

MA283: Linear Algebra

70% Exam

30% Continuous Assessment (Homework)

10% Optional Project (Bonus)

Robert Davidson

Contents

1	Review of Matrix Algebra	3
2	Systems of linear equations	6
2.1	Linear equations and Solution Sets	6
2.1.1	Interpreting Linear Systems as Matrix Equations	6
2.2	Elementary Row Operations	6
2.2.1	REF and Strategy	6
2.2.2	Row Reduced Echelon Form	7
2.3	Leading variables and free variables	7
2.4	Consistent and Inconsistent Systems	7
2.5	Possible Outcomes when solving a system of equations	7
2.6	Elementary Row Operations as Matrix Transformations	8
2.6.1	Multiplying a Row by a Non-Zero Scalar	8
2.6.2	Switching Two Rows	8
2.6.3	Adding a Multiple of One Row to Another	8
2.7	EROs and Inverses	10
3	Vector Spaces and Subspace Structure	11
3.1	The Image and Kernel of a Linear Transformation	11
3.2	Subspaces	11
3.3	The span : how to make subspaces	11
3.4	Spanning sets	12
3.5	Linear Dependence and Linear Independence	13
3.5.1	Test for linear independence	13
3.6	Finite Dimensional Spaces	13
3.7	Basis	14
3.8	Steinitz Replacement Lemma	14
3.9	Recap of span, linear independence and basis	14
3.10	Consequences of the replacement theorem	15
3.11	Every basis has the same number of elements	15
3.12	Row rank and column rank	16

1 Review of Matrix Algebra

Matrix Addition

If a matrix has m rows and n columns, we say it is $m \times n$. **Two matrices can only be added if they have the same size..** In this case, we just add the entries in each position.

The $m \times n$ **zero** matrix is a matrix with all entries equal to 0. It is the **Identity element** for matrix addition (adding it to any matrix does not change the matrix)

Matrix Multiplication by a Scalar

This simply means multiplying each entry of the matrix by the scalar. For example:

$$\alpha \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \alpha & 2\alpha \\ 3\alpha & 4\alpha \end{bmatrix}$$

Remark: Now that we have addition and scalar multiplication, we can subtract matrices ($A - B = A + (-1)B$), provided they are the same size.

Vector Space

With these operations of addition and scalar multiplication, the set of $m \times n$ matrices is a vector space. A **vector space** algebraic structure whose elements can be added, subtracted and multiplied by scalars.

Linear Combinations

Definition 1.1: Linear Combinations

Suppose v_1, v_2, \dots, v_k are elements that can be added together and multiplied by scalars.

A Linear Combination of v_1, v_2, \dots, v_k is an expression of the form:

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$$

where $\alpha_i \in \mathbb{R}$ are scalars, called **coefficients**.

Matrix-Vector Multiplication

Definition 1.2

Let A be a $m \times n$ matrix, and \mathbf{v} be a column vector with n entries ($n \times 1$ matrix).

Then the matrix vector product Av is the column vector, with m entries, obtained by taking the linear combination of the columns of A with the entries of \mathbf{v} as coefficients.

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix} = 7 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 9 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 41 \\ 33 \end{bmatrix}.$$

Remark: Av , if defined, has the same number of rows as A and the same number of columns as \mathbf{v} .

Matrix-Matrix Multiplication

Definition 1.3

Let A and B be matrices of size $m \times p$ and $p \times n$, respectively. Write v_1, \dots, v_n for the columns of B . Then the product AB is the $m \times n$ matrices whose columns are Av_1, \dots, Av_n .

The entry at row i and column j of the matrix A is given by A_{ij} . The entry in the i, j position of the product AB is the i th entry of the vector Av_j , where the vector v_j is the j th column of B . In other words, the entry in the i, j position of the product AB is given by:

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \dots + A_{ip}B_{pj} = \sum_{k=1}^p A_{ik}B_{kj}$$

Definition 1.4

If A is $m \times p$ with rows u_1, \dots, u_m and B is $p \times n$ with columns v_1, \dots, v_n , then the product AB is:

$$AB = \begin{bmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & \dots & u_1 \cdot v_n \\ u_2 \cdot v_1 & u_2 \cdot v_2 & \dots & u_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m \cdot v_1 & u_m \cdot v_2 & \dots & u_m \cdot v_n \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} \quad AB = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

For matrices A and B , the products AB and BA are generally not equal, even if they are both defined and even if both have the same size.

