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Analytic Discussion of Spatially Closed Friedman Universes with Cosmological Constant and Radiation Pressure

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We derive explicit formulas for various quantities of interest in the universes described in the title, and discuss the interpretation of cosmological constant in quantum field theories.

Contents.

1. INTRODUCTION

For spatially homogeneous, isotropic, pressureless universes the Einstein field equations reduce to a single ordinary non-linear differential equation derived by

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Friedman [1] in 1922. The early papers, including Friedman's, and Lemaître's [2], consider Friedman's equation with a non-zero cosmological term Λ; this equation can be found in most textbooks on the subject (see, e.g., [4–6, 17]).

While it is generally known that the solution to this differential equation involves elliptic functions, it seems that not many people have studied it in detail analytically (see, however, [2, 3]). One of our aims here is to write explicit expressions for various quantities of interest, considered as functions of dimensionless parameter time.

The evaluation of the formulae is easy, so that they can be used as an alternative to the direct numerical integration of Friedman's equation with cosmological term [53, 55]).

Furthermore, no essential changes are required in order to take into account pressure due to massless particles (neutrinos and radiation) (see Section 5).

We cannot discuss here the long history of cosmological models with $\Lambda \neq 0$; for well-known reasons the case $\Lambda = 0$ has received most attention, in particular, as far as comparison with observation is concerned ([58, 59]); see, however, [12, 13, 21, 52].

Our motivation for this work comes from the papers [7–11], which compare a class of models to observational data, and propose a set of cosmological parameters. Following [7] to [11], we mostly consider spatially closed universes and pay particular attention to the case where Λ is larger than the critical value corresponding to Einstein's stationary universe (see [12, 13, 14]).

A non-zero cosmological constant has an interpretation as vacuum expectation value of a quantum field theoretical stress-energy tensor (see Zel'dovich [15, 16]). We shall give here a survey of work relating to this.

2. Friedman Equation with Cosmological Constant and Radiation Pressure

2.A. *The Functions $R(\tau)$ and $T(\tau)$*

We consider Friedman–Lemaître models with positive spatial curvature ($k = +1$), with cosmological term Λ and with radiation pressure. The average local energy density splits into three parts, namely:

(i) Vacuum contribution

$$\rho_{\text{vac}} = \frac{\Lambda}{8\pi G}$$

independent of time.

(ii) Radiation (massless bosons) giving $\rho_{\text{rad}}(t)$ such that

$$R^4(t)\,\rho_{\text{rad}}(t) = \text{const.} = \frac{3}{8\pi G}\,C_r,$$

where $R(t)$ (with dimension of length) is the spatial radius of curvature.

(iii) $\rho_m(t)$ due to averaged pressureless matter, and satisfying

$$R^3(t)\,\rho_m(t) = \text{const.} = \frac{3}{8\pi G}\,C_m.$$

The evolution of $R(t)$ is determined by Friedman's equation:

$$\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 = \frac{C_r}{R^4} + \frac{C_m}{R^3} - \frac{1}{R^2} + \frac{\Lambda}{3} \tag{2.1}$$

(see, e.g., [6]).

It is convenient to transform (2.1) to a form where the r.h.s. is a polynomial in R. This is done by introducing the dimensionless *parameter time* τ (or conformal time) defined, up to an additive constant, by

$$d\tau = \frac{dt}{R}. \tag{2.2}$$

This variable is natural geometrically (it realizes the conformal flatness of universes that we study and gives three-dimensional geodesic distances) as well as analytically (it is the natural uniformizing parameter for the study of the curve $t \to R(t)$).

In terms of τ, Eq. (2.1) reads

$$\left(\frac{dR}{d\tau}\right)^2 = C_r + C_m R - R^2 + \frac{\Lambda}{3} R^4.$$

The quantities C_n, C_m, $\Lambda/3$ have dimensions L^2, L, L^{-2}, respectively. In order to replace R by a dimensionless quantity, we use matter to define a scale of length (see, e.g., [5, 6]). Introduce

$$\Lambda_c = \frac{4}{9c_m^2} \tag{2.3}$$

so that $\Lambda_c^{-1/2} = 4\pi G \rho_m R^3$ is a constant length. If the universe is, spatially, the 3-sphere S_3, then its total mass is $M = 4\pi^2 R^3 \rho_m$ and so $2GM = \pi \Lambda_c^{-1/2}$.

The natural dimensionless parameters (independent of time) are then

$$\lambda = \frac{\Lambda}{\Lambda_c} \tag{2.4}$$

and

$$\alpha = C_r \Lambda_c. \tag{2.5}$$

We shall always assume that $\alpha > 0$.

We shall find it convenient to replace $R(\tau)$ as basic object by the reduced black-body temperature $T(\tau)$, defined by

$$T(\tau)=\frac{1}{\Lambda_c^{1/2}R(\tau)}. \tag{2.6}$$

$T(\tau)$ is proportional to the temperature of the background black-body radiation. In terms of $T(\tau)$, Eq. (2.1) becomes

$$\left(\frac{dT}{d\tau}\right)^2=\alpha T^4+\frac{2}{3}T^3-T^2+\frac{\lambda}{3}. \tag{2.7}$$

We shall write

$$Q_{\alpha,\lambda}(T)=\alpha T^4+\frac{2}{3}T^3-T^2+\frac{\lambda}{3} \tag{2.8}$$

and

$$Q_\lambda(T)=Q_{0,\lambda}(T). \tag{2.9}$$

Equation (2.7) determines a unique $T(\tau)$ except for the freedom of choice for the origin of τ, and an ambiguity of sign in $dT/d\tau$; these matters will be discussed in Sections 4, 5.

We are interested only in solutions $T(\tau)$ of (2.7) that are real on the real τ-axis. This means

$$Q_{\alpha,\lambda}(T(\tau))\geqslant 0 \tag{2.10}$$

when τ is real.

2.B. *Other Quantities, Expressed in Terms of* $T(\tau)$

Other quantities of interest can be expressed in terms of $T(\tau)$. We shall consider, in particular, the following ones

(a) *The Hubble function* $H(\tau)$: The definition $H=R^{-1}(dR/dt)$ gives

$$H(\tau)=-\Lambda_c^{1/2}\frac{dT}{d\tau}. \tag{2.11}$$

(b) *Dimensionless densities*: Equation (2.1), divided by H^2, can be written as

$$1+k_S(\tau)=\alpha_S(\tau)+\Omega(\tau)+\lambda_S(\tau) \tag{2.12}$$

(see [7]), where

(b1) $\alpha_S(\tau)$ is the dimensionless radiation density

$$\alpha_S(\tau)=\frac{C_n}{H^2R^4}=\frac{8\pi G\rho_{\text{rad}}}{3H^2}=\frac{\alpha\Lambda_c T^4}{H^2}. \tag{2.13}$$

(b2) $\Omega(\tau)$ is the dimensionless matter density

$$\Omega(\tau) = \frac{C_m}{H^2 R^3} = \frac{8\pi G \rho_m}{3H^2} = \frac{2\Lambda_C T^3}{3H^2}. \tag{2.14}$$

(b3) $\lambda_S(\tau)$ is the dimensionless "vacuum density"

$$\lambda_S(\tau) = \frac{\Lambda}{3H^2} = \frac{8\pi G \rho_{\text{vac}}}{3H^2} = \frac{\lambda \Lambda_C}{3H^2} \tag{2.15}$$

and $k_S(\tau)$ is the "reduced 3-curvature"

$$k_S(\tau) = \frac{1}{R^2 H^2} = \frac{\Lambda_C T^2}{H^2}. \tag{2.16}$$

(c) *Deceleration function*: The definition $q = -R(dR/dt)^{-2}\,(d^2R/dt^2)$ can be rewritten as

$$q(\tau) = -1 - T\frac{d}{d\tau}\left[\left(\frac{dT}{d\tau}\right)^{-1}\right] = \frac{3\alpha T^4 + T^3 - \lambda}{3(\alpha T^4 + (2/3)T^3 - T^2 + \lambda/3)}. \tag{2.17}$$

(d) *Redshift function*: If τ refers to the moment of emission and $\tau_0 > \tau$ to the moment of observation, then the redshift is

$$z = \frac{T(\tau)}{T(\tau_0)} - 1. \tag{2.18}$$

It is natural to introduce the difference

$$\delta = \tau_0 - \tau \tag{2.19}$$

so that

$$z(\delta; \tau_0) = \frac{T(\tau_0 - \delta)}{T(\tau_0)} - 1. \tag{2.20}$$

(e) *Luminosity distance*: (see, e.g., [17]). It can be defined as

$$\begin{aligned} D_L(\delta; \tau_0) &= R(\tau_0)(1 + z(\delta; \tau_0)) \sin\delta \\ &= \Lambda_C^{-1/2} \frac{T(\tau_0 - \delta)}{T(\tau_0)^2} \sin\delta. \end{aligned} \tag{2.21}$$

(f) *Apparent size distance*: It is

$$D_S(\delta; \tau_0) = \Lambda_C^{-1/2} \frac{\sin\delta}{T(\tau_0 - \delta)}. \tag{2.22}$$

(g) *Cosmic time*: If we choose $t(0)=0$, we have, by (2.2),

$$t(\tau)=\int_0^\tau R(\tau')\,d\tau'=\Lambda_C^{-1/2}\int_0^\tau \frac{d\tau'}{T(\tau')}=-\Lambda_C^{-1/2}\int_T^\infty \frac{dT'}{T'\sqrt{Q_{\alpha,\lambda}(T')}}. \tag{2.23}$$

(h) *Dimensionless time*: It will be defined as

$$[Ht](\tau)=H(\tau)\,t(\tau). \tag{2.24}$$

3. General Properties of the Reduced Blackbody Temperature $T(\tau)$

3.A. *Elliptic Functions and Weierstrass Functions*

It is well known (see, e.g., [18]) that the solution of an equation of the form (2.7) is necessarily meromorphic and doubly periodic, i.e., an elliptic function. This property is necessarily shared by any *rational* function of T; such as the Hubble function H, the deceleration function q, and the redshift function z, considered as function of either one of its arguments. It gives rise to explicit formulas and expansions that will be discussed later.

Some of the quantities introduced above, such as time $t(\tau)$ or the apparent size distance $D_L(\delta)$ are *not* elliptic. Nevertheless, they can be expressed with the help of Weierstrass functions that will be studied in this section.

An intuitive understanding of the behaviour of $T(\tau)$ can be obtained by interpreting (2.7) as the energy equation of a one-dimensional mechanical system, which will also be introduced in this section. (Compare, e.g., [6].)

We shall start with some reminders concerning elliptic functions and Weierstrass functions (See, e.g., [18]).

Let L be a lattice in the complex plane, i.e., a set that can be generated additively by two non-zero complex numbers with non-real ratio. A function $f(\tau)$ is called elliptic with respect to L if it is meromorphic, not identically a constant, and if $f(\tau)=f(\tau+a)$ for all $\tau\in\mathbb{C}$ and all $a\in L$.

The following statements, derived by Liouville about 150 years ago, are at the root of all explicit formulas that will be given in this paper.

(i) Let f and g be elliptic with respect to the same lattice L. Assume that f and g have the same poles and the same principal parts (coefficients of negative powers in expansion around the poles). Then $f-g$ is a constant.

(ii) If f is an elliptic function and if a is a complex number (including ∞) then the number of solutions $\{\tau_j\}$, of the equation $f(\tau)=a$, within any fundamental periodicity cell is independent of a, if multiplicities are properly counted. This number is called the *order* of f. Thus, e.g., an elliptic function of order 2 takes every complex value exactly twice in each fundamental periodicity cell.

(iii) The order of an elliptic function is at least 2. Notice that a rational function of f (hence an elliptic function) will have the same order as f, if and only if, the transformation is fractionally linear; for example, $T(\tau)$ is of order 2 but $q(\tau)$ and $\Omega(\tau)$ are of order 6.

Given a lattice L, the *Weierstrass $\wp$-function* $\wp(\tau)$ corresponding to L is defined as the elliptic function (of order 2) that has a pole of second order at 0 (and consequently also at all other points of L), with principal part $1/\tau^2$, and is such that $1/\tau^2 - \wp(\tau)$ vanishes at $\tau = 0$. Notice that $\wp(\tau)$ is an even function of τ.

The Weierstrass ζ-function (with respect to L) can be defined by

$$\frac{d\zeta}{d\tau} = -\wp(\tau) \tag{3.1}$$

and the requirement that $1/\tau - \zeta(\tau)$ vanish at $\tau = 0$. It is meromorphic and has poles of first-order at points of L; the principal part at $\tau = 0$ is $1/\tau$. The function $\zeta(\tau)$ is *not* elliptic, since it is not periodic. However, if $a \neq b$, then the difference $\zeta(\tau - a) - \zeta(\tau - b)$ is elliptic of order 2, with poles at $\tau = a$ and $\tau = b$. Notice that $\zeta(\tau)$ is an odd function of τ.

The Weierstrass σ-function (with respect to L) can be defined as an entire function $\sigma(\tau)$ such that $\sigma(0) = 0$ and

$$\frac{1}{\sigma(\tau)} \frac{d\sigma}{d\tau} = \zeta(\tau). \tag{3.2}$$

It is *not* elliptic, since it is not periodic. However, if a_1, a_2, b_1, b_2 are complex numbers such that $a_1 + a_2 - b_1 - b_2 = 0$ then $\sigma(\tau - a_1)\,\sigma(\tau - a_2)/\sigma(\tau - b_1)\,\sigma(\tau - b_2)$ is an elliptic function of order 2 with poles at b_1, b_2 and zeroes at a_1, a_2. Notice that $\sigma(\tau)$ is an odd function of τ.

Finally it should be noticed that the sume of the zeroes of an elliptic function inside a periodicity cell is equal to the sum of the poles inside the same cell, up to an element of the lattice.

3.B. *Transformation of Equation for $T(\tau)$ into Canonical Form: Invariants and Discriminants*

The first general statement about $T(\tau)$ is:

PROPOSITION. *$T(\tau)$ is elliptic of order 2, with respect to some lattice L. It has thus two poles and two zeroes in every fundamental periodicity cell.*

In order to obtain informations about the lattice L (i.e., about the periods of $T(\tau)$), it is convenient to transform Eq. (2.7) to a canonical form by a fractional linear transformation.

We consider first the case $\alpha = 0$ (no pressure) which is somewhat simpler.

PROPOSITION. *The transformation*

$$\begin{aligned} T &= 6y + \tfrac{1}{2}, \\ y &= \tfrac{1}{6}(T - \tfrac{1}{2}), \end{aligned} \tag{3.3}$$

transforms the equation

$$\left(\frac{dT}{d\tau}\right)^2 = Q_{0,\lambda}(T) = \frac{2}{3}T^3 - T^2 + \frac{\lambda}{3} \tag{3.3'}$$

into the canonical Weierstrass form

$$\left(\frac{dy}{d\tau}\right)^2 = 4y^3 - g_2 y - g_3 \stackrel{\text{def}}{=} P_\lambda(y) \tag{3.4}$$

with

$$g_2 = \frac{1}{12}, \tag{3.5}$$

$$g_3 = 6^{-3}(1-2\lambda). \tag{3.6}$$

In the general case ($\alpha \neq 0$), the transformation to the Weierstrass canonical form runs as follows:

PROPOSITION. *Let T_j be any one of the (possibly complex) roots of the equation*

$$Q_{\alpha,\lambda}(T) = \alpha T^4 + \frac{2}{3}T^3 - T^2 + \frac{\lambda}{3} = 0. \tag{3.7}$$

Then the fractional linear transformation

$$y = \frac{Q'_{\alpha,\lambda}(T_j)}{4}\frac{1}{T-T_j} + \frac{Q''_{\alpha,\lambda}(T_j)}{24} \tag{3.8}$$

brings (2.7) *to the form*

$$\left(\frac{dy}{d\tau}\right)^2 = 4y^3 - g_2 y - g_3 \stackrel{\text{def}}{=} P_{\alpha,\lambda}(y) \tag{3.9}$$

with

$$g_2 = \frac{1}{12} + \frac{\alpha\lambda}{3}, \tag{3.10}$$

$$g_3 = 6^{-3}(1-2\lambda) - \frac{\alpha\lambda}{18}. \tag{3.11}$$

The three roots of the equation

$$P_{\alpha,\lambda}(y) = 0 \tag{3.12}$$

add up to zero. They are given by

$$-\frac{A+B}{2} \pm i\frac{\sqrt{3}}{2}(A-B) \tag{3.13}$$

$$\text{and } A+B, \tag{3.14}$$

where

$$A=\tfrac{1}{2}[g_3+\sqrt{-\Delta_{\alpha,\lambda}/27}]^{1/3}, \tag{3.15}$$

$$B=\tfrac{1}{2}[g_3-\sqrt{-\Delta_{\alpha,\lambda}/27}]^{1/3}, \tag{3.16}$$

and where $\Delta_{\alpha,\lambda}$ is the discriminant

$$\Delta_{\alpha,\lambda}=g_2^3-27g_3^2=2^{-4}3^{-3}\lambda(\lambda-\lambda_+)(\lambda-\lambda_-), \tag{3.17}$$

where

$$\lambda_\pm(\alpha)=\frac{1}{32\alpha^3}[24\alpha^2+12\alpha+1\pm(8\alpha+1)^{3/2}]. \tag{3.18}$$

As $\alpha\to 0$, we have $\lambda_+\to\infty$, and $\lambda_-\to 1$; to first order in α, we have

$$\lambda_+\sim\frac{1}{16\alpha^3}(1+12\alpha), \tag{3.19}$$

$$\lambda_-\sim 1-3\alpha. \tag{3.20}$$

Consequently

$$\Delta_\lambda \overset{\text{def}}{=} \Delta_{0,\lambda}=2^{-4}3^{-3}\lambda(1-\lambda). \tag{3.21}$$

If $\Delta_{\alpha,\lambda}=0$, then two zeroes of $P_{\alpha,\lambda}(y)$ coincide; then, also, two zeroes of $Q_{\alpha,\lambda}(T)$ coincide. This is the situation where the fundamental periodicity cell becomes infinite, and the elliptic functions degenerate to trigonometric or hyperbolic functions (see Section 7C).

