Visualizing the Solution of a System with Three Equations in Three Unknowns

From the opener for Week 9:

$$\begin{array}{rcrcrcr}
\chi_0 & - & 2\chi_1 & + & 4\chi_2 & = & -1 \\
\chi_0 & & & & & = & 2 \\
\chi_0 & + & 2\chi_1 & + & 4\chi_2 & = & 3
\end{array}$$

From the opener for Week 9:

$$\begin{array}{rcrcrcr} \chi_0 & - & 2\chi_1 & + & 4\chi_2 & = & -1 \\ \chi_0 & & & & & = & 2 \\ \chi_0 & + & 2\chi_1 & + & 4\chi_2 & = & 3 \end{array}$$

Solution:

$$\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -0.25 \end{pmatrix}$$

From the opener for Week 9:

$$\begin{array}{rcrcrcr} \chi_0 & - & 2\chi_1 & + & 4\chi_2 & = & -1 \\ \chi_0 & & & & & = & 2 \\ \chi_0 & + & 2\chi_1 & + & 4\chi_2 & = & 3 \end{array}$$

Solution:

$$\left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \chi_2 \end{array}\right) = \left(\begin{array}{r} 2 \\ 1 \\ -0.25 \end{array}\right)$$

Important:

$$\left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \chi_2 \end{array}\right) = \left(\begin{array}{r} 2 \\ 1 \\ -0.25 \end{array}\right)$$

solves each of the individual equations.

Visualize a plane of solutions corresponding to a single linear equation

$$\chi_0 - 2\chi_1 + 4\chi_2 = -1$$

As an appended system:

$$\begin{array}{cccc} \boxed{1} & -2 & 4 & -1 \\ \uparrow & \uparrow & \uparrow & \\ \text{dependent variable} & \text{free variable} & \text{free variable} & \end{array}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & -2 & 4 & -1 \end{array}\right).$$

Identify a specific solution

$$\chi_0 - 2\chi_1 + 4\chi_2 = -1$$

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$$x_s = \left(\begin{array}{c} \chi_0 \\ 0 \\ 0 \end{array} \right) \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify a specific solution

$$\chi_0 - 2\chi_1 + 4\chi_2 = -1$$

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Substituting $\chi_1 = \chi_2 = 0$ into the equation gives us

$$\chi_0 - 2(0) + 4(0) = -1.$$

Identify a specific solution

$$\chi_0 - 2\chi_1 + 4\chi_2 = -1$$

$$x_s = \begin{pmatrix} \chi_0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

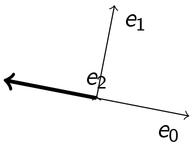
Substituting $\chi_1 = \chi_2 = 0$ into the equation gives us

$$\chi_0 - 2(0) + 4(0) = -1.$$

or $\chi_0 = -1$ so that

$$x_s = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Visualizing the specific solution



Identifying a basis in the null space

$$\chi_0 - 2\chi_1 + 4\chi_2 = 0.$$

Identifying a basis in the null space

$$\chi_0 - 2\chi_1 + 4\chi_2 = 0.$$

If we write this as an appended system, we get

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & -2 & 4 & 0 \end{array}\right) \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{dependent variable} & \text{free variable} & \text{free variable} \end{array} \end{array}$$

Identify linearly independent vectors in the null space

$$\chi_0 - 2\chi_1 + 4\chi_2 = 0.$$

Identify linearly independent vectors in the null space

$$\chi_0 - 2\chi_1 + 4\chi_2 = 0.$$

We look for solutions in the form

$$x_{n_0} = \begin{pmatrix} \chi_0 \\ 1 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

and

$$x_{n_1} = \begin{pmatrix} \chi_0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify linearly independent vectors in the null space

$$\chi_0 - 2\chi_1 + 4\chi_2 = 0.$$

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$$x_{n_0} = \left(\begin{array}{c} \chi_0 \\ 1 \\ 0 \end{array} \right) \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify linearly independent vectors in the null space

$$\chi_0 - 2\chi_1 + 4\chi_2 = 0.$$

$$x_{n_0} = \begin{pmatrix} \chi_0 \\ 1 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Substituting $\chi_1 = 1$ and $\chi_2 = 0$ into the equation gives us

$$\chi_0 - 2(1) + 4(0) = 0.$$

or $\chi_0 = 2$.

