most powerful unbiased test against the two-sided alternative, $\beta \neq 0$. This finding is established as follows. The log-likelihood function under H_2 is

$$\log L \equiv \log L(\beta, \theta_0 | X_1, X_2, \ldots, X_M)$$
$$= -M \log \Gamma(\xi) - \xi M \log \theta_0 - \xi\beta \sum_{i=1}^{M} (i-1) + (\xi - 1) \sum_{i=1}^{M} \log X_i - \frac{1}{\theta_0} \sum_{i=1}^{M} [X_i / e^{\beta(i-1)}] \ , \quad (2.7)$$

where $\equiv$ means "is defined as." We obtain

$$\frac{\partial \log L}{\partial \beta} = -\xi \sum_{i=1}^{M} (i-1) + \frac{1}{\theta_0} \sum_{i=1}^{M} [(i-1) X_i / e^{\beta(i-1)}] \ . \quad (2.8)$$

Replacing, in (2.8), the unknown value of θ_0 by its maximum likelihood estimator

$$\hat{\theta}_0 = \sum_{i=1}^{M} [X_i / e^{\beta(i-1)}] / (\xi M) \ , \quad (2.9)$$

and β by its null value, zero, we obtain:

$$\left.\frac{\partial \log L}{\partial \beta}\right|_{\theta_0 = \hat{\theta}_0, \beta = 0} = -\xi M(M-1)/2 + \xi M [\sum_{i=1}^{M} (i-1) X_i] / (\sum_{i=1}^{M} X_i)$$
$$= \xi M (M-1)(T - 0.5) \ , \quad (2.10)$$

where T has been defined in (2.4). It is well-known that a test using (2.10), or, equivalently, T^* as defined in (2.6), tends to the likelihood ratio test as $M \to \infty$ and possesses properties as claimed here. The interested reader is referred to Lehmann (1959, Ch. 7, Sec. 12 and 13) for the general theory and to Cox and Lewis (1966, pp. 45–47 and 153) for a detailed study of the special case of a Poisson process (i.e., $\xi = 1$), under H_2. However, the properties of the T^* test at a moderate size of shift, δ/θ_0 (under H_1), and a moderate length of series, M (under H_2), are difficult to obtain analytically. Thus, reliance on Monte Carlo results regarding power is necessary.

Using an Edgeworth expansion (see Johnson and Kotz 1972, Ch. 12, Sec. 4.2), we obtained approximate critical points for T^*, under H_0, at several moderate sample sizes for both the scaled $\chi^2(1)$ and exponential variables. The numerical results are displayed in Table 1. (As one can see from the entries, the normal approximation appears sufficiently accurate for most practical situations.)

As indicated by some Monte Carlo results, the power of the test under H_1 at a given M depends on the size of shift, δ/θ_0, and the location of the change-point. Some numerical illustrations are given in Table 2 for both the scaled $\chi^2(1)$ and the exponential variables. The test appears to have higher power when k, the change-point, is in the mid portion of the series than when k is near either end. For a given shift size, the power values observed when $k = i$ are, in general, not equal to those when $k = M - i$, $i = 1, 2, \ldots, M - 1$. It is clear from Table 2 that, for all combinations of shift sizes and locations, T^* provides higher power for the exponential than for the scaled $\chi^2(1)$. We thus conjecture that the power is an increasing function of the value of ξ, for a given set of M, δ/θ_0, and k.

Two clarifying remarks are in order. First, the above method is valid only for gamma random variables where the shape parameter ξ is assumed *known* and *fixed*. When this is not the case, other techniques are needed. Second, the low power values reported in Table 2, for M equal

1. Approximate Critical Values of the T^* Test for the Scaled $\chi^2(1)$ and Exponential Variables

M	$c_{.01}$[a]	$c_{.025}$	$c_{.05}$	$c_{.10}$	$c_{.25}$
		a. Scaled $\chi^2(1)$ variable			
10	2.272	1.944	1.650	1.299	0.693
15	2.289	1.949	1.648	1.294	0.687
20	2.298	1.952	1.647	1.290	0.684
25	2.304	1.953	1.647	1.289	0.682
30	2.308	1.954	1.646	1.287	0.680
∞	2.326	1.960	1.645	1.282	0.674
		b. Exponential variable			
10	2.296	1.951	1.647	1.291	0.684
15	2.307	1.954	1.647	1.288	0.681
20	2.312	1.956	1.646	1.286	0.679
25	2.315	1.956	1.646	1.285	0.678
30	2.317	1.957	1.646	1.285	0.677
∞	2.326	1.960	1.645	1.282	0.674

[a] We define $\int_{c_\alpha}^{\infty} f(z)dz = \alpha$, where $f(z)$ is the approximate probability density function of T^*.