We shall say that α, λ is *critical* if $\Delta_{\alpha,\lambda}=0$. The critical lines in the α,λ plane are shown in Fig. 1.

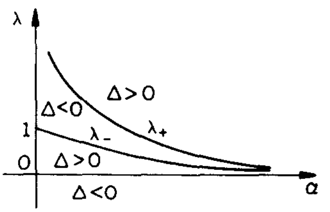

FIGURE 1

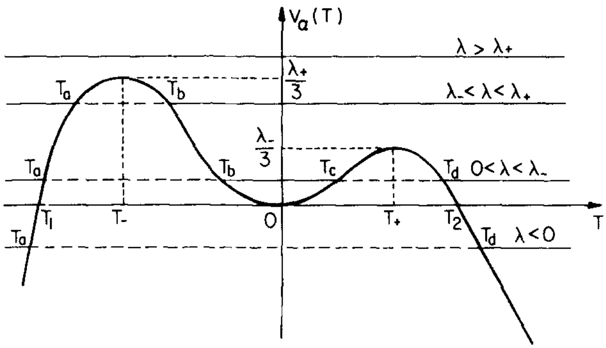

FIGURE 2

3.C. *The Associated Mechanical System*

Equation (2.7) can be written as

$$\left(\frac{dT}{d\tau}\right)^2 + V_\alpha(T) = \frac{\lambda}{3} \tag{3.22}$$

with

$$V_\alpha(T) = -\alpha T^4 - \frac{2}{3}T^3 + T^2 = -Q_{\alpha,\lambda}(T) + \frac{\lambda}{3}. \tag{3.23}$$

This is the energy equation of a one-dimensional mechanical system with "coordinate" T, potential $V_\alpha(T)$; (shown in Fig. 2) and total energy $\lambda/3$. The kinetic energy $(dT/d\tau)^2$ being non-negative, the associated mechanical system describes a horizontal line in the $(V(T), T)$ plane but never penetrates under the curve $V_\alpha(T)$, indeed, that would correspond to τ imaginary.

$$T_{\frac{1}{2}} = \frac{1}{3\alpha}\left[-1 \mp \sqrt{1+9\alpha}\right]. \tag{3.24}$$

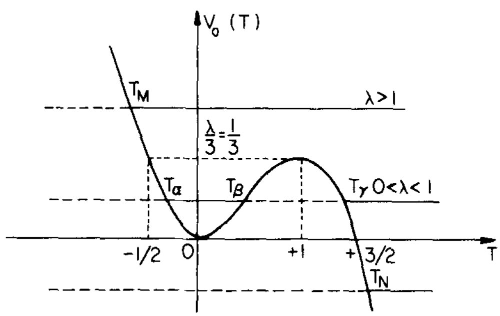

FIGURE 3

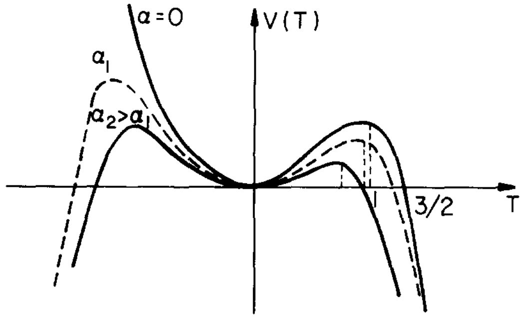

FIGURE 4

The extrema of $V_\alpha(T)$ are at $T=0$ and at

$$T_\pm = \frac{1}{4\alpha}\,[-1 \pm \sqrt{1+8\alpha}]. \tag{3.25}$$

As $\alpha \to 0$, the left-hand bump of V_α tends to $-\infty$ and the curve becomes

$$V_0(T) = -\tfrac{2}{3}T^3 + T^2 \tag{3.26}$$

given by Fig. 3.

Notice that the curve $V_0(T)$ is symmetric with respect to its inflection point $\{T=\frac{1}{2},\ V(T)=\frac{1}{6}\}$, corresponding to $\lambda=\frac{1}{2}$. The symmetry with respect to this point transforms "imaginary" paths corresponding to λ into real paths corresponding to $1-\lambda$; we have so

$$T_\lambda(i\tau) = T_{1-\lambda}(\tau), \tag{3.27}$$

which will be verified later on explicit expressions for $T(\tau)$. In order to compare easily the behaviour of $T(\tau)$ for $\alpha=0$ and for $\alpha \neq 0$, we plotted on Fig. 4 the potential $V_\alpha(T)$ for $\alpha=0$ and for finite values α_1, α_2 with $\alpha_2 > \alpha_1$.

4. EXPLICIT EXPRESSIONS FOR $T(\tau)$; NO RADIATION PRESSURE

Since $T(\tau)$ is an elliptic function of order 2, it is determined by the position of its two zeroes within any fundamental periodicity cell. It is convenient to discuss separately the various possible cases.

4.A. *Case* $\lambda > 1$ $(\alpha = 0)$

We have now $\alpha=0$, $\lambda>1$, and consequently $g_2 = 1/12 > 0$, $\Delta_\lambda < 0$ and $g_3 < 0$. The three roots (Eqs. (3.13), (3.14)) can be written

$$e_1 = a - ib,$$
$$e_2 = -2a, \tag{4.1}$$
$$e_3 = a + ib,$$

with $a > 0$ and $b > 0$.

Notice that e_2 in (4.1) is given by (3.14)–(3.16), where the real cubic root has to be chosen:

$$e_2 = -\frac{1}{12}\,[-1 + 2\lambda - 2\sqrt{\lambda(\lambda-1)}]^{1/3} - \frac{1}{12}\,[-1 + 2\lambda + 2\sqrt{\lambda(\lambda-1)}]^{1/3}.$$

We have then

$$T_M = 6e_2 + \tfrac{1}{2} < -\tfrac{1}{2} < 0, \tag{4.2}$$

where T_M is the minimum value of $T(\tau)$. It is taken at the point where $dT/d\tau = 0$, i.e., by Eq. (3.4), at the point $y = e_2$. The behaviour of T is shown by Fig. 5.

Its qualitative features can be read off the mechanical model just introduced. In order to obtain a closed expression for $T(\tau)$ we start with a discussion of its periods. Quite generally (see, e.g., [20]), the case $\Delta_\lambda < 0$ corresponds to a fundamental periodicity cell in the shape of a rhombus of the complex τ plane. For historical reasons, a set of fundamental periods is denoted by $2\omega_1$, $2\omega_2$. It is convenient to introduce also $\omega_3 = -\omega_1 - \omega_2$. The periodicity cell is then described in Fig. 6, where

$$\omega_r = \omega_2 = \int_{e2}^{\infty} \frac{dy}{\sqrt{P_\lambda(y)}} \tag{4.3}$$

and where

$$\omega_i = \mathrm{Im}(-\omega_3) = \int_{-\infty}^{e2} \frac{dy}{\sqrt{|P_\lambda(y)|}}. \tag{4.4}$$

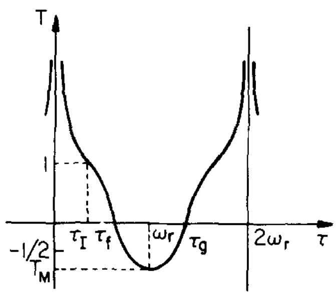

FIGURE 5

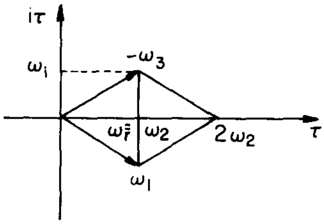

FIGURE 6

Because of $g_3 < 0$ we have $\omega_i < \omega_r$. Numerical values of $\omega_r = \omega_r(\lambda)$ and $\omega_i = \omega_i(\lambda)$ are given in Table I. (see Section 8.A).

The two zeroes of $T(\tau)$, $0 < \tau_f < \tau_g < 2\omega_r$, are given by

$$\tau_f = \int_0^\infty \frac{dT}{\sqrt{Q_\lambda(T)}} = \int_{-1/12}^\infty \frac{dy}{\sqrt{P_\lambda(y)}}, \tag{4.5}$$

$$\tau_g = \tau_f + 2\tau_c, \tag{4.6}$$

where Q_λ and P_λ are defined respectively by (2.9) and (3.4) and where

$$\tau_c = \int_{T_M}^0 \frac{dT}{\sqrt{Q_\lambda(T)}} = \int_{e_2}^{-1/12} \frac{dy}{\sqrt{P_\lambda(y)}}. \tag{4.7}$$

Notice that

$$\tau_f + \tau_g = 2\omega_r, \tag{4.8}$$

i.e., that τ_f and τ_g are placed symmetrically with respect to ω_r. It is now easy to give formulas for the function $T(\tau)$. We have, e.g.,

$$T(\tau) = 6\left[\wp(\tau; g_2, g_3) + \frac{1}{12}\right]. \tag{4.9}$$

The origin of τ has been chosen at the second order singularity of T ("the big bang"). Formula (4.9) can be derived very simply by examining Eq. (3.3′) in a neighbourhood of the singularity ($\tau = 0$), where $(dT/d\tau)^2 \sim (2/3)\,T^3$ and by applying the theorem of Liouville quoted in Section 3.A above. It shows that $T = 6\wp(\tau) +$ const. $= 6[\wp(\tau) - \wp(\tau_f)]$; by (3.3) and Fig. 3, we see that $\wp(\tau_f) = -1/12$.

Alternatively we can write

$$\begin{aligned}\sqrt{\frac{\lambda}{3}}\,\frac{1}{T(\tau)} &= \zeta(\tau_f - \tau) - \zeta(\tau_f) - \zeta(\tau_g - \tau) + \zeta(\tau_g)\\ &= \zeta(\tau + \tau_f) - \zeta(\tau - \tau_f) - 2\zeta(\tau_f),\end{aligned} \tag{4.10}$$

displaying the poles and principal parts of $1/T(\tau)$.

An expression that involves only the position of the poles and zeroes of T but not the behaviour of $T(\tau)$ in the neighbourhood of singularities can be obtained from Liouville's theorem (Section 3.A). It reads

$$T(\tau) = C \cdot \frac{\sigma(\tau - \tau_f)\,\sigma(\tau - \tau_g)}{\sigma^2(\tau)} \tag{4.11}$$

with

$$C = -T_M \cdot \frac{\sigma^2(\omega_r)}{\sigma^2(\tau_c)}. \tag{4.12}$$

A similar discussion, which we omit, will give other explicit formulas to be found in the sections that follow.

Integrating (2.2) we obtain

$$\left(\frac{\Lambda}{3}\right)^{1/2} t(\tau) = \ln\left[\frac{\sigma(\tau_f + \tau)}{\sigma(\tau_f - \tau)}\right] - 2\tau\zeta(\tau_f) \tag{4.13}$$

Notice that $t(\tau)$ is not elliptic. It has a logarithmic singularity at $\tau = \tau_f$. Between τ_f and τ_g, $t(\tau)$ is not real. The origin $\tau = 0$ is a regular point (a zero) for $t(\tau)$.

Physically, a spatially closed Friedman universe with $\lambda > 1$ (i.e., with cosmological constant positive and larger than Λ_c) starts with a big bang, expands slowing down until it reaches an inflection point at $\tau_I = \int_1^\infty dT/\sqrt{Q_\lambda(T)}$, $T = 1$. Afterwards, the "repulsion" due to vacuum energy (the cosmological constant) becomes predominant, and the universes expands and cools down even faster, until $t = +\infty$ (which corresponds to $\tau = \tau_f$) In a neighbourhood to the left of $\tau = \tau_f$, we have

$$\left(\frac{\Lambda}{3}\right)^{1/2} t(\tau) \sim -\ln(\tau_f - \tau), \tag{4.14}$$

$$T(\tau) \sim \left(\frac{\lambda}{3}\right)^{1/2} (\tau_f - \tau) \tag{4.15}$$

$$\left(\frac{\Lambda}{3}\right)^{1/2} R(\tau) \sim \frac{1}{\tau_f - \tau}, \tag{4.16}$$

i.e.,

$$R(t) \sim \left(\frac{3}{\Lambda}\right)^{1/2} e^{(\Lambda/3)^{1/2} t}, \tag{4.17}$$

i.e., the universe approaches the empty, spatially closed, de Sitter universe.

In Section 8 we shall study the universe with $\alpha = 0$ $\lambda > 1$ in more detail. An analysis of physics and astrophysics in a very old universe is made in [19].

4.B. *Case* $\lambda < 0$ $(\alpha = 0)$

We have now $\alpha = 0$, $\lambda < 0$, and consequently $g_2 = 1/12 > 0$, $\Delta_\lambda < 0$ and $g_3 > 0$ by (3.11), (3.21). The three roots (3.13), (3.14) are now written

$$\begin{aligned} e_1 &= -a + ib, \\ e_2 &= 2a, \\ e_3 &= -a - ib, \end{aligned} \tag{4.18}$$

with $a > 0$, $b > 0$. We have then

$$T_N = 6e_2 + \tfrac{1}{2} > 0, \tag{4.19}$$

where T_N is the minimal value of $T(\tau)$. The behaviour of T is shown by Fig. 7, with ω_2 and ω_i still given by (4.3), (4.4), respectively. We have now, however, $\omega_i > \omega_r = \omega_2$. The expression for $T(\tau)$ is still

$$T(\tau) = 6\left(\wp(\tau; g_2, g_3) + \frac{1}{12}\right). \tag{4.20}$$

The relation with case (4.A) can be described by

$$T_\lambda(i\tau) = T_{1-\lambda}(\tau), \tag{4.21}$$

which can be also obtained from the associated mechanical system (3.27)

In this case (4.B), τ_f and τ_g are imaginary. Consequently, the expressions (4.10), (4.11), (4.13) while still true, are not very convenient. An expression for $t(\tau)$ that involves only real quantities is

$$\sqrt{\Lambda_c}\, t(\tau) = \int_0^\tau \frac{dx}{6\wp(x) + \frac{1}{2}}. \tag{4.22}$$

The universe starts with a big bang at $t = \tau = 0$; it achieves its maximum size at $\tau = \omega_r$ corresponding to $t_{\max} = t(\omega_r)$ and starts contracting again. Since the singularities of cosmic time are now off the real axis, the collapse occurs at a finite time $t_p = t(2\omega_r)$. This is the Phoenix universe of Lemaître [2].

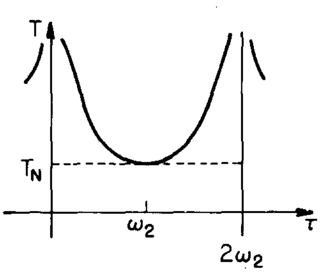

FIGURE 7

4.C. *Case* $0 < \lambda < 1$ $(\alpha = 0)$

By (3.11), (3.21), we have now $\Delta_\lambda > 0$. If $\lambda < 1/2$, then $g_3 > 0$; if $\lambda > 1/2$ then $g_3 < 0$. The three roots (3.13), (3.14) are now real and can be written as follows, with $a > 0$, $b > 0$:

$$\begin{aligned} e_1 &= 2a, \\ e_2 &= -a + b\sqrt{3}, \\ e_3 &= -a - b\sqrt{3}, \end{aligned} \tag{4.24}$$

i.e., $e_3 < e_2 < e_1$. Call $T_{\alpha,\beta,\gamma} = 6e_{3,2,1} + 1/2$ then $T_\alpha < T_\beta < T_\gamma$ (see Fig. 3). The motion of the system (3.22) can occur either in the infinite interval $]T_\gamma, +\infty[$ or in the finite interval $[T_\alpha, T_\beta]$.

4.C.1. In the first subcase, T is given by

$$T(\tau) = 6\left(\wp(\tau; g_2, g_3) + \frac{1}{12}\right) \tag{4.25}$$

and its behaviour, shown by Fig. 8, is qualitatively as the one for $\lambda < 0$ with

$$\omega_1 = \int_{T_\gamma}^{\infty} \frac{dT}{\sqrt{Q_\lambda(T)}} = \int_{e_1}^{\infty} \frac{dy}{\sqrt{P_\lambda(y)}}. \tag{4.26}$$

The universe starts with $T = \infty$, grows and recontracts to a point within a finite time; it is still a Phoenix universe. This function $t(\tau)$ has complex logarithmic singularities (off the real axis) and is given by (4.22).

