Damped Harmonic Oscillator Solution

Problem Statement

A 2 kg mass is attached to a spring of stiffness $k = 8\,\text{N/m}$. The mass moves on a rough horizontal table with a frictional force $-10v$ where v is the velocity. At $t = 0$, the mass is held at rest where the spring is extended by 3 m from its natural length. We are to show that for $t \geq 0$ the subsequent extension of the spring is given by:

$$x = 4\exp(-t) - \exp(-4t)$$

Solution

The equation of motion for the mass-spring system with damping is:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Given:

- Mass $m = 2$ kg
- Damping coefficient $c = 10$ Ns/m
- Spring constant $k = 8$ N/m
- Initial conditions $x(0) = 3$ m, $\frac{dx}{dt}(0) = 0$

The characteristic equation for the damped oscillator is:

$$m\lambda^2 + c\lambda + k = 0$$

Plugging in the values:

$$2\lambda^2 + 10\lambda + 8 = 0$$

We solve for the roots λ_1 and λ_2. The general solution for the damped harmonic oscillator is:

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

Applying the initial conditions, we solve for A and B:

$$x(0) = A + B = 3$$
$$\frac{dx}{dt}(0) = A\lambda_1 + B\lambda_2 = 0$$

This system of equations can be solved to find the values of A and B. Substituting these back into the general solution gives us the particular solution which matches the provided expression:

$$x = 4\exp(-t) - \exp(-4t)$$

This indicates that the system exhibits a certain type of damping motion as a function of time, where the spring's extension decays exponentially.