2. Power Values of the T^* Test for the Scaled $\chi^2(1)$ and Exponential Variables (M = 30; Test Size: 0.05, Right-sided)

	Scaled $\chi^2(1)$ variable			Exponential variable		
k[a]	η[b] = 2.0	η = 3.0	η = 4.0	η = 2.0	η = 3.0	η = 4.0
1	.062	.065	.068	.064	.070	.073
3	.084	.097	.108	.103	.136	.156
5	.107	.145	.174	.156	.235	.287
7	.142	.212	.262	.224	.371	.460
9	.180	.288	.367	.298	.508	.634
11	.219	.373	.482	.378	.637	.785
13	.254	.448	.584	.436	.737	.881
15	.282	.511	.677	.477	.797	.921
17	.301	.549	.727	.496	.819	.936
19	.307	.558	.736	.490	.813	.933
21	.297	.538	.708	.463	.776	.913
23	.264	.486	.641	.409	.711	.862
25	.221	.396	.541	.325	.596	.760
27	.165	.277	.387	.228	.416	.570
29	.091	.134	.174	.107	.178	.242

[a] The point of shift under H_1.
[b] $\eta = (\theta_0 + \delta)/\theta_0$.

to 30 and some moderate sizes of shift, mainly reflect the lack of information regarding both the location of the change-point and the initial level of the scale parameter, as assumed in the test problem formulated (see Section 2.1).

3. TESTS FOR HETEROGENEITY IN THE SCALE PARAMETER BASED ON KNOWN TIMES OF POSSIBLE CHANGE

The test investigated in Section 2 is designed for a small change, occurring at an unknown point, in the scale parameter of a serially independent gamma sequence. When the possible change-point or change-points are in fact known precisely, conventional tests for scale heterogeneity apply. Nevertheless, even when the change-point is unknown, the sequence can still be judiciously (sometimes arbitrarily, however) segmented and these tests used to compare the scale parameters.

In this section we summarize representative techniques useful for detecting variance shift (for both normal and nonnormal variables) and the change of occurrence rate in a Poisson process.

3.1 Tests for Variance Heterogeneity

3.1.1. Bartlett's Test: A modified version of the log-likelihood ratio test is due to Bartlett (1937). For K groups of data each containing n_i, presumably, normal random variables, $i = 1, 2, \ldots, K$, Bartlett's test statistic is

$$B = -\frac{1}{c}\sum_{i=1}^{K}(n_i - 1)\log_e(s_i^2/s^2) , \quad (3.1)$$

where

$$s_i^2 = \sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2/(n_i - 1) ,$$

$$s^2 = \sum_{i=1}^{K}(n_i - 1)s_i^2/\sum_{i=1}^{K}(n_i - 1) ,$$

$$c = 1 + \frac{1}{3(K-1)}\left[\sum_{i=1}^{K}\frac{1}{(n_i - 1)} - \frac{1}{\sum_{i=1}^{K}(n_i - 1)}\right].$$

The distribution of B under H_0: $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_K^2$, where σ_i^2 denotes the variance of the ith group, has been shown to be closely approximated by $\chi^2(K - 1)$. Validity of this test is known to be sensitive (in the sense that the actual significance level differs from the nominal) to the departure from normality (cf. Gartside 1972, Layard 1973, and Brown and Forsythe 1974).

3.1.2. Bartlett-Kendall Test: A test aiming at reducing sensitivity to nonnormality is due to Bartlett and Kendall (1946). In this test, the n_i observations in the ith group, as previously described, are further divided randomly into J_i subgroups, each containing m observations. Let

$$G = [\sum_{i}^{K} J_i(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2/(K - 1)]/$$
$$[\sum_{i=1}^{K}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i\cdot})^2/\sum_{i=1}^{K}(J_i - 1)] , \quad (3.2)$$

where

s_{ij}^2 = the sample variance of the jth subgroup in the ith group,

$Y_{ij} = \log_e s_{ij}^2$,

and

$$\bar{Y}_{i\cdot} = \sum_{j=1}^{J_i} Y_{ij}/J_i , \quad \bar{Y}_{\cdot\cdot} = \frac{\sum\sum Y_{ij}}{\sum J_i}.$$

Under H_0, G is approximately distributed, for large J_i, $i = 1, 2, \ldots, K$, as an F random variable with degrees of freedom $(K - 1)$ and $\sum_{i=1}^{K}(J_i - 1)$.

3.1.3. A Jackknife Test: The test statistic is defined as

$$J = [\sum_{i=1}^{K} n_i(\bar{U}_{i\cdot} - \bar{U}_{\cdot\cdot})^2/(K - 1)]/$$
$$[\sum_{i=1}^{K}\sum_{j=1}^{n_i}(U_{ij} - \bar{U}_{i\cdot})^2/\sum_{i=1}^{K}(n_i - 1)] , \quad (3.3)$$

where

$$U_{ij} = n_i \log_e s_i^2 - (n_i - 1)\log_e s_{i(j)}^2 ,$$

$$s_{i(j)}^2 = \frac{1}{(n_i - 2)}\sum_{k\neq j}(X_{ik} - \bar{X}_{i(j)})^2 ,$$

and

$$\bar{X}_{i(j)} = \left(\frac{1}{n_i - 1}\right)\sum_{k\neq j} X_{ik} .$$

Under H_0, J is approximately distributed, for large n_i, $i = 1, 2, \ldots, K$, as an F variable with degrees of freedom $(K - 1)$ and $\sum_{i=1}^{K}(n_i - 1)$. This test has been shown to be robust under certain types of nonnormality (cf. Layard 1973).