4.C.2. In the second subcase, the motion of the mechanical system (3.22) occurs in the interval $[T_\alpha, T_\beta]$; then the function $T(\tau)$ has no singularity on the real axis. We can use our freedom of choice for the origin $\tau = 0$ so that the function $T(\tau)$ has the shape given by Fig. 9. The corresponding formula is

$$\frac{1}{T(\tau)} - \frac{1}{T_\alpha} = \sqrt{\frac{3}{\lambda}}\,[\zeta(\tau - \tau_d) - \zeta(\tau + \tau_d) + 2\zeta(\tau_d)], \tag{4.27}$$

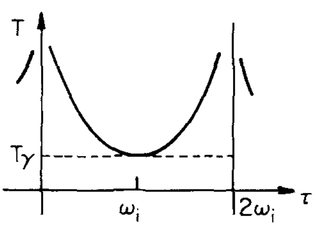

FIGURE 8

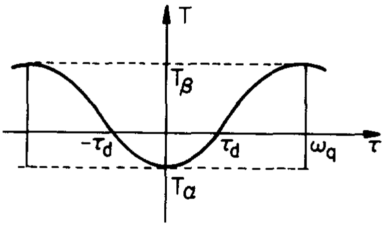

FIGURE 9

where

$$\tau_d = \int_{T_\alpha}^{0} \frac{dT}{\sqrt{Q_\lambda(T)}} = \int_{e_3}^{1/12} \frac{dy}{\sqrt{P_\lambda(y)}} \tag{4.28}$$

and

$$\omega_q = \int_{T_\alpha}^{T_\beta} \frac{dT}{\sqrt{Q_\lambda(T)}} = \int_{e_3}^{e_2} \frac{dy}{\sqrt{P_\lambda(y)}}. \tag{4.29}$$

If we consider this universe, e.g., starting with $\tau = -\omega_q$, we see that it starts with a finite size and expands forever, since $\tau = -\tau_d$ corresponds to $t = \infty$. For the sake of simplicity, we stick to the convention $t(0) = 0$, even though the moment $t = 0$ lies now in the unphysical region $T < 0$.

The expression for $t(\tau)$ is then

$$\left(\frac{\Lambda}{3}\right)^{1/2} t(\tau) = \tau \left[\left(\frac{\lambda}{3}\right)^{1/2} \frac{1}{T_\alpha} + 2\zeta(\tau_d) \right] + \ln \frac{\sigma(\tau_d - \tau)}{\sigma(\tau_d + \tau)}. \tag{4.30}$$

4.D. *Degenerate Cases*

4.D.1. *Case* $\alpha = 0$, $\lambda = 1$. This is a degenerate case, with $\Delta_\lambda = 0$, $g_2 = 1/12$, $g_3 = -1/6^3$. Consequently the Weierstrass functions can be expressed in terms of trigonometric of hyperbolic functions. On Fig. 3, we can distinguish three subcases.

(1) The point associated to the one-dimensional system arrives from the right (on Fig. 3) and reaches the local maximum with zero "velocity" $dT/d\tau = 0$ after an infinitely long interval of conformal time. The function $T(\tau)$ is

$$T(\tau) = 1 + \frac{3}{\operatorname{ch}\tau - 1} \tag{4.31}$$

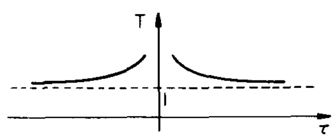

FIGURE 10

and is described on Fig. 10. The expression for he cosmic time is

$$t(\tau)=\tau-2\sqrt{3}\,\text{Argth}\left(\frac{1}{\sqrt{3}}\,\text{th}\,\frac{\tau}{2}\right)$$
$$=\tau-\frac{1}{\sqrt{3}}\ln\left|\frac{1+\dfrac{1}{\sqrt{3}}\,\text{th}\,\dfrac{\tau}{2}}{1-\dfrac{1}{\sqrt{3}}\,\text{th}\,\dfrac{\tau}{2}}\right|. \tag{4.32}$$

This case is obtained naturally as the limit $\Delta_\lambda \to 0$ of subcase 4.C.1 of the case $0<\lambda<1$, with $g_3<0$, i.e., $\lambda \to 1$ from below.

(2) The points associated to the one-dimensional system arrives from the left. The function $T(\tau)$ is given by

$$\frac{1}{T(\tau)}+\frac{1}{2}=\sqrt{\frac{3}{\lambda}}\cdot\frac{1}{2}\left[\coth\frac{\tau-\tau_d}{2}-\coth\frac{\tau+\tau_d}{2}+2\coth\frac{\tau_d}{2}\right], \tag{4.33}$$

where $\tau_d=\int_{-1/2}^{0}(dT/\sqrt{Q_\lambda(T)})$, which can be obtained as limiting case of subcase 4.C.2 of the case $0<\lambda<1$, with $g_3<0$, i.e., $\lambda\to 1$ from below.

The behaviour of this cold universe is described on Fig. 11.

The expression for the cosmic time is

$$\left(\frac{\Lambda}{3}\right)^{1/2} t(\tau)=\tau\left[\coth\frac{\tau_d}{2}-\frac{1}{2}\left(\frac{\lambda}{3}\right)^{1/2}\right]+\ln\frac{\text{sh}\,\dfrac{\tau d-\tau}{2}}{\text{sh}\,\dfrac{\tau d+\tau}{2}}. \tag{4.34}$$

(3) The point associated to the one-dimensional system stays in unstable equilibrium at the top of the local maximum in Fig. 3. The function $T(\tau)$ is now constant:

$$T(\tau)=1 \qquad \text{and} \qquad t=\frac{\tau}{\sqrt{\Lambda_c}}. \tag{4.35}$$

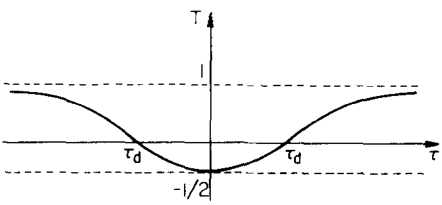

FIGURE 11

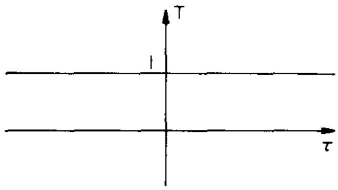

FIGURE 12

4.D.2. *Case* $\alpha = 0$, $\lambda = 0$. This is the case most often considered. One has $\Delta_\lambda = 0$, $g_2 = 1/12$, $g_3 = 1/6^3 > 0$, $e_2 = 1/6$ and

$$T(\tau) = \frac{3}{1 - \cos \tau}. \tag{4.36}$$

It resembles cases 4.B and 4.C.1; it has no inflection point. The cosmic time $t(\tau)$ is given by (4.37)

$$\sqrt{\Lambda_c}\, t(\tau) = \tfrac{1}{3}(\tau - \sin \tau). \tag{4.37}$$

Notice that $T(\tau)$, corresponding to the case where the point associated to the one-dimensional system stays in stable equilibrium at the dip of the local minimum at the origin, is also a solution of Friedman's equations. This is an infinite static universe.

5. Explicit Expressions for $T(\tau)$: Case with Radiation Pressure

5.A. *Generalities*

We now study Eq. (2.7) in general. The main qualitative feature that distinguishes the general $\alpha \neq 0$ from the special case $\alpha = 0$ is that, for $\alpha \neq 0$, the two poles of $T(\tau)$ are distinct. Consequently, each one of them is of first order, as can also be seen directly from the differential equation (Eq. (2.7)).

In contrast to the case $\alpha = 0$ and as can be seen from Fig. 2, there are now three critical values of λ namely 0, λ_- and λ_+. We discuss separately the different intervals in λ.

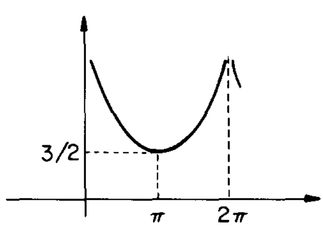

FIGURE 13

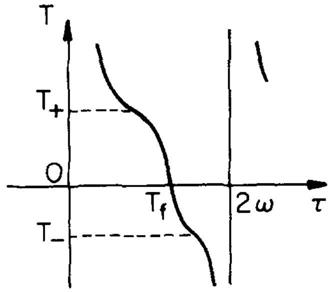

FIGURE 14

5.B. *Case* $\lambda > \lambda_+$

We have then $g_3 < 0$, $\Delta > 0$. The polynomial $P_{\alpha,\lambda}(y)$ has three real roots; the polynomial $Q_{\alpha,\lambda}(T)$ has no real roots. The movement of the associated mechanical system takes place between $T = +\infty$ and $T = -\infty$. The function $T(\tau)$ is described by Fig. 14. One of the poles of $T(\tau)$ is real (at $\tau = 0$). The other one is complex. Similarly $T(\tau)$ has one real and one complex zero τ_c in any fundamental periodicity cell.

We have

$$\sqrt{\frac{\lambda}{3}} \cdot T^{-1}(\tau) = [\zeta(\tau - \tau_c) + \zeta(\tau_c) - \zeta(\tau - \tau_f) + \zeta(\tau_f)] \tag{5.1}$$

with

$$\tau_f = \int_0^{\infty} \frac{dT}{\sqrt{Q_{\alpha,\lambda}(T)}}, \tag{5.2}$$

$$2\omega = \int_{-\infty}^{+\infty} \frac{dT}{\sqrt{Q_{\alpha,\lambda}(T)}}. \tag{5.3}$$

This case corresponds to a universe dominated by the interplay between cosmological constant and radiation pressure.

5.C. *Case* $\lambda_- < \lambda < \lambda_+$

By [7–11], this is the case of physical interest. Here $\Delta_{\alpha,\lambda} < 0$, $g_3 < 0$. The polynomial $Q_{\alpha,\lambda}(T)$ has two negative roots, T_α and T_b, with $T_a < T_b < 0$. (The root T_α and T_b, with $T_a < T_b < 0$. (The root T_a goes to $-\infty$ as $\alpha \to 0$; cf. Fig. 4). The mechanical system (Fig. 2) shows that $T(\tau)$ has the shape of Fig. 15.

$T(\tau)$ and $R(\tau)$ are given by

$$\sqrt{\frac{\Lambda}{3}} R(\tau) = \sqrt{\frac{\lambda}{3}} \frac{1}{T(\tau)} = \zeta(\tau - \tau_g) + \zeta(\tau_g) - \zeta(\tau - \tau_f) - \zeta(\tau_f), \tag{5.4}$$

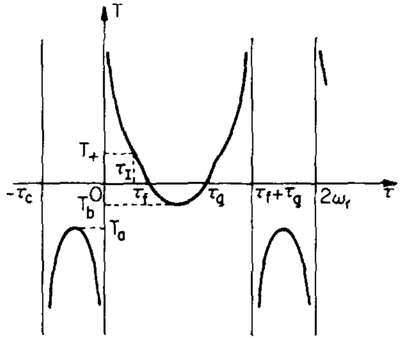

FIGURE 15

where

$$\tau_f = \int_0^{\infty} \frac{dT}{\sqrt{Q_{\alpha,\lambda}(T)}}, \tag{5.5}$$

$$\tau_g = \tau_f + 2\int_{T_b}^{0} \frac{dT}{\sqrt{Q_{\alpha,\lambda}(T)}}, \tag{5.6}$$

$$2\omega_r = \tau_f + \tau_g + 2\int_{-\infty}^{T_a} \frac{dT}{\sqrt{Q_{\alpha,\lambda}(T)}} = \tau_f + \tau_g + \tau_c. \tag{5.7}$$

In this case, which is particularly important by [7–11], an elementary periodicity cell is the rhombus 0, $\omega_r - i\omega_i$, $2\omega_r$, $\omega_r + i\omega_i$, where ω_i is given by

$$\omega_i = \int_{T_a}^{T_b} \frac{dT}{\sqrt{|Q_{\alpha,\lambda}(T)|}}. \tag{5.8}$$

Alternative expressions for $T(\tau)$ (compare Section 4.A) are

$$T(\tau) = c \cdot \frac{\sigma(\tau - \tau_f)\,\sigma(\tau - \tau_g)}{\sigma(\tau)\,\sigma(\tau - \tau_f - \tau_g)}, \tag{5.9}$$

where

$$c = T_b \frac{\sigma^2\left(\dfrac{\tau_f + \tau_g}{2}\right)}{\sigma^2\left(\dfrac{\tau_g - \tau_f}{2}\right)} \tag{5.10}$$

and

$$T(\tau) = T_a + \frac{1}{\sqrt{a}}\left[\zeta(\tau) - \zeta(\tau + \tau_c) + 2\zeta\left(\frac{\tau_c}{2}\right)\right]. \tag{5.11}$$

The cosmic time $t(\tau)$ is given by

$$\left(\frac{\Lambda}{3}\right)^{1/2} t(\tau) = [\zeta(\tau_g) - \zeta(\tau_f)]\tau + \ln \frac{\sigma(\tau_f)\,\sigma(\tau - \tau_g)}{\sigma(\tau_g)\,\sigma(\tau - \tau_f)}. \tag{5.12}$$

Equations (5.4) and (5.12) give a parametric representation of the curve $t \to R(t)$.

The universes starts with a big bang at $\tau = 0$; its qualitative behaviour resembles the zero radiation pressure case (4.A); however τ_f and τ_g are not symmetric with respect to ω_r, and there is an unphysical branch of $T(\tau)$ between $\tau_f + \tau_g$ and $2\omega_r$. Moreover, the inflection point is obtained at

$$\tau_I = \int_{T_+}^{\infty} \frac{dT}{\sqrt{Q_{\alpha,\lambda}(T)}}, \qquad T = T_+.$$

5.D. *Case* $0 < \lambda < \lambda_-$

We have now $\Delta_{\alpha,\lambda} > 0$; if $\lambda < 1/2(1+\alpha)$ then $g_3 > 0$; if $\lambda > 1/2(1+\alpha)$, then $g_3 < 0$. The three roots of $P_{\alpha,\lambda}(y)$ and the four roots of $Q_{\alpha,\lambda}(T)$ are all real, (see Fig. 2); we write: $T_a < T_b < 0 < T_c < T_d$. The motion of the associated mechanical system (3.22) can occur either in $]-\infty, T_a] \cup [T_d, +\infty[$ or in $[T_b, T_c]$.

5.D.1. In the first subcase, $T(\tau)$ is described by Fig. 16, and given by

$$T(\tau) = T_d + \frac{1}{\sqrt{\alpha}}\left[\zeta(\tau) - \zeta(\tau - \tau_h) - 2\zeta\left(\frac{\tau_h}{2}\right)\right] \tag{5.13}$$

$$= T_a + \frac{1}{\sqrt{\alpha}}\left[\zeta(\tau) - \zeta(\tau + \tau_c) + 2\zeta\left(\frac{\tau_c}{2}\right)\right], \tag{5.14}$$

with

$$\tau_h = 2\int_{T_d}^{\infty} \frac{dT}{\sqrt{Q_{\alpha,\lambda}(T)}}, \tag{5.15}$$

$$2\omega_1 = \tau_h + 2\int_{-\infty}^{T_a} \frac{dT}{\sqrt{Q_{\alpha,\lambda}(T)}} = \tau_h + \tau_c. \tag{5.16}$$

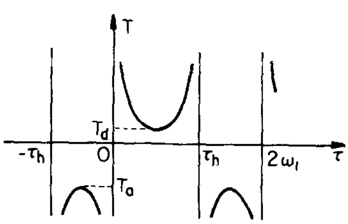

FIGURE 16

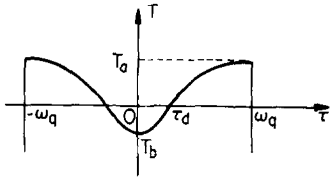

FIGURE 17

This universe re-contracts to a point after a finite interval of cosmic time. Its time function $t(\tau)$ has no real singularity.

5.D.2. The second subcase is described by Fig. 17, and given by

$$\frac{1}{T}=\frac{1}{T_b}+\sqrt{\frac{3}{\lambda}}\,[\zeta(\tau-\tau_d)-\zeta(\tau+\tau_d)+2\zeta(\tau_d)] \tag{5.17}$$

with

$$\tau_d=\int_{T_b}^{0}\frac{dT}{\sqrt{Q_{\alpha,\lambda}(T)}}. \tag{5.18}$$

5.E. *Case* $-\infty<\lambda<0$

Now $\Delta_{\alpha,\lambda}<0$ and $g_3>0$. The polynomial $Q_{\alpha,\lambda}(T)$ has two real zeroes $T_\alpha<0<T_d$. The discussion and the formulas are the same as in subcase 5.D.1 of the preceding case. The maximal "size" of the universe, corresponding to $R=1/\sqrt{\Lambda_c}\,T_d$ is now smaller than before.