Linear Transformations**Definition 1.5**

Let m and n be positive integers.

A **linear transformation** T from \mathbb{R}^n to \mathbb{R}^m is a function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that satisfies:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$
- $T(\lambda \mathbf{u}) = \lambda T(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

Matrix of a Linear Transformation

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is the linear transformation:

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 7 \end{bmatrix}$$

Then for the vector in \mathbb{R}^3 with entries a, b, c :

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = aT \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + bT \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + cT \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -6 \\ -3 & 4 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Where the 2×3 matrix M_T is called the **standard matrix** of A . A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be completely represented by an $m \times n$ matrix M_T .

Understanding the Matrix Representation

- The columns of matrix M_T are the images of the standard basis vectors e_1, e_2, \dots, e_n under T .
- For any vector $v \in \mathbb{R}^n$, we calculate $T(v)$ by multiplying: $M_T \cdot v$.
- Therefore, matrix-vector multiplication is simply evaluating a linear transformation.

Correspondence: Any $m \times n$ matrix A defines a linear transformation $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by: $T_A(v) = Av$. Linear transformations include rotations, reflections and scaling

Efficiency of Representation: A remarkable property of linear transformations is their information efficiency:

- To completely define $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, we need only mn values.
- These values are the coordinates of the n transformed basis vectors in \mathbb{R}^m .
- This differs fundamentally from general continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$, which cannot be fully determined by their values at finitely many points.

Matrix multiplication is composition

Suppose that $T : \mathbb{R}^n \rightarrow \mathbb{R}^p$ and $S : \mathbb{R}^p \rightarrow \mathbb{R}^m$ are linear transformations. Then the composition $S \circ T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is also a linear transformation from \mathbb{R}^n to \mathbb{R}^m defined for $\mathbf{v} \in \mathbb{R}^n$ by:

$$S \circ T(\mathbf{v}) = S(T(\mathbf{v}))$$

To see how that the $m \times n$ matrix $M_{S \circ T}$ depends on the matrix $M_S(m \times p)$ and $M_T(p \times n)$ we look at the definition of $M_{S \circ T}$:

- The first column has coordinates $S \circ T(e_1) = S(T(e_1))$
- $T(e_1)$ is first column of M_T
- Then $S(T(e_1))$ is the matrix-vector product $M_S \cdot M_T(e_1)$
- Same for all other columns $\implies M_{S \circ T} = M_S \cdot M_T$

Thus, we conclude **matrix multiplication is composition of linear transformations**.

2 Systems of linear equations

2.1 Linear equations and Solution Sets

A linear equation in the variables x and y is an equation of the form

$$2x + y = 3$$

If we replace x and y with some numbers, the statement **becomes true or false**.

Definition 2.1: Solution to a linear equation

A pair, $(x_0, y_0) \in \mathbb{R}$, is a solution to a linear equation if setting $x = x_0$ and $y = y_0$ **makes the equation true**.

Definition 2.2: Solution set

The **solution set** is the set of all solutions to a linear equation.

$$a_1X_1 + a_2X_2 + \dots + a_nX_n = b \quad \text{where } a_i, b \in \mathbb{R}$$

is an **affine hyperplane** in \mathbb{R}^n ; geometrically resembles a copy of \mathbb{R}^{n-1} inside \mathbb{R}^n .

2.1.1 Interpreting Linear Systems as Matrix Equations

$$\begin{array}{rrcrcl} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 0 \end{bmatrix}$$

2.2 Elementary Row Operations

To solve a system of linear equations we associate an **augmented matrix** to the system of equations. For example:

$$\begin{array}{rrcrcl} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array} \right]$$

To solve, we can perform the following **Elementary Row Operations (EROs)**:

1. Multiply a row by a non-zero constant.
2. Add a multiple of one row to another row.
3. Swap two rows.

The goal of these operations is to transform the augmented matrix into **row echelon form (REF)** or **reduced row echelon form (RREF)**.