3.1.4. Layard's χ^2 Test: The test suggested by Layard (1973), which uses a sample kurtosis measure to correct the bias due to potential nonnormality, is

$$S' = \sum_{i=1}^{K}(n_i - 1)[\log_e s_i^2$$
$$- \sum_{i=1}^{K}(n_i - 1)\log_e s_i^2/\sum_{i=1}^{K}(n_i - 1)]^2/\hat{\tau}^2 , \quad (3.4)$$

where

$$\hat{\tau}^2 = 2 + [1 - (1/\bar{n})]\hat{\gamma} , \quad \bar{n} = \sum_{i=1}^{K} n_i/K ,$$

$$\hat{\gamma} = \{(\sum_{i=1}^{K} n_i)\sum_i\sum_j(X_{ij} - \bar{X}_{i\cdot})^4/$$
$$[\sum_i\sum_j(X_{ij} - \bar{X}_{i\cdot})^2]^2\} - 3.0 .$$

In (3.4), $\hat{\gamma}$ is a sample kurtosis measure. For large n_i, $i = 1, 2, \ldots, K$, under H_0, S' is approximately a $\chi^2(K - 1)$ variable. This test is also robust under certain types of nonnormality.

3.1.5. Modified Levene's Test (due to Brown and Forsythe 1974): This test uses trimmed means for the calculation of the test statistic to reduce the sensitivity to nonnormality. The test statistic is

$$W_\alpha = [\sum_{i=1}^{K} n_i(\bar{Z}_{i\cdot} - \bar{Z}_{\cdot\cdot})^2/(K-1)]/ [\sum_{i=1}^{K}\sum_{j=1}^{n_i} (Z_{ij} - \bar{Z}_{i\cdot})^2/\sum_{i=1}^{K} (n_i - 1)] \ , \quad (3.5)$$

where

$$Z_{ij} = |X_{ij} - \tilde{X}_i^{(\alpha)}| \ ,$$

$\tilde{X}_i^{(\alpha)}$ = mean of the observations of the ith group after trimming the largest and the smallest $\alpha \times 100$ percent values ,

$$\bar{Z}_{i\cdot} = \sum_{j=1}^{n_i} Z_{ij}/n_i \ ,$$

and

$$\bar{Z}_{\cdot\cdot} = \sum_i \sum_j Z_{ij}/\sum_i n_i \ .$$

This statistic is approximately distributed (under H_0) as an $F(K-1, \sum (n_i - 1))$ for large n_i. An α value of 0.50 (note: $\tilde{X}_i^{(0.50)}$ = median of the ith group of data) has been shown, based on some Monte Carlo results, to be most adequate for testing variance heterogeneity in skewly distributed random variables, and $\alpha = 0.1$ to be most useful for variables with symmetrical fat-tailed distributions, including the Cauchy and Student's t distributions (cf. Brown and Forsythe 1974).

3.2 Tests for Heterogeneity of Occurrence Rates in Poisson Processes

Tests for comparing occurrence rates of Poisson processes can be based on either occurrence counts or times between occurrences.

3.2.1. A Dispersion Test: A test for comparing the occurrence rates of K Poisson processes, based on a χ^2 argument, is defined as

$$d = \sum_{i=1}^{K} [(n_i - t_i\hat{\lambda})^2/(t_i\hat{\lambda})] \ , \quad (3.6)$$

where n_i = the number of events observed from the ith process over a period of length t_i, and

$$\hat{\lambda} = (\sum_{i=1}^{K} n_i)/(\sum_{i=1}^{K} t_i)$$

(i.e., the mean occurrence rate). For the special case of equal t_i, the function d reduces to

$$d = \sum (n_i - \bar{n})^2/\bar{n} \ , \quad (3.6a)$$

where $\bar{n} = \sum_{i=1}^{K} n_i/K$ is the mean number of events per process. The distribution of d, under the hypothesis of homogeneous occurrence rate, tends to $\chi^2(K-1)$ for large values of t_i (cf. Cox and Lewis 1966, Sec. 9.3).

3.2.2. A Test Based on Times between Occurrences: Instead of preassigning fixed time intervals for the K Poisson processes in question, here we record each individual process up to the n_ith event, $i = 1, 2, \ldots, K$, and the times between consecutive occurrences are used for the test. It can be readily derived that the conventional logarithmic likelihood ratio in this case is

$$H \equiv -2 \log [L(\hat{\theta})/L(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_K)] = 2 [\sum_{i=1}^{K} n_i(\log \hat{\theta} - \log \hat{\theta}_i)] \ , \quad (3.7)$$

where

$$\hat{\theta}_i = \sum_{j=1}^{n_i} X_{ij}/n_i \ , \quad i = 1, 2, \ldots, K \ ,$$

$$\hat{\theta} = \sum_{i=1}^{K}\sum_{j=1}^{n_i} X_{ij}/\sum_{i=1}^{K} n_i \ ,$$

and X_{ij} = the jth interoccurrence time of the ith process. The distribution of H, under the homogeneity assumption, tends to $\chi^2(K-1)$ for large n_i, $i = 1, 2, \ldots, K$.

