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Tests for Variance Shift at an Unknown Time Point

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SUMMARY

Two tests for variance shift in a sequence of independent normal random variables, when the initial level of variance is unknown, are investigated in this article. The first is a locally most powerful test, and the second is a test based upon cusums of χ^2 values. Distribution functions of the two test statistics are approximated through the use of Edgeworth expansions and/or the beta distribution by matching the first few moments. Critical points of both test statistics are tabulated for various sample sizes. Powers of the two tests are compared using a Monte Carlo example. An illustration of the application of the tests to stock market price analysis is provided.

Keywords: VARIANCE SHIFT; DIRICHLET VARIABLES; EDGEWORTH EXPANSION; THE BETA DISTRIBUTION; MONTE CARLO METHODS

1. INTRODUCTION

IN statistical studies of practical events, one may often encounter problems of testing that can be formulated as follows. Assume a sequence of M consecutively observed independent normal random variables, denoted by $Z_1, Z_2, \ldots, Z_M$, all having a known and constant mean μ. Further, let the variances of these variables be represented by $\sigma_1^2, \sigma_2^2, \ldots, \sigma_M^2$, respectively. The problem is then to test

$$H_0\colon \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_M^2 = \sigma_0^2 \quad (\sigma_0^2 \text{ is } \textit{unknown})$$

against

$$H_1\colon \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2 = \sigma_0^2; \quad \sigma_{k+1}^2 = \sigma_{k+2}^2 = \ldots = \sigma_M^2 = \sigma_0^2 + \delta, \quad |\delta| > 0,$$

where k is unknown ($k = 1, 2, \ldots, M-1$); δ is unknown and $(\sigma_0^2 + \delta)\, \varepsilon (0, \infty)$. In reality, the variables Z_i defined above may be a sequence of rates of return per unit time from an investment, or a series of deviations from the target level observed from a quality-control process. In those cases, examination of the constancy of the variance value is important, and the shift in variance is often something of great practical concern (cf. Sharpe, 1970; Hansen, 1973, Chapter 8).

To date, to the author's best knowledge, no readily usable test statistic, or related test criteria, for the test problem described above has yet been reported in the literature. In this article, we investigate two methods relevant for the test problem specified and compare their powers and merits in practical applications. An illustration of the use of the tests is provided in detecting a variance change in the U.S. stock-market return series during the period 1971–74, which is suspected to have occurred in connection with the Watergate events.

2. THE LOCALLY MOST POWERFUL TEST FOR SMALL VARIANCE SHIFT

In connection with the variables described in the previous section, we further define

$$X_i = (Z_i - \mu)^2, \quad i = 1, 2, \ldots, M. \tag{2.1}$$

Clearly, the X's, except a constant, are distributed as a χ^2 variable with one degree of freedom, denoted by $\chi^2(1)$. Following a derivation method suggested by Chernoff and Zacks (1964), and Kander and Zacks (1966, pp. 1197–8), we obtained a test statistic which is locally most powerful as $\delta/\sigma_0^2 \to 0$, that is,

$$T = \left\{\sum_{i=1}^{M}(i-1)X_i\right\} \Big/ \left\{(M-1)\sum_{i=1}^{M}X_i\right\}, \quad 0 \leqslant T \leqslant 1. \tag{2.2}$$

Some useful characteristics of this test statistic, under the null hypothesis of no variance shift, are listed below.

$$\mu_1(T) = E(T) = \tfrac{1}{2}; \quad \mu_2(T) = \operatorname{var}(T) = (M+1)/\{6(M-1)(M+2)\};$$

$$\gamma_1(T) = \mu_3(T)/\{\mu_2(T)\}^{1\cdot 5} = 0;$$

and

$$\gamma_2(T) = \mu_4(T)/\{\mu_2(T)\}^2 - 3\cdot 0 = \frac{3(M+2)\{5(M-1)M(M+1)+12(3M^2-7)\}}{5(M-1)(M+1)(M+4)(M+6)} - 3\cdot 0$$

$$\approx -2\cdot 4/(M-1), \quad \text{for large } M. \tag{2.3}$$

These formulae indicate that, under H_0, the distribution of the test statistic T is symmetrical about the mean $\frac{1}{2}$ and is less peaked than the normal distribution. Further, the distribution of T tends to the normal as the sample size, M, becomes large. (Incidentally, we point out that T is a linear function of Dirichlet variables; this feature helped provide expressions given in (2.3).)