2.2.1 REF and Strategy

We say a matrix is in **row echelon form (REF)** if:

- The first non zero entry in each row is a 1 (called the **leading 1**).
- If a column has a leading 1, then all entries below it are 0.
- The leading 1 in each row is to the right of the leading 1 in the previous row.
- All rows of 0s are at the bottom of the matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Example of REF

We have produced a new system of equations. This is easily solved by back substitution.

Concept 2.1: Strategy for Obtaining REF

- Get a 1 as the top left entry
- Use this 1 to clear the entries below it
- Move to the next column and repeat
- Continue until all leading 1s are in place
- Use back substitution to solve the system

2.2.2 Row Reduced Echelon Form

A matrix is in **reduced row echelon form** (RREF) if:

- It is in REF
- The leading 1 in each row is the only non-zero entry in its column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Example of RREF

2.3 Leading variables and free variables

We'll start by an example:

$$\begin{array}{rrrrrrrrcl} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 & \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 & \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 & \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{array} \right]$$

Solving this system of equations, we get:

$$\text{RREF: } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \Rightarrow \begin{array}{rcl} x_1 + 2x_4 & = & 4 \\ x_2 - x_4 & = & 2 \\ x_3 + x_4 & = & 2 \end{array} \Rightarrow \begin{array}{rcl} x_1 & = & 4 - 2x_4 \\ x_2 & = & 2 + x_4 \\ x_3 & = & 2 - x_4 \end{array}$$

This RREF tells us how the **leading variables** (x_1, x_2, x_3) depend on the **free variable** (x_4). The free variable can take any value in \mathbb{R} . We write the solution set as:

$$x_1 = 4 - 2t, \quad x_2 = 2 + t, \quad x_3 = 2 - t, \quad x_4 = t \quad \text{where } t \in \mathbb{R}$$

$$(x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t); \quad t \in \mathbb{R}$$

Definition 2.3: Leading and Free Variables

- **Leading variable** : A variable whose columns in the RREF contain a leading 1
- **Free variable** : A variable whose columns in the RREF do not contain a leading 1

2.4 Consistent and Inconsistent Systems

Consider the following system of equations:

$$\begin{array}{rrrrrrcl} 3x & + & 2y & - & 5z & = & 4 & \\ x & + & y & - & 2z & = & 1 & \\ 5x & + & 3y & - & 8z & = & 6 & \end{array} \Rightarrow \left[\begin{array}{ccc|c} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (\text{REF})$$

We can see the last row of the REF is:

$$0x + 0y + 0z = 1$$

This equation clearly has no solution, and hence the system has no solutions. We say the system is **inconsistent**. Alternatively, we say the system is **consistent** if it has at least one solution.

2.5 Possible Outcomes when solving a system of equations

- The system may be **inconsistent** (no solutions) - i.e:

$$[0 \ 0 \ \dots \ 0 \ | \ a] \quad a \neq 0$$

- The system may be **consistent** which occurs if:

- **Unique Solutions** each column (aside from the rightmost) contains a single leading 1. - i.e:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

- **Infinitely many solutions** at least one variable does not appear as a leading 1 in any row, making it a free variable - i.e:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

2.6 Elementary Row Operations as Matrix Transformations

Elementary row operations may be interpreted as **matrix multiplication**. To see this, first we introduce the **Identity matrix**:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The I_m Identity matrix is an $m \times m$ matrix with 1s on the diagonal and 0s elsewhere. We also introduce the $E_{i,j}$ matrix which has 1 in the (i, j) position and 0s elsewhere. For example:

$$E_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then:

$$I_3 + 4E_{1,2} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing a row operation on A is the same as multiplying A by an appropriate matrix E on the left. These matrices are called **elementary matrices**. They are **always invertible**, and **their inverses are also elementary matrices**. The statement:

"every matrix can be reduced to RREF through EROs"

is equivalent to saying that

"for every matrix A with m rows, there exists a $m \times m$ matrix B which is a product of elementary matrices such that BA is in RREF."