5.F. *Degenerate Cases*

5.F.1. *Case* $\alpha\neq 0$, $\lambda=0$. This is the universe without cosmological constant, with matter and with radiation pressure. It can be obtained as a limit from the case $\lambda<0$ (see Section 5.E). We have here $g_3>0$. The behaviour of $T(\tau)$ is shown on Fig. 18. Here T_1 and T_2 are given by formula (3.24); the real period is $2\omega=2\pi$, and the imaginary period is infinite. We have

$$T(\tau)=T_2+\frac{1}{\sqrt{\alpha}}\left[\frac{1}{2}\operatorname{cotg}\frac{\tau}{2}+\frac{1}{2}\operatorname{cotg}\frac{\tau_h-\tau}{2}-\operatorname{cotg}\frac{\tau_h}{4}\right] \tag{5.19}$$

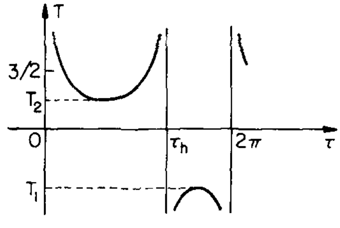

FIGURE 18

with

$$\tau_h = 2\int_{T_2}^{\infty} \frac{dT}{\sqrt{Q_{\alpha,0}(T)}}. \tag{5.20}$$

Another special case here is

$$T(\tau) = 0, \tag{5.21}$$

describing an infinite static universe.

5.F.2. *Case* $\lambda = \lambda_-$. The polynomial $Q_{\alpha,\lambda}(T)$ has two real simple roots T_a and T_b, and a real double root T_+ (given by (3.25)); with $T_a < T_b < 0 < T_+$. We have $g_3 < 0$. There are three subcases, depending on whether the motion of the associated mechanical system belongs to $]-\infty, T_a] \cup]T_+, \infty[$ or $[T_b, T_+[$ or to the one point set $\{T_+\}$.

(1) In the first subcase, the motion is a limiting case of subcase 5.D.1 of case $0 < \lambda < \lambda_-$ as $\lambda \to \lambda_-$. In that limit $\tau_h \to \infty$ and $\omega_1 \to \infty$ but $\tau_c = 2\omega_i - \tau_h$ stays finite. The function $T(\tau)$ is described by Fig. 19.

It is given by

$$T(\tau) = T_a + \sqrt{\frac{3c_-}{\alpha}}\left[\coth\sqrt{3c_-}\,\tau - \coth\sqrt{3c_-}\,(\tau+\tau_c) + 2\coth\frac{\sqrt{3c_-}\,\tau_c}{2}\right], \tag{5.22}$$

where

$$c_- = \frac{1}{12}\left[1 + \frac{\alpha\lambda_-}{4}\right]^{1/2} \tag{5.23}$$

and τ_c is given by (5.16). The real period is infinite and the imaginary period is $i\pi/\sqrt{12c_-}$.

(2) In the second subcase, the motion is in $[T_b, T_+[$. It is obtained as limiting case of subcase 5.D.2 of case $0 < \lambda < \lambda_-$ as $\lambda \to \lambda_-$. The curve $T(\tau)$ has the shape of Fig. 20, with

$$\frac{1}{T} = \frac{1}{T_b} + \sqrt{\frac{3}{\lambda_-}}\sqrt{3c_-}\,[\coth\sqrt{3c_-}\,(\tau-\tau_d) - \coth\sqrt{3c_-}\,(\tau+\tau_d) + 2\coth\sqrt{3c_-}\,\tau_d], \tag{5.24}$$

where c is given by (5.23) and τ_d by (5.18).

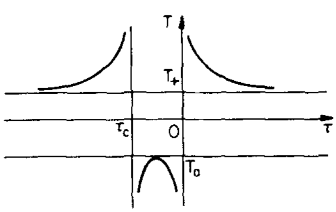

FIGURE 19

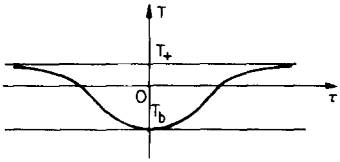

FIGURE 20

(3) The third (special) subcase is given by

$$T(\tau) = T_+ = \frac{-1 + \sqrt{1 + 8\alpha}}{4\alpha} \tag{5.25}$$

and corresponds to a static universe generalizing the Einstein universe $T = 1$. We see that there exists a stationary universe for every $\alpha > 0$ provided λ is chosen to be $\lambda_-(\alpha)$. (given by (3.18)).

5.F.3. *Case* $\lambda = \lambda_+$. This corresponds to $g_3 < 0$. It is obtained as the limit $\lambda \to \lambda_+$ of case 5.C, where $\lambda_- < \lambda < \lambda_+$. In this limit $\omega_r \to \infty$ and $\tau_g \to \infty$ but τ_f stays finite.

The function $T(\tau)$ has the appearance given by Fig. 21. Taking the limit $\tau_g \to \infty$ in (5.4) while keeping $\tau_g - \tau$ positive, we obtain

$$\sqrt{\frac{\lambda_+}{3}} \cdot \frac{1}{T} = \sqrt{3c_+}\,[\coth \sqrt{3c_+}\,(\tau_f - \tau) - \coth \sqrt{3c_+}\,\tau_f], \tag{5.26}$$

where

$$\tau_f = \int_0^\infty \frac{dT}{\sqrt{Q_{\alpha,\lambda}(T)}} \tag{5.27}$$

and

$$c_+ = \frac{1}{12}\left[1 + \frac{\alpha\lambda_+}{4}\right]. \tag{5.28}$$

There is also an unphysical stationary solution $T(\tau) = T_- < 0$.

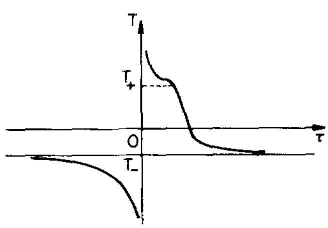

FIGURE 21

6. Density and Deceleration Parameters

6.A. *Relations between Cosmological Parameters and Observable Quantities*

6.A.1. *Case without radiation pressure* ($\alpha = 0$). *Dimensionless quantities*: If there is no radiation pressure, a Friedman universe is determined by the single parameter λ. A further dimensionless parameter is needed to specify a time of observation (the present-day universe); we choose τ_0, the present-day value of the conformal time.

The values of all dimensionless quantities at the moment τ are functions of λ and τ. Instead of specifying λ and τ, we may specify the values of any two independent functions of them. Traditionally, the choice is $\Omega(\tau)$ and $q(\tau)$, where $\Omega(\tau)$ is defined by (2.14) and $q(\tau)$ is defined by (2.17).

Given Ω and q, we can compute

$$T = \frac{3}{2} \frac{\Omega}{\frac{3\Omega}{2} - q - 1}, \tag{6.1}$$

$$\lambda_s = \frac{\Omega}{2} - q, \tag{6.2}$$

$$k_s = \frac{3\Omega}{2} - q - 1, \tag{6.3}$$

and

$$\lambda = \frac{27}{4} \frac{\lambda_s \Omega^2}{k_s^3} = \frac{27}{4} \frac{\left(\frac{\Omega}{2} - q\right) \Omega^2}{\left(\frac{3\Omega}{2} - q - 1\right)^3}. \tag{6.4}$$

The dimensionless age $[Ht](\tau)$ is given by

$$[Ht](\tau) = \left[\frac{2}{3} T^3 - T^2 + \frac{\lambda}{3}\right]^{1/2} \cdot \int_0^{\tau} \frac{d\tau'}{T(\tau')}. \tag{6.4'}$$

From the known value of T and from the expressions for $T(\tau)$ given in Section 4, one can obtain τ. For example, if the value of λ, obtained from (6.4) satisfies the inequality $\lambda > 1$ then τ can be found as the smaller of the two solutions of

$$\wp(\tau) + \frac{1}{12} = \frac{T}{6}, \tag{6.5}$$

where T is given by (6.1).

Equivalently, τ is given by

$$\tau = \tau_f - \int_{-1/12}^{-1/12 + T/6} P_\lambda(y)^{-1/2}\, dy = \tau_f + \int_0^T Q_\lambda(T')^{-1/2}\, dT'. \tag{6.6}$$

Quantities with dimension: If the value of H (a scale) is known from observation, then (2.11) (3.3) give

$$\Lambda_c = \frac{H^2}{\frac{2}{3}T^3 - T^2 + \frac{\lambda}{3}}, \tag{6.7}$$

where T and λ are given in terms of Ω, q by (6.1) and (6.4), respectively.

Consequently, all other quantities such as $R(\tau)$ and $t(\tau)$ are now determined by (2.6), (2.23). Notice that in the special case (usually considered) $\lambda = 0$, we have:

$$\Omega = 2q, \qquad \lambda_s = 0, \qquad k_s = 2q - 1, \qquad \text{and} \qquad T = \frac{3q}{2q-1}.$$

6.A.2. *Case with radiation pressure* ($\alpha \neq 0$). Since our knowledge of α comes from measuring ρ_{rad} in terrestrial units it is not possible, now, to determine dimensionless parameters independently of scale (H).

Suppose H has been obtained by suitable measurements, suppose also that measurements of blackbody radiation have given its temperature $\tilde{T}$, so that

$$\rho_{\mathrm{rad}} = 4\sigma \tilde{T}^4, \tag{6.8}$$

where

$$\sigma = \frac{\pi^2 k^4}{60} \tag{6.9}$$

is the Stefan–Boltzmann constant (in units $\hbar = c = 1$). We have also

$$\rho_{\mathrm{rad}} = \frac{3}{8\pi G}\alpha \Lambda_c T^4 \tag{6.10}$$

and consequently

$$\tilde{T}^4 = \frac{3}{8\pi G}\frac{\alpha\Lambda_c}{4\sigma}T^4. \tag{6.11}$$

However, by (2.13)

$$\alpha \Lambda_c T^4 = \alpha_s H^2. \tag{6.12}$$

Consequently

$$\alpha_s = \frac{32\pi G \sigma \tilde{T}^4}{3H^2}. \tag{6.13}$$

Now, given Ω, q and α_s, we can compute

$$T = \frac{3}{2} \frac{\Omega}{\left[\frac{3}{2}\Omega - q - 1 + 2\alpha_s\right]}, \tag{6.14}$$

$$\lambda_s = -q + \frac{\Omega}{2} + \alpha_s, \tag{6.15}$$

$$k_s = \frac{3}{2}\Omega - q - 1 + 2\alpha_s, \tag{6.16}$$

$$\lambda = \frac{27}{4} \frac{\lambda_s \Omega^2}{k_s^3} = \frac{27}{4} \frac{\left(-q + \frac{\Omega}{2} + \alpha_s\right)\Omega^2}{\left(\frac{3}{2}\Omega - q - 1 + 2\alpha_s\right)^3}, \tag{6.17}$$

$$\alpha = \frac{4}{9} \frac{\alpha_s k_s}{\Omega^2}. \tag{6.18}$$

If the value of λ corresponds to the case 5.C, the value of τ (corresponding to the present), is now given by

$$\tau = \tau_f + \int_0^T \frac{dT'}{\sqrt{Q_{\alpha,\lambda}(T')}}, \tag{6.19}$$

where T is given by (6.14) and τ_f by (5.5).

Finally, formulas (2.11) and (2.7) give

$$\Lambda_c = \frac{H^2}{\alpha T^4 + \frac{2}{3}T^3 - T^2 + \frac{\lambda}{3}}, \tag{6.20}$$

where T, λ and α are given by (6.14), (6.17) and (6.18), respectively.

Other quantities with dimension such as $R(\tau)$ and $t(\tau)$ are now determined by (2.6) and (2.23). Notice that in this paragraph we neglect a possible contribution from neutrinos.

6.B. *The Behaviour of $q(\tau)$ and $\Omega(\tau)$*

As we mentioned above, (Section 3A) $q(\tau)$ and $\Omega(\tau)$ are elliptic functions of order six. Their period parallelogram is, of course, the same as that of $T(\tau)$. In this section, we shall discuss their behaviour, limiting ourselves to solutions with a big bang, and considering mostly the "physical region" where $T(\tau)$ is positive.

6.B.1. *Case $\alpha = 0$.* We distinguish between the ever-expanding universe ($\lambda > 1$) and the universe which will eventually recontract ($\lambda < 1$). (Remember that all universes discussed here are spatially closed,) The formulas expressing Ω and q in terms of $T(\tau)$ are

$$\Omega(\tau) = \frac{\frac{2}{3} T^3}{\frac{2}{3} T^3 - T^2 + \frac{\lambda}{3}} = \frac{2}{3} \frac{T^3}{\left(\frac{dT}{d\tau}\right)^2}, \tag{6.21}$$

$$q(\tau) = \frac{T^3 - \lambda}{3\left[\frac{2}{3} T^3 - T^2 + \frac{\lambda}{3}\right]} = -1 + T \frac{\frac{d^2 T}{d\tau^2}}{\left(\frac{dT}{d\tau}\right)^2}. \tag{6.22}$$

Notice that:

(1) Ω and q have a pole at ω_r, where T has a minimum.

(2) Ω vanishes at the zeroes of T while q vanishes at the points where $T^3 = \lambda$. The function $1 + q$ vanishes at the points where $T^3 = \lambda$. The function $1 + q$ vanishes at the zeroes of T and at the inflection points of T.

(3) In the neighbourhood of the big bang $t = 0$, Ω behaves as

$$\Omega \sim 1 + \frac{\tau^2}{4} \sim 1 + \frac{3}{2T}, \tag{6.23}$$

while q behaves as

$$q \sim \frac{1}{2} + \frac{3}{8} \tau^2 \sim \frac{1}{2} + \frac{9}{4T}. \tag{6.24}$$

Consequently $\Omega > 1$ for sufficiently small τ. A straighforward calculation of $d\Omega/d\tau$ shows that if $\lambda < 1$, then Ω is monotonically increasing in the interval $0 < \tau < \omega$, has a pole at $\tau = \omega$ and is bigger than one throughout the physical region $0 < \tau < 2\omega$ (see Fig. 22).

If $\lambda > 1$, then Ω has a maximum

$$\Omega_{\max} = \frac{\sqrt{\lambda}}{\sqrt{\lambda} - 1}, \tag{6.25}$$

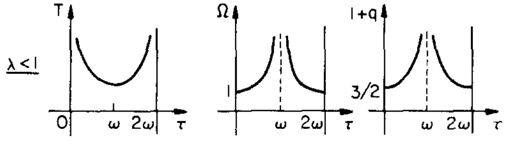

FIGURE 22

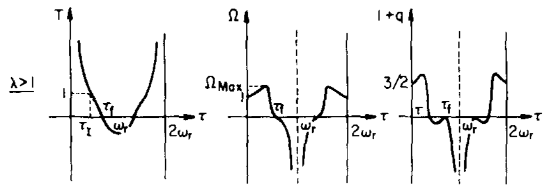

FIGURE 23

corresponding to the reduced temperature $T = \sqrt{\lambda} > 1$; this occurs after the inflection point $T = 1$. Afterwards Ω decreases and vanishes at $\tau = \tau_f$. (See Fig. 23.)

We would like to point out the fact that *if* $\lambda < 1$ (in particular if $\lambda = 0$), then an observation of $\Omega < 1$ implies that the universe is not spatially closed. However, if $\lambda > 1$, then $\Omega < 1$ does not allow us to conclude anything about the spatial openness of the universe.

If $\lambda < 1$, then the behaviour of $1 + q$ is qualitatively similar to the behaviour of Ω. (see Fig. 22).

If $\lambda > 1$, the behaviour of $1 + q$ is more complicated (see Fig. 23). Notice that $1 + q = 0$ at $\tau = \tau_I$ corresponding to $T(\tau_I) = 1$ (inflection point) and at the end of time τ_f.

We shall now discuss the question of determining the parameters of a Friedman universe (without radiation) from independent measurements of Ω and q. Such a universe can be described by

(1) Its spatial curvature ($k = \pm 1$, or 0).

(2) The "type" of the Friedmann solution (value of λ).

(3) Its conformal age (present-day value of τ).

Finally,

(4) A unit of length (or of time) that allows a change from cosmological quantities to the standard meter.

Independent measures of Ω and q allow us to determine 1, 2, and 3, as we shall see, while a measure of the present day value $H_0 = H(\tau_0)$ of Hubble's function (i.e., of Λ_c) gives an answer to 4.

The discussion of 1, 2, 3 which we shall give now, is of course not new. (Compare, e.g., [6]).

Given present-day values of Ω and of q, the constant λ is determined by (6.4), i.e.,

$$\lambda = \frac{27}{4} \frac{\left(\frac{\Omega}{2} - q\right) \Omega^2}{\left[\frac{3}{2}\Omega - q - 1\right]^3}. \tag{6.26}$$

The present-day value of τ is found by inverting the function $T(\tau)$ using the formulas of Section 4, with T given by (6.1).

In order to discuss the spatial curvature of the universe, notice that, in general k_s is defined not by (2.16) but by $k_s(\tau) = kR^{-2}H^{-2}$, where k is ± 1 or 0. Formula (6.3) holds in all cases, and consequently k is the sign of $(3\Omega/2) - q - 1$.

Returning to the case $k = +1$, notice that the sign of λ is the same as the sign of $\Omega - 2q$.

Finally, by (6.26) we see that $\lambda > 1$ is equivalent to

$$\frac{27}{4}\Omega^2 - \frac{9}{2}\Omega(q+1)^2 + (q+1)^3 > 0. \tag{6.27}$$

The quation obtained by setting the l.h.s. of (6.27) equal to zero has two roots, $\Omega_+(q)$ and $\Omega_-(q)$; the curve $\Omega_-(q)$ lies entirely in the region of spatially open universes. We consider only

$$\Omega_+(q) = \frac{(q+1)^3}{3}\left[1 + \left(1 - \frac{4}{3(1+q)}\right)^{1/2}\right]. \tag{6.28}$$

The condition $\Omega > \Omega_+(q)$ is equivalent to $\lambda > 1$.