Identify linearly independent vectors in the null space

$$\chi_0 - 2\chi_1 + 4\chi_2 = 0.$$

Identify linearly independent vectors in the null space

$$\chi_0 - 2\chi_1 + 4\chi_2 = 0.$$

$$x_{n_1} = \begin{pmatrix} \chi_0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify linearly independent vectors in the null space

$$\chi_0 - 2\chi_1 + 4\chi_2 = 0.$$

$$x_{n_1} = \begin{pmatrix} \chi_0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Substituting $\chi_1 = 0$ and $\chi_2 = 1$ into the equation gives us

$$\chi_0 - 2(0) + 4(1) = 0.$$

or $\chi_0 = -4$.

Identify linearly independent vectors in the null space

Thus two (nonunique) linearly independent vectors in the null space are given by

$$x_{n_0} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad x_{n_1} = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}.$$

Identify linearly independent vectors in the null space

Thus two (nonunique) linearly independent vectors in the null space are given by

$$x_{n_0} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad x_{n_1} = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}.$$

They are linearly independent:

$$If \begin{pmatrix} 2 & -4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Identify linearly independent vectors in the null space

Thus two (nonunique) linearly independent vectors in the null space are given by

$$x_{n_0} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad x_{n_1} = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}.$$

They are linearly independent:

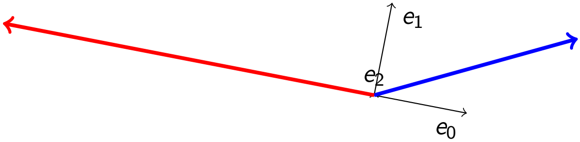
$$\mathit{If} \begin{pmatrix} 2 & -4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

then

$$\mathit{If} \left(\begin{array}{cc} 2 & -4 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right) \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{array}{c} \begin{pmatrix} 2 & -4 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array}$$

Thus, $\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

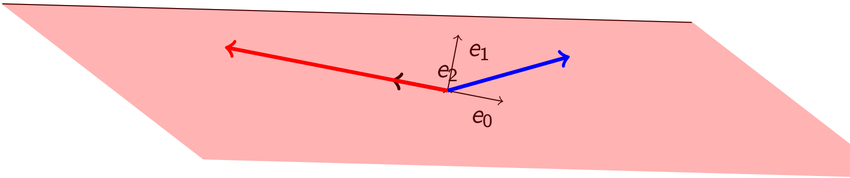
Visualizing the vectors in the null space



Towards visualizing the plane of solutions

$$x_{\mathrm{general}} = x_s + \alpha x_{n_0} + \beta x_{n_1}$$

Visualizing the plane of solutions



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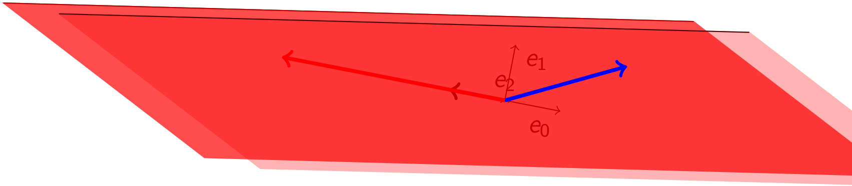
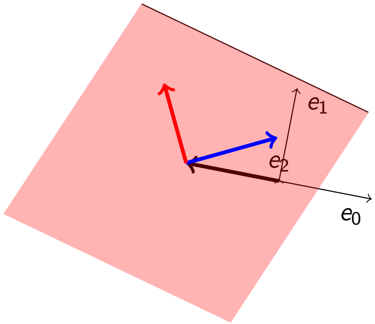
Visualizing the plane of solutions



Visualizing the vectors in the null space



Creating orthogonal vectors (later this week)

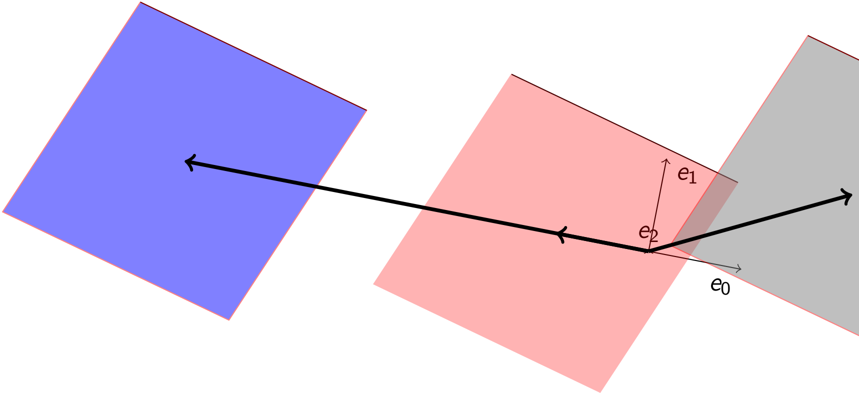
Orthogonal (perpendicular) is better!