2.6.1 Multiplying a Row by a Non-Zero Scalar

When multiplying row i of matrix A by a scalar $\alpha \neq 0$, we can use the matrix:

$$I_m + (\alpha - 1)E_{i,i}$$

This works because it modifies only the (i, i) entry of the identity matrix to be α while keeping all other entries unchanged. When multiplied with A , it scales row i by α and leaves all other rows intact.

Example: If $\alpha = 5$ and $i = 2$, then:

$$I_3 + 4E_{2,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad (I_3 + 4E_{2,2})A = \begin{bmatrix} 1 & 2 & 3 \\ 20 & 25 & 30 \\ 7 & 8 & 9 \end{bmatrix}$$

2.6.2 Switching Two Rows

To swap rows i and k , we use:

$$S = I_m + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k}$$

This works by:

- Removing the 1's at positions (i, i) and (k, k) from the identity matrix
- Adding 1's at positions (i, k) and (k, i)

Example: Swapping rows 1 and 3:

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad SA = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

2.6.3 Adding a Multiple of One Row to Another

To replace row k with row $k + \alpha \times$ row i , use:

$$I_m + \alpha E_{k,i}$$

This adds α times row i to row k while leaving all other rows unchanged because:

- For any row $j \neq k$, the corresponding row in this matrix is just the standard basis row
- Row k becomes the sum of the standard basis row k plus α times the standard basis row i

Example: Adding 3 times row 1 to row 2:

$$I_3 + 3E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad (I_3 + 3E_{2,1})A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \\ 7 & 8 & 9 \end{bmatrix}$$

Example 2.1

Write the inverse of an elementary matrix and show it is an elementary matrix.

Multiplying a row by a nonzero scalar:

- **Operation:** Multiply row i by $\alpha \neq 0$.
- **Elementary Matrix:** $E = I_m + (\alpha - 1)E_{i,i}$
- **Inverse:** To reverse the operation, multiply row i by $1/\alpha$. Hence, the inverse is

$$E^{-1} = I_m + \left(\frac{1}{\alpha} - 1\right) E_{i,i} = I_m + \frac{1 - \alpha}{\alpha} E_{i,i}.$$

Swapping two rows:

- **Operation:** Swap rows i and k .
- **Elementary Matrix:** $S = I_m - E_{i,i} - E_{k,k} + E_{i,k} + E_{k,i}$
- **Inverse:** Since swapping the same two rows twice returns them to their original positions,

$$S^{-1} = S.$$

Adding a multiple of one row to another:

- **Operation:** Add α times row i to row k .
- **Elementary Matrix:** $E = I_m + \alpha E_{k,i}$
- **Inverse:** To undo the operation, subtract α times row i from row k . Therefore,

$$E^{-1} = I_m - \alpha E_{k,i}.$$

Example 2.2

Prove that every invertible matrix in $M_n(\mathbb{R})$ is a product of elementary matrices.

Let A be an invertible matrix in $M_n(\mathbb{R})$. Since A is invertible, we can use Gaussian elimination to transform A into the identity matrix I_n .

Let E_1, E_2, \dots, E_k be the elementary matrices corresponding to the row operations used in the elimination process. Then, we have:

$$\text{Multiplying a row by a scalar: } I_n + (\alpha - 1)E_{i,i}$$

$$\text{Swapping two rows: } I_n + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k}$$

$$\text{Adding a multiple of one row to another: } I_n + \alpha E_{k,i}$$

Applying these in sequence to A gives:

$$E_k \cdots E_2 E_1 A = I_n$$

Since $E_k \cdots E_2 E_1 = I_n$, we can multiply both sides by $(E_k \cdots E_2 E_1)^{-1}$ on the left to obtain:

$$A = (E_k \cdots E_2 E_1)^{-1} I_n = (E_k \cdots E_2 E_1)^{-1}$$

Using the property:

$$(E_k \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1},$$

we can express A as a product of elementary matrices:

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Since each E_i is an elementary matrix, its inverse is also an elementary matrix. Therefore, A can be expressed as a product of elementary matrices.