3.3 Remarks

All techniques presented in (3.1) through (3.7) require the data to be separated into groups of homogeneous observations on an a priori basis. When information needed for such separation of data in a sequence of observations is not available or is not free of argument, the methods, in principle, become infeasible without the introduction of certain subjective judgments, or arbitrariness, into the prior grouping process. Consequently, the test results depend to some extent on how the sequence is segmented. It is possible, in many cases, to produce a predetermined conclusion, regardless of the nature of the data, by choice of the method of segmentation. Because of this, a test for parameter shift in a sequence of observations using the methods based on segmentation, although helpful for preliminary data analysis, lacks the solid ground needed for a clear-cut conclusion. The T^* test discussed in Section 2 is free of this problem. However, many of the techniques for grouped data (e.g., test statistics described in (3.2)–(3.5)) have been proved to be robust under certain types of nonnormality. This is a virtue the T^* test lacks (the T^* test is sensitive to misspecification of the shape parameter ξ, due to the dependency of var(T) on ξ, and is suspected to offer little protection against misuse in the case of nongamma variables).

On balance, we suggest that for analysis of real-world data, both types of test, the T^* and the tests using (3.1)–(3.7), be used with precautions regarding their respective shortcomings. In using the tests for grouped data, it is imperative that a reasonable decision concerning the segmentation of the series be provided, keeping in mind that a large group size is needed to meet the asymptotic theory underlying the methods and to raise the test power, and that a small group size reduces the chance of involving heterogeneous data in a single group.

Note that another significant advantage of the T^* test is its simple and unified manner in working through, among others, the cases of normal and Poisson variables. Few other existing test schemes possess this feature.

Further, the T^* test, in the case of Poisson variables, is an addition to the existing standard procedures for testing the Poisson model assumption for the overall series.

When faced with the formality of significance testing, one may find difficulty in determining the overall significance level of the results from multiple tests. In the spirit of data analysis, however, it seems sensible to examine different aspects of the data by a variety of tests to help illuminate the nature of the data. In this spirit, we employ the various tests discussed in Sections 2 and 3 for the applications presented in Sections 4 and 5.

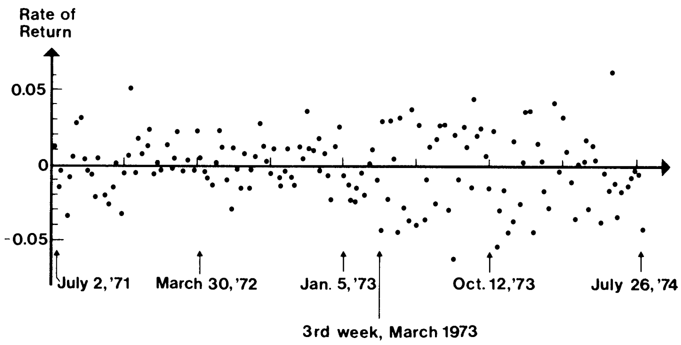
4. ANALYSIS OF STOCK-MARKET PRICES

Since the establishment of a general financial model, due to Sharpe (1964), Lintner (1965), and Mossin (1966), that relates expected return to risk associated with an investment, the measurement of risk has been an important element in various applications. To date, however, little agreement has been reached among researchers concerning the concept and the particular statistical method useful for measuring risk. The discussions were initiated by Mandelbrot (1963) and Fama (1965), followed by Press (1967), Praetz (1972), Clark (1973), Blattberg and Gonedes (1974), Officer (1972), Hsu, Miller, and Wichern (1974), Barnea and Downes (1973), Rosenberg (1972), etc. Most of these articles approach the risk problem by assuming that rates of return of an investment, especially those associated with stock prices, are distributed as a random variable, either constructed by letting the variance of a normal variable follow a random process, or simply belonging to a specific class of distribution models, mostly fat-tailed compared to the normal, with a set of presumably *fixed* parameter values. As noted by Hsu, Miller, and Wichern (1974), however, the phenomena of nonrandom shifts in variance, quite often discontinuous step shifts, have not been adequately considered and incorporated in the models previously suggested. Wichern, Miller, and Hsu (1976) addressed this issue more constructively. Our purpose here, however, is not to attempt to settle these arguments. Rather, we analyze a recent stock-market price series using the techniques developed or summarized in the preceding sections and hope to cast new light on the problems.

Guided by considerations to be stated, we feel that the U.S. stock-market price series from July 1971 through August 1974 is worthy of a thorough investigation to try to illustrate a variance shift (or shifts) in real situations. First, the period in question contains both bullish (up to early 1973) and bearish (from early 1973) stock markets. Second, the period covers times of both relatively peaceful and disturbed (due to Watergate events) political atmosphere in the U.S. Third, the Arab oil embargo and a steady climb of prime interest rates in the U.S. occurred during the later part of this period, which may have had some impact on stock prices. For these reasons, we collected weekly closing values of the Dow-Jones Industrial Average from July 1, 1971 through August 2, 1974 (the endpoint was arbitrarily set at the week before former President Nixon resigned, and the starting date was conveniently assigned at roughly one year before the Watergate break-in), with 162 observations in all. This series is given in Appendix A.

