Explanations (sketch): Fusion graphs for Lie groups

A Dynkin diagram G encodes, among other things, the structure of a finite dimensional simply connected compact Lie group, the structure of a complex Lie algebra, and the structure of the enveloping algebra of the later (an infinite dimensional associative algebra).

The associated Lie group G has infinitely many (classes of equivalent) finite dimensional irreducible representations. They can be added and tensor multiplied and decomposed into irreducible components. We obtain a ring. This ring is special in the sense that, considered as an algebra over the integers, it comes with a special basis (the irreducible representations) for which structure constants are non negative integers.

The above (linear) basis has infinitely many elements, but the ring of representations can be, in some sense, generated (algebraically) by a finite number of irreducible representations called the fundamental representations. There are r of these generators when G is of rank r. Each one is associated to a node of the Dynkin diagram.

From the classical to the quantum: The choice of a positive integer k determines a finite truncation of the above general picture. Those remaining representations surviving at level k are called "integrable", and there is only a finite number or them (how to build this category of representations is not our concern here, it is enough to know that it can be done in a consistent way). We still have a ring. It is called the fusion ring of G at level k. We still have generators, but when k is too small not all fundamental representations are necessarily integrable: what happens when k grows is that more and more irreducible representations appear. Loosely speaking, the classical theory summarized in the previous paragraph is recovered when k goes to infinity. When k is finite, the number of simple (ie irreducible) objects in this category of representations is finite. The notion of dimension of a representation still makes sense, and it behaves as usual under sums and products, but it is usually not an integer.

The fusion ring (often called "the fusion algebra") is, as before, a Z-module and still comes with a special basis (the integrable representations $m, n, p, \ldots$): the structure constants, in this basis, are non-negative integers N_{mnp} encoding the multiplication $m \otimes n = \sum_p N_{mnp}\, p$. The multiplication is known as soon as the fusion matrices $N_m = (N_m)_{np} = N_{mnp}$ are determined. It is enough to know them when m is a generator, ie an integrable fundamental representation. These are the fundamental fusion matrices. Examples: In the case $G = SU(2)$, at any level k, there is only one such matrix. In the case $G = SU(3)$, at any level k, there are two such matrices. For $G = E_6$ there are only two generators at level 1, five at level 2 and 6 when the level is bigger than 2.

A fusion matrix has non negative integer coefficients. It can be encoded by a graph. The matrix is the adjacency matrix of the graph. The fusion graph of f describes the multiplication by f : the result of a multiplication $f \otimes p$ is the sum of neighbours (vertices) of p on the graph. It is enough to know the fundamental fusion graphs: those associated with integrable fundamental representations.

In the quantum case, one or several of the integrable representations may be "more fundamental" than others, in the sense that for finite k, all other integrable fundamental representations can be expressed polynomially in terms of the basic ones, but the coefficients of these polynomias are (huge) rational numbers, they are not integers in general. Such representations are sometimes called "basic". The representation that classically corresponds to the fundamental representation of smallest dimension is always "basic".

What we give on the web pages are pictures of the fundamental fusion graphs, for all possible choices of G and for several levels k, starting from $k = 1$. When the level is small, the number of graphs may be smaller than the rank of G. When k is big enough, this number is equal to the rank. Notice that the shape of the basic graph(s), when k grows, is quite predictible.

Remark: The fusion graph of A_1 (ie the group $SU(2)$) at level k happens to coincide with the Dynkin diagram A_{k+1}. This important observation can be generalized. Read the section: Modules for fusion graphs.

For the advanced reader: The data (G, k) defines a modular category $\mathcal{A}_k(G)$ (a special case of monoidal category) that can be builded in terms of integrable representations of affine Lie algebras at the chosen level, or in terms of quantum groups at some root of unity. It has a finite number of simple objects. The fusion ring is the Grothendieck ring of this category.