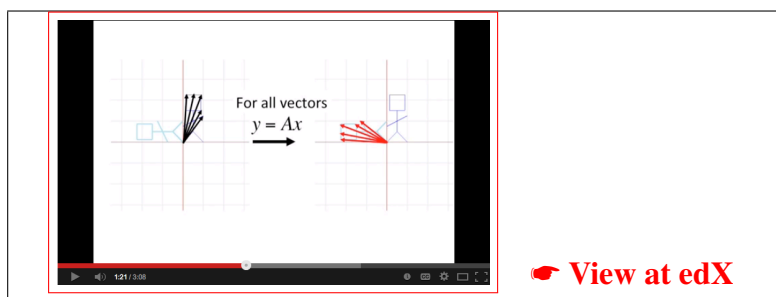


# Matrix-Vector Operations

## 3.1 Opening Remarks

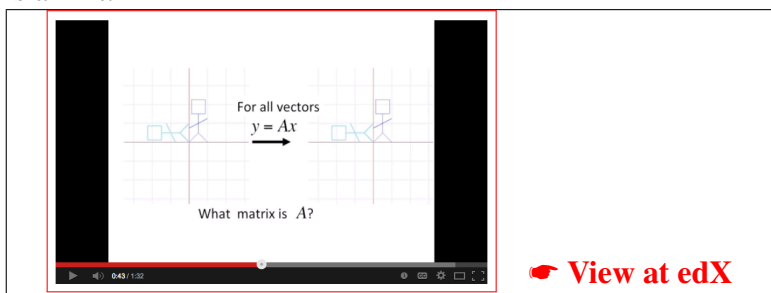
### 3.1.1 Timmy Two Space



**Homework 3.1.1.1** Click on the below link to open a browser window with the “Timmy Two Space” exercise. This exercise was suggested to us by our colleague Prof. Alan Cline. It was first implemented using an IPython Notebook by Ben Holder. During the Spring 2014 offering of LAFF on the edX platform, one of the participants, Ed McCardell, rewrote the activity as the below webpage.

- [Timmy! on the web.](#)

If you get really frustrated, here is a hint:



### 3.1.2 Outline Week 3

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### 3.1.3 What You Will Learn

Upon completion of this unit, you should be able to

- Recognize matrix-vector multiplication as a linear combination of the columns of the matrix.
  - Given a linear transformation, determine the matrix that represents it.
  - Given a matrix, determine the linear transformation that it represents.
  - Connect special linear transformations to special matrices.
  - Identify special matrices such as the zero matrix, the identity matrix, diagonal matrices, triangular matrices, and symmetric matrices.
  - Transpose a matrix.
  - Scale and add matrices.
  - Exploit properties of special matrices.
  - Extrapolate from concrete computation to algorithms for matrix-vector multiplication.
  - Partition (slice and dice) matrices with and without special properties.
  - Use partitioned matrices and vectors to represent algorithms for matrix-vector multiplication.
  - Use partitioned matrices and vectors to represent algorithms in code.
-

## 3.2 Special Matrices

### 3.2.1 The Zero Matrix

Algorithm (set  $A$  to the zero matrix)

```

Algorithm:  $[A] := \text{SET\_TO\_ZERO}(A)$ 
Partition  $A \rightarrow (A_L \mid A_R)$ 
where  $A_L$  has 0 columns
while  $n(A_L) < n(A)$  do
  Repartition
   $(A_L \mid A_R) \rightarrow (A_0 \mid a_1 \mid A_2)$ 
   $a_1 := 0$ 
  Continue with
   $(A_L \mid A_R) \leftarrow (A_0 \mid a_1 \mid A_2)$ 
endwhile
                    
```

0	0	$a_{0,2}$	$a_{0,3}$
0	0	$a_{1,2}$	$a_{1,3}$
0	0	$a_{2,2}$	$a_{2,3}$
0	0	$a_{3,2}$	$a_{3,3}$

[View at edX](#)

**Homework 3.2.1.1** Let  $L_0 : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be the function defined for every  $x \in \mathbb{R}^n$  as  $L_0(x) = 0$ , where 0 denotes the zero vector “of appropriate size”.  $L_0$  is a linear transformation.

True/False

We will denote the matrix that represents  $L_0$  by 0, where we typically know what its row and column sizes are from context (in this case,  $0 \in \mathbb{R}^{m \times n}$ ). If it is not obvious, we may use a subscript ( $0_{m \times n}$ ) to indicate its size, that is,  $m$  rows and  $n$  columns.

By the definition of a matrix, the  $j$ th column of matrix 0 is given by  $L_0(e_j) = 0$  (a vector with  $m$  zero components). Thus, the matrix that represents  $L_0$ , which we will call the zero matrix, is given by the  $m \times n$  matrix

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It is easy to check that for any  $x \in \mathbb{R}^n$ ,  $0_{m \times n}x_n = 0_m$ .

**Definition 3.1** A matrix  $A \in \mathbb{R}^{m \times n}$  equals the  $m \times n$  zero matrix if all of its elements equal zero.

Throughout this course, we will use the number 0 to indicate a scalar, vector, or matrix of “appropriate size”.

In Figure 3.1, we give an algorithm that, given an  $m \times n$  matrix  $A$ , sets it to zero. Notice that it exposes columns one at a time, setting the exposed column to zero.

MATLAB provides the function “zeros” that returns a zero matrix of indicated size. You are going to write your own, to help you understand the material.

**Algorithm:**  $[A] := \text{SET\_TO\_ZERO}(A)$

**Partition**  $A \rightarrow \left( A_L \mid A_R \right)$   
**where**  $A_L$  has 0 columns

**while**  $n(A_L) < n(A)$  **do**

**Repartition**

$\left( A_L \mid A_R \right) \rightarrow \left( A_0 \mid a_1 \mid A_2 \right)$   
**where**  $a_1$  has 1 column

---

---

$a_1 := 0$  (Set the current column to zero)

---

---

**Continue with**

$\left( A_L \mid A_R \right) \leftarrow \left( A_0 \mid a_1 \mid A_2 \right)$

**endwhile**

Figure 3.1: Algorithm for setting matrix  $A$  to the zero matrix.

Check if the files `laff_zerov.m` and `laff_onev.m` are in directory

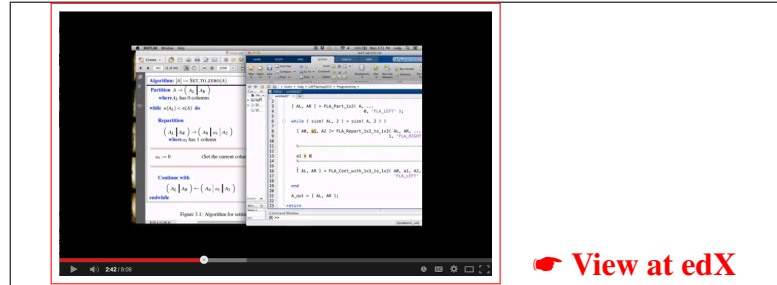
LAFFSpring2015  $\rightarrow$  Programming  $\rightarrow$  laff  $\rightarrow$  vecvec.

If not, download them into that directory from [HERE](#). Make sure you name the files correctly. Also, make sure that the path to the `laff` subdirectory is added in MATLAB, so that the various routines from the `laff` library that we are about to use will be found by MATLAB: To do this, in MATLAB, click

HOME  $\rightarrow$  Set Path  $\rightarrow$  Add with Subfolders

and then browse until you find the `laff` subdirectory.

**Homework 3.2.1.2** With the FLAME API for MATLAB (FLAME@lab) implement the algorithm in Figure 3.1. You will use the function `laff_zerov( x )`, which returns a zero vector of the same size and shape (column or row) as input vector `x`. Since you are still getting used to programming with M-script and FLAME@lab, you may want to follow the instructions in this video:



Some links that will come in handy:

- **Spark**  
(alternatively, open the file [LAFFSpring2015/Spark/index.html](http://LAFFSpring2015/Spark/index.html))
- **PictureFLAME**  
(alternatively, open the file [LAFFSpring2015/PictureFLAME/PictureFLAME.html](http://LAFFSpring2015/PictureFLAME/PictureFLAME.html))

You will need these in many future exercises. Bookmark them!

**Homework 3.2.1.3** In the MATLAB Command Window, type

```
A = zeros( 5, 4 )
```

What is the result?

**Homework 3.2.1.4** Apply the zero matrix to Timmy Two Space. What happens?

1. Timmy shifts off the grid.
2. Timmy disappears into the origin.
3. Timmy becomes a line on the x-axis.
4. Timmy becomes a line on the y-axis.
5. Timmy doesn't change at all.

## 3.2.2 The Identity Matrix

[View at edX](#)

**Homework 3.2.2.1** Let  $L_I : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the function defined for every  $x \in \mathbb{R}^n$  as  $L_I(x) = x$ .  $L_I$  is a linear transformation.

True/False

We will denote the matrix that represents  $L_I$  by the letter  $I$  (capital “I”) and call it the identity matrix. Usually, the size of the identity matrix is obvious from context. If not, we may use a subscript,  $I_n$ , to indicate the size, that is: a matrix that has  $n$  rows and  $n$  columns (and is hence a “square matrix”).

Again, by the definition of a matrix, the  $j$ th column of  $I$  is given by  $L_I(e_j) = e_j$ . Thus, the identity matrix is given by

$$I = \left( \begin{array}{c|c|c|c} e_0 & e_1 & \cdots & e_{n-1} \end{array} \right) = \left( \begin{array}{c|c|c|c} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{array} \right).$$

Here, and frequently in the future, we use vertical lines to indicate a partitioning of a matrix into its columns. (Slicing and dicing again!) It is easy to check that  $Ix = x$ .

**Definition 3.2** A matrix  $I \in \mathbb{R}^{n \times n}$  equals the  $n \times n$  identity matrix if all its elements equal zero, except for the elements on the diagonal, which all equal one.

The diagonal of a matrix  $A$  consists of the entries  $\alpha_{0,0}$ ,  $\alpha_{1,1}$ , etc. In other words, all elements  $\alpha_{i,i}$ .

Throughout this course, we will use the capital letter  $I$  to indicate an identity matrix “of appropriate size”.

We now motivate an algorithm that, given an  $n \times n$  matrix  $A$ , sets it to the identity matrix.