The above discussion is summarized in Fig. 24, where for the sake of completeness we have included some results pertaining to the spatially open case.

Notice that $q + 1 < 0$ implies that the universe is spatially closed $(k = +1)$ and that $\lambda > 1$, independently of Ω.

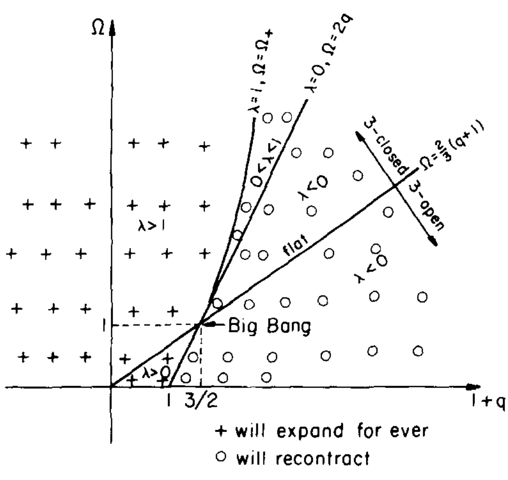

FIGURE 24

6.B.2. *Case $\alpha \neq 0$.* We have now

$$\Omega(\tau) = \frac{\frac{2}{3}T^3}{\alpha T^4 + \frac{2}{3}T^3 - T^2 + \frac{\lambda}{3}}, \tag{6.29}$$

$$q(\tau) = \frac{1}{3}\frac{3\alpha T^4 + T^3 - \lambda}{\alpha T^4 + \frac{2}{3}T^3 - T^2 + \frac{\lambda}{3}}. \tag{6.30}$$

The behaviour of Ω and q as $\tau \to 0$ is of course now different from that in the case $\alpha = 0$ (see (6.23) and (6.24)). We have now

$$\Omega \sim \frac{2}{3\alpha}\frac{1}{T} \sim \frac{2}{3}\frac{\tau}{\sqrt{\alpha}}, \tag{6.31}$$

which does not go continuously into the expression (6.23) as $\alpha \to 0$

$$q \sim 1 + \frac{1}{3\alpha T} \sim 1 + \frac{\tau}{3\sqrt{\alpha}}. \tag{6.32}$$

The discussion can schematically described by Fig. 25.

In the physically interesting case $\lambda_- < \lambda < \lambda_+$, $\Omega_{\max}$ in Fig. 25 corresponds to the reduced temperature

$$T = \left[\frac{-1 + \sqrt{1 + 4\alpha\lambda}}{2\alpha}\right]^{1/2} > 1.$$

The spatial curvature of the universe depends on the sign of (6.16).

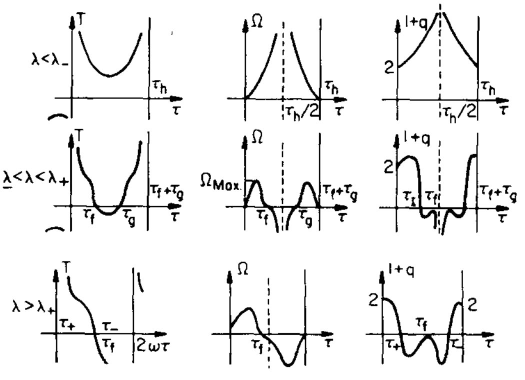

FIGURE 25

7. Remarks on Numerical Evaluation and Approximate Expressions

Friedman's equation (2.1) or (2.7) can, of course, be integrated numerically. If one prefers to use the formulae of this paper and evaluate them, the procedure splits naturally into two parts:

(a) Determination of parameters, such as ω_r, ω_i, τ_f,... corresponding to a given model.

(b) Evaluation of Weierstrass functions.

We shall discuss the second question first and give three methods for the effective calculation of Weierstrass functions in the range of interest. (As far as we know, the most general Weierstrass functions are not tabulated and are not available on computer programs because they contain too many parameters.)

7.A. *Duplication Procedure*

Given the invariants g_2, g_3 (which can be found, e.g., by (3.10), (3.11)) a convenient recursive procedure starts with the duplication formulas (see, e.g., [18]).

$$\wp(2\tau) = -2\wp(\tau) + \frac{(6\wp^2(\tau) - \frac{1}{2}g^2)^2}{4(4\wp^3(\tau) - g_2\wp(\tau) - g_3)}, \tag{7.1}$$

$$\zeta(2\tau) = 2\zeta(\tau) - \frac{6\wp^2(\tau) - \frac{1}{2}g_2}{2(4\wp_3(\tau) - g_2\wp(\tau) - g_3)^{1/2}}, \tag{7.2}$$

$$\sigma(2\tau) = (\sigma(\tau))^4 \cdot (4\wp^3(\tau) - g_2\wp(\tau) - g_3)^{1/2}. \tag{7.3}$$

These formulas allow the calculation of Weierstrass functions starting with their values for small arguments. For small enough τ, the Weierstrass functions are well approximated by their dominant terms

$$\wp(\tau) \sim \frac{1}{\tau^2}, \tag{7.4}$$

$$\zeta(\tau) \sim \frac{1}{\tau}, \tag{7.5}$$

$$\sigma(\tau) \sim \tau. \tag{7.6}$$

This iterative method is well adapted to use on a programmable pocket calculator.

7.B. *Reduction to Jacobi Functions*

Weierstrass functions can be expressed in terms of Jacobi functions, which are available in most computer libraries.

If $\Delta_{\alpha,\lambda} < 0$ and if e_2 denotes the real root of $P_{\alpha,\lambda}(y)$ we have

$$\wp(x) = e_2 + H_2 \frac{1 + \mathrm{cn}(x'|m)}{1 - \mathrm{cn}(x'|m)}, \tag{7.7}$$

$$\wp'(x) = -4H_2^{3/2}\mathrm{sn}(x'|m)\,\mathrm{dn}(x|m)[1 - \mathrm{cn}(x'|m)]^{-2}, \tag{7.8}$$

with

$$H_2^2 = 3e_2^2 - \frac{1}{4} g_2, \tag{7.9}$$

$$x' = 2x\sqrt{H_2}, \tag{7.10}$$

$$m = \frac{1}{2} - \frac{3}{4}\frac{e_2}{H_2}. \tag{7.11}$$

If $\Delta_{\alpha,\lambda} > 0$ and if the roots of $P_{\alpha,\lambda}(y)$ are ordered by

$$e_1 > e_2 > e_3$$

then

$$\wp(x) = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2(x'|m)}, \tag{7.12}$$

$$\wp'(x) = -2(e_1 - e_3)^{3/2}\,\mathrm{cn}(x'|m)\,\mathrm{dn}(x'|m)\,\mathrm{sn}^3(x'|m), \tag{7.13}$$

with

$$x' = (e_1 - e_3)^{1/2}\,x \tag{7.14}$$

and

$$m = \frac{e_2 - e_3}{e_1 - e_3} \tag{7.15}$$

(see [20]).

In order to evaluate expressions containing ζ we can use, e.g., the formula

$$\zeta(n-a) - \zeta(n-b) - \zeta(a) - \zeta(b) = \frac{1}{2}\left[\frac{\wp'(x) + \wp'(a)}{\wp(x) + \wp(a)} - \frac{\wp'(x) + \wp'(b)}{\wp(x) + \wp(b)}\right]. \tag{7.16}$$

7.C. *Fourier Expansions*

For the sake of definiteness, we restrict ourselves to the case $\Delta_{\alpha,\lambda} < 0$, where the fundamental parallelogram is a rhombus. Assume that ω_r and ω_i are given. They can be calculated by (4.7), (5.7) and (4.4), (5.8), respectively.

The method consists of subtracting from a Weierstrass function a trigonometric function with the same singularities and principal parts on the real axis. The remainder can then be expanded in a Fourier series that converges exponentially fast on a strip around the real axis. See (7.21) and (7.23).

Similarly, one can subtract singularities on the imaginary axis, and obtain alternative expressions that converge for $|\mathrm{Re}\,\tau| < \omega_r$, which is amply sufficient for our purposes. See (7.22) and (7.24).

The formulae below follow, e.g., from the discussion in Chapter 4 of Ref. [50]. They are apparently not listed in the standard references [18, 20].

Define

$$s = -e^{-\pi\omega_i/\omega_r}, \tag{7.17}$$

$$a_n = \frac{s^n}{1-s^n} \qquad (n = 1, 2,...), \tag{7.18}$$

and

$$h = -e^{-\pi\omega_r/\omega_i}, \tag{7.19}$$

$$b_n = \frac{h^n}{1-h^n} \qquad (n = 1, 2,...). \tag{7.20}$$

Then $\wp(\tau) = \wp(\tau \mid \omega_r + i\omega_i, 2\omega_r)$ is given by

$$\wp(\tau) = \left(\frac{\pi}{\omega_r}\right)^2 \left[\left(2\sin\frac{\pi\tau}{2\omega_r}\right)^{-2} - \frac{1}{12} + 2\sum_{n=1}^{\infty} na_n\left(1-\cos\frac{n\pi\tau}{\omega_r}\right)\right]. \tag{7.21}$$

It is also given by

$$\wp(\tau) = \left(\frac{\pi}{\omega_i}\right)^2 \left[\left(2\sinh\frac{\pi\tau}{2\omega_i}\right)^{2} + \frac{1}{12} + 2\sum_{n=1}^{\infty} nb_n\left(\cosh\frac{n\pi\tau}{\omega_i}-1\right)\right]. \tag{7.22}$$

The expression (7.22) is convenient in the discussion of the limit $\lambda \to 1$, when $\omega_r \to \infty$ (see Section 4D). An analogous expression for $\Delta_{\alpha,\lambda} > 0$, can be used to discuss the limit $\lambda \to 0$.

The expressions for ζ are

$$\zeta(\tau) = \left(\frac{\pi}{\omega_r}\right)\left[\frac{1}{2}\cot\frac{\pi\tau}{2\omega_r} + \left(\frac{1}{12} - 2\sum_{n=1}^{\infty} na_n\right)\left(\frac{\pi\tau}{\omega_r}\right) + 2\sum_{n=1}^{\infty} a_n \sin\frac{n\pi\tau}{\omega_r}\right] \tag{7.23}$$

and

$$\zeta(\tau) = \left(\frac{\pi}{\omega_i}\right)\left[\frac{1}{2}\coth\frac{\pi\tau}{2\omega_i} - \left(\frac{1}{12} - 2\sum_{n=1}^{\infty} nb_n\right)\left(\frac{\pi\tau}{\omega_i}\right) - 2\sum_{n=1}^{\infty} b_n \sinh\frac{n\pi\tau}{\omega_i}\right]. \tag{7.24}$$

Notice the appearance of a term linear in τ.

Finally, σ is expressed in terms of θ-series that converge even faster than (7.21) to (7.24): we have

$$\sigma(\tau) = Ce^{\zeta(\omega_r)\tau^2/2\omega_r} \sum_{n=0}^{\infty} (-1)^n s^{n(n+1)/2} \sin \frac{(2n+1)\pi\tau}{2\omega_r} \tag{7.25}$$

with

$$C = \frac{2\omega_r}{\pi} \left(\sum_{n=0}^{\infty} (2n+1)(-1)^n s^{n(n+1)/2} \right)^{-1}. \tag{7.26}$$

Notice that C drops out of expressions involving only ratios of σ at different arguments, and is immaterial for the purposes of this paper (see, e.g., (5.9) or (5.12)). The number $\zeta(\omega_r)$, appearing in (7.25), is evaluated by (7.23) to

$$\zeta(\omega_r) = \frac{\pi^2}{\omega_r} \left(\frac{1}{12} - 2 \sum_{n=1}^{\infty} na_n \right). \tag{7.27}$$

We have also

$$\sigma(\tau) = C'e^{-\tilde{\zeta}(\omega_i)\tau^2/2\omega_i} \sum_{n=0}^{\infty} (-1)^n h^{n(n+1)/2} \sinh \frac{(2n+1)\pi\tau}{2\omega_i} \tag{7.28}$$

with

$$C' = \frac{2\omega_i}{\pi} \left(\sum_{n=0}^{\infty} (2n+1)(-1)^n h^{n(n+1)/2} \right)^{-1} \tag{7.29}$$

and

$$\tilde{\zeta}(\omega_i) = \frac{\pi^2}{\omega_i} \left[\frac{1}{12} - 2 \sum_{n=1}^{\infty} nb_n \right],$$

i.e.,

$$\omega_r \tilde{\zeta}(\omega_i) + \omega_i \zeta(\omega_r) = \pi. \tag{7.30}$$

7.D. *Determination of Parameters*

For any given model, the values of the periods and τ_f, τ_g, etc., are given by integrals shown in previous sections. In some cases, it is possible to recast these integrals into the form of a standard elliptic integral. For example, if one wants to compute the present-day value τ_0 of the conformal time, given the knowledge of the reduced temperature T_0 (and supposing that we are interested in the case where $\lambda > 1$ with $\alpha = 0$), we can proceed in one of the following ways.

— Find the smallest positive root of the equation

$$\wp(\tau_0) + \frac{1}{12} = \frac{T_0}{6}. \tag{7.31}$$

— Compute the integral

$$\tau_0 = \int_\infty^{T_0} \frac{dT}{\sqrt{Q_\lambda(T)}}. \tag{7.32}$$

— Compute the standard elliptic integral [60]

$$\tau_0 = \frac{1}{2\sqrt{H_2}} \int_0^\varphi \frac{d\theta}{[1 - m \sin^2 \theta]^{1/2}}, \tag{7.33}$$

where

$$\varphi = \operatorname{Arc\,cos} \left\{ \frac{\frac{1}{H_2}\left[\frac{T_0}{6} - e_2\right] - 1}{\frac{1}{H_2}\left[\frac{T_0}{6} - e_2\right] + 1} \right\}, \tag{7.34}$$

where H_2, m are defined in (7.9), (7.11) and e_2 in (3.14), (4.1).

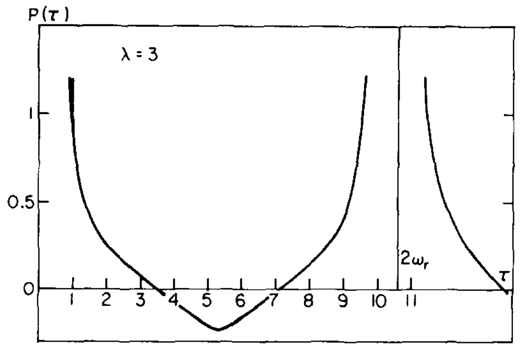

FIGURE 26

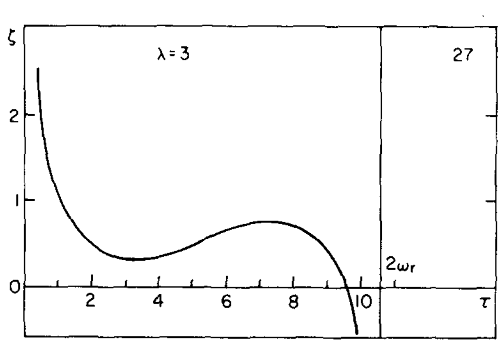

FIGURE 27

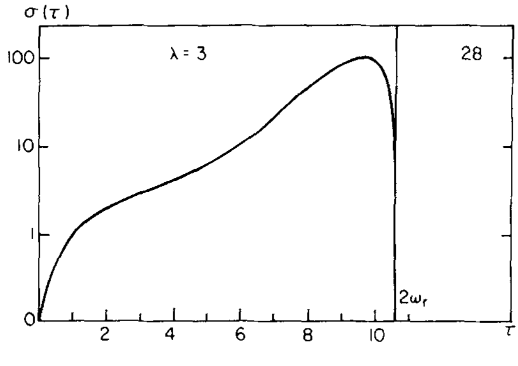

FIGURE 28

The curves $\wp(\tau)$, $\zeta(\tau)$ and $\sigma(\tau)$ are given in Figs. 26, 27, 28, respectively, for values of g_2, g_3 given by (3.5), (3.6), and with $\lambda = 3$ (which is close to the values given in [7–11]).

8. Numerical Evaluation of Some Models (with $\alpha = 0$ and $\lambda > 1$)

8.A. $\tau_f(\lambda)$, $\omega_r(\lambda)$, $\omega_i(\lambda)$

By the preceding paragraphs, the quantities of interest can easily be calculated if we know the periods ω_r, ω_i and τ_f, as functions of λ. These quantities, as well as the real zero e_2 of $P_\lambda(y)$ and $h = -e^{-\pi(\omega_r/\omega_i)}$, $s = -e^{-\pi(\omega_i/\omega_r)}$ are given in Table I and Fig. 29.