2.7 EROs and Inverses

Elementary Row Operations can be used to find the inverse of a square matrix. Consider a square matrix $A \in M_n(\mathbb{F})$ (that is, an $n \times n$ matrix over a field \mathbb{F}). If A is invertible, let

$$A^{-1} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}$$

be its inverse, where each \mathbf{v}_i is the i th column of A^{-1} . By definition of the matrix inverse, we have

$$A A^{-1} = A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} = I_n,$$

the $n \times n$ identity matrix. This implies that

$$A\mathbf{v}_i = \mathbf{e}_i, \quad \text{for each } i = 1, 2, \dots, n,$$

where \mathbf{e}_i is the i th column of I_n (which has a 1 in the i th row and 0 everywhere else). In other words, each column \mathbf{v}_i of A^{-1} is the unique solution to the linear system

$$A\mathbf{v}_i = \mathbf{e}_i.$$

To find A^{-1} effectively, we form the augmented matrix $[A \mid I_n]$ and apply EROs to transform A into I_n . When this is achieved, the augmented portion becomes A^{-1} . Thus, we have

$$\text{RREF}([A \mid I_n]) = [I_n \mid A^{-1}].$$

Example 2.3

Find A^{-1} if $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$.

We form a 3×6 matrix $A' = [A \mid I_3]$:

$$A' = \begin{bmatrix} 3 & 4 & -1 & | & 1 & 0 & 0 \\ 1 & 0 & 3 & | & 0 & 1 & 0 \\ 2 & 5 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

We apply the following EROs to A' :

- $R_1 \leftrightarrow R_2$
- $R_2 \rightarrow R_2 - 3R_1$
- $R_3 \rightarrow R_3 - 2R_1$
- $R_3 \rightarrow R_3 + R - 2$
- $R_3 \leftrightarrow R_2$
- $R_3 \rightarrow R_3 - 4R_2$
- $R_3 \times (-\frac{1}{10})$
- $R_1 \rightarrow R_1 - 3R_3$

To obtain:

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & | & -1 & 1 & 1 \\ 0 & 0 & 1 & | & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

That is:

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

It is easily checked that $AA^{-1} = I_3$.

3 Vector Spaces and Subspace Structure

3.1 The Image and Kernel of a Linear Transformation

$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the linear transformation defined with:

$$M_T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix}$$

The **image** of T is the subset of \mathbb{R}^3 consisting of all elements $T(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^3$. This is the set of all vectors of the form:

$$a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

In matrix terms, this is the **column space** of M_T .

The **kernel** of T is the set of all vectors $\mathbf{v} \in \mathbb{R}^3$ such that $T(\mathbf{v}) = \mathbf{0}$. This is the set of all column vectors, whose entries, a, b, c satisfies:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The kernel is a line and the image is a plane

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & | & 0 \\ 0 & -5 & 5 & | & 0 \\ 1 & 1 & 1 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The kernel (or nullspace) is $(2, 1, 1)t, t \in \mathbb{R}$, which is a line in \mathbb{R}^3 . The fact that $(-2, 1, 1)$ is in the kernel of T , means that column 3 of M_T is a linear combination of columns 1 and 2.

$$-2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

It follows that every linear combination of all three columns of M_T is just a linear combination of columns 1 and 2.

The column space of M_T is:

$$\left\{ a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

3.2 Subspaces

Definition 3.1

A non empty subset \mathbf{V} of \mathbb{R}^n is a **subspace** if:

- **Closed under addition:** $u + v \in \mathbf{V}, u, v \in \mathbf{V}$
- **Closed under scalar multiplication:** $\alpha u \in \mathbf{V}, u \in \mathbf{V}, \alpha \in \mathbb{R}$

Examples of subspaces

- $\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$ is not a subspace of \mathbb{R}^3 . the $[1, 0, 0]$ and $(0, 1, 0)$ vectors are in the set, but their sum $(1, 1, 0)$ is not in the set.
- $\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot (1, 2, 3) = 0\}$ is a subspace of \mathbb{R}^3 .
- $\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot (1, 2, 3) \neq 0\}$ is not a subspace of \mathbb{R}^3 .
- The kernel of any linear transformation is a subspace of \mathbb{R}^n .
- The image of any linear transformation is a subspace of \mathbb{R}^n .