We’ll start by trying to closely mirror the `Set_to_zero` algorithm from the previous unit:

**Algorithm:**  $[A] := \text{SET\_TO\_IDENTITY}(A)$

**Partition**  $A \rightarrow \left( A_L \mid A_R \right)$   
**where**  $A_L$  has 0 columns

**while**  $n(A_L) < n(A)$  **do**

**Repartition**

$\left( A_L \mid A_R \right) \rightarrow \left( A_0 \mid a_1 \mid A_2 \right)$   
**where**  $a_1$  has 1 column

---

---

$a_1 := e_j$  (Set the current column to the correct unit basis vector)

---

---

**Continue with**

$\left( A_L \mid A_R \right) \leftarrow \left( A_0 \mid a_1 \mid A_2 \right)$

**endwhile**

The problem is that our notation doesn't keep track of the column index,  $j$ . Another problem is that we don't have a routine to set a vector to the  $j$ th unit basis vector.

To overcome this, we recognize that the  $j$ th column of  $A$ , which in our algorithm above appears as  $a_1$ , and the  $j$ th unit basis vector can each be partitioned into three parts:

$$a_1 = a_j = \begin{pmatrix} a_{01} \\ \alpha_{11} \\ a_{21} \end{pmatrix} \quad \text{and} \quad e_j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where the 0's refer to vectors of zeroes of appropriate size. To then set  $a_1 (= a_j)$  to the unit basis vector, we can make the assignments

$$\begin{aligned} a_{01} &:= 0 \\ \alpha_{11} &:= 1 \\ a_{21} &:= 0 \end{aligned}$$

The algorithm in Figure 3.2 very naturally exposes exactly these parts of the current column.

**Why is it guaranteed that  $\alpha_{11}$  refers to the diagonal element of the current column?**

**Answer:**  $A_{TL}$  starts as a  $0 \times 0$  matrix, and is expanded by a row and a column in every iteration. Hence, it is always square. This guarantees that  $\alpha_{11}$  is on the diagonal.

MATLAB provides the routine "eye" that returns an identity matrix of indicated size. But we will write our own.



**Algorithm:**  $[A] := \text{SET\_TO\_IDENTITY}(A)$

**Partition**  $A \rightarrow \left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$

**where**  $A_{TL}$  is  $0 \times 0$

**while**  $m(A_{TL}) < m(A)$  **do**

**Repartition**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

**where**  $\alpha_{11}$  is  $1 \times 1$

---

---

set current column to appropriate unit basis vector

$a_{01} := 0$                       set  $a_{01}$ 's components to zero

$\alpha_{11} := 1$

$a_{21} := 0$                       set  $a_{21}$ 's components to zero

---

---

**Continue with**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

**endwhile**

Figure 3.2: Algorithm for setting matrix  $A$  to the identity matrix.





Check if the file `laff_onev` is in directory

LAFSpring2015  $\rightarrow$  Programming  $\rightarrow$  laff  $\rightarrow$  vecvec

(in file `laff_onev.m`). If not, download it into that file from [HERE](#) and place it in that directory, in file `laff_onev.m`.

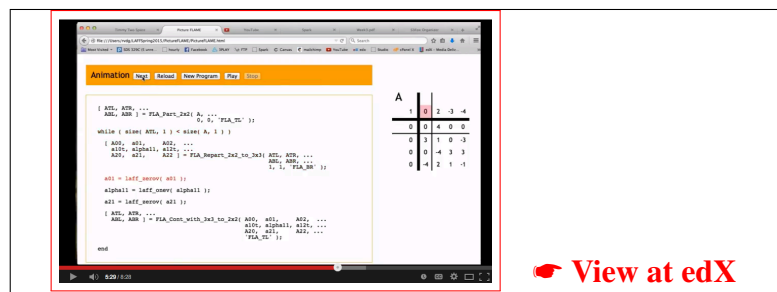
**Homework 3.2.2.2** With the FLAME API for MATLAB (FLAME@lab) implement the algorithm in Figure 3.2. You will use the functions `laff_zerov( x )` and `laff_onev( x )`, which return a zero vector and vector of all ones of the same size and shape (column or row) as input vector `x`, respectively. Try it yourself! (Hint: in Spark, you will want to pick Direction TL→BR.) Feel free to look at the below video if you get stuck.

Some links that will come in handy:

-  **Spark**  
(alternatively, open the file  [LFAFFSpring2015/Spark/index.html](http://LFAFFSpring2015/Spark/index.html))
-  **PictureFLAME**  
(alternatively, open the file  [LFAFFSpring2015/PictureFLAME/PictureFLAME.html](http://LFAFFSpring2015/PictureFLAME/PictureFLAME.html))

You will need these in many future exercises. Bookmark them!

VIDEO HERE!!!!!!



**Homework 3.2.2.3** In the MATLAB Command Window, type

`A = eye( 4, 4 )`

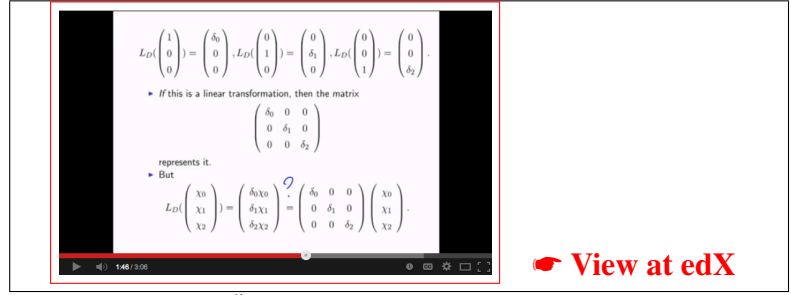
What is the result?

**Homework 3.2.2.4** Apply the identity matrix to Timmy Two Space. What happens?

1. Timmy shifts off the grid.
2. Timmy disappears into the origin.
3. Timmy becomes a line on the x-axis.
4. Timmy becomes a line on the y-axis.
5. Timmy doesn't change at all.

**Homework 3.2.2.5** The trace of a matrix equals the sum of the diagonal elements. What is the trace of the identity  $I \in \mathbb{R}^{n \times n}$ ?

## 3.2.3 Diagonal Matrices



Let  $L_D : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the function defined for every  $x \in \mathbb{R}^n$  as

$$L_D \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} \delta_0 x_0 \\ \delta_1 x_1 \\ \vdots \\ \delta_{n-1} x_{n-1} \end{pmatrix},$$

where  $\delta_0, \dots, \delta_{n-1}$  are constants.

Here, we will denote the matrix that represents  $L_D$  by the letter  $D$ . Once again, by the definition of a matrix, the  $j$ th column of  $D$  is given by

$$L_D(e_j) = L_D \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \delta_0 \times 0 \\ \vdots \\ \delta_{j-1} \times 0 \\ \delta_j \times 1 \\ \delta_{j+1} \times 0 \\ \vdots \\ \delta_{n-1} \times 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \delta_j \times 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \delta_j \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \delta_j e_j.$$

This means that

$$D = \left( \delta_0 e_0 \mid \delta_1 e_1 \mid \cdots \mid \delta_{n-1} e_{n-1} \right) = \begin{pmatrix} \delta_0 & 0 & \cdots & 0 \\ 0 & \delta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_{n-1} \end{pmatrix}.$$

**Definition 3.3** A matrix  $A \in \mathbb{R}^{n \times n}$  is said to be diagonal if  $\alpha_{i,j} = 0$  for all  $i \neq j$  so that

$$A = \begin{pmatrix} \alpha_{0,0} & 0 & \cdots & 0 \\ 0 & \alpha_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n-1,n-1} \end{pmatrix}.$$

Let  $L_D : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the function defined for every  $x \in \mathbb{R}^n$  as

$$L_D \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} \delta_0 x_0 \\ \delta_1 x_1 \\ \vdots \\ \delta_{n-1} x_{n-1} \end{pmatrix},$$

where  $\delta_0, \dots, \delta_{n-1}$  are constants.

Here, we will denote the matrix that represents  $L_D$  by the letter  $D$ . Once again, by the definition of a matrix, the  $j$ th column of  $D$  is given by

$$L_D(e_j) = L_D \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \delta_0 \times 0 \\ \vdots \\ \delta_{j-1} \times 0 \\ \delta_j \times 1 \\ \delta_{j+1} \times 0 \\ \vdots \\ \delta_{n-1} \times 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \delta_j \times 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \delta_j \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \delta_j e_j.$$

This means that

$$D = \left( \delta_0 e_0 \mid \delta_1 e_1 \mid \cdots \mid \delta_{n-1} e_{n-1} \right) = \begin{pmatrix} \delta_0 & 0 & \cdots & 0 \\ 0 & \delta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_{n-1} \end{pmatrix}.$$

**Definition 3.3** A matrix  $A \in \mathbb{R}^{n \times n}$  is said to be diagonal if  $\alpha_{i,j} = 0$  for all  $i \neq j$  so that

$$A = \begin{pmatrix} \alpha_{0,0} & 0 & \cdots & 0 \\ 0 & \alpha_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n-1,n-1} \end{pmatrix}.$$

**Homework 3.2.3.1** Let  $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$  and  $x = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ . Evaluate  $Ax$ .

**Homework 3.2.3.2** Let  $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ . What linear transformation,  $L$ , does this matrix represent? In particular, answer the following questions:

- $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ . What are  $m$  and  $n$ ?
- A linear transformation can be described by how it transforms the unit basis vectors:

$$L(e_0) = \begin{pmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{pmatrix}; L(e_1) = \begin{pmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{pmatrix}; L(e_2) = \begin{pmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{pmatrix}$$

$$\bullet L\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{pmatrix}$$

An algorithm that sets a given square matrix  $A$  to a diagonal matrix that has as its  $i$ th diagonal entry the  $i$ th entry of vector  $x$  is given in Figure 3.3.

**Homework 3.2.3.3** Implement a function

```
[ A_out ] = DiagonalMatrix_unb( A, x )
```

based on Figure 3.3.

**Homework 3.2.3.4** In the MATLAB Command Window, type

```
x = [ -1; 2; -3 ]
A = diag( x )
```

What is the result?