$$q_0 = x_{n_0}/\|x_{n_0}\|_2 \quad \text{and} \quad q_1 = x_{n_1}^{\perp}/\|x_{n_1}^{\perp}\|_2,$$

where $x_{n_1}^{\perp}$ equals the component of x_{n_1} orthogonal to q_0. We compute this using the Gram-Schmidt algorithm:

- $\rho_{0,0} := \|x_{n_0}\|_2$.
- $q_0 := x_{n_0}/\rho_{0,0}$.
- $\rho_{0,1} := q_0^T x_{n_1}$. $x_{n_1}^{\perp} := x_{n_1} - \rho_{0,1} q_0$.
- $\rho_{1,1} := \|x_{n_1}^{\perp}\|_2$.
- $q_1 := x_{n_1}^{\perp}/\rho_{1,1}$.

Visualizing the vectors in the null space



Visualize a plane of solutions corresponding to a single linear equation

New equation:

$$\chi_0 + (0)\chi_1 + (0)\chi_2 = 2$$

As an appended system:

$$\begin{array}{cccc} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \end{array}\right) \\ \uparrow & \uparrow & \uparrow & \\ \text{dependent variable} & \text{free variable} & \text{free variable} & \end{array}$$

Identify a specific solution

$$\chi_0 + (0)\chi_1 + (0)\chi_2 = 2$$

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$$x_s = \begin{pmatrix} \chi_0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify a specific solution

$$\chi_0 + (0)\chi_1 + (0)\chi_2 = 2$$

$$x_s = \left(\begin{array}{c} \chi_0 \\ 0 \\ 0 \end{array} \right) \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Substituting $\chi_1 = \chi_2 = 0$ into the equation gives us

$$\chi_0 + (0)(0) + (0)(0) = 2.$$

Identify a specific solution

$$\chi_0 + (0)\chi_1 + (0)\chi_2 = 2$$

$$x_s = \begin{pmatrix} \chi_0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

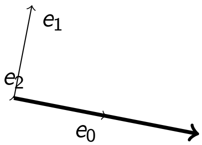
Substituting $\chi_1 = \chi_2 = 0$ into the equation gives us

$$\chi_0 + (0)(0) + (0)(0) = 2.$$

or $\chi_0 = 2$ so that

$$x_s = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

Drawing the specific solution



Identify linearly independent vectors in the null space

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We look for solutions in the form

$$x_{n_0} = \left(\begin{array}{c} \chi_0 \\ 1 \\ 0 \end{array} \right) \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

and

$$x_{n_1} = \left(\begin{array}{c} \chi_0 \\ 0 \\ 1 \end{array} \right) \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

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Identify linearly independent vectors in the null space

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$$x_{n_0} = \begin{pmatrix} \chi_0 \\ 1 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Substituting $\chi_1 = 1$ and $\chi_2 = 0$ into the equation gives us

$$\chi_0 + (0)(0)(1) + (0)(0) = 0.$$

or $\chi_0 = 0$.

Identify linearly independent vectors in the null space

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Identify linearly independent vectors in the null space

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Substituting $\chi_1 = 0$ and $\chi_2 = 1$ into the equation gives us

$$\chi_0 + (0)(0) + (0)(1) = 0.$$

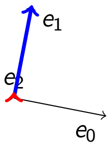
or $\chi_0 = 0$.