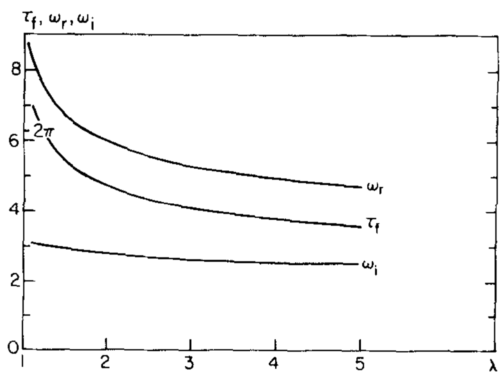

FIGURE 29

TABLE I

LAMBDA	$E2$	TAUF	OMEGAREEL	OMEGAIMAG	$\|H\|$	$\|S\|$
0.110E+01	−0.170E+00	0.702E+01	0.832E+01	0.310E+01	0.218E−03	0.310E+00
0.120E+01	−0.174E+00	0.631E+01	0.761E+01	0.307E+01	0.418E−03	0.281E+00
0.130E+01	−0.177E+00	0.590E+01	0.719E+01	0.304E+01	0.589E−03	0.265E+00
0.140E+01	−0.180E+00	0.561E+01	0.689E+01	0.300E+01	0.728E−03	0.255E+00
0.150E+01	−0.183E+00	0.539E+01	0.667E+01	0.297E+01	0.872E−03	0.246E+00
0.160E+01	−0.186E+00	0.521E+01	0.648E+01	0.295E+01	0.995E−03	0.240E+00
0.170E+01	−0.189E+00	0.506E+01	0.632E+01	0.292E+01	0.110E−02	0.235E+00
0.180E+01	−0.191E+00	0.493E+01	0.619E+01	0.289E+01	0.120E−02	0.230E+00
0.190E+01	−0.194E+00	0.482E+01	0.608E+01	0.287E+01	0.129E−02	0.227E+00
0.200E+01	−0.196E+00	0.472E+01	0.597E+01	0.285E+01	0.138E−02	0.223E+00
0.210E+01	−0.199E+00	0.463E+01	0.588E+01	0.283E+01	0.146E−02	0.221E+00
0.220E+01	−0.201E+00	0.455E+01	0.579E+01	0.281E+01	0.154E−02	0.218E+00
0.230E+01	−0.203E+00	0.448E+01	0.571E+01	0.280E+01	0.164E−02	0.215E+00
0.240E+01	−0.206E+00	0.441E+01	0.565E+01	0.278E+01	0.169E−02	0.213E+00
0.250E+01	−0.208E+00	0.435E+01	0.558E+01	0.286E+01	0.173E−02	0.212E+00
0.260E+01	−0.210E+00	0.429E+01	0.552E+01	0.275E+01	0.183E−02	0.209E+00
0.270E+01	−0.212E+00	0.424E+01	0.546E+01	0.273E+01	0.189E−02	0.207E+00
0.280E+01	−0.214E+00	0.419E+01	0.541E+01	0.271E+01	0.190E−02	0.207E+00
0.290E+01	−0.216E+00	0.414E+01	0.536E+01	0.269E+01	0.194E−02	0.206E+00
0.300E+01	−0.218E+00	0.410E+01	0.531E+01	0.268E+01	0.200E−02	0.204E+00
0.310E+01	−0.220E+00	0.406E+01	0.526E+01	0.267E+01	0.204E−02	0.203E+00
0.320E+01	−0.221E+00	0.402E+01	0.522E+01	0.267E+01	0.212E−02	0.201E+00
0.330E+01	−0.223E+00	0.398E+01	0.518E+01	0.265E+01	0.213E−02	0.201E+00
0.340E+01	−0.225E+00	0.395E+01	0.514E+00	0.263E+01	0.216E−02	0.200E+00
0.350E+01	−0.227E+00	0.391E+01	0.511E+01	0.262E+01	0.220E−02	0.199E+00
0.360E+01	−0.229E+00	0.388E+01	0.507E+01	0.262E+01	0.226E−02	0.198E+00
0.370E+00	−0.230E+00	0.385E+01	0.504E+01	0.260E+01	0.227E−02	0.198E+00
0.380E+01	−0.232E+00	0.382E+01	0.501E+01	0.259E+01	0.230E−02	0.197E+00
0.390E+01	−0.234E+00	0.380E+01	0.498E+01	0.258E+01	0.233E−02	0.196E+00
0.400E+01	−0.235E+00	0.377E+01	0.495E+01	0.257E+01	0.236E+02	0.196E+00
0.410E+01	−0.237E+00	0.374E+01	0.492E+01	0.256E+01	0.239E−02	0.195E+00
0.420E+01	−0.238E+00	0.372E+01	0.489E+01	0.255E+01	0.242E−02	0.194E+00
0.430E+01	−0.240E+00	0.369E+01	0.487E+01	0.254E+01	0.244E−02	0.194E+00
0.440E+01	−0.241E+00	0.367E+01	0.484E+01	0.253E+01	0.247E−02	0.193E+00
0.450E+01	−0.243E+00	0.365E+01	0.482E+01	0.252E+01	0.249E−02	0.193E+00
0.460E+01	−0.244E+00	0.363E+00	0.479E+01	0.251E+01	0.250E−02	0.192E+00
0.470E+01	−0.246E+00	0.361E+01	0.477E+01	0.250E+01	0.252E−02	0.192E+00
0.480E+01	−0.247E+00	0.359E+00	0.475E+01	0.251E+01	0.259E−02	0.191E+00
0.490E+01	−0.249E+00	0.357E+01	0.473E+01	0.249E+01	0.257E−02	0.191E+00

8.B. *Plots*

In order to make concrete the discussion above, we give now curves corresponding to various values of λ, $\lambda = 1.6$ and $\lambda = 3$. This curves are associated to the functions $T(\tau)$, $\Omega(\tau)$, $1 + q(\tau)$, $\lambda_s(\tau)$, $k_s(\tau)$, $[Ht](\tau)$, $\sqrt{\Lambda/3}\,R(\tau)$, $\sqrt{\Lambda/3}\,t(\tau)$ and $(H/\sqrt{\Lambda})(\tau)$; (Figs. 30–37). Tables of these quantities, corresponding to a range of values of λ, are available in microfiche form from the authors.

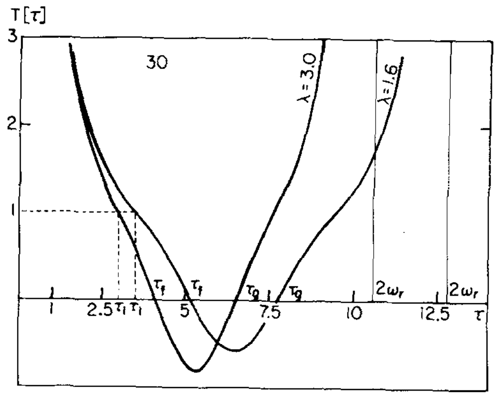

FIGURE 30

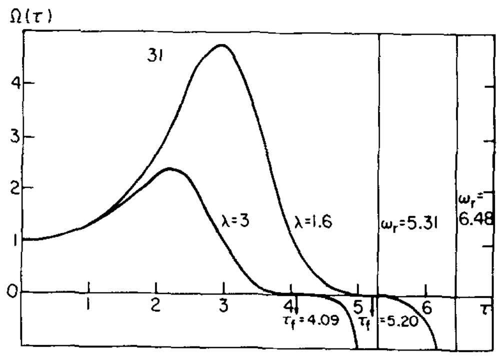

FIGURE 31

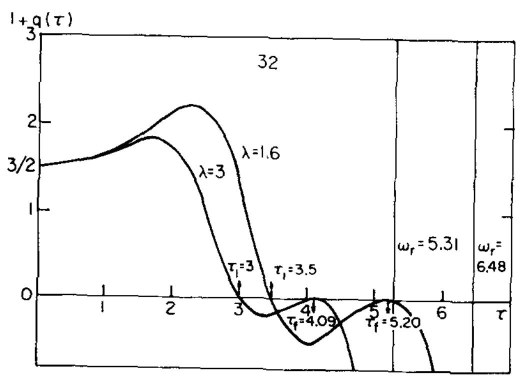

FIGURE 32

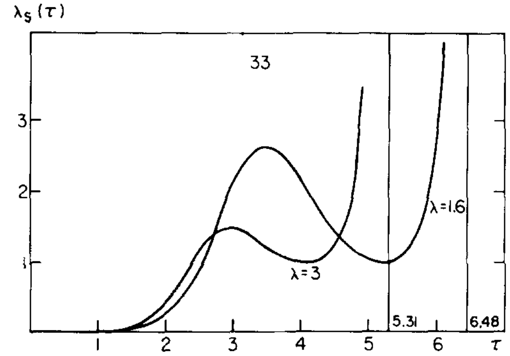

FIGURE 33

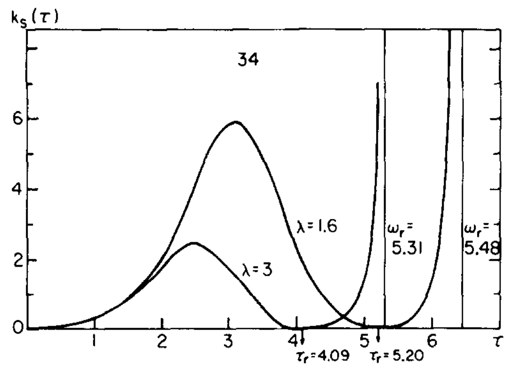

FIGURE 34

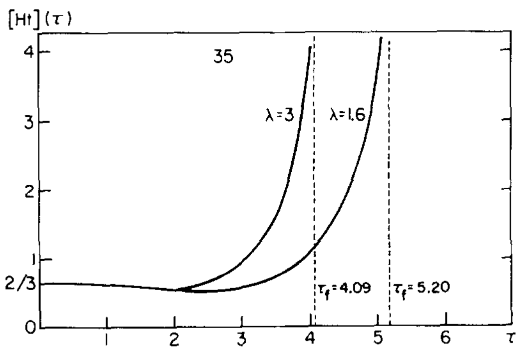

FIGURE 35

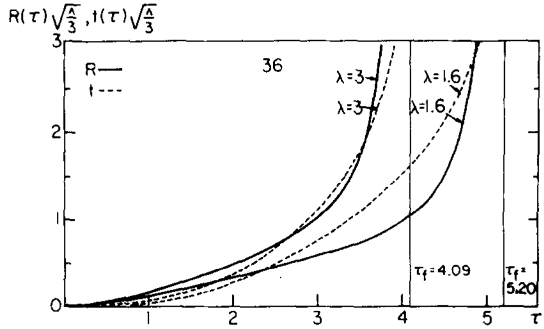

FIGURE 36

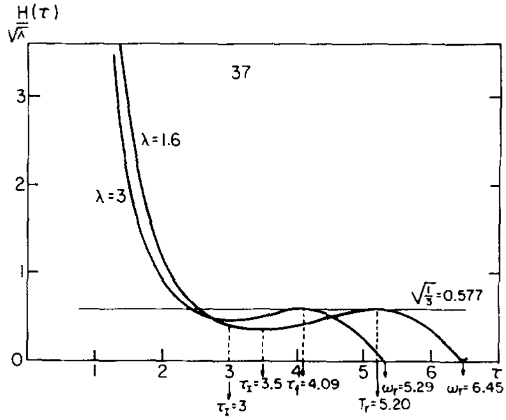

FIGURE 37

Notice that we have, corresponding to the above case

	$\lambda = 3$	$\lambda = 1.6$
τ_f	4.09	5.20
ω_r	5.31	6.48
ω_i	2.68	2.95
τ_I	2.90	3.50
e_2	−0.218	−0.186
g_2	$0.833 \; 10^{-1}$	$0.833 \; 10^{-1}$
g_3	$-0.231 \; 10^{-1}$	$-0.102 \; 10^{-1}$

8.C. *A Special Case* [9–11]

According to [9–11] and as discussed in Section 10, values of special physical interest are

$$\Omega = 0.08 \qquad \text{and} \qquad q = -1.12,$$

then, using formulas given in Sections 2.B, 4.A and 6.A, one finds that these values correspond to

reduced cosmological constant	$\lambda = 3.625 \ (>1)$
reduced temperature	$T(\tau_0) = 0.50$
dimensionless vacuum density	$\lambda_s(\tau_0) = 1.16$
reduced 3-curvature	$k_s(\tau_0) = 0.24$
today's value of confomal time	$\tau_0 = 3.40 \quad (>1)$

(Our notations and terminology differ those in [7]. See remarks in Introduction.)
The dynamics of the universe is then ruled by the equation

$$T(\tau) = 6\left[\wp(\tau; g_2, g_3) + \frac{1}{12}\right]$$

with Weierstrass invariants $g_2 = 1/12$ and $g_3 = -0.2894 \ 10^{-1}$.
Other quantities of interest are

the real root e_2 of $P_\lambda(y)$	$e_2 = -0.229$
the real half period	$\omega_r = 5.06$
the imaginary half period	$\omega_i = 2.61$
the quantities $h = -e^{-\pi(\omega_r/\omega_i)}$, $s = -e^{-\pi(\omega_i/\omega_r)}$	$\lvert h\rvert = 0.226 \cdot 10^{-2}$, $\lvert s\rvert = 0.198$
the end of conformal time	$\tau_f = 3.88$
the value of τ at the inflection point	$\tau_I = 2.89$
the values of Weierstrass functions for	$\tau = \tau_0$:

$$\wp(\tau_0) = 0.1286 \ 10^{-2}, \qquad \zeta(\tau_0) = 0.3318, \qquad \sigma(\tau_0) = 3.4211.$$

Finally, the following values for radius, cosmic time, Hubble contant and dimensionless age:

$$R(\tau_0)\sqrt{\frac{\Lambda}{3}} = 3.75, \qquad t(\tau_0)\sqrt{\frac{\Lambda}{3}} = 1.90,$$

$$\frac{H(\tau_0)}{\sqrt{\Lambda}} = 0.535, \qquad [Ht](\tau_0) = 1.76.$$

If we now use $H_0 \equiv H(\tau_0) = 100 \text{ km sec}^{-1} \text{ Mpc}^{-1}$ ($H_0^{-1} = 9.778 \ 10^9 yr$) [21] and the previous value of $H_0/\sqrt{\Lambda}$, we find $\Lambda = 4.08\ \ 10^{-56}\ \text{cm}^{-2}$ then $\Lambda_c = 1.13\ \ 10^{-56}\ \text{cm}^{-2}$ (since $\lambda = \Lambda/\Lambda_c$), and finally

$$R(\tau_0) = 3.21\ \ 10^{28}\ \text{cm} = 3.39\ \ 10^{10}\ lyr,$$

$$t(\tau_0) = 1.63\ \ 10^{28}\ \text{cm} = 1.72\ \ 10^{10}\ yr.$$

8.D. *Temperature in* K

Although the parameter α associated to radiation pressure is very small, it is not strictly zero since we measure experimentally the remnant blackbody radiation $\tilde{T} = 2.7$ K. Using formula (6.13), with

$$\sigma = 59.8\ \text{cm}^{-1}\ \text{K}^{-1} \qquad \text{and} \qquad G = 2.61\ \ 10^{-66}\ \text{cm}^2 \qquad (\hbar = c = 1)$$

and $H_0 = 100\ \text{km sec}^{-1}\ \text{Mpc}^{-1}$, we find $\alpha_s(\tau_0) = 2.38\ \ 10^{-5}$. Then, the value of α is obtained from (6.18) using values of k_s and Ω given in the previous section

$$\alpha = 3.97\ \ 10^{-4}.$$

(Notice that we neglect here the contribution of neutrinos to radiation.)

Then, the evolution of $\tilde{T}$ with conformal time τ given by (6.11) is the same as the one of T, up to a proportionality factor

$$\frac{\tilde{T}}{T} = \frac{\tilde{T}(\tau_0)}{T(\tau_0)} = \frac{0.50}{2.7} = 5.40 = \left[\frac{3}{8\pi G}\frac{\alpha\Lambda_c}{4\sigma}\right]^{1/4}.$$

This evolution of $\tilde{T}$ (in K) is shown on Fig. 38.

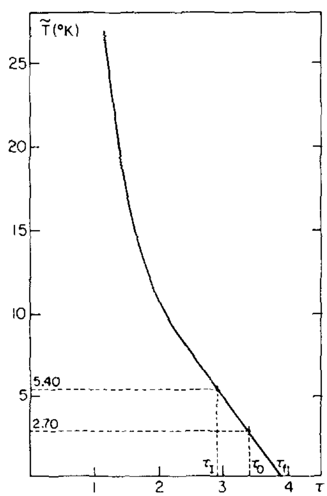

FIGURE 38

9. Redshift-Distance Formulae

9.A. *Case* $\alpha = 0$, $\lambda > 1$

Until now, we have used the time independent length $\Lambda_c^{-1/2}$ to bring Friedman's equation (without radiation pressure) to a dimensionless form, which depended on the single parameter $\lambda = \Lambda/\Lambda_c$. We shall now discuss the redshift, z, as function of the difference

$$\delta = \tau_0 - \tau \tag{9.1}$$

between parameter time, τ, of emitter, and parameter time, τ_0, of the observer. That relationship depends on τ_0 since the redshift is given by

$$z = \frac{T(\tau)}{T(\tau_0)} - 1 = \frac{R(\tau_0)}{R(\tau)} - 1. \tag{9.2}$$

From (9.2), it is clear that z (considered as function of τ for fixed τ_0), is elliptic of order two. Its zeroes in the periodicity cell (Fig. 6) are placed symmetrically with respect to ω_r. They are τ_0 and $\tau_1 = 2\omega_r - \tau_0$; this corresponds to $\delta_0 = 0$ and $\delta_1 = 2\tau_0 - 2\omega_r$. The function (9.2) has a second-order pole at $\tau = 0$ (corresponding to $\delta = \tau_0$), with principal part $6/T(\tau_0)\,\tau^2$. Finally $z(\tau_f) = -1$, since $T(\tau_f) = 0$.