**Algorithm:**  $[A] := \text{SET\_TO\_DIAGONAL\_MATRIX}(A, x)$

**Partition**  $A \rightarrow \left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right), x \rightarrow \left( \begin{array}{c} x_T \\ x_B \end{array} \right)$

**where**  $A_{TL}$  is  $0 \times 0$ ,  $x_T$  has 0 elements

**while**  $m(A_{TL}) < m(A)$  **do**

**Repartition**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right), \left( \begin{array}{c} x_T \\ x_B \end{array} \right) \rightarrow \left( \begin{array}{c} x_0 \\ \chi_1 \\ x_2 \end{array} \right)$$

**where**  $\alpha_{11}$  is  $1 \times 1$ ,  $\chi_1$  is a scalar

---

---

$a_{01} := 0$

$\alpha_{11} := \chi_1$

$a_{21} := 0$

---

---

**Continue with**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right), \left( \begin{array}{c} x_T \\ x_B \end{array} \right) \leftarrow \left( \begin{array}{c} x_0 \\ \chi_1 \\ x_2 \end{array} \right)$$

**endwhile**

Figure 3.3: Algorithm that sets  $A$  to a diagonal matrix with the entries of  $x$  on its diagonal.

In linear algebra an element-wise vector-vector product is not a meaningful operation: when  $x, y \in \mathbb{R}^n$  the product  $xy$  has no meaning. However, MATLAB has an “element-wise multiplication” operator “ $\cdot$ .” Try

```
x = [-1; 2; -3]
y = [1; -1; 2]
x .* y
diag( x ) * y
```

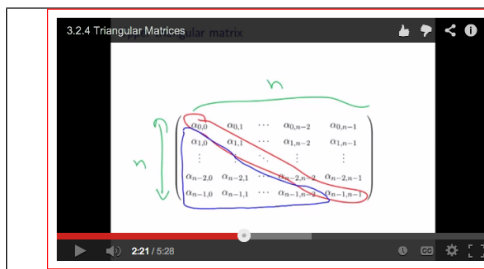
Conclude that element-wise multiplication by a vector is the same as multiplication by a diagonal matrix with diagonal elements equal to the elements of that vector.

**Homework 3.2.3.5** Apply the diagonal matrix  $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$  to Timmy Two Space. What happens?

1. Timmy shifts off the grid.
2. Timmy is rotated.
3. Timmy doesn't change at all.
4. Timmy is flipped with respect to the vertical axis.
5. Timmy is stretched by a factor two in the vertical direction.

**Homework 3.2.3.6** Compute the trace of  $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ .

### 3.2.4 Triangular Matrices



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**Homework 3.2.4.1** Let  $L_U : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be defined as  $L_U\left(\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} 2\chi_0 - \chi_1 + \chi_2 \\ 3\chi_1 - \chi_2 \\ -2\chi_2 \end{pmatrix}$ . We have proven for similar functions that they are linear transformations, so we will skip that part. What matrix,  $U$ , represents this linear transformation?

A matrix like  $U$  in the above practice is called a triangular matrix. In particular, it is an *upper* triangular matrix.

The following defines a number of different special cases of triangular matrices:

**Definition 3.4 (Triangular matrix)**

A matrix  $A \in \mathbb{R}^{n \times n}$  is said to be

<i>lower triangular</i>	$\alpha_{i,j} = 0$ if $i < j$	$\begin{pmatrix} \alpha_{0,0} & 0 & \cdots & 0 & 0 \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & \alpha_{n-2,n-2} & 0 \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & \alpha_{n-1,n-1} \end{pmatrix}$
<i>strictly lower triangular</i>	$\alpha_{i,j} = 0$ if $i \leq j$	$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \alpha_{1,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & 0 & 0 \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & 0 \end{pmatrix}$
<i>unit lower triangular</i>	$\alpha_{i,j} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \end{cases}$	$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \alpha_{1,0} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & 1 & 0 \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & 1 \end{pmatrix}$
<i>upper triangular</i>	$\alpha_{i,j} = 0$ if $i > j$	$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ 0 & \alpha_{1,1} & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_{n-2,n-2} & \alpha_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & \alpha_{n-1,n-1} \end{pmatrix}$
<i>strictly upper triangular</i>	$\alpha_{i,j} = 0$ if $i \geq j$	$\begin{pmatrix} 0 & \alpha_{0,1} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ 0 & 0 & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$
<i>unit upper triangular</i>	$\alpha_{i,j} = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \end{cases}$	$\begin{pmatrix} 1 & \alpha_{0,1} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ 0 & 1 & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$

If a matrix is either lower or upper triangular, it is said to be triangular.

**Homework 3.2.4.2** A matrix that is both lower and upper triangular is, in fact, a diagonal matrix.

Always/Sometimes/Never

**Algorithm:**  $[A] := \text{SET\_TO\_LOWER\_TRIANGULAR\_MATRIX}(A)$

**Partition**  $A \rightarrow \left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$

**where**  $A_{TL}$  is  $0 \times 0$

**while**  $m(A_{TL}) < m(A)$  **do**

**Repartition**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

**where**  $\alpha_{11}$  is  $1 \times 1$

---

---

set the elements of the current column above the diagonal to zero

$a_{01} := 0$                       set  $a_{01}$ 's components to zero

---

---

**Continue with**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

**endwhile**

Figure 3.4: Algorithm for making a matrix  $A$  a lower triangular matrix by setting the entries above the diagonal to zero.

**Homework 3.2.4.3** A matrix that is both strictly lower and strictly upper triangular is, in fact, a zero matrix.

Always/Sometimes/Never

The algorithm in Figure 3.4 sets a given matrix  $A \in \mathbb{R}^{n \times n}$  to its lower triangular part (zeroing the elements above the diagonal).

**Homework 3.2.4.4** In the above algorithm you could have replaced  $a_{01} := 0$  with  $a_{12}^T := 0$ .

Always/Sometimes/Never



**Homework 3.2.4.5** Consider the following algorithm.

**Algorithm:**  $[A] := \text{SET\_TO\_???\_TRIANGULAR\_MATRIX}(A)$

**Partition**  $A \rightarrow \left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$

**where**  $A_{TL}$  is  $0 \times 0$

**while**  $m(A_{TL}) < m(A)$  **do**

**Repartition**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

**where**  $\alpha_{11}$  is  $1 \times 1$

????

**Continue with**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

**endwhile**

Change the ???? in the above algorithm so that it sets  $A$  to its

- Upper triangular part. (`Set_to_upper_triangular_matrix_unb`)
- Strictly upper triangular part. (`Set_to_strictly_upper_triangular_matrix_unb`)
- Unit upper triangular part. (`Set_to_unit_upper_triangular_matrix_unb`)
- Strictly lower triangular part. (`Set_to_strictly_lower_triangular_matrix_unb`)
- Unit lower triangular part. (`Set_to_unit_lower_triangular_matrix_unb`)

The MATLAB functions `tril` and `triu`, when given an  $n \times n$  matrix  $A$ , return the lower and upper triangular parts of  $A$ , respectively. The strictly lower and strictly upper triangular parts of  $A$  can be extracted by the calls `tril( A, -1 )` and `triu( A, 1 )`, respectively. We now write our own routines that sets the appropriate entries in a matrix to zero.

**Homework 3.2.4.6** Implement functions for each of the algorithms from the last homework. In other words, implement functions that, given a matrix  $A$ , return a matrix equal to

- the upper triangular part. (`Set_to_upper_triangular_matrix`)
- the strictly upper triangular part. (`Set_to_strictly_upper_triangular_matrix`)
- the unit upper triangular part. (`Set_to_unit_upper_triangular_matrix`)
- strictly lower triangular part. (`Set_to_strictly_lower_triangular_matrix`)
- unit lower triangular part. (`Set_to_unit_lower_triangular_matrix`)

(Implement as many as you enjoy implementing. Then move on.)

**Homework 3.2.4.7** In MATLAB try this:

```
A = [ 1,2,3;4,5,6;7,8,9 ]
tril( A )
tril( A, -1 )
tril( A, -1 ) + eye( size( A ) )
triu( A )
triu( A, 1 )
triu( A, 1 ) + eye( size( A ) )
```

**Homework 3.2.4.8** Apply  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  to Timmy Two Space. What happens to Timmy?

1. Timmy shifts off the grid.
2. Timmy becomes a line on the x-axis.
3. Timmy becomes a line on the y-axis.
4. Timmy is skewed to the right.
5. Timmy doesn't change at all.

### 3.2.5 Transpose Matrix

$$(A^T)^T = A$$

$$A = \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix}$$

$$B = A^T = \begin{pmatrix} a_{0,0} & a_{1,0} & a_{2,0} \\ a_{0,1} & a_{1,1} & a_{2,1} \\ a_{0,2} & a_{1,2} & a_{2,2} \\ a_{0,3} & a_{1,3} & a_{2,3} \end{pmatrix}$$

$$B^T = (A^T)^T = \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix}$$

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**Definition 3.5** Let  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times m}$ . Then  $B$  is said to be the transpose of  $A$  if, for  $0 \leq i < m$  and  $0 \leq j < n$ ,  $\beta_{j,i} = \alpha_{i,j}$ . The transpose of a matrix  $A$  is denoted by  $A^T$  so that  $B = A^T$ .

We have already used  $^T$  to indicate a row vector, which is consistent with the above definition: it is a column vector that has been transposed.

**Homework 3.2.5.1** Let  $A = \begin{pmatrix} -1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 2 \\ 3 & 1 & -1 & 3 \end{pmatrix}$  and  $x = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$ . What are  $A^T$  and  $x^T$ ?

Clearly,  $(A^T)^T = A$ .

Notice that the columns of matrix  $A$  become the rows of matrix  $A^T$ . Similarly, the rows of matrix  $A$  become the columns of matrix  $A^T$ .

The following algorithm sets a given matrix  $B \in \mathbb{R}^{n \times m}$  to the transpose of a given matrix  $A \in \mathbb{R}^{m \times n}$ :

**Algorithm:**  $[B] := \text{TRANPOSE}(A, B)$

**Partition**  $A \rightarrow \left( A_L \mid A_R \right), B \rightarrow \left( \begin{array}{c} B_T \\ B_B \end{array} \right)$

**where**  $A_L$  has 0 columns,  $B_T$  has 0 rows

**while**  $n(A_L) < n(A)$  **do**

**Repartition**

$$\left( A_L \mid A_R \right) \rightarrow \left( A_0 \mid a_1 \mid A_2 \right), \left( \begin{array}{c} B_T \\ B_B \end{array} \right) \rightarrow \left( \begin{array}{c} B_0 \\ \frac{b_1^T}{B_2} \end{array} \right)$$

**where**  $a_1$  has 1 column,  $b_1$  has 1 row

---


$$b_1^T := a_1^T$$

(Set the current row of  $B$  to the current column of  $A$ )

---

**Continue with**

$$\left( A_L \mid A_R \right) \leftarrow \left( A_0 \mid a_1 \mid A_2 \right), \left( \begin{array}{c} B_T \\ B_B \end{array} \right) \leftarrow \left( \begin{array}{c} B_0 \\ \frac{b_1^T}{B_2} \end{array} \right)$$

**endwhile**

The  $^T$  in  $b_1^T$  is part of indicating that  $b_1^T$  is a row. The  $^T$  in  $a_1^T$  in the assignment changes the column vector  $a_1$  into a row vector so that it can be assigned to  $b_1^T$ .

