Solutions of Damped Harmonic Motion

Question

Solve the differential equation of damped harmonic motion

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + x = 0$$

for *i* $b = \sqrt{5}$, *ii* $b = 2$ and *iii* $b = \sqrt{3}$. In each case find the complete solution subject to the initial condition $x(0) = 0$, $\frac{dx}{dt}(0) = 1$, and sketch the solution. Comment on the type of damping observed.

Solution

The characteristic equation of the differential equation is:

$$m^2 + bm + 1 = 0$$

Case (i): $b = \sqrt{5}$

The characteristic equation becomes:

$$m^2 + \sqrt{5}m + 1 = 0$$

Solving for m yields complex roots:

$$m = \frac{-\sqrt{5} \pm i}{2}$$

indicating that the system is underdamped. The motion will be oscillatory with decreasing amplitude over time. The general solution is:

$$x(t) = e^{-\sqrt{5}t/2}(C_1 \cos(t/2) + C_2 \sin(t/2))$$

Applying the initial conditions $x(0) = 0$, $\frac{dx}{dt}(0) = 1$, we find $C_1 = 0$ and $C_2 = 2$. Thus, the solution is:

$$x(t) = 2e^{-\sqrt{5}t/2} \sin(t/2)$$

Case (ii): $b = 2$

The characteristic equation becomes:

$$m^2 + 2m + 1 = 0$$

This has a repeated real root $m = -1$, indicating that the system is critically damped. The general solution is:

$$x(t) = (C_1 + C_2 t)e^{-t}$$

Applying the initial conditions $x(0) = 0$, $\frac{dx}{dt}(0) = 1$, we find $C_1 = 0$ and $C_2 = 1$. Thus, the solution is:

$$x(t) = te^{-t}$$

Case (iii): $b = \sqrt{3}$

The characteristic equation becomes:

$$m^2 + \sqrt{3}m + 1 = 0$$

Solving for m yields complex roots:

$$m = \frac{-\sqrt{3} \pm i}{2}$$

indicating that the system is underdamped. The motion will be oscillatory with decreasing amplitude over time. The general solution is:

$$x(t) = e^{-\sqrt{3}t/2}(C_1 \cos(t/2) + C_2 \sin(t/2))$$

Applying the initial conditions $x(0) = 0$, $\frac{dx}{dt}(0) = 1$, we find $C_1 = 0$ and $C_2 = 2$. Thus, the solution is:

$$x(t) = 2e^{-\sqrt{3}t/2} \sin(t/2)$$