Since the exact distribution of T, under H_0, except when $M = 2$, is very difficult to obtain, two approximation methods were combined to compute numerical critical points. The first is the Edgeworth expansion (up to the terms containing the fourth cumulant), and the second is to match the first three moments of T using a one-parameter beta distribution. More specifically, the Edgeworth expansion was found to have provided very precise results for the cases

TABLE 1

Critical values for the T-test in the standardized scale

M	$s_{0\cdot 01}$†	$s_{0\cdot 025}$	$s_{0\cdot 05}$	$s_{0\cdot 10}$	$s_{0\cdot 25}$
2	1·414	1·410	1·397	1·345	1·000
3	1·901	1·771	1·613	1·370	0·803
4	2·083	1·868	1·643	1·343	0·749
5	2·170	1·906	1·650	1·325	0·724
6	2·219	1·925	1·652	1·314	0·710
7	2·251	1·938	1·652	1·306	0·700
8	2·260	1·940	1·651	1·303	0·697
9	2·266	1·942	1·650	1·301	0·695
10	2·272	1·944	1·650	1·299	0·693
15	2·289	1·949	1·648	1·294	0·687
20	2·298	1·952	1·647	1·290	0·684
25	2·304	1·953	1·647	1·289	0·682
30	2·308	1·954	1·646	1·287	0·680
∞	2·326	1·960	1·645	1·282	0·674

†We define $\int_{c_\alpha}^{1} f(z)\,dz = \alpha$, where $f(z)$ is the approximate probability density function of T, and $s_\alpha = (c_\alpha - 0\cdot 5)/\{\operatorname{var}(T)\}^{\frac{1}{2}}$.

of moderate and large M (the error magnitudes on the cumulative density functions are in the order of $M^{-\frac{3}{2}}$ (cf. Cramér, 1962, Theorem 26), and, in our cases, are believed to be restrained to the third decimal place for $M > 10$), while the beta approximation is more desirable for small M, as the range of the variable, $[0, 1]$, can be readily matched. An *ad hoc* rule was specified, based on some Monte Carlo results, and used to weigh the numerical results from these two methods. The resulting critical points of T, in the standardized scale, for various test and sample sizes are displayed in Table 1. From the entries, it is noted that a sample size of 25 appears large enough for the normal approximation to apply. More details about the procedure which provided these approximations are available in Hsu (1976).

3. A Test Using Cusums of χ^2 Variables

When one is solely interested in the test problem formulated in Section 1, the T test discussed above may be ideal. However, in many practical situations information about the nature, the potential time-point and the size of variance shift is also required. A conventional cusum scheme is seen to fit this need well. A test method which is closely incorporated with the cusum method is described below.

A cusum chart of the χ^2 variables is constructed by plotting the variable $W_k = \sum X_i$ (summing over 1 to k) against k, $k = 1, 2, \ldots, M$ (see (2.1) for the definition of X_i). As one can readily see, the slope of the graph represents the size of the variance and can thus provide indications about the potential existence of either a step-change or gradual changes at the value of the variance. When a step-change is suspected and the change-point can be judiciously located, a conventional F ratio, contingent upon the identified change-point, is known to be

$$Q_k = [(W_M - W_k)/(M-k)]/(W_k/k), \tag{3.1}$$

where k represents the specified change-point, $k = 1, 2, \ldots, M-1$. Letting the observed value of Q_k be denoted by q_k, the significance level of q_k, as usual, is determined by the value of

$$g_k = \int_0^{q_k} \mathrm{F}(t \mid M-k, k)\, dt, \tag{3.2}$$

or $1 - g_k$, depending on the direction of the suspected change. (In (3.2), the term $F(t \mid a, b)$ represents an F distribution with a and b degrees of freedom.) The F test thus constructed is the most powerful for testing H_0 against an adequately specified one-sided shift, at the prescribed k. However, when it is necessary to test the constancy of variance *without* specifying the potential change-point, a convenient test statistic which takes advantage of the computed g_k values expressed above is proposed to be

$$G = \sum_{k=1}^{M-1} g_k/(M-1), \quad 0 \leqslant G \leqslant 1. \tag{3.3}$$

It can be shown that, under the null hypothesis H_0, G is symmetrically distributed about the mean $\frac{1}{2}$ and tends to a non-normal equilibrium distribution as $M \to \infty$. Since the numerical distributions of the G statistic, under H_0, except for $M = 2$, are extremely difficult to obtain, extensive Monte Carlo experiments were performed. Analysis of the computer-generated results, after several cycles of trial and error, provided the following approximation procedure for the probability density functions of G, for various values of M, that is,