Using an approach common in finance, we transformed the values of the index into a series of rates of return; i.e., $R_t = (P_{t+1} - P_t)/P_t$, where P_t are index values at week t, $t = 1, 2, \ldots, 161$. The R_t sequence is plotted in Figure A. The time series plot suggests that the second

A. Weekly Rates of Return Computed Based on Values of the Dow-Jones Industrial Average



portion of the series, roughly from mid-March 1973, exhibits a larger variation than does the first portion—a step-change in variance appears to take place at the separation point in question. To examine this impression more rigorously, we performed a T^* test on the R_t series by assuming that the values are distributed as *independent, normal* random variables. (These assumptions will be reexamined after the major analysis is done.) Further, in the absence of a "true" mean value, we feel that, since the sample size of 161 is fairly large and the mean level of the R_t series does not seem to change visibly over the period investigated, the sample mean over the entire period (-0.0008) could be a reasonable substitute for the "true" value. Upon this substitution and following (2.4)–(2.6), the computed value of T^*, for the squared R_t, is 3.521, far greater than the critical point, 2.326, at the 0.01 right-sided significance level. This confirms our impression obtained from inspecting the time series. To investigate further, we estimated the point of shift (defined as k in Section 2.1) using the standard maximum likelihood method and found that it was the 89th point in the data of R_t; that is, the shift occurred during the week March 19–23, 1973 (the third full week of March 1973). The R_t series was separated into two subseries at this estimated change-point, and tests were run to examine normality and serial independence for each of the two subseries individually, and to test the difference between the average returns over the two periods. The results are reported in Table 3.

Explanations of the test statistics used and interpretations of the numerical results follow: (1) $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are the sample measures of γ_1 and γ_2, respectively (see (2.5) for the definitions, and D'Agostino and Pearson 1973 for

3. Results of Tests for Examining Assumptions Underlying T^*

Test statistic	Pre-change period (1st–89th week)		Post-change period (90th–161st week)	
	Test value	Significance level	Test value	Significance level
$\hat{\gamma}_1$	0.35	>.60[b]	0.09	>.80[b]
γ_2	0.54	.20[b]	−0.91	.02[b]
$\hat{\alpha}_{.96}$	1.81	>.25[c]	2.00	>.50[c]
R	5.47	>.50[b]	4.49	>.50[b]
Q_{20}	18.97	>.50[a,d]	31.48	.05[a,d]
s^2	0.00025	—	0.00079	—
$\bar{x}$	0.00101	—	−0.00303	—

[a] Right-sided.
[b] Two-sided.
[c] Left-sided.
[d] The significance level indicated is determined based on an asymptotic approximation. For a moderate-sized sample such as the one encountered here, the exact significance level might be substantially lower. The reader is referred to Davies, Triggs, and Newbold (1977) for a related discussion.

the significance levels); none of these results is significant at a two percent, two-sided significance level. (2) $\alpha_{0.96}$ is an estimate of the characteristic exponent of the stable variables (see Fama and Roll (1968, 1971) for the definition and the corresponding sampling distribution; a value sufficiently close to two implies compatibility with normality); our numerical estimates are well within the significance limits under the normality assumption. (3) R = (range)/(sample standard deviation) is the studentized range statistic (see David, Hartley, and Pearson 1954 for the significance levels), which has been shown by Fama and Roll (1971) to be most powerful, among several alternative tests, in testing normality against the nonnormal stable alternatives; our results, again, are well within the significance limits under normality. (4) Autocorrelation coefficients for up to lag 20 were computed for both subseries and none was found significantly different from zero, judged by the large-sample standard error $(1/\sqrt{n})$ at the two percent, two-sided level. In addition, a portmanteau test statistic $Q_k = n \sum_{i=1}^{k} r_i^2$, where the r_i are the autocorrelation coefficients at lag i, and k is the lag up to which the autocorrelation coefficient function is considered, is known to be *approximately* distributed as $\chi^2(k)$ under the independence assumption (cf. Box and Jenkins 1970, pp. 289–291); none of our results appears significant at less than the five percent level. (5) The values of the sample variance, s^2, are reported in the second to last row of Table 3 for both subseries; their ratio is estimated to be $0.00079/0.00025 = 3.2$, which is significantly different from 1 at a level less than 0.0005 when the separation point is assumed to be the true change-point and the F test criterion is used. (6) Finally, the sample means for the two periods were compared. The computed statistic $t = (\bar{x}_1 - \bar{x}_2)/(s_1^2/n_1 + s_2^2/n_2)^{\frac{1}{2}}$, where $\bar{x}_i$, s_i^2, and n_i, $i = 1, 2$, are sample means, variances, and sample sizes for the first and second period, respectively, is equal to 1.09. Since the sample sizes for both periods are fairly large, this test value can be roughly compared with the standard normal distribution; the two-sided significance level is found to be approximately 0.28. This confirms the stability of the mean value and justifies the use of the overall mean.

The results reported in Table 3 establish the plausibility of the assumptions underlying the T^* test for the particular series in question. One may still argue, however, that mild departures from normality may not always be detected by these tests, and that the normality assumption (in our case, the assumption of $\xi = \frac{1}{2}$) may not be plausible for other stock price series. To remedy this weakness, we supplement the T^* test by the tests described in (3.1)–(3.5). From an examination of the total number of observations available and the group size required for a reliable asymptotic approximation, we decided to divide the first 160 observations of R_t, discarding the last value, into four groups of 40 consecutive observations. Further, $m = 5$ was determined for the calculation of the Bartlett-Kendall statistic (see (3.2)).

