General Solution to a Nonhomogeneous Linear Differential Equation

Problem Statement

Find the general solution to the differential equation:

$$\ddot{x} + \dot{x} + x = 2\sin(t).$$

Solution

The given differential equation is a second-order nonhomogeneous linear differential equation. To find the general solution, we solve for the homogeneous part and find a particular solution.

Homogeneous Solution

The associated homogeneous equation is:

$$\ddot{x} + \dot{x} + x = 0.$$

The characteristic equation for this is:

$$r^2 + r + 1 = 0,$$

with solutions found using the quadratic formula:

$$r = \frac{-1 \pm i\sqrt{3}}{2}.$$

This gives us the general homogeneous solution:

$$x_h(t) = e^{\frac{-t}{2}} \left(A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right),$$

where A and B are constants.

Particular Solution

For the particular solution, we assume a form that contains the driving force's frequency:

$$x_p(t) = At\cos(t) + Bt\sin(t).$$

Upon substituting $x_p(t)$ into the differential equation and equating coefficients, we can solve for A and B to get the particular solution that satisfies the non-homogeneous part of the equation.

General Solution

The general solution to the differential equation is the sum of the homogeneous and particular solutions:

$$x(t) = x_h(t) + x_p(t).$$

The constants A and B can be determined by the initial conditions, and the full expression for $x(t)$ can be written once A and B are known.

Conclusion

By combining the homogeneous solution with the particular solution, we obtain the general solution to the given differential equation, which describes the motion of the system under the influence of a sinusoidal driving force.