$$f(x,y) = \frac{x^2}{x^2+y^2}$$

→ quotient rule: $\frac{u'v - uv'}{v^2}$

- $\partial/\partial x \left[\frac{x^2}{x^2+y^2}\right]$

→ $u = x^2$ → $v = x^2+y^2$

→ $u_x = 2x$ → $v_x = 2x$

→ $v^2 = (x^2+y^2)^2$

$$= \frac{2x(x^2+y^2) - x^2(2x)}{(x^2+y^2)^2} = \frac{2x^3 + 2xy^2 - 2x^3}{(x^2+y^2)^2}$$

$$= \frac{2xy^2}{(x^2+y^2)^2}$$

- $\partial/\partial y \left[x^2/x^2+y^2\right]$

$u = x^2$ v x^2+y^2

$u_y = 0$ v_y $2y$

v^2 (x^2+y^2)

$$\frac{(0)(x^2+y^2) - (x^2)(2y)}{(x^2+y^2)^2} = -\frac{2x^2y}{(x^2+y^2)^2}$$

$$f(x,y,z) = x^2 y^3 z \sin\left(\tfrac{1}{1+x^2}\right)$$

→ Product Rule : $uv' + vu'$

→ Chain Rule : $\partial f/\partial u \cdot \partial u/\partial x$

→ Reciprocal Rule : $\frac{1}{v} = \frac{-v'}{v^2}$

• $\partial f/\partial x$:

→ $u \quad x^2y^3z$ | $v \quad \sin\left(\frac{1}{1+x^2}\right)$

→ $u_x \quad 2xy^3z$ | v_x : $\partial g/\partial h \cdot \partial h/\partial x$

→ $g = \sin(h)$

→ $g_h = \cos(h)$

→ $h = \frac{1}{1+x^2}$

→ h_x

→ $a \quad 1+x^2$

$a_x \quad 2x$

$a^2 \quad (1+x^2)^2$

$h_x = \frac{-2x}{(1+x^2)^2}$

$$v_x = \cos\left(\tfrac{1}{1+x^2}\right) \cdot \left(-\frac{2x}{(1+x^2)^2}\right)$$

$$= -\frac{2x}{(1+x^2)^2} \cos\left(\frac{1}{1+x^2}\right)$$

$$\partial f/\partial x = x^2y^3z\left(-2x/(1+x^2)^2\right)\cos\left(\tfrac{1}{1+x^2}\right) + \sin\left(\tfrac{1}{1+x^2}\right)2xy^3z$$

$$= \frac{-2x^2y^3z}{(1+x^2)^2}\cos\left(\frac{1}{1+x^2}\right) + 2xy^3z \cdot \sin\left(\frac{1}{1+x^2}\right)$$

$$= 2xy^3z\left[\frac{-x}{(1+x^2)^2}\cos\left(\frac{1}{1+x^2}\right) + \sin\left(\frac{1}{1+x^2}\right)\right]$$

$$= 2xy^3z\left[\frac{-x\cos\left(\frac{1}{1+x^2}\right)}{(1+x^2)^2} + \sin\left(\frac{1}{1+x^2}\right)\right]$$

$$\frac{\partial f}{\partial x} = 2xy^3z\left[\sin\left(\frac{1}{1+x^2}\right) - \frac{x\cos\left(\frac{1}{1+x^2}\right)}{(1+x^2)}\right]$$

• $\partial f/\partial y$

→ $u = x^2y^3z$ → $v = \sin\left(\frac{1}{1+x^2}\right)$

→ $u_y = 3x^2y^2z$ → $v_y = 0$

$$\partial f/\partial y = x^2y^3z(0) + 3x^2y^2z \cdot \sin\left(\tfrac{1}{1+x^2}\right)$$

$$= 3x^2y^2z\sin\left(\tfrac{1}{1+x^2}\right)$$

• $\partial f/\partial z$:

→ $u = x^2y^3z$ → $v = \sin\left(\frac{1}{1+x^2}\right)$

→ $u_z = x^2y^3$ → $v_z = 0$

$$\partial f/\partial z = x^2y^3z(0) + 3x^2y^3\left[\sin\left(\tfrac{1}{1+x^2}\right)\right]$$