Identify linearly independent vectors in the null space

Thus two (nonunique) linearly independent vectors in the null space are given by

$$x_{n_0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad x_{n_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

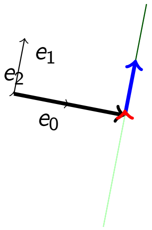
Visualizing the vectors in the null space



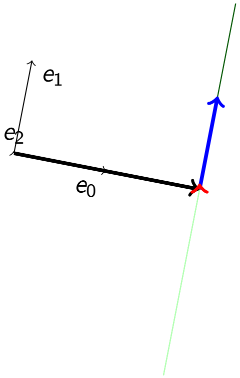
Towards visualizing the plane of solutions

$$x_{\text{general}} = x_s + \alpha x_{n_0} + \beta x_{n_1}$$

Visualizing the plane of solutions



Visualizing the vectors in the null space



Creating orthogonal vectors (later this week)

Orthogonal (perpendicular) is better!

$$q_0 = x_{n_0} / \|x_{n_0}\|_2 \quad \text{and} \quad q_1 = x_{n_1}^{\perp} / \|x_{n_1}^{\perp}\|_2,$$

where $x_{n_1}^{\perp}$ equals the component of x_{n_1} orthogonal to q_0. We compute this using the Gram-Schmidt algorithm:

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- $\rho_{1,1} := \|x_{n_1}^{\perp}\|_2$.
- $q_1 := x_{n_1}^{\perp} / \rho_{1,1}$.

Visualize a plane of solutions corresponding to a single linear equation

$$\chi_0 + 2\chi_1 + 4\chi_2 = 3$$

As an appended system:

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \end{array}\right) \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{dependent variable} & \text{free variable} & \text{free variable} \end{array} \end{array}$$

Identify a specific solution

$$\chi_0 + 2\chi_1 + 4\chi_2 = 3$$

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$$\chi_0 + 2\chi_1 + 4\chi_2 = 3$$

$$x_s = \begin{pmatrix} \chi_0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify a specific solution

$$\chi_0 + 2\chi_1 + 4\chi_2 = 3$$

$$x_s = \left(\begin{array}{c} \chi_0 \\ 0 \\ 0 \end{array} \right) \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Substituting $\chi_1 = \chi_2 = 0$ into the equation gives us

$$\chi_0 + 2(0) + 4(0) = 3.$$

Identify a specific solution

$$\chi_0 + 2\chi_1 + 4\chi_2 = 3$$

$$x_s = \left(\begin{array}{c} \chi_0 \\ 0 \\ 0 \end{array} \right) \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

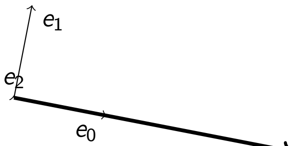
Substituting $\chi_1 = \chi_2 = 0$ into the equation gives us

$$\chi_0 + 2(0) + 4(0) = 3.$$

or $\chi_0 = 3$ so that

$$x_s = \left(\begin{array}{c} 3 \\ 0 \\ 0 \end{array} \right).$$

Drawing a plane



Identify linearly independent vectors in the null space

$$\chi_0 + 2\chi_1 + 4\chi_2 = 0$$

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If we write this as an appended system, we get

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ \uparrow & \uparrow & \uparrow & \\ \text{dependent variable} & \text{free variable} & \text{free variable} & \end{array} \right)$$

Identify linearly independent vectors in the null space

$$\chi_0 + 2\chi_1 + 4\chi_2 = 0$$

Identify linearly independent vectors in the null space

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We look for solutions in the form

$$x_{n_0} = \begin{pmatrix} \chi_0 \\ 1 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

and

$$x_{n_1} = \begin{pmatrix} \chi_0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify linearly independent vectors in the null space

$$\chi_0 + 2\chi_1 + 4\chi_2 = 0$$

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$$x_{n_0} = \left(\begin{array}{c} \chi_0 \\ 1 \\ 0 \end{array} \right) \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify linearly independent vectors in the null space

$$\chi_0 + 2\chi_1 + 4\chi_2 = 0$$

$$x_{n_0} = \left(\begin{array}{c} \chi_0 \\ 1 \\ 0 \end{array} \right) \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Substituting $\chi_1 = 1$ and $\chi_2 = 0$ into the equation gives us

$$\chi_0 + 2(1) + 4(0) = 0$$

or $\chi_0 = -2$.

Identify linearly independent vectors in the null space

$$\chi_0 + 2\chi_1 + 4\chi_2 = 0$$

Identify linearly independent vectors in the null space

$$\chi_0 + 2\chi_1 + 4\chi_2 = 0$$

$$x_{n_1} = \begin{pmatrix} \chi_0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify linearly independent vectors in the null space

$$\chi_0 + 2\chi_1 + 4\chi_2 = 0$$

$$x_{n_1} = \begin{pmatrix} \chi_0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Substituting $\chi_1 = 0$ and $\chi_2 = 1$ into the equation gives us

$$\chi_0 + 2(0) + 4(1) = 0$$

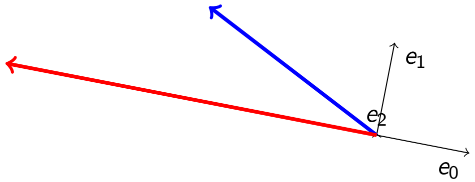
or $\chi_0 = -4$.

