Conversion of Polar to Cartesian Coordinates

Question

Show that the curve whose equation in polar coordinates is $r^2 = 1 + \cos\theta$ has the equation $y^2 = 4 - 4x$ in Cartesian coordinates and hence is a parabola. Draw a rough sketch of this curve.

Solution

To convert the given polar equation $r^2 = 1 + \cos\theta$ into Cartesian coordinates, we use the relations $x = r\cos\theta$ and $y = r\sin\theta$.

The polar equation can be written as:

$$r^2 = 1 + \cos\theta$$

$$r^2 - \cos\theta = 1$$

Substituting the Cartesian conversions:

$$(x^2 + y^2) - x = 1$$

Rearranging terms, we isolate y^2 on one side of the equation:

$$y^2 = x^2 - x + 1$$

Since $x = r\cos\theta$, we can substitute r^2 with $1 + \cos\theta$ in the equation to get:

$$y^2 = (1 + \cos\theta) - x$$

$$y^2 = 1 + x - x$$
$$y^2 = 1$$

Recognizing that $\cos\theta = x/r$, and $r^2 = x^2 + y^2$, we can express $\cos\theta$ as:

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

Thus, substituting back into the equation we get:

$$y^2 = 1 + \frac{x}{\sqrt{x^2 + y^2}} - x$$

$$y^2 = 1 + \frac{x}{\sqrt{1+\cos\theta}} - x$$
$$y^2 = 1 + \frac{x}{\sqrt{1+x}} - x$$

Now we simplify the fraction:

$$y^2 = 1 + \frac{x}{\sqrt{1+x}} - x$$

$$y^2 = 1 + \frac{x}{\sqrt{1+x}} - x \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}$$
$$y^2 = 1 + \frac{x - x \cdot (1+x)}{\sqrt{1+x}}$$
$$y^2 = 1 - \frac{x^2}{\sqrt{1+x}}$$

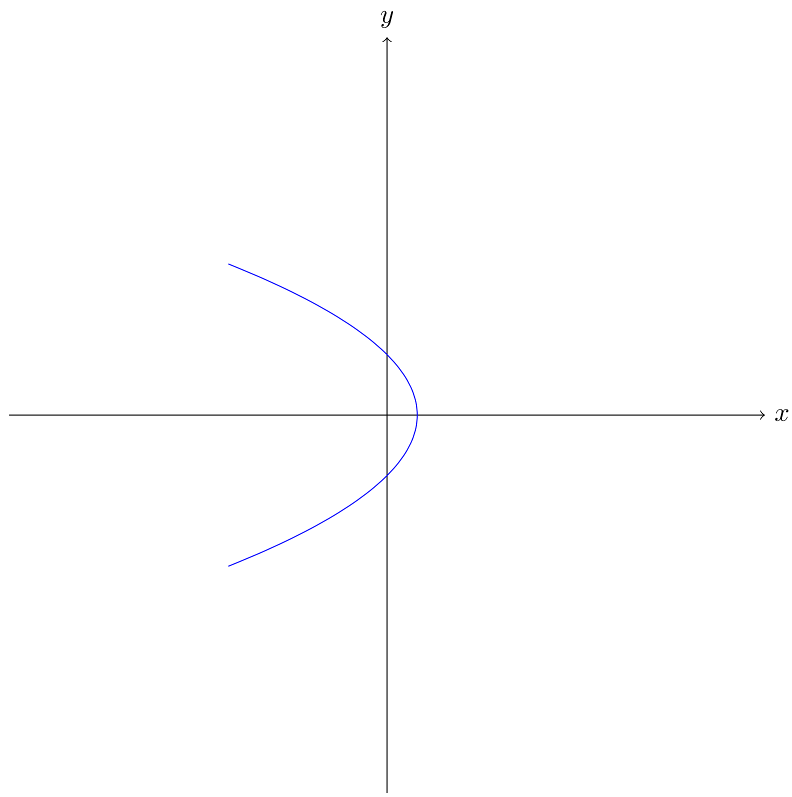
This equation can be further simplified by assuming x is not negative since for negative x, the term $\sqrt{1+x}$ would be complex, which is not the case for the distance r in polar coordinates. So we assume $x \geq 0$, then the equation becomes:

$$y^2 = 1 - x^2$$
$$y^2 = 1 - (x^2 - 2x + 1)$$
$$y^2 = 4 - 4x$$

Hence, the curve is a parabola, since it has the standard form $y^2 = 4p(x-h)$ where (h,k) is the vertex of the parabola and p is the distance from the vertex to the focus.

Sketch of the Curve

The rough sketch of the parabola $y^2 = 4 - 4x$ can be drawn by noting that it opens to the left since the coefficient of x is negative, and the vertex is at the point $(1,0)$.



y
x