This allows us to write down a multitude of expressions for z; we give just a few.

— *Redshift formulas without radiation pressure*

(a) Formula displaying zeroes and poles of z:

$$z(\delta, \tau_0) = \frac{\sigma(\delta)\,\sigma(2\tau_0 - \delta)}{\sigma^2(\tau_0 - \delta)}\, C_1 \tag{9.3}$$

with

$$C_1 = \frac{\sigma^2(\tau_f)}{\sigma(\tau_f - \tau_0)\,\sigma(\tau_f + \tau_0)}. \tag{9.3a}$$

(b) Formula displaying pole and principal part of z:

$$z(\delta, \tau_0) = C_2 \left[\wp(\delta + \tau_0) + \frac{1}{12}\right] - 1 \tag{9.4}$$

with

$$C_2 = \left[\wp(\tau_0) + \frac{1}{12}\right]^{-1} \tag{9.4a}$$

(c) Formula displaying the two poles and principal part of $1/z$:

$$[z(\delta, \tau_0)]^{-1} = A[\zeta(\delta) - \zeta(\delta - 2\tau_0)] + B, \tag{9.5}$$

where

$$A = \frac{1}{\zeta(\tau_0 + \tau_f) - \zeta(\tau_f - \tau_0) - 2\zeta(\tau_0)}, \tag{9.5a}$$

$$B = \frac{2\varphi(\tau_0)}{\zeta(\tau_0 + \tau_f) - \zeta(\tau_f - \tau_0) - 2\zeta(\tau_0)}, \tag{9.5b}$$

With the help of the parameters Ω, k_s, λ_s, one can easily invert the formulae (9.3) to (9.5), i.e., express δ as function of z. Using (9.2), (3.3) and definitions (2.13) to (2.16), one finds:

$$\left[\frac{d}{d\tau}(1+z)\right]^2 = \frac{1}{k_s(\tau_0)}\left[\Omega(\tau_0)(1+z)^3 - k_s(\tau_0)(1+z)^2 + \lambda_s(\tau_0)\right] \tag{9.6}$$

then

$$\delta = \sqrt{k_s(\tau_0)} \int_1^{1+z} \frac{1}{\sqrt{\Omega(\tau_0)\, x^3 - k_s(\tau_0)\, x^2 + \lambda_s(\tau_0)}}. \tag{9.7}$$

9.B. *The Case* $\alpha \neq 0$, $\lambda_- < \lambda < \lambda_+$

From the results of Section 5.C, it can be seen that the two simple poles of $R(\tau)$—hence the two poles of z—are obtained for the values $\tau = 0$ and $\tau = \tau_f + \tau_g$; the two simple zeroes of z are obtained when $R(\tau_0) = R(\tau)$ that is $\tau = \tau_0$ and $\tau = \tau_f + \tau_g - \tau_0$; as it should, the sum of poles is equal to the sum of the zeroes (up to the lattice). This allows us to write the following formula, displaying zeroes and poles of z:

$$z = \frac{\sigma(\tau - \tau_0)\, \sigma(\tau + \tau_0 - \tau_f - \tau_g)}{\sigma(\tau)\, \sigma(\tau - \tau_f - \tau_g)}\, C \tag{9.8}$$

with

$$C = \frac{\sigma(\tau_f)\, \sigma(\tau_g)}{\sigma(\tau_f - \tau_0)\, \sigma(\tau_0 - \tau_g)}. \tag{9.9}$$

Introducing the variable δ defined in (9.1) we can write also

$$z = C\, \frac{\sigma(\delta)\, \sigma(2\tau_0 - \delta - \tau_f - \tau_g)}{\sigma(\delta - \tau_0)\, \sigma(\tau_0 - \delta - \tau_f - \tau_g)}. \tag{9.10}$$

With the help of the parameters α_s, Ω, k_s, λ_s, one can invert the preceeding formulae, i.e., express δ as function of z:

$$\delta = \sqrt{k_s(\tau_0)} \int_1^{1+z} \frac{dx}{\sqrt{\alpha_s(\tau_0)\, x^4 + \Omega_s(\tau_0)\, x^3 - k_s(\tau_0)\, x^2 + \lambda_s(\tau_0)}}. \tag{9.11}$$

9.C. *Luminosity Distance and Apparent Size Distance*

These two quantities are defined in (2.21) and (2.22). From the explicit formulae given for $T(\tau)$ in Sections 4 and 5, we can easily find explicit expressions for these two quantities. For example, supposing $\alpha = 0$ and $\lambda > 1$, we find

$$D_L(\delta, \tau_0) = \Lambda_c^{-1/2} \frac{\wp(\tau_0 - \delta) + \dfrac{1}{12}}{6\left[\wp(\tau_0) + \dfrac{1}{12}\right]^2} \sin \delta, \tag{9.12}$$

$$D_S(\delta, \tau_0) = \Lambda_c^{-1/2} \frac{\sin \delta}{6\left[\wp(\tau_0 - \delta) + \dfrac{1}{12}\right]}. \tag{9.13}$$

Notice that D_L and D_S exhibit an interesting interplay between the period $2\omega_2$ associated to the lattice and the period 2π associated with the function $\sin \delta$. This is made even more explicit if we use (7.21) for the expression of $\wp(\tau)$; we see, e.g., that

$$\begin{aligned}
\wp(\tau_0 - \delta) \sin \delta = \left(\frac{\pi}{\omega_k}\right)^2 \Bigg\{ & \frac{\sin \delta}{\left[2 \sin \dfrac{\pi}{2\omega_r}(\delta - \tau_0)\right]^2} + \left(-\frac{1}{12} + 2 \sum_{n=1}^{\infty} n a_n\right) \sin \delta \\
& - \sum_{n=1}^{\infty} n a_n \sin\left[\delta\left(1 - \frac{n\pi}{\omega_r}\right) + \frac{n\pi\tau_0}{\omega_r}\right] \\
& - \sum_{n=1}^{\infty} n a_n \sin\left[\delta\left(1 + \frac{n\pi}{\omega_r}\right) - \frac{n\pi\tau_0}{\omega_r}\right] \Bigg\}.
\end{aligned}$$

We distinguish a singular contribution given by the first term, a contribution periodic in δ, and two non-harmonic series that look like Fourier series for large values of n.

9.D. *Numerical Evaluation of z, D_L D_S*

In order to make concrete the formula exhibited in the above paragraph, we plot on Figs. 39, 40, the quantities z, D_L and D_S as functions of δ with the parameters of Section 8.C. For small enough δ: $D_L \sqrt{\Lambda_c} \sim D_S \sqrt{\Lambda_c} \sim \delta$. Notice that in Fig. 40 two different unit scales are used for D_L and D_S.

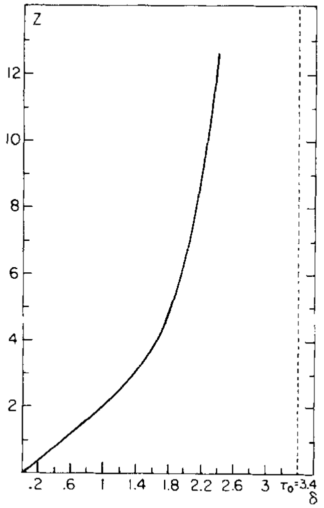

FIGURE 39

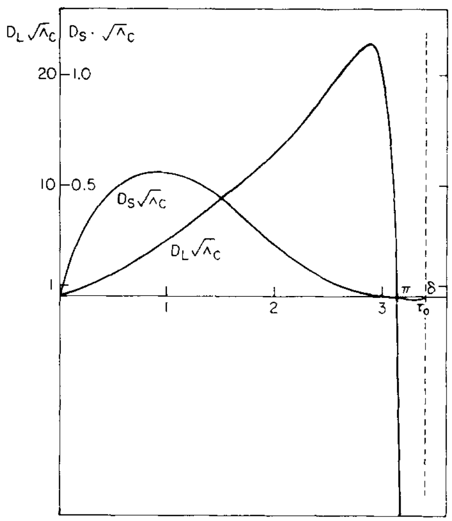

FIGURE 40

10. LARGE-SCALE GEOMETRY

In this section, motivated directly by [8–11], we discuss the appearance of large-scale features of the universe (in particular of orbits under natural group actions) to observers that record sighting directions (or rather, angles between sighting directions) and redshifts.

In other words, we want to compare the "sky charts" of cosmologically distinct observers looking at a hypothetical persistent feature in the spatial distribution of galaxies. (Such a feature is compatible with spatial constance of averaged matter density. See [7–11]).

(1) The discussion is simplified by the fact that $S^{(3)}$, the three-dimensional unit sphere, has a group structure. This can be seen by associating to each $X \in S^{(3)}$, $X = \{X_0, X_1, X_2, X_3\}$ the quaternion

$$X = X_0 + iXK_1 + jX_2 + kX_3$$

and using the multiplication rule $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

It is convenient also to write X in a form analogous to the representation $z = e^{i\varphi} = \cos\varphi + i\sin\varphi$ of a complex number of unit norm. Let $X \in S^{(3)}$, $X \neq \pm 1$. Define $\delta = \delta(X)$ by

$$\delta = \operatorname{arc\,cos}(X_0) \qquad (0 < \delta < \pi). \tag{10.1}$$

Geometrically, δ is the geodesic distance, on $S^{(3)}$, between the quaternions 1 and X.

Consider now a three-dimensional vector X of unit length defined by

$$\hat{X}_l = \frac{1}{\sin\delta} X_l \qquad (l = 1, 2, 3). \tag{10.2}$$

Geometrically, $\hat{X}$ is the sighting direction from 1 to X along the geodesic. We have, quaternionically

$$X = \cos\delta + \hat{X}\sin\delta = e^{\delta\hat{X}}. \tag{10.3}$$

The geodesic distance between $T \in S^{(3)}$ and $X \in S^{(3)}$ is

$$\delta(T, X) = \operatorname{arc\,cos}(\bar{T}X)_0; \tag{10.4}$$

here $\bar{T}X$ is the quaternionic product of $\bar{T} = T_0 - iT_1 - jT_2 - kT_3$ with X.

The sighting direction from T (e.g., earth) to X (e.g., a quasar) is given by

$$\hat{Y}_l = [\sin\delta(T, X)]^{-1}\,(\bar{T}X)_l \qquad (l = 1, 2, 3). \tag{10.5}$$

For fixed T, sighting directions are defined only up to a fixed three-dimensional rotation.

Notice that $\hat{Y}$ is not defined if $\bar{T}X = -1$, i.e., if X is antipodal to T.

(2) In the models of universe studied here, it is natural to specify events by pairs $\{\tau, T\}$, where τ is a value of parameter time and T a point on $S^{(3)}$.

An observer at $\{\tau_0, T\}$, looking at a photon emitted at $\{\tau_0 - \delta, X\}$, (on the backward light cone of $\{T_0, T\}$, will record a redshift $z = z(\delta, \tau_0)$ given by (7.3), and the sighting direction $\hat{Y}$, where

$$Y = \bar{T}X = \cos\delta + \hat{Y}\sin\delta. \tag{10.6}$$

(3) The above—or spherical trigonometry— can be used to describe "cosmic parallels" [8–11]. Fix a point $P \in S^{(3)}$, and consider the "cosmic latitude" of X with respect to P (a "pole") defined by

$$l_P(X) = \frac{n}{2} - \operatorname{arc\,cos}(\bar{P}X)_0. \tag{10.7}$$

Let T be another point on $S^{(3)}$, and α the angle between the sighting direction from T to P and the sighting direction from T to X. If $l_P(T)$ is the cosmic latitude of T, and δ the geodesic distance (on $S^{(3)}$), between T and X, then, writing $\bar{P}X = (\bar{P}T)(\bar{T}X)$, one finds:

$$\sin l_P(X) = \cos\delta \sin l_P(T) - \sin\delta \cos\alpha \cos l_P(T), \tag{10.8}$$

$$\cos\alpha = \frac{-\sin l_P(X)}{\sin\delta \cos l_P(T)} + \cot\delta \operatorname{tg} l_P(T). \tag{10.9}$$

In particular, if $l_P(X) = 0$ (cosmic equator), we have

$$\begin{aligned} \cos\alpha &= +\cot\delta \operatorname{tg} l_P(T), \\ \delta &= +\operatorname{arctg}\left(\frac{\operatorname{tg} l_P(T)}{\cos\alpha}\right). \end{aligned} \tag{10.10}$$

(4) *Geodesic Planes.* In [9], the cosmological parameters were determined from the hypothesis that an observed large-scale feature in the distribution of quasars (the empty zone (S)) corresponds to an "equator" of $S^{(3)}$. An equator is determined by the corresponding poles $\pm P$; it is the two-dimensional manifold of points $X \in S^{(3)}$ such that

$$\delta(X, P) = \delta(X, -P) = \frac{\pi}{2}. \tag{10.11}$$

(5) *An Example.* As an application of the previous relations (specially useful in the following), notice that if X is a quasar sitting on an equatorial $S^{(2)}$-sphere of $S^{(3)}$ with respect to a pole P, its redshift as recorded by P is (see (9.3), (9.3a))

$$z\left(\delta = \frac{\pi}{2}\right) = \frac{\sigma\left(\frac{\pi}{2}\right)\sigma\left(2\tau_0 - \frac{\pi}{2}\right)}{\sigma^2\left(\tau_0 - \frac{\pi}{2}\right)} C_1. \tag{10.12}$$

Now if $l_P(T)$ is the cosmic latitude of the earth T with respect to this equatorial sphere and α the sighting angle between the pole P and the quasar X, then, the same quasar X will have with respect to the earth a redshift Z equal to

$$Z = + \frac{\sigma\left[\operatorname{arctg}\left(\dfrac{\operatorname{tg} l_P}{\cos\alpha}\right)\right]\sigma\left[2\tau_0 - \operatorname{arctg}\left(\dfrac{\operatorname{tg} l_P}{\cos\alpha}\right)\right]}{\sigma^2\left[\tau_0 - \operatorname{arctg}\dfrac{\operatorname{tg} l_P}{\cos\alpha}\right]} C_1. \tag{10.13}$$

(Here we restrict ourselves to the case discussed in Sections 4.A and 5.C.) We suppose also $\tau_0 > \pi/2$ since in the opposite case, the light coming from X cannot be recorded by P.

(6) *Comment.* The previous discussion can be generalized to the case where S^3 is replaced by any of the coset spaces:

S^3/(subgroup of $SU(2)$ operating without fixed point) (see [57]).

11. Significance of the Cosmological Constant in Quantum Field Theories

11.A. *General Topics*

The energy momentum tensor appearing on the right-hand side of Einsteins equation

$$\mathbb{G} = 8\pi G T \tag{11.1}$$

can be split into three parts: A first one T_{MAT} giving the contribution of matter, another one T_{RAD} giving the contribution of radiation, and a third part T_{VAC} giving the contribution of the vacuum itself.

$$T = T_{\mathrm{MAT}} + T_{\mathrm{RAD}} + T_{\mathrm{VAC}}. \tag{11.2}$$

By "vacuum," we mean the lowest state of the field configuration. The fact that a vacuum carries an energy is not surprising. This is reminiscent of the zero-point energy $\frac{1}{2}\hbar\omega$ of the harmonic oscillator. A non-interacting field is usually considered as an infinite set of harmonic oscillators of all possible frequencies; hence, in order to compute the vacuum energy density, one has to compute the sum

$$\sum_\alpha \tfrac{1}{2}\hbar\omega_\alpha. \tag{11.3}$$

Obviously, this quantity is divergent; in the same way, one can compute the other components of T_{VAC} (i.e., the vacuum pressure). In particle physics, this (infinite) zero-point energy is harmless since one only measures energy differences. Then, one simply subtracts in all cases the vacuum contribution; an equivalent way to obtain the same results is to introduce a "normal ordering" for the product of operators. It has been shown [15] that such a procedure is not compulsory and that one could as well renormalize the vacuum energy to a finite number. In such case, one obtains:

$$8\pi G T^{\mu\nu}_{\rm VAC} = \Lambda g^{\mu\nu}. \tag{11.4}$$

The interpretation is clear: the cosmological constant Λ can be obtained by adding the contribution of all possible quantum fields to the vacuum energy. If we had a unified field theory at our disposal, we would have just to add all possible vacuum–vacuum Feynman diagrams and find the value of the cosmological constant! Unfortunately, such is not the case. However, a number of papers have been written in the past few years (in the reviews of field theory and particle physics) about the cosmological constant and we would like to say a few words about the discussions that go on. Our purpose is neither to sum up all that has been written on that subject nor to present new ideas but to guide the reader among several aspects of this problem. We do not pretend to be exhaustive but hope that such a discussion will be useful for those (astrophysicists, cosmologists or others...) who are not aware of the possible field-theoretical approaches to the cosmological constant.

11.B. *The Cosmological Constant and the Casimir Effect*

Let us first recall what is the Casimir effect [22]. As we said before, the vacuum energy, as computed for a free scalar field (the simplest example) is a divergent quantity; more precisely, one finds [15] an energy ε_0 equal to

$$\varepsilon_0 = (\text{constant}) \cdot p_0^4 + [\text{constant}]\, p_0^2 + [\text{constant}]\, \text{Log}\, p_0, \tag{11.5}$$

where p_0 is a momentum cut-off.