3.3 The span : how to make subspaces

Definition 3.2

Let $S = \{v_1, \dots, v_k\}$ be any finite subset of \mathbb{R}^n

The subset of \mathbb{R}^n consisting of all linear combinations of the elements of S is a subspace of \mathbb{R}^n and is called the **span** of S and is denoted by $\langle(S)\rangle$.

Proof that $\langle S \rangle$ is a subspace of \mathbb{R}^n

- **Closed under addition:**

Let $u, v \in \langle S \rangle$. Then $u = a_1v_1 + a_2v_2 + \cdots + a_kv_k$ and $v = b_1v_1 + b_2v_2 + \cdots + b_kv_k$ for some $a_i, b_i \in \mathbb{R}$. We see that:

$$u + v = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \cdots + (a_k + b_k)v_k$$

So S is closed under addition.

- **Closed under scalar multiplication:**

Let $u \in \langle S \rangle$ and $\alpha \in \mathbb{R}$. We need to show that cu is a linear combination of v_1, \dots, v_k . We have $u = a_1v_1 + a_2v_2 + \cdots + a_kv_k$ for some $a_i \in \mathbb{R}$. Then:

$$cu = c(a_1v_1 + a_2v_2 + \cdots + a_kv_k) = (ca_1)v_1 + (ca_2)v_2 + \cdots + (ca_k)v_k$$

so $cu \in \langle S \rangle$.

3.4 Spanning sets

Definition 3.3

Let V be a subspace of \mathbb{R}^n .

A subset S of V is a **spanning set** for V if $\langle S \rangle = V$.

This means that every element of V can be expressed as a linear combination of the elements of S .

Example

The set $\{e_1, e_2, e_3\}$ is a spanning set of \mathbb{R}^3 . We know that:

$$e_1 = [1, 0, 0], \quad e_2 = [0, 1, 0], \quad e_3 = [0, 0, 1]$$

We can represent every element of \mathbb{R}^3 as a linear combination of e_1, e_2, e_3 :

$$\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 2e_1 - 3e_2 + 4e_3$$

Remark A set S of three column vectors in \mathbb{R}^3 is a spanning set of \mathbb{R}^3 if and only if the three vectors are linearly independent. This occurs only if the 3×3 matrix whose columns are the three vectors has S as an **inverse**.

Questions about spanning sets

- Does \mathbb{R}^3 have a spanning set fewer than three vectors?
 - **No.** A spanning set for \mathbb{R}^3 must contain at least three linearly independent vectors, since the dimension of \mathbb{R}^3 is 3. Fewer than three vectors cannot span all of \mathbb{R}^3 .
- Does every spanning set of \mathbb{R}^3 have three vectors?
 - **No.** A spanning set can have more than three vectors, but not necessarily exactly three. Redundant vectors (linearly dependent ones) can be included, so a spanning set might have more than three vectors.
- Does every spanning set of \mathbb{R}^3 contain one with exactly three elements?
 - **Yes.** Every spanning set of \mathbb{R}^3 contains a basis, and since the dimension is 3, there exists a subset of exactly three linearly independent vectors that still span \mathbb{R}^3 .
- If V is a subspace of \mathbb{R}^3 does V have a spanning set with at most three elements?
 - **Yes.** Any subspace of \mathbb{R}^3 has a basis, and since \mathbb{R}^3 has dimension 3, the basis of any of its subspaces can have at most 3 elements. Hence, every subspace can be spanned by at most three vectors.
- If V is a proper subspace of \mathbb{R}^3 , does V have a spanning set with fewer than three elements?
 - **Yes.** A proper subspace of \mathbb{R}^3 has dimension less than 3, so it can be spanned by fewer than three vectors.

3.5 Linear Dependence and Linear Independence

Definition 3.4

A set of at least two vectors in \mathbb{R}^n is **linearly dependent** if one of its elements is a linear combination of the others.

A set of vectors in \mathbb{R}^n is **linearly independent** if it is not linearly dependent.

For a subset $\{v_1, \dots, v_k\}$ of \mathbb{R}^n , suppose that v_k is a linear combination of $\{v_1, \dots, v_{k-1}\}$. Then every linear combination of $\{v_1, \dots, v_k\}$ is **already a linear combination** of v_1, \dots, v_{k-1} :

$$\langle v_1, \dots, v_k \rangle = \langle v_1, \dots, v_{k-1} \rangle$$

If we are interested in the span of $\{v_1, \dots, v_k\}$, we can throw away v_k and this wouldn't change the span.