Identify linearly independent vectors in the null space

Thus two (nonunique) linearly independent vectors in the null space are given by

$$x_{n_0} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad x_{n_1} = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}.$$

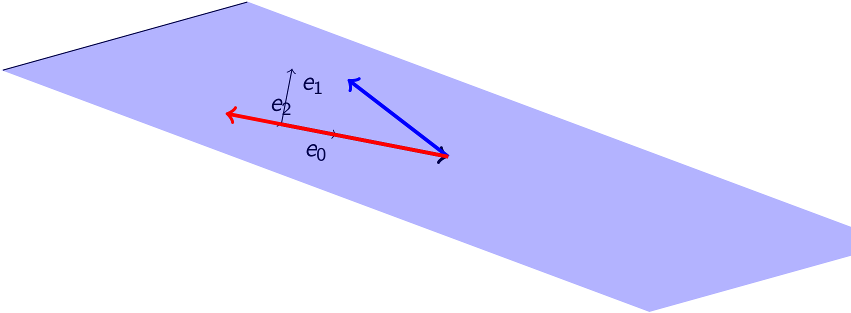
Visualizing the vectors in the null space



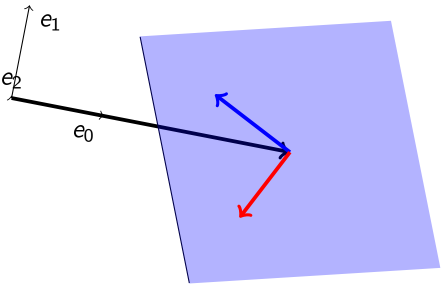
Towards visualizing the plane of solutions

$$x_{\mathrm{general}} = x_s + \alpha x_{n_0} + \beta x_{n_1}$$

Visualizing the plane of solutions



Visualizing the vectors in the null space



Creating orthogonal vectors (later this week)

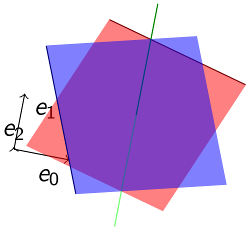
Orthogonal (perpendicular) is better!

$$q_0 = x_{n_0}/\|x_{n_0}\| \quad \text{and} \quad q_1 = x_{n_1}^{\perp}/\|x_{n_1}^{\perp}\|_2,$$

where $x_{n_1}^{\perp}$ equals the component of x_{n_1} orthogonal to q_0. We compute this using the Gram-Schmidt algorithm:

- $\rho_{0,0} := \|x_{n_0}\|_2$.
- $q_0 := x_{n_0}/\rho_{0,0}$.
- $\rho_{0,1} := q_0^T x_{n-1}$. $x_{n_1}^{\perp} := x_{n_1} - \rho_{0,1} q_0$.
- $\rho_{1,1} := \|x_{n_1}^{\perp}\|_2$.
- $q_1 := x_{n_1}^{\perp}/\rho_{1,1}$.

Visualizing the solution as the intersection of the planes



Adding lines where the planes meet

$$\begin{array}{rcrcrcr} \chi_0 & - & 2\chi_1 & + & 4\chi_2 & = & -1 \\ \chi_0 & + & (0)\chi_1 & + & (0)\chi_2 & = & 2 \end{array}$$

Adding lines where the planes meet

$$\begin{array}{rcrcrcr} \chi_0 & - & 2\chi_1 & + & 4\chi_2 & = & -1 \\ \chi_0 & + & (0)\chi_1 & + & (0)\chi_2 & = & 2 \end{array}$$

Appended system:

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & -1 \\ 1 & 0 & 0 & 2 \end{array} \right)$$

Adding lines where the planes meet

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 1 & 0 & 0 & 2 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ & 2 & -4 & 3 \end{array}\right)$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{dependent variable} & \text{dependent variable} & \text{free variable} \end{array}$$

Adding lines where the planes meet

We know a specific solution since we know the solution to the three equations in three unknowns.

