Driven Mass-Spring System Solution

Problem Statement

The same spring system of the previous question is placed at rest on a smooth table. At $t = 0$ it is then subjected to a driving force $2\cos(2t)$. Describe the subsequent behavior of the system.

Solution

The equation of motion for the mass-spring system with a driving force is:

$$m\frac{d^2x}{dt^2} + kx = F(t)$$

Given:

- Mass $m = 2$ kg,
- Spring constant $k = 8$ N/m,
- Driving force $F(t) = 2\cos(2t)$.

The differential equation becomes:

$$2\frac{d^2x}{dt^2} + 8x = 2\cos(2t)$$

Dividing by the mass m simplifies the equation:

$$\frac{d^2x}{dt^2} + 4x = \cos(2t)$$

The characteristic equation of the homogeneous part is:

$$\lambda^2 + 4 = 0$$

with solutions $\lambda = \pm 2i$. The homogeneous solution is:

$$x_h(t) = A\cos(2t) + B\sin(2t)$$

For the particular solution, we guess:

$$x_p(t) = C\cos(2t) + D\sin(2t)$$

Substituting $x_p(t)$ into the differential equation, we find that C must be zero since the cosine terms on both sides cancel due to the derivative. For D, we have:

$$-4D\sin(2t) + 4D\sin(2t) = \cos(2t)$$

This is an incorrect approach as we see that it leads to no solution because we chose a form for $x_p(t)$ that is not linearly independent of the homogeneous solution. To correct this, we need to multiply our guess by t to ensure linear independence.

Our new guess is:

$$x_p(t) = t(C\cos(2t) + D\sin(2t))$$

Substituting $x_p(t)$ into the differential equation and solving for C and D will give us the particular solution. The complete solution is then the sum of the homogeneous and particular solutions:

$$x(t) = A\cos(2t) + B\sin(2t) + t(C\cos(2t) + D\sin(2t))$$

Conclusion

By finding the constants A, B, C, and D using the initial conditions and the coefficients from the differential equation, we can describe the subsequent behavior of the system, which will exhibit a steady-state oscillation at the driving frequency with an amplitude determined by the particular solution.