**Homework 3.2.5.2** Consider the following algorithm.

**Algorithm:**  $[B] := \text{TRANPOSE\_ALTERNATIVE}(A, B)$

**Partition**  $A \rightarrow \left( \begin{array}{c} A_T \\ A_B \end{array} \right), B \rightarrow \left( \begin{array}{c|c} B_L & B_R \end{array} \right)$

**where**  $A_T$  has 0 rows,  $B_L$  has 0 columns

**while**  $m(A_T) < m(A)$  **do**

**Repartition**

$$\left( \begin{array}{c} A_T \\ A_B \end{array} \right) \rightarrow \left( \begin{array}{c} A_0 \\ \frac{a_1^T}{A_2} \end{array} \right), \left( \begin{array}{c|c} B_L & B_R \end{array} \right) \rightarrow \left( \begin{array}{c|c|c} B_0 & b_1 & B_2 \end{array} \right)$$

**where**  $a_1$  has 1 row,  $b_1$  has 1 column

**Continue with**

$$\left( \begin{array}{c} A_T \\ A_B \end{array} \right) \leftarrow \left( \begin{array}{c} A_0 \\ \frac{a_1^T}{A_2} \end{array} \right), \left( \begin{array}{c|c} B_L & B_R \end{array} \right) \leftarrow \left( \begin{array}{c|c|c} B_0 & b_1 & B_2 \end{array} \right)$$

**endwhile**

Modify the above algorithm so that it copies rows of  $A$  into columns of  $B$ .

**Homework 3.2.5.3** Implement functions

- `Transpose_unb( A, B )`
- `Transpose_alternative_unb( A, B )`

**Homework 3.2.5.4** The transpose of a lower triangular matrix is an upper triangular matrix.  
Always/Sometimes/Never

**Homework 3.2.5.5** The transpose of a strictly upper triangular matrix is a strictly lower triangular matrix.

Always/Sometimes/Never

**Homework 3.2.5.6** The transpose of the identity is the identity.

Always/Sometimes/Never

**Homework 3.2.5.7** Evaluate

$$\bullet \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T =$$

$$\bullet \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^T =$$

**Homework 3.2.5.8** If  $A = A^T$  then  $A = I$  (the identity).

True/False

### 3.2.6 Symmetric Matrices

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A matrix  $A \in \mathbb{R}^{n \times n}$  is said to be symmetric if  $A = A^T$ .

In other words, if  $A \in \mathbb{R}^{n \times n}$  is symmetric, then  $\alpha_{i,j} = \alpha_{j,i}$  for all  $0 \leq i, j < n$ . Another way of expressing this is that

$$A = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{0,n-2} & \alpha_{1,n-2} & \cdots & \alpha_{n-2,n-2} & \alpha_{n-2,n-1} \\ \alpha_{0,n-1} & \alpha_{1,n-1} & \cdots & \alpha_{n-2,n-1} & \alpha_{n-1,n-1} \end{pmatrix}$$

and

$$A = \begin{pmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdots & \alpha_{n-2,0} & \alpha_{n-1,0} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{n-2,1} & \alpha_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & \alpha_{n-2,n-2} & \alpha_{n-1,n-2} \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & \alpha_{n-1,n-1} \end{pmatrix}.$$

**Homework 3.2.6.1** Assume the below matrices are symmetric. Fill in the remaining elements.

$$\begin{pmatrix} 2 & \square & -1 \\ -2 & 1 & -3 \\ \square & \square & -1 \end{pmatrix}; \quad \begin{pmatrix} 2 & \square & \square \\ -2 & 1 & \square \\ -1 & 3 & -1 \end{pmatrix}; \quad \begin{pmatrix} 2 & 1 & -1 \\ \square & 1 & -3 \\ \square & \square & -1 \end{pmatrix}.$$

**Homework 3.2.6.2** A triangular matrix that is also symmetric is, in fact, a diagonal matrix.  
Always/Sometimes/Never

The nice thing about symmetric matrices is that only approximately half of the entries need to be stored. Often, only the lower triangular or only the upper triangular part of a symmetric matrix is stored. Indeed: Let  $A$  be symmetric, let  $L$  be the lower triangular matrix stored in the lower triangular part of  $A$ , and let  $\tilde{L}$  is the strictly lower triangular matrix stored in the strictly lower triangular part of  $A$ . Then  $A = L + \tilde{L}^T$ :

$$\begin{aligned} A &= \begin{pmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdots & \alpha_{n-2,0} & \alpha_{n-1,0} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{n-2,1} & \alpha_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & \alpha_{n-2,n-2} & \alpha_{n-1,n-2} \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & \alpha_{n-1,n-1} \end{pmatrix} \\ &= \begin{pmatrix} \alpha_{0,0} & 0 & \cdots & 0 & 0 \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & \alpha_{n-2,n-2} & 0 \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & \alpha_{n-1,n-1} \end{pmatrix} + \begin{pmatrix} 0 & \alpha_{1,0} & \cdots & \alpha_{n-2,0} & \alpha_{n-1,0} \\ 0 & 0 & \cdots & \alpha_{n-2,1} & \alpha_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha_{n-1,n-2} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_{0,0} & 0 & \cdots & 0 & 0 \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & \alpha_{n-2,n-2} & 0 \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & \alpha_{n-1,n-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \alpha_{1,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & 0 & 0 \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & 0 \end{pmatrix}^T. \end{aligned}$$

Let  $A$  be symmetric and assume that  $A = L + \tilde{L}^T$  as discussed above. Assume that only  $L$  is stored in  $A$  and that we would like to also set the upper triangular parts of  $A$  to their correct values (in other words, set the strictly upper triangular part of  $A$  to  $\tilde{L}$ ). The following algorithm performs this operation, which we will call “symmetrizing”  $A$ :

**Algorithm:**  $[A] := \text{SYMMETRIZE\_FROM\_LOWER\_TRIANGLE}(A)$

**Partition**  $A \rightarrow \left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$

**where**  $A_{TL}$  is  $0 \times 0$

**while**  $m(A_{TL}) < m(A)$  **do**

**Repartition**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

**where**  $\alpha_{11}$  is  $1 \times 1$

---

---

(set  $a_{01}$ 's components to their symmetric parts below the diagonal)

$$a_{01} := (a_{10}^T)^T$$


---

---

**Continue with**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

**endwhile**

**Homework 3.2.6.3** In the above algorithm one can replace  $a_{01} := a_{10}^T$  by  $a_{12}^T = a_{21}$ .  
Always/Sometimes/Never

**Homework 3.2.6.4** Consider the following algorithm.

**Algorithm:**  $[A] := \text{SYMMETRIZE\_FROM\_UPPER\_TRIANGLE}(A)$

**Partition**  $A \rightarrow \left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$

**where**  $A_{TL}$  is  $0 \times 0$

**while**  $m(A_{TL}) < m(A)$  **do**

**Repartition**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

**where**  $\alpha_{11}$  is  $1 \times 1$

?????

**Continue with**

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

**endwhile**

What commands need to be introduced between the lines in order to “symmetrize”  $A$  assuming that only its upper triangular part is stored initially.

**Homework 3.2.6.5** Implement functions

- `Symmetrize_from_lower_triangle_unb( A, B )`
- `Symmetrize_from_upper_triangle_unb( A, B )`

## 3.3 Operations with Matrices

### 3.3.1 Scaling a Matrix

Summary

$$\beta \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} = \begin{pmatrix} \beta\alpha_{0,0} & \beta\alpha_{0,1} & \cdots & \beta\alpha_{0,n-1} \\ \beta\alpha_{1,0} & \beta\alpha_{1,1} & \cdots & \beta\alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta\alpha_{m-1,0} & \beta\alpha_{m-1,1} & \cdots & \beta\alpha_{m-1,n-1} \end{pmatrix}$$

$\beta Ax = (\beta A)x = A(\beta x) = A\beta x$

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**Theorem 3.6** Let  $L_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation and, for all  $x \in \mathbb{R}^n$ , define the function  $L_B : \mathbb{R}^n \rightarrow \mathbb{R}^m$  by  $L_B(x) = \beta L_A(x)$ , where  $\beta$  is a scalar. Then  $L_B(x)$  is a linear transformation.

**Homework 3.3.1.1** Prove the above theorem.

Let  $A$  be the matrix that represents  $L_A$ . Then, for all  $x \in \mathbb{R}^n$ ,  $\beta(Ax) = \beta L_A(x) = L_B(x)$ . Since  $L_B$  is a linear transformation, there should be a matrix  $B$  such that, for all  $x \in \mathbb{R}^n$ ,  $Bx = L_B(x) = \beta(Ax)$ . Recall that  $b_j = Be_j$ , the  $j$ th column of  $B$ . Thus,  $b_j = Be_j = \beta(Ae_j) = \beta a_j$ , where  $a_j$  equals the  $j$ th column of  $A$ . We conclude that  $B$  is computed from  $A$  by scaling each column by  $\beta$ . But that simply means that each element of  $B$  is scaled by  $\beta$ .

The above motivates the following definition.