Adding lines where the planes meet

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Find vector in null space:

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ & 2 & -4 & 0 \end{array}\right)$$

Adding lines where the planes meet

We know a specific solution since we know the solution to the three equations in three unknowns.

Find vector in null space:

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ & 2 & -4 & 0 \end{array}\right)$$

$$x_n = \begin{pmatrix} \chi_0 \\ \chi_1 \\ 1 \end{pmatrix}$$

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$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ & 2 & -4 & 0 \end{array}\right)$$

$$x_n = \begin{pmatrix} \chi_0 \\ \chi_1 \\ 1 \end{pmatrix}$$

$$\begin{array}{cccc} \chi_0 & -2\chi_1 & = & -4 \\ & 2\chi_1 & = & 4 \end{array}$$

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Find vector in null space:

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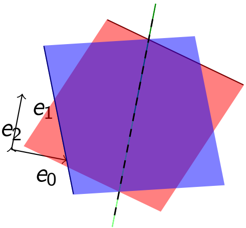
$$x_n = \begin{pmatrix} \chi_0 \\ \chi_1 \\ 1 \end{pmatrix}$$

$$\begin{array}{cccc} \chi_0 & -2\chi_1 & = & -4 \\ & 2\chi_1 & = & 4 \end{array}$$

$$x_n = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

Visualizing intersecting planes



Adding lines where the planes meet

$$\begin{array}{rcrcrcl} \chi_0 & + & (0)\chi_1 & + & (0)\chi_2 & = & 2 \\ \chi_0 & + & 2\chi_1 & + & 4\chi_2 & = & 3 \end{array}$$

Adding lines where the planes meet

$$
\begin{array}{rcrcrcr}
\chi_0 & + & (0)\chi_1 & + & (0)\chi_2 & = & 2 \\
\chi_0 & + & 2\chi_1 & + & 4\chi_2 & = & 3
\end{array}
$$

Appended system:

$$
\left(\begin{array}{ccc|c}
1 & 0 & 0 & 2 \\
1 & 2 & 4 & 3
\end{array}\right)
$$

Adding lines where the planes meet

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 1 & 2 & 4 & 3 \end{array}\right)$$

$$\downarrow$$

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & 4 & 1 \end{array}\right) \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{dependent variable} & \text{dependent variable} & \text{free variable} \end{array} \end{array}$$

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$$\begin{array}{lcr} \chi_0 \qquad\qquad & = & 0 \\ \qquad 2\chi_2 & = & -4 \end{array}$$

Adding lines where the planes meet

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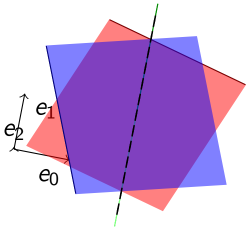
$$\begin{array}{ccc} \chi_0 & & = & 0 \\ & 2\chi_2 & = & -4 \end{array}$$

$$x_n = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

Visualizing intersecting planes



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$$\begin{array}{rcrcrcr} \chi_0 & - & 2\chi_1 & + & 4\chi_2 & = & -1 \\ \chi_0 & + & 2\chi_1 & + & 4\chi_2 & = & 3 \end{array}$$

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$$\begin{array}{rcrcrcr} \chi_0 & - & 2\chi_1 & + & 4\chi_2 & = & -1 \\ \chi_0 & + & 2\chi_1 & + & 4\chi_2 & = & 3 \end{array}$$

Appended system:

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & -1 \\ 1 & 2 & 4 & 3 \end{array}\right)$$

Adding lines where the planes meet

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 1 & 2 & 4 & 3 \end{array}\right)$$

$$\downarrow$$

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 4 & 0 & 4 \end{array}\right) \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ \text{dependent variable} & \text{dependent variable} & \text{free variable} & \end{array} \end{array}$$

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$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right)$$

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Find vector in null space:

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right)$$

$$x_n = \begin{pmatrix} \chi_0 \\ \chi_1 \\ 1 \end{pmatrix}$$

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$$x_n = \left(\begin{array}{c} \chi_0 \\ \chi_1 \\ 1 \end{array}\right)$$

$$\begin{array}{rrcr} \chi_0 & -2\chi_1 & = & -4 \\ 0 & 4\chi_1 & = & 0 \end{array}$$

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Find vector in null space:

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right)$$

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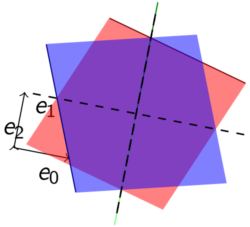
$$\begin{array}{cccc} \chi_0 & -2\chi_1 & = & -4 \\ 0 & 4\chi_1 & = & 0 \end{array}$$

$$x_n = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$

Visualizing intersecting planes



Adding one more equation

$$\begin{array}{rcrcrcr}
\chi_0 & - & 2\chi_1 & + & 4\chi_2 & = & -1 \\
\chi_0 & & & & & = & 2 \\
\chi_0 & + & 2\chi_1 & + & 4\chi_2 & = & 3 \\
\chi_0 & + & 4\chi_1 & + & 16\chi_2 & = & 9
\end{array}$$

Adding one more equation

$$\begin{array}{rcrcrcr}
\chi_0 & - & 2\chi_1 & + & 4\chi_2 & = & -1 \\
\chi_0 & & & & & = & 2 \\
\chi_0 & + & 2\chi_1 & + & 4\chi_2 & = & 3 \\
\color{red}{\chi_0} & \color{red}{+} & \color{red}{4\chi_1} & \color{red}{+} & \color{red}{16\chi_2} & \color{red}{=} & \color{red}{9}
\end{array}$$

Appended system:

$$\left(\begin{array}{rrr|r}
1 & -2 & 4 & -1 \\
1 & & & 2 \\
1 & 2 & 4 & 3 \\
1 & 4 & 16 & 9
\end{array}\right)$$

Visualize a plane of solutions corresponding to a single linear equation

$$\chi_0 + 4\chi_1 + 16\chi_2 = 9$$

As an appended system:

$$\left(\begin{array}{ccc|c} 1 & 4 & 16 & 9 \end{array}\right)$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{dependent variable} & \text{free variable} & \text{free variable} \end{array}$$

Identify linearly independent vectors in the null space

$$\chi_0 + 4\chi_1 + 16\chi_2 = 0.$$

Identify linearly independent vectors in the null space

$$\chi_0 + 4\chi_1 + 16\chi_2 = 0.$$

If we write this as an appended system, we get

$$\begin{array}{ccc}
\left(\begin{array}{ccc|c} 1 & 4 & 16 & 0 \end{array}\right) & & \\
\uparrow & \uparrow & \uparrow \\
\text{dependent variable} & \text{free variable} & \text{free variable}
\end{array}$$

Identify linearly independent vectors in the null space

$$\chi_0 + 4\chi_1 + 16\chi_2 = 0.$$

Identify linearly independent vectors in the null space

$$\chi_0 + 4\chi_1 + 16\chi_2 = 0.$$

We look for solutions in the form

$$x_{n_0} = \begin{pmatrix} \chi_0 \\ 1 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

and

$$x_{n_1} = \begin{pmatrix} \chi_0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify linearly independent vectors in the null space

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$$x_{n_0} = \begin{pmatrix} \chi_0 \\ 1 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify linearly independent vectors in the null space

$$\chi_0 + 4\chi_1 + 16\chi_2 = 0.$$

$$x_{n_0} = \begin{pmatrix} \chi_0 \\ 1 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{ Dependent variable to be computed} \\ \longleftarrow \text{ Free variable} \\ \longleftarrow \text{ Free variable} \end{array}$$

Substituting $\chi_1 = 1$ and $\chi_2 = 0$ into the equation gives us

$$\chi_0 + 4(1) + 16(0) = 0.$$

or $\chi_0 = -4$.

Identify linearly independent vectors in the null space

$$\chi_0 + 4\chi_1 + 16\chi_2 = 0.$$

Identify linearly independent vectors in the null space

$$\chi_0 + 4\chi_1 + 16\chi_2 = 0.$$

$$x_{n_1} = \begin{pmatrix} \chi_0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Identify linearly independent vectors in the null space

$$\chi_0 + 4\chi_1 + 16\chi_2 = 0.$$

$$x_{n_1} = \begin{pmatrix} \chi_0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Substituting $\chi_1 = 0$ and $\chi_2 = 1$ into the equation gives us

$$\chi_0 + 4(0) + 16(1) = 0.$$

or $\chi_0 = -16$.

Identify linearly independent vectors in the null space

Thus two (nonunique) linearly independent vectors in the null space are given by

$$x_{n_0} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad x_{n_1} = \begin{pmatrix} -16 \\ 0 \\ 1 \end{pmatrix}.$$

Visualizing the vectors in the null space