The computed results and their approximate significance levels are shown in Table 4. (Different trimming proportions, specifically $\alpha = 0.10$, 0.20, 0.30, 0.40, and 0.475 (in our case of $n_i = 40$, $\tilde{X}^{(0.475)}$ gives the median), were tried for the modified Levene's statistic, W_α, and the computed values were all found to be quite close, with $W_{0.475}$ being the smallest among them.) From entries in the last column of Table 4, we observe a unanimous rejection of the homogeneity assumption by all five tests. This evidence is consistent with the result of the T^* test reported and is in favor of a variance-shift assumption. Another trial using $n_i = 20$ and $K = 8$ was also performed for the five tests and yielded a similar conclusion (although with slightly less extreme significance levels).

4. Results of Tests for Variance Heterogeneity Based on Grouped Data

Test statistic	Test value	Null distribution	Significance level[a]
B	21.25	$\chi^2(3)$	<.001
G	5.15	$F(3,28)$	0.006
J	6.57	$F(3,156)$	<.001
S'	21.41	$\chi^2(3)$	<.001
$W_{.475}$	8.49	$F(3,156)$	<.001

[a] Right-sided.

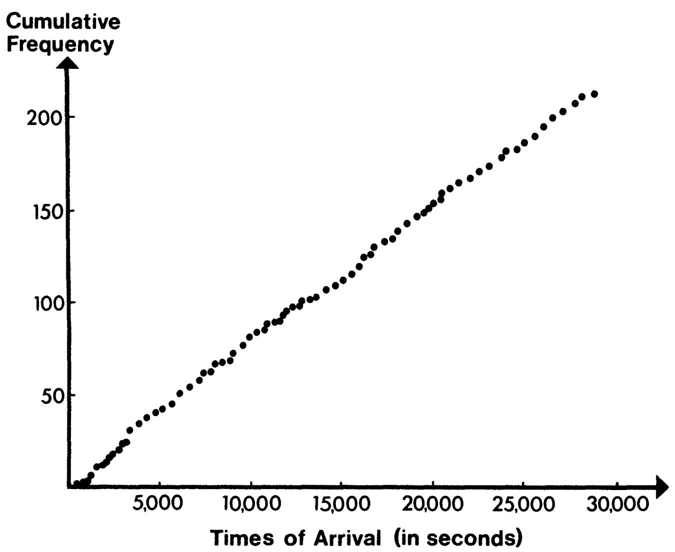
In theory, a normal sequence which contains a variance shift should reveal a fat-tailedness, compared with the normal, in its overall distribution. When a moderate-sized variance shift is contained in a sample sequence of limited length, however, the chance to observe a fat-tailed distribution is not overwhelming. In fact, tests for normality usually are not powerful in detecting contamination of normal variables that possess moderately heterogeneous variances (see Shapiro, Wilk, and Chen 1968). For the particular stock-market return series under study, it happens that the overall distribution, as indicated by some test results, appears to show no evidence of fat-tailedness.

To explain why and how the variance shift in stock-market return occurred during the period investigated, especially the abrupt fashion of its occurrence, would be something of general interest. Two observations have struck us in this regard: (1) the Watergate events caught the full attention of the U.S. public starting roughly in mid-March 1973 and persisting through the end of the sample period; and (2) prime interest rates in the U.S. rose steadily during the first part of 1973. Connections between the variance shift and the two events mentioned, however, are not at all clear.

5. ANALYSIS OF AIR TRAFFIC DENSITIES

In a Federal Aviation Administration–sponsored research project carried out at Princeton University in the last few years, a large-scale computer simulation model for air traffic control voice communications was constructed to enable experimentation with the control networks (see Hsu and Hunter 1975 for details). One important question that was raised in the process of the model construction was whether the air traffic densities observed in the historical data (which consist of hours of complete records of aircraft activities associated with some 100 air traffic controllers in the New York area) were constant over the sample periods. To answer this question, the test described in Section 2 was used to examine the extensive data sets. For the illustrative purpose of this article, we select a typical set of aircraft arrival (entrance) times collected from a low-altitude transitional control sector (identified as 454LT, which covered a specified volume of airspace located to the northwest of Newark airport) for the period from noon through 8 P.M. on April 30, 1969. The times of the aircraft arrivals at the sector are listed in Appendix B. There were 213 arrivals during the eight-hour sample period