Linear independence means that throwing away any element of the set **shrinks the span**

Example 3.1

$$\begin{array}{rrrrrr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 = 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 = 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 = 1 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & -18 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The three equations of the system form a linearly dependent set. One row was eliminated by adding a linear combination of the other two rows. Thus, all the information in the system was contained in the first two equations.

The non-zero rows of the RREF are linearly independent, they span the row space of the matrix. The row space is the subspace of \mathbb{R}^5 spanned by the rows of the matrix.

3.5.1 Test for linear independence

A set is linearly independent if none of its elements is a linear combination of the others. While this makes sense, to use it as a test would mean checking every element. We have an alternative formulation, which is easier to check:

"A set of vectors is linearly independent if the only way to write the zero vector as a linear combination of the vectors in the set is to use all zero coefficients."

To decide if the set $\{v_1, \dots, v_k\}$ is linearly independent, try to write the zero vector as a linear combination of the vectors in the set:

$$\sum_{i=1}^k \alpha_i v_i = \alpha_1 v_1 + \dots + \alpha_k v_k = 0 \quad \text{for } \alpha_i \in \mathbb{R}$$

If $\forall i \rightarrow \alpha_i = 0$, then the set is linearly independent. If not, the set is linearly dependent.

Example 3.2

Decide whether the set $\{[1, 0, 1], [1, 0, -1], [1, 1, 1]\}$ is linearly independent or dependent.

To solve, we use ERO and find:

$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{array} \right] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a = b = c = 0$$

The set is linearly independent

3.6 Finite Dimensional Spaces

Definition 3.5

A vector space V is finite dimensional if it contain a finite spanning set.

This means a set $\{v_1, \dots, v_k\}$ of elements, with the property that every element of V is a linear combination of v_1, \dots, v_k .

Examples

- \mathbb{R}^n is finite dimensional with $\{e_1, \dots, e_n\}$ as a spanning set. The dimension of \mathbb{R}^n is n .
- $M_{m \times n}(\mathbb{R})$ is finite dimensional, with $\{E_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq n}$ as a spanning set with mn elements.
- An example of an infinite dimensional space is the set, $\mathbb{R}[x]$, of all polynomials with real coefficients. This set is infinite dimensional because it contains an infinite number of linearly independent vectors, such as $\{1, x, x^2, \dots\}$.

3.7 Basis

Definition 3.6

A **basis** for a vector space is a **linearly independent spanning set**.

- A basis is a minimal spanning set, one in which every element is needed and does not contain a smaller spanning set.
- Example: $\{e_1, \dots, e_n\}$ is a basis for \mathbb{R}^n .
- $\{(1, 3), (1, 4)\}$ is a basis for \mathbb{R}^2 .
- If S is a finite spanning set of a vector space V , then S contains a basis of V . If S is not linearly independent, then some $v \in S$ is a linear combination of the other elements of S . Throwing away v leaves a smaller set that still spans V . This process can be repeated until a basis is obtained.

3.8 Steinitz Replacement Lemma

Lemma 3.1

Let V be a vector space that has a basis with n elements.

Then every linearly independent set with n elements in V is a basis for V .

Proof (for $n = 3$)

Suppose $B = \{b_1, b_2, b_3\}$ is a basis of V and let $\{y_1, y_2, y_3\}$ be a linearly independent subset of V .

Step 1.

$y_1 = a_1 b_1 + a_2 b_2 + a_3 b_3$ for scalars a_1, a_2, a_3 , not all zero. We can assume (after maybe relabelling the b_i), that $a_1 \neq 0$. Then

$$b_1 = a_1^{-1} y_1 - a_1^{-1} a_2 b_2 - a_1^{-1} a_3 b_3.$$

So $b_1 \in \langle y_1, b_2, b_3 \rangle$ and $\{y_1, b_2, b_3\}$ spans V . (Note that we have to use the fact that we can divide by non-zero scalars to write b_1 as a linear combination of y_1, b_2, b_3 .)

Step 2.