$$h(G) = \{B(\beta, \beta)\}^{-1}[G(1-G)]^{\beta-1}, \quad 0 \leqslant G \leqslant 1, \tag{3.4}$$

where

$$\beta = \{1 - 4\sigma_M^2(G)\}/\{8\sigma_M^2(G)\}, \; \sigma_M^2(G) = 0{\cdot}0393 + 0{\cdot}0206/m + 0{\cdot}0999/m^2 - 0.1445/m^3 + 0{\cdot}0662/m^4,$$

and $m = M-1$. In the formula, the expression for $\sigma_M^2(G)$, the variance of G at sample size M, was formulated as a Taylor series expansion of an unknown function and is a least-squares fit to the Monte Carlo data. The magnitudes of the fitted errors, relative to the empirical values, are less than 4 per cent for $\sigma_M^2(G)$, and are less than 17 per cent for the kurtosis measure, γ_2, in all cases investigated (17 different values of M, ranging from 2 to 100, were involved). Critical points for various test and sample sizes are displayed in Table 2.

TABLE 2

Approximate critical values for the G-test

M	$c_{0\cdot01}$†	$c_{0\cdot025}$	$c_{0\cdot05}$	$c_{0\cdot10}$	$c_{0\cdot25}$
2	0·989	0·973	0·947	0·895	0·746
3	0·964	0·935	0·898	0·838	0·698
4	0·948	0·914	0·874	0·813	0·680
5	0·938	0·902	0·860	0·800	0·671
6	0·931	0·894	0·852	0·792	0·665
7	0·927	0·889	0·847	0·787	0·662
8	0·924	0·885	0·843	0·783	0·660
9	0·921	0·883	0·840	0·781	0·658
10	0·920	0·881	0·838	0·779	0·657
15	0·915	0·876	0·833	0·774	0·654
20	0·913	0·873	0·831	0·772	0·652
30	0·911	0·871	0·829	0·770	0·651
100	0·909	0·869	0·826	0·768	0·650
∞	0·908	0·868	0·825	0·767	0·649

†We define $\int_{c_\alpha}^{1} h(G)\, dG = \alpha$, where $h(G)$ is the approximate probability density function of G, given by equation (3.4).

We conclude this section by pointing out that, when a significant departure from the null hypothesis H_0 is found, and a step-change in variance is suspected, the value of k which gives the maximum to $|g_k - \frac{1}{2}|$, denoted by g_k^*, $k = 1, 2, \ldots, M-1$, is a convenient estimate of the change-point.

4. A COMPARISON OF TEST POWERS

Relative performances of the two tests, T and G, are here illustrated by a Monte Carlo example with $M = 30$. Cases of various combinations of change-points and shift sizes were created, and each test was repeated 1000 times for each case to count the frequencies of rejection. The resulting power values are displayed in Table 3. Both tests behave similarly over the cases investigated: the power values are relatively high when the change-point is located in the middle portion of the sequence and are an increasing function of the size of shift, δ/σ_0^2. For cases where the change-point falls in the earlier portion (roughly the first quarter) of the series, the G test works slightly better than the T test. For the remaining cases, the situation is reversed.

In summary, although the T test appears to be slightly, and sometimes considerably, better than the G in most of the range of the potential change-point, the G test is more closely incorporated with the cusum scheme which is useful for investigating many other aspects of the variance-shift problem in practice.

TABLE 3

Comparison of power values of the T-and G-test
(M= 30; test size= 0·05, right-sided)

	Change-point	η† 2	3	4	Change-point	η 2	3	4
T	$k = 1$	0·061	0·065	0·068	$k = 17$	0·309	0·534	0·702
G		0·060	0·070	0·073		0·257	0·449	0·592
T	3	0·075	0·091	0·108	19	0·316	0·545	0·715
G		0·083	0·107	0·128		0·257	0·456	0·600
T	5	0·108	0·149	0·179	21	0·294	0·529	0·683
G		0·116	0·167	0·199		0·241	0·444	0·577
T	7	0·140	0·211	0·256	23	0·270	0·472	0·628
G		0·140	0·222	0·277		0·219	0·401	0·532
T	9	0·177	0·294	0·373	25	0·220	0·394	0·529
G		0·170	0·286	0·354		0·177	0·333	0·445
T	11	0·207	0·377	0·474	27	0·158	0·277	0·371
G		0·192	0·342	0·437		0·132	0·245	0·317
T	13	0·245	0·443	0·575	29	0·085	0·131	0·166
G		0·229	0·395	0·503		0·082	0·123	0·156
T	15	0·287	0·496	0·649				
G		0·244	0·427	0·557				

†$\eta = (\sigma_0^2+\delta)/\sigma_0^2$.