The situation is analoguous for the electromagnetic field. However, if one introduces two condensor plates in the electromagnetic vacuum (with the usual conditions of continuity for the **E** and **B** fields), the vacuum energy is changed by a small finite amount

$$\varepsilon - \varepsilon_0 = -\frac{\pi^2 \hbar c \Sigma}{720 a^3}, \tag{11.6}$$

where a is the distance between the two parallel condensor plates and Σ is their surface. Hence, one should observe an attractive force:

Force per unit of surface

$$\mathscr{F} = -\frac{\pi^2 \hbar c}{240 a^4}\left(= -\frac{0{,}013}{a^4_{\mu m}}\, dy_n/cm^2\right). \tag{11.7}$$

For a detailed calculation, see, for example, [23]. The Casimir effect has been studied for other configurations of conductors in [24]. It has been experimentally verified [25]. In the same way, the vacuum energy of a system of fields computed in a flat space-time differs by a finite amount from the same vacuum energy computed in a space-time endowed with a non-zero curvature or in a space-time possessing a different topology. This aspect has been emphasized in [26] and more recently in [16].

11.C. *Cosmological Constant and Spontaneously Broken Gauge Theories*

These days, it is believed that weak and electromagnetic interactions are ruled by a gauge theory based on the group $SU(2) \times U(1)$ [27]. Roughly speaking, the particles that mediate the interactions are associated to fields (called gauge fields) defined on space-time, with values in the Lie algebra of the above group. The situation is similar to the general relativity case in the sense that the theory is invariant under arbitrary change of coordinates in the "internal" space. The weak interactions being a short-range phenomenon are necessarily mediated by massive particles; in order to give a mass to the gauge fields (i.e., to the vector bosons W^+, W^- and Z—the photon remaining massless), the following mechanism, called Higgs mechanism, [28] is invoked.

One supposes the existence of a two-component charged scalar particle (called Higgs particle) associated to a field whose equation of motion can be classically derived from the following Lagrangian density:

$$\mathscr{L} = D_\mu \varphi D^\mu \varphi + \mu^2 \varphi^2 - \lambda \varphi^4 \qquad (\mu^2 > 0, \lambda > 0), \tag{11.8}$$

where D_μ is the covariant derivative with respect to the gauge group. In that case, the potential

$$V(\varphi) = -\mu\varphi^2 + \lambda\varphi^4 \tag{11.9}$$

has the shape shown on Fig. 41.

It is not possible develop a perturbative quantum field theory around the origin $\varphi = 0$, which is an unstable point; one has to shift the φ-field

$$\varphi = \hat{\varphi} + \frac{\sigma_0}{\sqrt{2}} \qquad \text{with} \qquad \frac{\sigma_0}{\sqrt{2}} = \frac{\mu}{\sqrt{2\lambda}}. \tag{11.10}$$

The effect of this shift in the previous lagrangian density is to give a mass to the gauge bosons (via the covariant derivative); the Higgs particle (not yet observed experimentally) will also aquire a mass (positive) equal to $\sqrt{2\mu^2}$ but what interests us

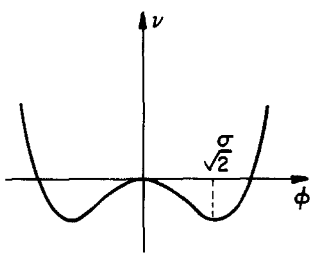

FIGURE 41

here is that the potential V (which has the dimensions of an energy density) is also shifted by a finite amount:

$$V = \hat{V} + \langle V \rangle_0$$

with

$$\hat{V} = -\mu^2 \hat{\varphi}^2 + \lambda \hat{\varphi}^4,$$

$$\langle V \rangle_0 = -\frac{\mu^2 \sigma_0^2}{2} + \frac{\lambda \sigma_0^4}{4} = -\frac{\mu^2 \sigma_0^2}{4} = -\frac{\lambda \sigma_0^4}{4}. \tag{11.11}$$

Moreover,

$$\sigma_0^2 = \frac{\mu^2}{\lambda} = \frac{4M_W^2}{g^2} = \frac{1}{G_F \sqrt{2}}, \tag{11.12}$$

where M_W is the mass of the W vector boson, g is the coupling constant associated to the $SU(2)$ gauge group and G_F is the Fermi weak interaction constant ($G_F = 1.16\ 10^{-5}$ GeV); one can write

$$\langle V \rangle_0 = -\frac{m_H^2}{8\pi G_F \sqrt{2}}, \tag{11.13}$$

where $m_H^2 = 2\mu^2$ is the mass of the Higgs particle.

The quantity $\langle V \rangle_0$ contributes to the vacuum energy and, consequently, to the cosmological constant. If one assumes that $\langle V \rangle_0$ is the only contribution to the cosmological constant and writes [29]

$$\Lambda = 8\pi G \langle V \rangle_0, \tag{11.14}$$

$$\Lambda = -\left(\frac{\pi G}{\sqrt{2}\, G_F}\right) m_H^2, \tag{11.15}$$

one reaches two conclusions:

Λ is negative.

m_H is very small, $m_H < 4.6\ 10^{-27} m_e$ since experimentally $|\Lambda| \leqslant 10^{-55}\ \mathrm{cm}^{-2}$.

This last result is completely excluded experimentally since such a small particle would lead to macroscopic forces which are not observed. As pointed out in [30], even if the Higgs particle is coupled with the proton as well as with the electron, with an opposite sign, there would be a macroscopic effect between the Higgs particle and atoms. The conclusion is then that $\langle V \rangle_0$ cannot be the only contribution to the cosmological constant.

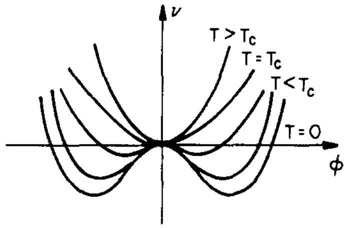

FIGURE 42

11.D. *Temperature Dependence of the Cosmological Constant*

Up to now, we have assumed that the cosmological constant is really a constant, i.e., that the tensors T_{VAC} and $T_{\mathrm{MAT}} + T_{\mathrm{RAD}}$ were separately conserved. However, nothing in field theory requires such a constraint: Only the sum of the three tensors has to be conserved. In today's particle physics, one usually assumes that the system of fields is at zero temperature; however, at the beginning of the universe, temperature effects were not negligible and, in that case, one has to use the formalism of quantum field theory at finite temperature (Gibbs average rather than Feynman average). It is noticeable that the behaviour of a system of fields like the one discussed in the preceding section is very much the same as the one of a ferromagnetic body below the Curie point. Indeed, it can be shown that the the potential $V(\varphi)$ depends on the temperature and that, above a critical temperature T_c, the symmetry-breaking phenomenon disappears, that is, the origin becomes an absolute minimum and no shift of the φ field is necessary. In that phase, the gauge bosons become massless and the weak interactions become long range (infinite) forces.

Figures 42, 43, illustrate the phenomena.

The dependence of V on temperature has been especially discussed in [31, 32]. It is clear on the previous curves that the value of V at the maximum, $\langle V \rangle_T$, i.e., the vacuum energy density of the system, changes with the temperature. This, in turn, induces a change in the cosmological constant that has been discussed in [33]; the discussion can be summed up as follows: At a very early stage of the universe, we can assume that T was above the critical value T_c, then $\langle V \rangle_{T > T_c} = 0$. The quantity $\langle V \rangle_{\mathrm{To\,day}} \simeq \langle V \rangle_0$ has been evaluated in the previous paragraph; if we assume that the

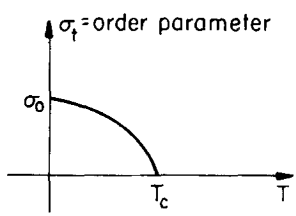

FIGURE 43

other contributions to the vacuum energy have not changed during this time, we find that

$$\begin{aligned}
&[\text{energy of the vacuum}]_{T>T_c} - [\text{energy of the vacuum}]_0 \\
&\quad = \{[\text{other contributions}] + \langle V\rangle_{T>T_c}\} \\
&\qquad - \{[\text{other contributions}] + \langle V\rangle_0\} \\
&\quad = -\langle V\rangle_0 . \qquad (11.16)
\end{aligned}$$

Hence

$$\begin{aligned}
\Lambda_{T>T_c} = \Lambda_{T_c} &= \Lambda_0 - 8\pi G\langle V\rangle_0 \\
&= \Lambda_0 + \left(\frac{m_H^2}{8\pi G_F\sqrt{2}}\right) 8\pi G. \qquad (11.17)
\end{aligned}$$

m_H is not known but experimental bounds exist:

$$\frac{m_H^2}{8\pi G_F\sqrt{2}} \geqslant 10^{21} \ \text{g/cm}^3. \qquad (11.18)$$

Finally

$$\Lambda_{T>T_c} = \Lambda_{T_c} > 10^{-6}\,\text{cm}^{-2}. \qquad (11.19)$$

This last result implies that if we believe in the $SU(2) \times U(1)$ theory of weak interactions and if the other contributions to the cosmological constant have not changed too much since the beginning of the expansion, the cosmological constant has changed by more than 49 orders of magnitude.

11.E. *False Vacuum, Vacuum Decay and Cosmological Constant*

The situation is actually more complex: The relevant quantity is not really the classical potential $V(\varphi)$ but the effective potential, that is the one which takes into account quantum corrections. It can be shown that the effective potential of the $SU(2) \times U(1)$ model has an additional minimum at the origin. The depth of this new minimum is a function of the masses of the fermions, vector bosons and Higgs particle. Moreover, the whole curve is a function of the temperature. The respective depths of the minima at the origin $\varphi = 0$ and at $\varphi = \sigma$ is of course of great importance for symmetry breaking; in principle the physical vacuum should correspond to the lowest minimum, however such is not necessarily the case: the decay from a false vacuum to a true vacuum takes time and the life-time of the metastable vacuum may appear longer than the age of the universe; for all these questions, see [34] and references therein. The dynamics of formation of bubbles of true vacuum have been investigated especially in [35, 36], where gravitationnal effects are also taken into account. (Notice that the value of the cosmological constant is different inside and

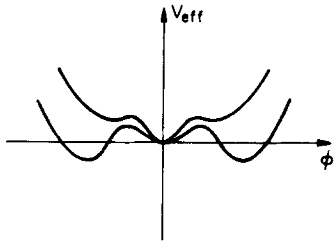

FIGURE 44

outside a bubble of true vacuum.) The behaviour of the effective potential in the Weinberg–Salam model for several choices of the Higgs mass m is shown in Fig. 44.

11.F. *Grand Unified Theories, Baryon Asymmetry and the Cosmological Constant*

In the previous three sections, our analysis was carried out in the context of the $SU(2) \times U(1)$ theory of electroweak interactions; on the other hand, strong interactions are supposed to be ruled by the laws of quantum chromodynamics (QCD) based on the Lie group $SU(3)$: eight massless vector gluons (field taking their value in the Lie algebra of $SU(3)$) mediate the interactions between quarks that carry an index belonging to the fundamental representation of $SU(3)$. One can try to develop a "grand unified theory" introducing new gauge fields coupling quarks and leptons. These new fields carry fractional electric charge and mediate super strong interactions that make the proton unstable. This kind of idea has just been discussed in [37] in connection with the problem of asymmetry with respect to the number of particles and antiparticles in the universe (C-symmetry). In the past few years, grand unified theories based on the gauge groups $SU(5)$ and $SO(10)$ have been proposed [38]. In order to explain the observed symmetries of today's physics, one has to "break" the original summetry; for example, starting with the 24 massless gauge bosons of $SU(5)$, one introduces a first symmetry breaking via a first set of 24 Higgs particles and obtain 12 very massive vector bosons (those wich lead to the proton decay) and 12 massless gauge bosons. Then, a new set of five Higgs particles whose potential looks like the one of the previous section breaks the remaining $SU(3) \times SU(2) \times U(1)$ symmetry and leaves eight massless gluons, the massless photon and three massive gauge bosons which mediate weak interactions. Other grand unified theories based on larger groups have been developed in the last years. In all cases one has to break the symmetry in several steps via a Higgs mechanism; also, the quantum corrections to the classical potential can be computed as well as its temperature dependence. In general, when the temperature is high enough, the biggest symmetry is restaured and all the gauge particles are massless; as the universe expands and cools down, one or several symmetry breakings occur whose detailed mechanism depends on the model. At each stage of the symmetry breaking mechanism, a bubble (or several bubbles) of new vacuum appears; this bubble rapidly grows and its wall traces out a hyperboloid in Minkowski space, asymptotic to the light cone. In a new vacuum there are new constants of nature and there is a new cosmological constant.

From the theoretical point of view, most of these grand unified theories are very aesthetic and appealing; from the experimental point of view, they all lead to the unstability of the proton ($\tau \sim 10^{31}$ *yrs*); several experiments are currently set up in order to measure its possible decay; however, Higgs particles, that are very massive, and responsible of the symmetry breaking have not yet been seen. In order to conclude this section let us point out that this kind of grand unified theories have been used in order to provide a scenario of cosmological baryon production [39], explanations for non-vanishing baryon asymmetry, and also a possible solution to the horizon and flatness problem [40].

11.G. *Supersymmetries, Supergravity and the Cosmological Constant*

In usual gauge theories (not supersymmetric), particles of different spins never appear in the same representations. It has been shown that is is possible to construct theories where such is not the case (see the review [41] and references therein). Supersymmetric theories are invariant under graded Lie groups [42], this invariance can be global or local: the lagrangian of the system is invariant under a supersymmetry transformation described by an infinitesimal anticommuting parameter α, which is independent of space-time in the global case and can be dependent of the position in the local case. If one starts with a lagrangian which is globally supersymmetric and tries to promote this invariance to a local one, one has to couple the matter fields to a bosonic gauge field of spin 2 (the graviton) and to a fermionic gauge field of spin 3/2 (the gravitino). In this last case, one speaks of supergravity theories [43]. A remarkable feature of unbroken supersymmetric theories is that the cosmological term vanishes; indeed, the sum of all vacuum diagrams vanishes identically to all orders of perturbation theory as a consequence of compensations among contributions involving different fields of the supermultiplet [44]. This can be easily understood as follows. If one quantizes a two-level system (it can be described by two anticommuting operators b and $b^\dagger$) one finds that the vacuum state of the system carries an energy equal to $-\frac{1}{2}\hbar\omega$; if one now constructs a supersymmetric oscillator by coupling this two-level system to a harmonic oscillator (whose vacuum energy is $\frac{1}{2}\hbar\omega$), one finds that the ground state of the supersymmetric system has zero energy. This property can be generalized for an arbitrary unbroken supersummetric theory (global or local).

Phenomenologically, supersymmetry is at most a badly broken symmetry; indeed, in exact supersymmetry, the particles of the same multiplet (bosons and fermions) have the same mass; such is not the case in nature. It is possible to construct a supersymmetric Higgs effect and consider its possible application to the construction of realistic models [45]. The first-order correction to the cosmological constant, as calculated in [46] contains quartic, quadratic and logarithmic divergences in general. Such divergences disappear in supersymmetric theories leaving at most a constant correction depending on the masses of the several particles when supersymmetry is spontaneously broken. In this kind of theories, mass formulas exist between the several members of the supermultiplets [47]; these relations allow one in some cases to cancel the left-over term in the cosmological constant. Although very appealing from a theoretical point of view, supergravity theories are not tested experimentally in the sense that they predict a big number of particles which have not been seen (however, they are not in conflict with experiment since one can argue that the unseen particles have a very high mass).

11.H. *Quantum Gravity and the Cosmological Term*

The importance of the Λ term is clear if one tries to quantize gravity itself (without using a sypersymmetric theory in a background flat space). For example, in [48], the foamlike structure of space-time at a small scale is described by introducing a Λ term as a Lagrange multiplier for the 4 volume (a regularizing box). There, a "volume canonical ensemble" is described by a partition function

$$Z[\Lambda] = \int \mathscr{D}[g] \exp(-I[g]), \tag{11.20}$$

where $\mathscr{D}[g]$ is a measure on the space of all metrics g and

$$I[g] = -\frac{1}{16\pi} \int (R - 2\Lambda)(g)^{1/2}\, d^4x \tag{11.21}$$

is the Euclidean action of the gravitationnal field including the Λ-term, and the path integral is taken over all metrics on all compact manifolds.

Other references discussing the influence of the cosmological constant in quantum gravity can be found in [49].

11.I. *Remarks*

All the aspects of the cosmological term have not been discussed above and we apologize for missing references; we have tried to convince the reader that, as far as the cosmological constant is concerned, there are many links between the world of particle physics and the one of cosmology.

Normally when something is strictly zero, there is a reason for it but there is no reason for the cosmological constant to vanish. Undoubtedly, and as can be guessed from the previous discussions, the computation of the present value of the cosmological constant is out of the reach of today's particle physics; however, we hope that a precise answer will appear in a not too far future.

Acknowledgments

The motivation for this work is a specific Friedman–Lemaître model described and confronted with data by H. H. Fliche, J. M. Souriau and R. Triay. We learned about the model—and about the subject—from the authors, to whom we express our warmest thanks.

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