Now $y_2 \in \langle y_1, b_2, b_3 \rangle$ and y_2 is not a scalar multiple of y_1 (because $\{y_1, y_2, y_3\}$ is linearly independent).

So b_2 (or b_3) has non-zero coefficient in any description of y_2 as a linear combination of y_1, b_2, b_3 .

Replace again: $\{y_1, y_2, b_2\}$ spans V .

Step 3.

Same reasoning: we can replace b_2 with y_3 to conclude $\{y_1, y_2, y_3\}$ spans V .

Conclusion: $\{y_1, y_2, y_3\}$ is a basis of V .

3.9 Recap of span, linear independence and basis

Let V be a vector space, e.g. $V = \mathbb{R}^n$ and S be a finite subset of V . Let V be a vector space (e.g. $V = \mathbb{R}^n$). Let S be a (finite) subset of V .

1. S is a spanning set of V (or S spans V) if every element of V is a linear combination of the elements of S .
2. The span of S , denoted $\langle S \rangle$, is the set of all linear combinations of elements of S , a subspace of V .
3. S is linearly independent if no element of S is a linear combination of the other elements of S .
Equivalently, if no proper subset of S spans $\langle S \rangle$.
4. S is a basis of V if S is linearly independent **AND** S spans V .
A basis is a minimal spanning set.
A basis is a maximal linearly independent set.
5. Every finite spanning set of V contains a basis of V .
6. Every linearly independent subset of V can be extended to a basis of V (we have not proved this yet!).

3.10 Consequences of the replacement theorem

Theorem 3.1

Let V be a vector space that has a basis with n elements.

Then every linearly independent set with n elements in V is a basis for V .

If V has a spanning set with n elements, a linearly independent set in V cannot have more than n elements.

If V has a linearly independent set with n elements, a spanning set in V must have at least n elements. More concisely:

Concept 3.1

The number of elements of a linearly independent set cannot exceed the number in a spanning set. Every spanning set has at least as many elements as the biggest independent set.

3.11 Every basis has the same number of elements

Let V be a finite dimensional vector space and let B and B' be the bases of V . Then:

- B is linearly independent and B' is a spanning set, so B has **at most** as many elements as B' .
- B is a spanning set and B' is linearly independent, so B has **at least** as many elements as B' .

It follows that B and B' have the same number of elements.

Definition 3.7

The dimension of V is the number of elements in a basis of V .

Note: Every vector space that has a finite spanning set has a finite basis (since we can discard elements from a finite spanning set until a basis remains).

Examples:

- The set $\{1, x, x^2, x^3\}$ is a basis for the vector space P_3 of all polynomials of degree at most 3 with real coefficients.

It is linearly independent because the only way to write the zero polynomial as

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

is by taking $a_0 = a_1 = a_2 = a_3 = 0$.

Another basis of P_3 , preferable for some applications, consists of the first four Legendre polynomials:

$$\{1, x, \frac{1}{2}(3x^2 - 1), \frac{1}{2}(5x^3 - 3x)\}.$$

- The **row space** of an $m \times n$ matrix is the subspace of \mathbb{R}^n spanned by its rows. When we reduce a matrix to row-reduced echelon form (RREF), we are computing a basis of its row space.
- In \mathbb{R}^2 , the reflection in the line $y = 2x$ sends:

$$(1, 0) \mapsto \left(-\frac{3}{5}, \frac{4}{5}\right), \quad (0, 1) \mapsto \left(\frac{4}{5}, \frac{3}{5}\right).$$

Its standard matrix is:

$$\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

The same reflection sends:

$$(1, 2) \mapsto (1, 2), \quad (2, -1) \mapsto (-2, 1).$$

It is easier to describe this transformation in terms of the basis:

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

3.12 Row rank and column rank

Let A be a $m \times n$ matrix.

The **row rank**, r of A , is the dimension of the row space of A - which is the subspace of \mathbb{R}^n spanned by the rows of A .

The **column rank**, c of A , is the dimension of the column space of A - which is the subspace of \mathbb{R}^m spanned by the columns of A . The column rank is the dimension of the image of the linear transformation whose matrix is A .

The row rank is at most m and the column rank at most n , but both can be less.