If  $A \in \mathbb{R}^{m \times n}$  and  $\beta \in \mathbb{R}$ , then

$$\beta \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} = \begin{pmatrix} \beta\alpha_{0,0} & \beta\alpha_{0,1} & \cdots & \beta\alpha_{0,n-1} \\ \beta\alpha_{1,0} & \beta\alpha_{1,1} & \cdots & \beta\alpha_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \beta\alpha_{m-1,0} & \beta\alpha_{m-1,1} & \cdots & \beta\alpha_{m-1,n-1} \end{pmatrix}.$$

An alternative motivation for this definition is to consider

$$\begin{aligned} \beta(Ax) &= \beta \begin{pmatrix} \alpha_{0,0}\chi_0 + \alpha_{0,1}\chi_1 + \cdots + \alpha_{0,n-1}\chi_{n-1} \\ \alpha_{1,0}\chi_0 + \alpha_{1,1}\chi_1 + \cdots + \alpha_{1,n-1}\chi_{n-1} \\ \vdots \\ \alpha_{m-1,0}\chi_0 + \alpha_{m-1,1}\chi_1 + \cdots + \alpha_{m-1,n-1}\chi_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \beta(\alpha_{0,0}\chi_0 + \alpha_{0,1}\chi_1 + \cdots + \alpha_{0,n-1}\chi_{n-1}) \\ \beta(\alpha_{1,0}\chi_0 + \alpha_{1,1}\chi_1 + \cdots + \alpha_{1,n-1}\chi_{n-1}) \\ \vdots \\ \beta(\alpha_{m-1,0}\chi_0 + \alpha_{m-1,1}\chi_1 + \cdots + \alpha_{m-1,n-1}\chi_{n-1}) \end{pmatrix} \\ &= \begin{pmatrix} \beta\alpha_{0,0}\chi_0 + \beta\alpha_{0,1}\chi_1 + \cdots + \beta\alpha_{0,n-1}\chi_{n-1} \\ \beta\alpha_{1,0}\chi_0 + \beta\alpha_{1,1}\chi_1 + \cdots + \beta\alpha_{1,n-1}\chi_{n-1} \\ \vdots \\ \beta\alpha_{m-1,0}\chi_0 + \beta\alpha_{m-1,1}\chi_1 + \cdots + \beta\alpha_{m-1,n-1}\chi_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \beta\alpha_{0,0} & \beta\alpha_{0,1} & \cdots & \beta\alpha_{0,n-1} \\ \beta\alpha_{1,0} & \beta\alpha_{1,1} & \cdots & \beta\alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta\alpha_{m-1,0} & \beta\alpha_{m-1,1} & \cdots & \beta\alpha_{m-1,n-1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = (\beta A)x. \end{aligned}$$

Since, by design,  $\beta(Ax) = (\beta A)x$  we can drop the parentheses and write  $\beta Ax$  (which also equals  $A(\beta x)$  since  $L(x) = Ax$  is a linear transformation).

Given matrices  $\beta \in \mathbb{R}$  and  $A \in \mathbb{R}^{m \times n}$ , the following algorithm scales  $A$  by  $\beta$ .

**Algorithm:**  $[A] := \text{SCALE\_MATRIX}(\beta, A)$

**Partition**  $A \rightarrow \left( A_L \mid A_R \right)$   
**where**  $A_L$  has 0 columns

**while**  $n(A_L) < n(A)$  **do**

**Repartition**

$\left( A_L \mid A_R \right) \rightarrow \left( A_0 \mid a_1 \mid A_2 \right)$   
**where**  $a_1$  has 1 column

---

$a_1 := \beta a_1$  (Scale the current column of  $A$ )

---

**Continue with**

$\left( A_L \mid A_R \right) \leftarrow \left( A_0 \mid a_1 \mid A_2 \right)$

**endwhile**

**Homework 3.3.1.2** Consider the following algorithm.

**Algorithm:**  $[A] := \text{SCALE\_MATRIX\_ALTERNATIVE}(\beta, A)$

**Partition**  $A \rightarrow \begin{pmatrix} A_T \\ A_B \end{pmatrix}$

**where**  $A_T$  has 0 rows

**while**  $m(A_T) < m(A)$  **do**

**Repartition**

$$\begin{pmatrix} A_T \\ A_B \end{pmatrix} \rightarrow \begin{pmatrix} A_0 \\ \frac{a_1^T}{A_2} \end{pmatrix}$$

**where**  $a_1$  has 1 row

?????

**Continue with**

$$\begin{pmatrix} A_T \\ A_B \end{pmatrix} \leftarrow \begin{pmatrix} A_0 \\ \frac{a_1^T}{A_2} \end{pmatrix}$$

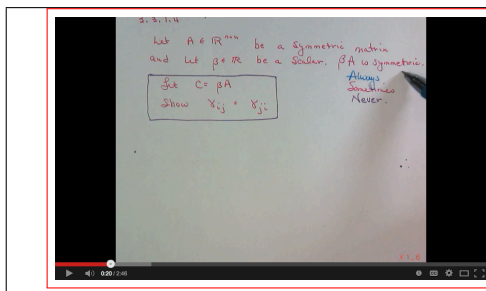
**endwhile**

What update will scale  $A$  one row at a time?

With MATLAB, when  $\beta$  is a scalar and  $A$  is a matrix, the simple command  $\beta * A$  will scale  $A$  by  $\beta$ .

**Homework 3.3.1.3** Implement function `Scale_matrix_unb( beta, A )`.

**Homework 3.3.1.4**

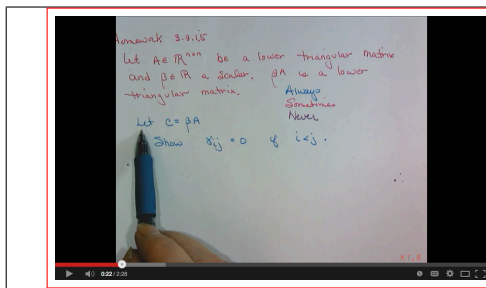


[View at edX](#)

Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix and  $\beta \in \mathbb{R}$  a scalar,  $\beta A$  is symmetric.

Always/Sometimes/Never

### Homework 3.3.1.5



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Let  $A \in \mathbb{R}^{n \times n}$  be a lower triangular matrix and  $\beta \in \mathbb{R}$  a scalar,  $\beta A$  is a lower triangular matrix.  
Always/Sometimes/Never

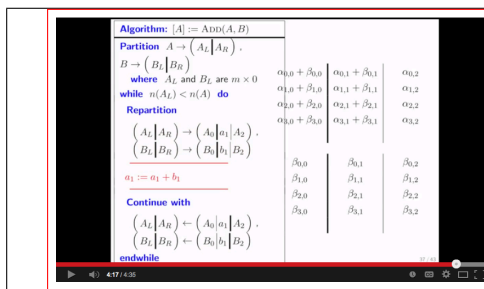
**Homework 3.3.1.6** Let  $A \in \mathbb{R}^{n \times n}$  be a diagonal matrix and  $\beta \in \mathbb{R}$  a scalar,  $\beta A$  is a diagonal matrix.

Always/Sometimes/Never

**Homework 3.3.1.7** Let  $A \in \mathbb{R}^{m \times n}$  be a matrix and  $\beta \in \mathbb{R}$  a scalar,  $(\beta A)^T = \beta A^T$ .

Always/Sometimes/Never

## 3.3.2 Adding Matrices



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**Homework 3.3.2.1** The sum of two linear transformations is a linear transformation. More formally: Let  $L_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $L_B : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be two linear transformations. Let  $L_C : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be defined by  $L_C(x) = L_A(x) + L_B(x)$ .  $L$  is a linear transformation.

Always/Sometimes/Never

Now, let  $A$ ,  $B$ , and  $C$  be the matrices that represent  $L_A$ ,  $L_B$ , and  $L_C$  in the above theorem, respectively. Then, for all  $x \in \mathbb{R}^n$ ,  $Cx = L_C(x) = L_A(x) + L_B(x)$ . What does  $c_j$ , the  $j$ th column of  $C$ , equal?

$$c_j = Ce_j = L_C(e_j) = L_A(e_j) + L_B(e_j) = Ae_j + Be_j = a_j + b_j,$$

where  $a_j$ ,  $b_j$ , and  $c_j$  equal the  $j$ th columns of  $A$ ,  $B$ , and  $C$ , respectively. Thus, the  $j$ th column of  $C$  equals the sum of the corresponding columns of  $A$  and  $B$ . That simply means that each element of  $C$  equals the sum of the corresponding elements of  $A$  and  $B$ .

If  $A, B \in \mathbb{R}^{m \times n}$ , then

$$\begin{aligned}
 A + B &= \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} + \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \\ \beta_{1,0} & \beta_{1,1} & \cdots & \beta_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \beta_{m-1,0} & \beta_{m-1,1} & \cdots & \beta_{m-1,n-1} \end{pmatrix} \\
 &= \begin{pmatrix} \alpha_{0,0} + \beta_{0,0} & \alpha_{0,1} + \beta_{0,1} & \cdots & \alpha_{0,n-1} + \beta_{0,n-1} \\ \alpha_{1,0} + \beta_{1,0} & \alpha_{1,1} + \beta_{1,1} & \cdots & \alpha_{1,n-1} + \beta_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} + \beta_{m-1,0} & \alpha_{m-1,1} + \beta_{m-1,1} & \cdots & \alpha_{m-1,n-1} + \beta_{m-1,n-1} \end{pmatrix}.
 \end{aligned}$$

Given matrices  $A, B \in \mathbb{R}^{m \times n}$ , the following algorithm adds  $B$  to  $A$ .

**Algorithm:**  $[A] := \text{ADD\_MATRICES}(A, B)$

**Partition**  $A \rightarrow \left( A_L \mid A_R \right), B \rightarrow \left( B_L \mid B_R \right)$   
**where**  $A_L$  has 0 columns,  $B_L$  has 0 columns

**while**  $n(A_L) < n(A)$  **do**

**Repartition**

$\left( A_L \mid A_R \right) \rightarrow \left( A_0 \mid a_1 \mid A_2 \right), \left( B_L \mid B_R \right) \rightarrow \left( B_0 \mid b_1 \mid B_2 \right)$   
**where**  $a_1$  has 1 column,  $b_1$  has 1 column

---

$a_1 := a_1 + b_1$  (Add the current column of  $B$  to the current column of  $A$ )

---

**Continue with**

$\left( A_L \mid A_R \right) \leftarrow \left( A_0 \mid a_1 \mid A_2 \right), \left( B_L \mid B_R \right) \leftarrow \left( B_0 \mid b_1 \mid B_2 \right)$

**endwhile**

**Homework 3.3.2.2** Consider the following algorithm.

**Algorithm:**  $[A] := \text{ADD\_MATRICES\_ALTERNATIVE}(A, B)$

**Partition**  $A \rightarrow \begin{pmatrix} A_T \\ A_B \end{pmatrix}, B \rightarrow \begin{pmatrix} B_T \\ B_B \end{pmatrix}$

**where**  $A_T$  has 0 rows,  $B_T$  has 0 rows

**while**  $m(A_T) < m(A)$  **do**

**Repartition**

$$\begin{pmatrix} A_T \\ A_B \end{pmatrix} \rightarrow \begin{pmatrix} A_0 \\ \frac{a_1^T}{A_2} \end{pmatrix}, \begin{pmatrix} B_T \\ B_B \end{pmatrix} \rightarrow \begin{pmatrix} B_0 \\ \frac{b_1^T}{B_2} \end{pmatrix}$$

**where**  $a_1$  has 1 row,  $b_1$  has 1 row

**Continue with**

$$\begin{pmatrix} A_T \\ A_B \end{pmatrix} \leftarrow \begin{pmatrix} A_0 \\ \frac{a_1^T}{A_2} \end{pmatrix}, \begin{pmatrix} B_T \\ B_B \end{pmatrix} \leftarrow \begin{pmatrix} B_0 \\ \frac{b_1^T}{B_2} \end{pmatrix}$$

**endwhile**

What update will add  $B$  to  $A$  one row at a time, overwriting  $A$  with the result?

