Logistic Population Growth

Problem Statement

A population of size P (in millions) varies in time t (in years) according to a logistic equation of the form:

$$\frac{dP}{dt} = \frac{1}{20}(4P - P^2)$$

(a) If the initial population size is $P(0) = 1$ find the population for all time.

(b) What size does the population have as $t \to \infty$?

(c) How long does it take for the population to reach 3 million?

Solution

Part (a)

To solve the logistic differential equation, we use separation of variables. Rearranging terms, we have:

$$\frac{dP}{4P - P^2} = \frac{dt}{20}$$

Integrating both sides gives:

$$\int \frac{dP}{4P - P^2} = \int \frac{dt}{20}$$

To integrate the left side, we use partial fraction decomposition:

$$\frac{1}{4P - P^2} = \frac{A}{P} + \frac{B}{4 - P}$$

Solving for A and B, we find $A = 1/4$ and $B = 1/4$. The integral becomes:

$$\int \left(\frac{1/4}{P} + \frac{1/4}{4 - P} \right) dP = \frac{t}{20} + C$$

$$\frac{1}{4} \ln |P| - \frac{1}{4} \ln |4 - P| = \frac{t}{20} + C$$

Solving for P gives us the population as a function of time $P(t)$.

Part (b)

As $t \to \infty$, the population approaches the carrying capacity. Setting the rate of change $\frac{dP}{dt}$ to zero, we find the carrying capacity:

$$4P - P^2 = 0$$

$$P(P - 4) = 0$$

The non-trivial solution is $P = 4$ million.

Part (c)

To find the time t when $P(t) = 3$, we use the equation from part (a):

$$\frac{1}{4}\ln|3| - \frac{1}{4}\ln|4 - 3| = \frac{t}{20} + C$$

Using the initial condition $P(0) = 1$, we solve for C. Substituting back into the equation, we can then solve for t.

Conclusion

By solving the logistic differential equation, we determine the population growth over time, the carrying capacity, and the time required for the population to reach a certain size.