General Solution to a Second-Order Inhomogeneous Differential Equation

Problem Statement

Find the general solution to the differential equation:

$$\ddot{x} + 4x = t - 1.$$

Solution

The given differential equation is second-order and linear with constant coefficients. To find the general solution, we separate it into two parts: the homogeneous solution and the particular solution.

Homogeneous Solution

For the homogeneous equation:

$$\ddot{x} + 4x = 0,$$

we find the characteristic equation:

$$m^2 + 4 = 0.$$

The solutions to the characteristic equation are $m = \pm 2i$, which gives us the general homogeneous solution:

$$x_h(t) = A\cos(2t) + B\sin(2t),$$

where A and B are constants.

Particular Solution

For the particular solution to the inhomogeneous equation, we guess a solution of the form:

$$x_p(t) = Ct + D.$$

Substituting $x_p(t)$ into the differential equation, we obtain:

$$4Ct + 4D = t - 1.$$

Matching coefficients, we find that $C = \frac{1}{4}$ and $D = -\frac{1}{4}$, thus giving us the particular solution:

$$x_p(t) = \frac{1}{4}t - \frac{1}{4}.$$

General Solution

The general solution to the differential equation is the sum of the homogeneous and particular solutions:

$$x(t) = x_h(t) + x_p(t) = A\cos(2t) + B\sin(2t) + \frac{1}{4}t - \frac{1}{4}.$$

This function describes the motion of the system for any given time t, where the constants A and B can be determined by the initial conditions.