When  $A$  and  $B$  are created as matrices of the same size, MATLAB adds two matrices with the simple command  $A + B$ . We'll just use that when we need it!

**Try this!** In MATLAB execute

```
A = [ 1, 2; 3, 4; 5, 6 ]
B = [ -1, 2; 3, -4; 5, 6 ]
C = A + B
```

**Homework 3.3.2.3** Let  $A, B \in \mathbb{R}^{m \times n}$ .  $A + B = B + A$ .

Always/Sometimes/Never

**Homework 3.3.2.4** Let  $A, B, C \in \mathbb{R}^{m \times n}$ .  $(A + B) + C = A + (B + C)$ .

Always/Sometimes/Never

**Homework 3.3.2.5** Let  $A, B \in \mathbb{R}^{m \times n}$  and  $\gamma \in \mathbb{R}$ .  $\gamma(A + B) = \gamma A + \gamma B$ .

Always/Sometimes/Never

**Homework 3.3.2.6** Let  $A \in \mathbb{R}^{m \times n}$  and  $\beta, \gamma \in \mathbb{R}$ .  $(\beta + \gamma)A = \beta A + \gamma A$ .

Always/Sometimes/Never

**Homework 3.3.2.7** Let  $A, B \in \mathbb{R}^{n \times n}$ .  $(A + B)^T = A^T + B^T$ .

Always/Sometimes/Never

**Homework 3.3.2.8** Let  $A, B \in \mathbb{R}^{n \times n}$  be symmetric matrices.  $A + B$  is symmetric.

Always/Sometimes/Never

**Homework 3.3.2.9** Let  $A, B \in \mathbb{R}^{n \times n}$  be symmetric matrices.  $A - B$  is symmetric.

Always/Sometimes/Never

**Homework 3.3.2.10** Let  $A, B \in \mathbb{R}^{n \times n}$  be symmetric matrices and  $\alpha, \beta \in \mathbb{R}$ .  $\alpha A + \beta B$  is symmetric.

Always/Sometimes/Never

**Homework 3.3.2.11** Let  $A, B \in \mathbb{R}^{n \times n}$ .

If  $A$  and  $B$  are lower triangular matrices then  $A + B$  is lower triangular.

True/False

If  $A$  and  $B$  are strictly lower triangular matrices then  $A + B$  is strictly lower triangular.

True/False

If  $A$  and  $B$  are unit lower triangular matrices then  $A + B$  is unit lower triangular.

True/False

If  $A$  and  $B$  are upper triangular matrices then  $A + B$  is upper triangular.

True/False

If  $A$  and  $B$  are strictly upper triangular matrices then  $A + B$  is strictly upper triangular.

True/False

If  $A$  and  $B$  are unit upper triangular matrices then  $A + B$  is unit upper triangular.

True/False

**Homework 3.3.2.12** Let  $A, B \in \mathbb{R}^{n \times n}$ .

If  $A$  and  $B$  are lower triangular matrices then  $A - B$  is lower triangular.

True/False

If  $A$  and  $B$  are strictly lower triangular matrices then  $A - B$  is strictly lower triangular. True/False

If  $A$  and  $B$  are unit lower triangular matrices then  $A - B$  is *strictly* lower triangular. True/False

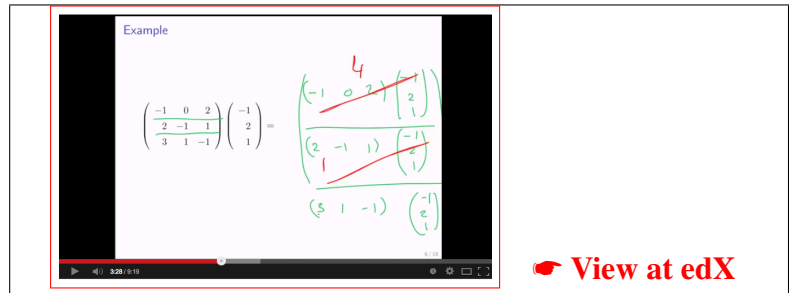
If  $A$  and  $B$  are upper triangular matrices then  $A - B$  is upper triangular. True/False

If  $A$  and  $B$  are strictly upper triangular matrices then  $A - B$  is strictly upper triangular. True/False

If  $A$  and  $B$  are unit upper triangular matrices then  $A - B$  is unit upper triangular. True/False

## 3.4 Matrix-Vector Multiplication Algorithms

### 3.4.1 Via Dot Products



#### Motivation

Recall that if  $y = Ax$ , where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ , and  $y \in \mathbb{R}^m$ , then

$$y = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{m-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0}\chi_0 + \alpha_{0,1}\chi_1 + \cdots + \alpha_{0,n-1}\chi_{n-1} \\ \alpha_{1,0}\chi_0 + \alpha_{1,1}\chi_1 + \cdots + \alpha_{1,n-1}\chi_{n-1} \\ \vdots \\ \alpha_{m-1,0}\chi_0 + \alpha_{m-1,1}\chi_1 + \cdots + \alpha_{m-1,n-1}\chi_{n-1} \end{pmatrix}.$$

If one looks at a typical row,

$$\alpha_{i,0}\chi_0 + \alpha_{i,1}\chi_1 + \cdots + \alpha_{i,n-1}\chi_{n-1}$$

one notices that this is just the dot product of vectors

$$\tilde{a}_i = \begin{pmatrix} \alpha_{i,0} \\ \alpha_{i,1} \\ \vdots \\ \alpha_{i,n-1} \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}.$$

In other words, the dot product of the  $i$ th row of  $A$ , viewed as a column vector, with the vector  $x$ , which one can visualize as

$$\begin{pmatrix} \psi_0 \\ \vdots \\ \psi_i \\ \vdots \\ \psi_{m-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{i,0} & \alpha_{i,1} & \cdots & \alpha_{i,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$$

The above argument starts to explain why we write the dot product of vectors  $x$  and  $y$  as  $x^T y$ .



**Example 3.7** Let  $A = \begin{pmatrix} -1 & 0 & 2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$  and  $x = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ . Then

$$\begin{aligned}
 Ax &= \begin{pmatrix} -1 & 0 & 2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \end{pmatrix} \\
 &= \begin{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}^T \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} (-1)(-1) + (0)(2) + (2)(1) \\ (2)(-1) + (-1)(2) + (1)(1) \\ (3)(-1) + (1)(2) + (-1)(1) \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -2 \end{pmatrix}
 \end{aligned}$$

#### Algorithm (traditional notation)

An algorithm for computing  $y := Ax + y$  (notice that we add the result of  $Ax$  to  $y$ ) via dot products is given by

```

for  $i = 0, \dots, m-1$ 
  for  $j = 0, \dots, n-1$ 
     $\Psi_i := \Psi_i + \alpha_{i,j} \chi_j$ 
  endfor
endfor

```

If initially  $y = 0$ , then it computes  $y := Ax$ .

Now, let us revisit the fact that the matrix-vector multiply can be computed as dot products of the rows of  $A$  with the vector  $x$ . Think of the matrix  $A$  as individual rows:

$$A = \begin{pmatrix} \tilde{a}_0^T \\ \tilde{a}_1^T \\ \vdots \\ \tilde{a}_{m-1}^T \end{pmatrix},$$

where  $\tilde{a}_i$  is the (column) vector which, when transposed, becomes the  $i$ th row of the matrix. Then

$$Ax = \begin{pmatrix} \tilde{a}_0^T \\ \tilde{a}_1^T \\ \vdots \\ \tilde{a}_{m-1}^T \end{pmatrix} x = \begin{pmatrix} \tilde{a}_0^T x \\ \tilde{a}_1^T x \\ \vdots \\ \tilde{a}_{m-1}^T x \end{pmatrix},$$

which is exactly what we reasoned before. To emphasize this, the algorithm can then be annotated as follows:

```

for  $i = 0, \dots, m-1$ 
    for  $j = 0, \dots, n-1$ 
         $\psi_i := \psi_i + \alpha_{i,j} \chi_j$ 
    endfor
endfor
    
```

$$\left. \vphantom{\begin{matrix} \text{for } i = 0, \dots, m-1 \\ \text{for } j = 0, \dots, n-1 \\ \psi_i := \psi_i + \alpha_{i,j} \chi_j \\ \text{endfor} \end{matrix}} \right\} \psi_i := \psi_i + \tilde{a}_i^T x$$

### Algorithm (FLAME notation)

We now present the algorithm that casts matrix-vector multiplication in terms of dot products using the FLAME notation with which you became familiar earlier this week:

**Algorithm:**  $y := \text{MVMULT\_N\_UNB\_VAR1}(A, x, y)$

**Partition**  $A \rightarrow \begin{pmatrix} A_T \\ A_B \end{pmatrix}, y \rightarrow \begin{pmatrix} y_T \\ y_B \end{pmatrix}$

**where**  $A_T$  is  $0 \times n$  and  $y_T$  is  $0 \times 1$

**while**  $m(A_T) < m(A)$  **do**

**Repartition**

$$\begin{pmatrix} A_T \\ A_B \end{pmatrix} \rightarrow \begin{pmatrix} A_0 \\ a_1^T \\ A_2 \end{pmatrix}, \begin{pmatrix} y_T \\ y_B \end{pmatrix} \rightarrow \begin{pmatrix} y_0 \\ \psi_1 \\ y_2 \end{pmatrix}$$