5. AN EXAMPLE OF U.S. STOCK-MARKET PRICES

In order to investigate the potential impacts of the Watergate events upon the U.S. stock market, we collected the weekly closing values of the Dow-Jones industrial average over the period from July 1st, 1971, to August 2nd, 1974 (the week before former President Nixon resigned), with 162 observations in all. According to a conventional financial consideration, the market index values were transformed into a series of rates of return (i.e. $R_t = (P_{t+1}/P_t)-1$, where P_t represent the index values in week t). A time series plot of R_t, $t = 1,2,\ldots,161$, is shown in Fig. 1. An inspection of the graph suggests that the variance of R_t was subjected to a

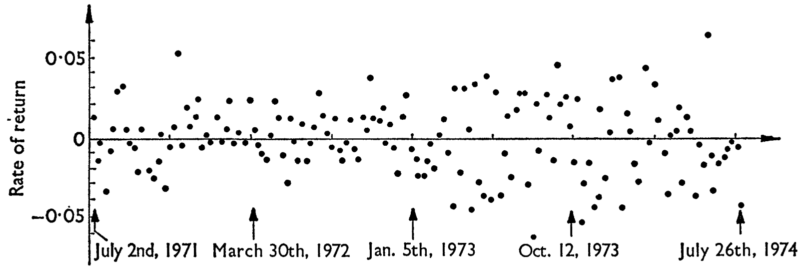

FIG. 1. Weekly rates of return computed as based upon the values of the Dow-Jones industrial average.

sudden increase occurring, roughly, in mid-March, 1973. (Here we note that the Watergate events held the full attention of the U.S. public, largely owing to enthusiastic reporting by the news media, starting from about February or March 1973). To examine this more rigorously, we assume that the R_t are serially independent and normally distributed with the means equal to zero throughout. (Letting the mean float and be non-zero only very slightly changes the results reported below.) The calculated values of the T and G statistics are:

$$T^* = (T-0{\cdot}5)/\{\text{var}(T)\}^{0{\cdot}5} = 3{\cdot}578 > s_{0{\cdot}01} \quad (=2{\cdot}326) \quad \text{and} \quad G = 0{\cdot}909 > c_{0{\cdot}01} \quad (\approx 0{\cdot}9085).$$

Both tests suggest that the null hypothesis of there being no variance shift in the stock-market return series should be rejected at the 0·01, right-sided, level. Further, by choosing the k value which maximizes g_k^*, $k = 1, 2, \ldots, 161$, we estimated the change-point to be the 89th week in the data, that is the third week of March 1973. Upon separation of the series into two sub-series at this point, the variance ratio of the post- to the pre-change observations was calculated to be $(0{\cdot}79\times10^{-3})/(0{\cdot}24\times10^{-3}) = 3{\cdot}24$. Moreover, tests were run to examine the hypothesized normality and serial independence underlying both tests, for each of the two sub-series individually, and the results were found to confirm these assumptions.

Financial implications of this finding, along with results of a more extensive and thorough investigation, will be reported elsewhere.

References

CHERNOFF, H. and ZACKS, S. (1964). Estimating the current mean of a normal distribution which is subjected to changes in time. *Ann. Math. Statist.*, **35**, 999–1018.

CRAMÉR, H. (1962). *Random Variables and Probability Distributions*, 2nd ed. Cambridge: University Press.

HANSEN, B. L. (1963). *Quality Control: Theory and Applications*. Englewood Cliffs, N.J.: Prentice-Hall.

HSU, D. A. (1976). A test for parameter shift in a sequence of gamma random variables. Civil Engineering Research Report No. 76–TR–15, Civil Engineering Department, Princeton University.

KANDER, Z. and ZACKS, S. (1966). Test procedures for possible changes in parameters of statistical distributions occurring at unknown time points. *Ann. Math. Statist.*, **37**, 1196–210.

SHARPE, W. F. (1970). *Portfolio Theory and Capital Markets*. New York: McGraw-Hill.