B. Plot of Cumulative Arrival Counts against Times of Arrival

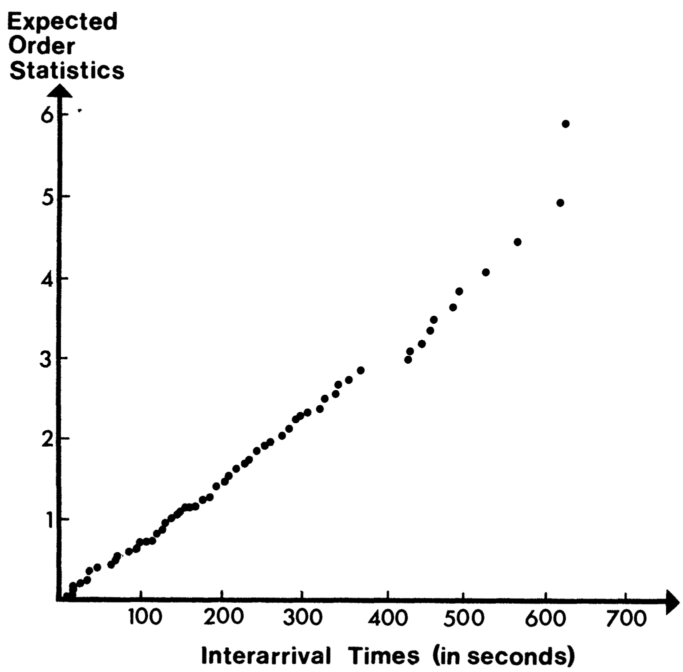


and, if we assume that the last arrival in the preceding period occurred at time zero, 213 aircraft interarrival times. A plot of the cumulative frequencies against times of arrival is given in Figure B. The linear configuration formed by the sample points suggests a homogeneous arrival stream. To investigate more rigorously, the Poisson model assumption is examined below.

Since there were airways passing through in many different directions and at various altitudes in the sector investigated, the aircraft arrival process forms a superposition of processes and may well be Poisson (cf. Cox and Lewis 1966, Ch. 8). Based on this assumption, we specified an exponential model and calculated the T^* value for the interarrival times—T^* equals 1.232, well within the significance bounds. That is, the aircraft were arriving at an approximately constant rate during the period studied.

To reinforce this test result, the Poisson assumption can be examined graphically. As a first stage of this examination, following the suggestion by Cox (1968, p. 276) and Healy (1968), we plotted the interarrival times, ordered by their magnitudes, against the expected order statistics of an exponential variable with unit mean (cf. Cox and Lewis 1966, p. 27). The graph is shown in Figure C. The plotted sample points, except for several

C. Plot of Interarrival Times against the Expected Order Statistics of an Exponential Variable with Unit Mean



of the largest observations, form a nearly straight line, indicating rough consistency between the exponential model and the observed interarrival times. (Since the very last portion of the sample order statistics is known to be subject to larger variations, a visible departure from a straight line in this region is expected under the postulated model.)

5. Results of Tests for Distributions, Independence, and Homogeneity under the Poisson Model

Test statistic	Test value	Null distribution	Significance level
(1) Kolmogorov-Smirnov test ($\sqrt{n}D_n$)	.569	K-S	>.50[a]
(2) Cramér-Von Mises test ($n\omega_n^2$)	.085	C-V	>.50[a]
(3) Anderson-Darling's modified test (W_n^2)	.060	A-D	>.50[a]
(4) Moran test (ℓ_n)	182.300	$\chi^2(212)$	≈.15[b]
(5) Exponential ordered score test for the autocorrelation of lag 1 (R_1*)	−.405	$N(0,1)$	≈.70[b]
(6) Portmanteau test for autocorrelation function up to lag 20 (Q_{20})	26.263	$\chi^2(20)$	≈.17[a,c]
(7) Dispersion test (d_8)	4.500	$\chi^2(7)$	≈.75[a]
(8) Log-likelihood ratio test (H)	2.500	$\chi^2(4)$	≈.65[a]

[a] Right-sided.
[b] Two-sided.
[c] See footnote d of Table 3.

Formal tests for the Poisson structure were subsequently performed on the data, and the results are reported in Table 5. Explanations of the test statistics follow. Since, under the Poisson assumption, arrival times are distributed as a uniform variable over the interval between the start of the sample period and the time of the observed final arrival, tests (1)–(3) are used to examine this feature. The Kolmogorov-Smirnov test statistic measures the maximum discrepancy between the theoretical (uniform) and empirical distributions, while the Cramér-Von Mises statistic measures the squares of the discrepancies between the two distributions, integrated over the appropriate sample interval. Anderson-Darling's modified version of the Cramér-Von Mises statistic is to increase the test power in detecting discrepancies near the ends of the sample period. Finally, the Moran test is the likelihood ratio test for testing exponentiality (for the interarrival times) against the alternative of nonexponential gamma variables. All of these four tests are described in detail in Cox and Lewis (1966, pp. 145, 147, 150, 161, and a table on p. 258). None of the computed results of these four tests suggests a departure from the assumed distributions.

Tests for serial correlation were also performed on the interarrival times, including (1) a test for the autocorrelation of lag 1 by replacing the observed values by the expected order statistics of an exponential variable, and (2) a test of autocorrelation function based on both individual coefficients and a portmanteau statistic Q_{20}. The statistic R_1* used for the former test is a standardized variable of the R_1' explained in Cox and Lewis (1966, p. 167), while the latter test is described in Section 4 associated with Table 3. None of the calculated R_1*, Q_{20}, and the individual autocorrelation coefficients, up to lag 20, was significant at a five percent, appropriately sided level.