**where**  $a_1$  is a row

---


$$\psi_1 := a_1^T x + \psi_1$$


---

**Continue with**

$$\begin{pmatrix} A_T \\ A_B \end{pmatrix} \leftarrow \begin{pmatrix} A_0 \\ a_1^T \\ A_2 \end{pmatrix}, \begin{pmatrix} y_T \\ y_B \end{pmatrix} \leftarrow \begin{pmatrix} y_0 \\ \psi_1 \\ y_2 \end{pmatrix}$$

**endwhile**

**Homework 3.4.1.1** Implement function `Mvmult_n_unb_var1( A, x, y )`.

### 3.4.2 Via AXPY Operations

for  $j = 0, \dots, n-1$   
 for  $i = 0, \dots, m-1$   
 $y_j := y_j + a_{i,j} x_i$   
 endfor  
endfor

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### Motivation

Note that, by definition,

$$Ax = \begin{pmatrix} \alpha_{0,0}\chi_0 + \alpha_{0,1}\chi_1 + \cdots + \alpha_{0,n-1}\chi_{n-1} \\ \alpha_{1,0}\chi_0 + \alpha_{1,1}\chi_1 + \cdots + \alpha_{1,n-1}\chi_{n-1} \\ \vdots \\ \alpha_{m-1,0}\chi_0 + \alpha_{m-1,1}\chi_1 + \cdots + \alpha_{m-1,n-1}\chi_{n-1} \end{pmatrix} =$$

$$\chi_0 \begin{pmatrix} \alpha_{0,0} \\ \alpha_{1,0} \\ \vdots \\ \alpha_{m-1,0} \end{pmatrix} + \chi_1 \begin{pmatrix} \alpha_{0,1} \\ \alpha_{1,1} \\ \vdots \\ \alpha_{m-1,1} \end{pmatrix} + \cdots + \chi_{n-1} \begin{pmatrix} \alpha_{0,n-1} \\ \alpha_{1,n-1} \\ \vdots \\ \alpha_{m-1,n-1} \end{pmatrix}.$$

**Example 3.8** Let  $A = \begin{pmatrix} -1 & 0 & 2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$  and  $x = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ . Then

$$\begin{aligned} Ax &= \begin{pmatrix} -1 & 0 & 2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + (2) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + (1) \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} (-1)(-1) \\ (-1)(2) \\ (-1)(3) \end{pmatrix} + \begin{pmatrix} (2)(0) \\ (2)(-1) \\ (2)(1) \end{pmatrix} + \begin{pmatrix} (1)(2) \\ (1)(1) \\ (1)(-1) \end{pmatrix} \\ &= \begin{pmatrix} (-1)(-1) + (0)(2) + (2)(1) \\ (2)(-1) + (-1)(2) + (1)(1) \\ (3)(-1) + (1)(2) + (-1)(1) \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -2 \end{pmatrix} \end{aligned}$$

### Algorithm (traditional notation)

The above suggests the alternative algorithm for computing  $y := Ax + y$  given by

```

for  $j = 0, \dots, n-1$ 
  for  $i = 0, \dots, m-1$ 
     $\psi_i := \psi_i + \alpha_{i,j} \chi_j$ 
  endfor
endfor

```

If we let  $a_j$  denote the vector that equals the  $j$ th column of  $A$ , then

$$A = \left( a_0 \mid a_1 \mid \cdots \mid a_{n-1} \right)$$

and

$$\begin{aligned}
 Ax &= \chi_0 \underbrace{\begin{pmatrix} \alpha_{0,0} \\ \alpha_{1,0} \\ \vdots \\ \alpha_{m-1,0} \end{pmatrix}}_{a_0} + \chi_1 \underbrace{\begin{pmatrix} \alpha_{0,1} \\ \alpha_{1,1} \\ \vdots \\ \alpha_{m-1,1} \end{pmatrix}}_{a_1} + \cdots + \chi_{n-1} \underbrace{\begin{pmatrix} \alpha_{0,n-1} \\ \alpha_{1,n-1} \\ \vdots \\ \alpha_{m-1,n-1} \end{pmatrix}}_{a_{n-1}} \\
 &= \chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}.
 \end{aligned}$$

This is emphasized by annotating the algorithm as follows:

```

for  $j = 0, \dots, n-1$ 
  for  $i = 0, \dots, m-1$ 
     $\psi_i := \psi_i + \alpha_{i,j} \chi_j$ 
  endfor
endfor

```

$$\left. \vphantom{\begin{array}{l} \text{for } j = 0, \dots, n-1 \\ \text{for } i = 0, \dots, m-1 \\ \psi_i := \psi_i + \alpha_{i,j} \chi_j \\ \text{endfor} \end{array}} \right\} y := \chi_j a_j + y$$

#### Algorithm (FLAME notation)

Here is the algorithm that casts matrix-vector multiplication in terms of AXPYs using the FLAME notation:

**Algorithm:**  $y := \text{MVMULT\_N\_UNB\_VAR2}(A, x, y)$

**Partition**  $A \rightarrow \left( A_L \mid A_R \right), x \rightarrow \begin{pmatrix} x_T \\ x_B \end{pmatrix}$

**where**  $A_L$  is  $m \times 0$  and  $x_T$  is  $0 \times 1$

**while**  $m(x_T) < m(x)$  **do**

**Repartition**

$\left( A_L \mid A_R \right) \rightarrow \left( A_0 \mid a_1 \mid A_2 \right), \begin{pmatrix} x_T \\ x_B \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ \frac{\chi_1}{x_2} \end{pmatrix}$

**where**  $a_1$  is a column

$y := \chi_1 a_1 + y$

**Continue with**

$\left( A_L \mid A_R \right) \leftarrow \left( A_0 \mid a_1 \mid A_2 \right), \begin{pmatrix} x_T \\ x_B \end{pmatrix} \leftarrow \begin{pmatrix} x_0 \\ \frac{\chi_1}{x_2} \end{pmatrix}$

**endwhile**

**Homework 3.4.2.1** Implement function `Mvmult_n_unb_var2( A, x, y )`.  
(Hint: use the function `laff_dots( x, y, alpha )` that updates  $\alpha := x^T y + \alpha$ .)

### 3.4.3 Compare and Contrast

$$\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} + := \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} \\ \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} \\ \alpha_{3,0} & \alpha_{3,1} & \alpha_{3,2} \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}$$

$\rightarrow$  for  $i = 0, \dots, m-1$   $\rightarrow$  for  $j = 0, \dots, n-1$   
 $\rightarrow$  for  $j = 0, \dots, n-1$   $\rightarrow$  for  $i = 0, \dots, m-1$   
 $\psi_i := \psi_i + \alpha_{i,j} \chi_j$   $\psi_j := \psi_j + \alpha_{i,j} \chi_i$   
 endfor endfor endfor endfor

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### Motivation

It is always useful to compare and contrast different algorithms for the same operation.

**Algorithms (traditional notation)**

Let us put the two algorithms that compute  $y := Ax + y$  via “double nested loops” next to each other:

<pre> <b>for</b> <math>j = 0, \dots, n - 1</math>   <b>for</b> <math>i = 0, \dots, m - 1</math>     <math>\psi_i := \psi_i + \alpha_{i,j} \chi_j</math>   <b>endfor</b> <b>endfor</b> </pre>	<pre> <b>for</b> <math>i = 0, \dots, m - 1</math>   <b>for</b> <math>j = 0, \dots, n - 1</math>     <math>\psi_i := \psi_i + \alpha_{i,j} \chi_j</math>   <b>endfor</b> <b>endfor</b> </pre>
--	--

On the left is the algorithm based on the dot product and on the right the one based on AXPY operations. Notice that these loops differ only in that the order of the two loops are interchanged. This is known as “interchanging loops” and is sometimes used by compilers to optimize nested loops. In the enrichment section of this week we will discuss why you may prefer one ordering of the loops over another.

The above explains, in part, why we chose to look at  $y := Ax + y$  rather than  $y := Ax$ . For  $y := Ax + y$ , the two algorithms differ only in the order in which the loops appear. To compute  $y := Ax$ , one would have to initialize each component of  $y$  to zero,  $\psi_i := 0$ . Depending on where in the algorithm that happens, transforming an algorithm that computes  $y := Ax$  elements of  $y$  at a time (the inner loop implements a dot product) into an algorithm that computes with columns of  $A$  (the inner loop implements an AXPY operation) gets trickier.

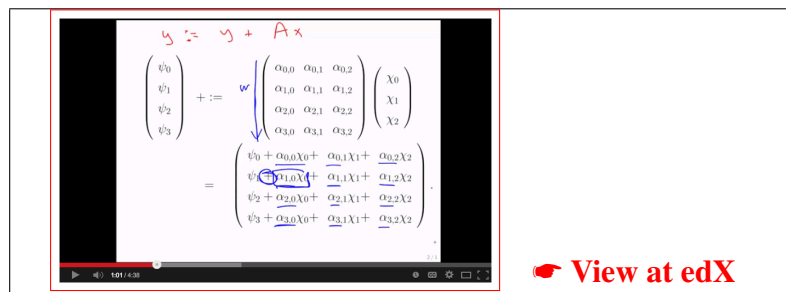
**Algorithms (FLAME notation)**

Now let us place the two algorithms presented using the FLAME notation next to each other:

<b>Algorithm:</b> $y := \text{MVMULT\_N\_UNB\_VAR1}(A, x, y)$	<b>Algorithm:</b> $y := \text{MVMULT\_N\_UNB\_VAR2}(A, x, y)$
<p><b>Partition</b> <math>A \rightarrow \begin{pmatrix} A_T \\ A_B \end{pmatrix}, y \rightarrow \begin{pmatrix} y_T \\ y_B \end{pmatrix}</math></p> <p><b>where</b> <math>A_T</math> is <math>0 \times n</math> and <math>y_T</math> is <math>0 \times 1</math></p> <p><b>while</b> <math>m(A_T) &lt; m(A)</math> <b>do</b></p> <p><b>Repartition</b></p> $\begin{pmatrix} A_T \\ A_B \end{pmatrix} \rightarrow \begin{pmatrix} A_0 \\ a_1^T \\ A_2 \end{pmatrix}, \begin{pmatrix} y_T \\ y_B \end{pmatrix} \rightarrow \begin{pmatrix} y_0 \\ \psi_1 \\ y_2 \end{pmatrix}$ <hr style="border: 0.5px solid red;"/> $\psi_1 := a_1^T x + \psi_1$ <hr style="border: 0.5px solid red;"/> <p><b>Continue with</b></p> $\begin{pmatrix} A_T \\ A_B \end{pmatrix} \leftarrow \begin{pmatrix} A_0 \\ a_1^T \\ A_2 \end{pmatrix}, \begin{pmatrix} y_T \\ y_B \end{pmatrix} \leftarrow \begin{pmatrix} y_0 \\ \psi_1 \\ y_2 \end{pmatrix}$ <p><b>endwhile</b></p>	<p><b>Partition</b> <math>A \rightarrow (A_L   A_R), x \rightarrow \begin{pmatrix} x_T \\ x_B \end{pmatrix}</math></p> <p><b>where</b> <math>A_L</math> is <math>m \times 0</math> and <math>x_T</math> is <math>0 \times 1</math></p> <p><b>while</b> <math>m(x_T) &lt; m(x)</math> <b>do</b></p> <p><b>Repartition</b></p> $(A_L   A_R) \rightarrow (A_0   a_1   A_2), \begin{pmatrix} x_T \\ x_B \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ \chi_1 \\ x_2 \end{pmatrix}$ <hr style="border: 0.5px solid red;"/> $y := \chi_1 a_1 + y$ <hr style="border: 0.5px solid red;"/> <p><b>Continue with</b></p> $(A_L   A_R) \leftarrow (A_0   a_1   A_2), \begin{pmatrix} x_T \\ x_B \end{pmatrix} \leftarrow \begin{pmatrix} x_0 \\ \chi_1 \\ x_2 \end{pmatrix}$ <p><b>endwhile</b></p>

The algorithm on the left clearly accesses the matrix by rows while the algorithm on the right accesses it by columns. Again, this is important to note, and will be discussed in enrichment for this week.

### 3.4.4 Cost of Matrix-Vector Multiplication



Consider  $y := Ax + y$ , where  $A \in \mathbb{R}^{m \times n}$ :

$$y = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{m-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0}\chi_0 + \alpha_{0,1}\chi_1 + \cdots + \alpha_{0,n-1}\chi_{n-1} + \psi_0 \\ \alpha_{1,0}\chi_0 + \alpha_{1,1}\chi_1 + \cdots + \alpha_{1,n-1}\chi_{n-1} + \psi_1 \\ \vdots \\ \alpha_{m-1,0}\chi_0 + \alpha_{m-1,1}\chi_1 + \cdots + \alpha_{m-1,n-1}\chi_{n-1} + \psi_{m-1} \end{pmatrix}.$$

Notice that there is a multiply and an add for every element of  $A$ . Since  $A$  has  $m \times n = mn$  elements,  $y := Ax + y$ , requires  $mn$  multiplies and  $mn$  adds, for a total of  $2mn$  floating point operations (flops). This count is the same regardless of the order of the loops (i.e., regardless of whether the matrix-vector multiply is organized by computing dot operations with the rows or axpy operations with the columns).

## 3.5 Wrap Up

### 3.5.1 Homework

No additional homework this week. You have done enough...



### 3.5.2 Summary

#### Special Matrices

Name	Represents linear transformation	Has entries
Zero matrix, $0_{m \times n} \in \mathbb{R}^{m \times n}$	$L_0 : \mathbb{R}^n \rightarrow \mathbb{R}^m$ $L_0(x) = 0$ for all $x$	$0 = 0_{m \times n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$
Identity matrix, $I \in \mathbb{R}^{n \times n}$	$L_I : \mathbb{R}^n \rightarrow \mathbb{R}^n$ $L_I(x) = x$ for all $x$	$I = I_{n \times n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$
Diagonal matrix, $D \in \mathbb{R}^{n \times n}$	$L_D : \mathbb{R}^n \rightarrow \mathbb{R}^n$ if $y = L_D(x)$ then $\psi_i = \delta_i \chi_i$	$D = \begin{pmatrix} \delta_0 & 0 & \cdots & 0 \\ 0 & \delta_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_{n-1} \end{pmatrix}$

# Triangular matrices

$A \in \mathbb{R}^{n \times n}$ is said to be...	if ...	
<i>lower</i> triangular	$\alpha_{i,j} = 0$ if $i < j$	$\begin{pmatrix} \alpha_{0,0} & 0 & \cdots & 0 & 0 \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & \alpha_{n-2,n-2} & 0 \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & \alpha_{n-1,n-1} \end{pmatrix}$
<i>strictly</i> lower triangular	$\alpha_{i,j} = 0$ if $i \leq j$	$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \alpha_{1,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & 0 & 0 \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & 0 \end{pmatrix}$
<i>unit</i> lower triangular	$\alpha_{i,j} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \end{cases}$	$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \alpha_{1,0} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & 1 & 0 \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & 1 \end{pmatrix}$
<i>upper</i> triangular	$\alpha_{i,j} = 0$ if $i > j$	$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ 0 & \alpha_{1,1} & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_{n-2,n-2} & \alpha_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & \alpha_{n-1,n-1} \end{pmatrix}$
<i>strictly</i> upper triangular	$\alpha_{i,j} = 0$ if $i \geq j$	$\begin{pmatrix} 0 & \alpha_{0,1} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ 0 & 0 & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$
<i>unit</i> upper triangular	$\alpha_{i,j} = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \end{cases}$	$\begin{pmatrix} 1 & \alpha_{0,1} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ 0 & 1 & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$

**Transpose matrix**

$$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{m-2,0} & \alpha_{m-2,1} & \cdots & \alpha_{m-2,n-2} & \alpha_{m-2,n-1} \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-2} & \alpha_{m-1,n-1} \end{pmatrix}^T = \begin{pmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdots & \alpha_{m-2,0} & \alpha_{m-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{m-2,1} & \alpha_{m-1,1} \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{0,n-2} & \alpha_{1,n-2} & \cdots & \alpha_{m-2,n-2} & \alpha_{m-1,n-2} \\ \alpha_{0,n-1} & \alpha_{1,n-1} & \cdots & \alpha_{m-2,n-1} & \alpha_{m-1,n-1} \end{pmatrix}$$

**Symmetric matrix**

Matrix  $A \in \mathbb{R}^{n \times n}$  is symmetric if and only if  $A = A^T$ :

$$A = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n-2,0} & \alpha_{n-2,1} & \cdots & \alpha_{n-2,n-2} & \alpha_{n-2,n-1} \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-2} & \alpha_{n-1,n-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdots & \alpha_{n-2,0} & \alpha_{n-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{n-2,1} & \alpha_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{0,n-2} & \alpha_{1,n-2} & \cdots & \alpha_{n-2,n-2} & \alpha_{n-1,n-2} \\ \alpha_{0,n-1} & \alpha_{1,n-1} & \cdots & \alpha_{n-2,n-1} & \alpha_{n-1,n-1} \end{pmatrix} = A^T$$

**Scaling a matrix**

Let  $\beta \in \mathbb{R}$  and  $A \in \mathbb{R}^{m \times n}$ . Then

$$\begin{aligned} \beta A &= \beta \left( a_0 \mid a_1 \mid \cdots \mid a_{n-1} \right) = \left( \beta a_0 \mid \beta a_1 \mid \cdots \mid \beta a_{n-1} \right) \\ &= \beta \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} = \begin{pmatrix} \beta \alpha_{0,0} & \beta \alpha_{0,1} & \cdots & \beta \alpha_{0,n-1} \\ \beta \alpha_{1,0} & \beta \alpha_{1,1} & \cdots & \beta \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \beta \alpha_{m-1,0} & \beta \alpha_{m-1,1} & \cdots & \beta \alpha_{m-1,n-1} \end{pmatrix} \end{aligned}$$

**Adding matrices**

Let  $A, B \in \mathbb{R}^{m \times n}$ . Then

$$\begin{aligned} A + B &= \left( a_0 \mid a_1 \mid \cdots \mid a_{n-1} \right) + \left( b_0 \mid b_1 \mid \cdots \mid b_{n-1} \right) = \left( a_0 + b_0 \mid a_1 + b_1 \mid \cdots \mid a_{n-1} + b_{n-1} \right) \\ &= \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} + \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \\ \beta_{1,0} & \beta_{1,1} & \cdots & \beta_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \beta_{m-1,0} & \beta_{m-1,1} & \cdots & \beta_{m-1,n-1} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \alpha_{0,0} + \beta_{0,0} & \alpha_{0,1} + \beta_{0,1} & \cdots & \alpha_{0,n-1} + \beta_{0,n-1} \\ \alpha_{1,0} + \beta_{1,0} & \alpha_{1,1} + \beta_{1,1} & \cdots & \alpha_{1,n-1} + \beta_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} + \beta_{m-1,0} & \alpha_{m-1,1} + \beta_{m-1,1} & \cdots & \alpha_{m-1,n-1} + \beta_{m-1,n-1} \end{pmatrix}$$

- Matrix addition commutes:  $A + B = B + A$ .
- Matrix addition is associative:  $(A + B) + C = A + (B + C)$ .
- $(A + B)^T = A^T + B^T$ .

### Matrix-vector multiplication

$$\begin{aligned} Ax &= \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0}\chi_0 + \alpha_{0,1}\chi_1 + \cdots + \alpha_{0,n-1}\chi_{n-1} \\ \alpha_{1,0}\chi_0 + \alpha_{1,1}\chi_1 + \cdots + \alpha_{1,n-1}\chi_{n-1} \\ \vdots \\ \alpha_{m-1,0}\chi_0 + \alpha_{m-1,1}\chi_1 + \cdots + \alpha_{m-1,n-1}\chi_{n-1} \end{pmatrix} \\ &= \left( a_0 \mid a_1 \mid \cdots \mid a_{n-1} \right) \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1} \\ &= \begin{pmatrix} \tilde{a}_0^T \\ \tilde{a}_1^T \\ \vdots \\ \tilde{a}_{m-1}^T \end{pmatrix} x = \begin{pmatrix} \tilde{a}_0^T x \\ \tilde{a}_1^T x \\ \vdots \\ \tilde{a}_{m-1}^T x \end{pmatrix} \end{aligned}$$