$$\partial f/\partial z = x^2y^3\sin\left(\tfrac{1}{1+x^2}\right)$$

$f(x,y) = 5x^2y - 4xy^3$

i) $\vec{v} = \langle 3,4 \rangle$

→ Normalize $\vec{v}$ to find unit vector in same direction $(\hat{u})$

→ $\hat{u} = \frac{\vec{v}}{\|v\|}$

↳ $\|v\| = \sqrt{(3)^2 + (-4^2)} = \sqrt{9+16} = \sqrt{25} = 5$

$\hat{u} = \langle 3/5, -4/5 \rangle$

→ Compute gradient of function $(\nabla f(x,y))$

→ $\nabla f(x,y) = (\partial f/\partial x, \partial f/\partial y)$

→ $\partial f/\partial x = \partial/\partial x\, [5x^2y - 4xy^3]$

$\partial f/\partial x = 10xy - 4y^3$

→ $\partial f/\partial y = \partial/\partial y\, [5x^2y - 4xy^3]$

$\partial f/\partial y = 5x^2 - 12xy^2$

$\nabla f(x,y) = (10xy - 4y^3, 5x^2 - 12xy^2)$

→ Compute dot product

→ $D_x = \nabla f \cdot \hat{u}$

$= (10xy - 4y^3)(3/5) + (-4/5)(5x^2 - 12xy^2)$

$= 6xy - 12/5\, y^3 - 4x^2 + 48/5\, xy^2$

$= 30xy - 20x^2 - 12y^3 + 48xy^2$

ii) $v = \langle 1, 2 \rangle$

→ Normalize $\vec{v}$

$\hat{u} = \vec{v}/\|\vec{v}\|$

↳ $\|\vec{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\hat{u} = \langle 1/\sqrt{3}, 2/\sqrt{3} \rangle$

→ Compute gradient of $f(x,y)$

→ $\nabla f(x,y) = (\partial f/\partial x, \partial f/\partial y)$

from before

→ $\nabla f(x,y) = (10xy - 4y^3, 5x^2 - 12xy^2)$

→ Compute dot product:

→ $D_x = \nabla f \cdot \hat{u}$

$= 1/\sqrt{3}\, (10xy - 4y^3) + 2/\sqrt{3}\, (5x^2 - 12xy^2)$

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a) $\alpha = xy\,dx + y^2 dy - z\,dz$

$p = (1,0,1)$

$v_p = \langle 2,1,0 \rangle$

→ Evaluate 1-form α at point p

$\alpha_p = (1)(0)dx + (0)^2 dy - (1)dz$

$\alpha_p = -1dz$

$\alpha_p = -dz$

→ Vector representation:

$v_p = 2\,\partial/\partial x + 1\,\partial/\partial y + 0\,\partial/\partial z$

$v_p = 2\,\partial/\partial x + \partial/\partial y$

→ Apply 1-form:

$\alpha_p(v_p) = -dz\,(2\,\partial/\partial x + \partial/\partial y)$

$\downarrow \quad = 0 + 0$

$\alpha_p(v_p) = 0$

b) $\alpha = x^2 dx + y^2 dy + z^2 dz$

$p = (2,1,1)$

$v_p = \langle 1,1,1 \rangle$

→ Coordinate representation:

$\alpha_p = 2^2 dx + 1^2 dy + 1^2 dz$

$\alpha_p = 4dx + dy + dz$

→ Vector representation:

$v_p = 1 \cdot \partial/dx + 1\,\partial/dy + 1\,\partial/dz$

$v_p = \partial/dx + \partial/dy + \partial/dz$

→ Apply 1-form

$\alpha_p(v_p) = (4dx + dy + dz) \cdot (\partial/dx + \partial/dy + \partial/dz)$

$= 4dx(\partial/dx + \partial/dy + \partial/dz) + dy(\partial/dx + \partial/dy + \partial/dz) + dz(\partial/dx + \partial/dy + \partial/dz)$

$= 4(1+0+0) + (0+1+0) + (0+0+1)$

$= 4 + 2$

$\alpha_p(v_p) = 6$

Find the differential of $f(x,y) = x^2y^3$

→ Differential of a function:

$df = \partial f/\partial x \, dx + \partial f/\partial y \, dy$

→ $\partial f/\partial x$:

↳ $\partial/\partial x [x^2y^3]$

$= 2xy^3$

→ $\partial f/\partial y$:

↳ $\partial/\partial y [x^2y^3]$

$= 3x^2y^2$

→ Thus, the differential of f:

$df = 2xy^3 \, dx + 3x^2y^2 \, dy$

Find the df of $f(x,y,z) = x + y^2 + z^3$

→ $\partial/\partial x [x + y^2 + z^3] = 1$

→ $\partial/\partial y [x + y^2 + z^3] = 2y$

→ $\partial/\partial z [x + y^2 + z^3] = 3z^2$

↓

→ Thus:

$df = dx + 2y \, dy + 3z^2 dz$