The dispersion test (see (3.6)) was then applied to the arrival counts by separating the eight-hour period into eight convenient one-hour intervals. The frequency counts in each hour were obtained for test. The likelihood ratio test described in (3.7) was also performed on the series of interarrival times by dividing the entire series into five segments of consecutive observations, each of the first four containing 40 observations and the last segment containing 53. None of the computed results using (3.6) and (3.7), listed as items (7) and (8) in Table 5, casts doubt on the homogeneity of occurrence rate under the Poisson assumption.

This observation justified the use of the communications responses (e.g., the communications loadings per unit time) in this eight-hour time interval as outcomes of a steady traffic flow. Similar conclusions for the majority of the available data sets provided a basis for the construction and subsequent validation of the simulation model (see Hsu and Hunter 1975, Ch. 1–3).

APPENDIX A

The following list gives the weekly closing values of the Dow-Jones Industrial Average from July 1, 1971 through August 2, 1974, studied in this article. The tabulated values are read one row at a time. The data were extracted from *Daily Stock Price Record: New York Stock Exchange*, published quarterly by Standard and Poor's Co., New York, N.Y.

890.19	901.80	888.51	887.78	858.43	850.61
856.02	880.91	908.15	912.75	911.00	908.22
889.31	893.98	893.91	874.85	852.37	839.00
840.39	812.94	810.67	816.55	859.59	856.75
873.80	881.17	890.20	910.37	906.68	907.44
906.38	906.68	917.59	917.52	922.79	942.43
939.87	942.88	942.28	940.70	962.60	967.72
963.80	954.17	941.23	941.83	961.54	971.25
961.39	934.45	945.06	944.69	929.03	938.06
922.26	920.45	926.70	951.76	964.18	965.83
959.36	970.05	961.24	947.23	943.03	953.27
945.36	930.46	942.81	946.42	984.12	995.26
1,005.57	1,025.21	1,023.43	1,033.19	1,027.24	1,004.21
1,020.02	1,047.49	1,039.36	1,026.19	1,003.54	980.81
979.46	979.23	959.89	961.32	972.23	963.05
922.71	951.01	931.07	959.36	963.20	922.19
953.87	927.89	895.17	930.84	893.96	920.00
888.55	879.82	891.71	870.11	885.99	910.90
936.71	908.87	852.38	871.84	863.49	887.57
898.63	886.36	927.90	947.10	971.25	978.63
963.73	987.06	935.28	908.42	891.33	854.00
822.25	838.05	815.65	818.73	848.02	880.23
841.48	855.47	859.39	843.94	820.40	820.32
855.99	851.92	878.05	887.83	878.13	846.68
847.54	844.81	859.90	834.64	845.90	850.44
818.84	816.65	802.17	853.72	843.09	815.39
802.41	791.77	787.23	787.94	784.57	752.58

APPENDIX B

The following aircraft arrival times are for the control sector 454LT studied in this article. The sample period was 16/00/00-24/00/00 GMT (noon–8 P.M. New York time) on April 30, 1969. Tabulated values are in seconds from the start of the sample period. The data were originally collected by the Federal Aviation Administration, National Aviation Facilities Experimental Center, Atlantic City, New Jersey.

467	761	792	812	926	1,100	1,147
1,163	1,398	1,462	1,487	1,749	1,865	2,004
2,177	2,208	2,279	2,609	2,682	2,733	2,818
2,837	2,855	2,868	3,089	3,209	3,223	3,233
3,272	3,399	3,595	3,634	3,650	3,851	4,176
4,304	4,391	4,453	4,539	4,748	4,839	5,049
5,202	5,355	5,551	5,598	5,640	5,702	5,935
6,000	6,192	6,435	6,474	6,600	6,810	6,824
7,168	7,181	7,202	7,218	7,408	7,428	7,720
7,755	7,835	7,958	8,307	8,427	8,754	8,819
8,904	8,938	8,980	9,048	9,237	9,268	9,513
9,635	9,750	9,910	9,929	10,167	10,254	10,340
10,624	10,639	10,669	10,889	11,386	11,515	11,651
11,727	11,737	11,844	11,928	12,168	12,657	12,675
12,696	12,732	13,092	13,281	13,536	13,556	13,681
13,710	14,008	14,151	14,601	14,877	14,927	15,032
15,134	15,213	15,491	15,589	15,600	15,631	15,674
15,797	15,953	16,089	16,118	16,215	16,394	16,503
16,515	16,537	16,570	16,597	16,619	16,693	17,314
17,516	17,646	17,770	17,897	17,913	17,922	18,174
18,189	18,328	18,345	18,499	18,521	18,588	19,117
19,150	19,432	19,662	19,758	19,789	19,831	19,978
20,119	20,312	20,346	20,449	20,455	20,604	20,675
20,817	20,898	21,245	21,386	21,562	22,022	22,056
22,095	22,182	22,554	22,764	22,955	22,993	23,025
23,117	23,321	23,341	23,650	23,766	23,879	23,888
24,458	24,889	24,930	24,967	25,224	25,312	25,477
25,498	25,712	25,721	25,884	25,919	25,985	26,196
26,459	26,468	26,494	26,505	26,554	26,906	27,003
27,437	27,661	27,675	27,697	27,721	27,734	27,802
27,971	28,116	28,746				

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