Robert Davidson

ST1112: Statistics

70% Exam

30% Continuous Assessment (3 parts)

Contents

1 Inferential Statistics

The ultimate goal in statistical inference is to estimate population parameters (like the mean μ) based on sample statistics (like the sample mean $\bar{X}$).

1.1 Part 1

1.1.1 Probability vs Statistics

- **Probability** deals with known underlying processes: one starts with a model (like proportion of red vs. green jelly beans in a jar) and computes probability of specific outcomes
- **Statistics** works in reverse: one observes outcomes (sample data) and attempts to infer the underlying process or population parameters (e.g. proportion of red jellybeans)

1.1.2 Definitions and Concepts

Definition 1.1: Population

A **population** is the complete set of items (or individuals) of interest.

Definition 1.2: Sample

A **sample** is a subset of that population, intended to represent the population

For example the sample mean $\bar{X}$ is an estimate of the population mean μ.

Definition 1.3: Population Mean (μ)

μ represents the central tendency of a population distribution.

$$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$$

where N is the population size and x_i are the individual values in the population.

μ is sometimes called the expected value or average.

Definition 1.4: Population standard deviation (σ)

σ measures the dispersion or spread of values around the mean in a population.

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}$$

where N is the population size and x_i are the individual values in the population.

Concept 1.1: Sampling Variation

When we take multiple samples from the same population, each sample's mean $\bar{X}$ will be different. This is variability is called **sampling variation**.

Larger sample sizes tend to reduce this variation, that is as n gros, the sample mean $\bar{X}$ becomes a better estimate of the population mean μ.

Concept 1.2: Sampling Distributions

The sample mean itself is a **random variable** because different samples yield different mean values.

The distribution of all possible sample means (of a given sample size n) is called the **sampling distribution** of the sample mean ($\bar{X}$).

Definition 1.5: Expected Value of the Sample Mean

$$E(\bar{X}) = \mu$$

This means if you averaged all possible sample means, you would get the population mean μ.

Definition 1.6: Standard Error of the Mean

$$SD(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

where σ is the population standard deviation and n is the sample size.

This value is called the **standard error** of the mean and measures how much the sample mean $\bar{X}$ fluctuates around the population mean μ.

Definition 1.7: Central Limit Theorem

$$\bar{X} \sim N\left(\mu, \frac{\sigma^m}{n}\right)$$

where $\bar{X}$ is the sample mean, μ is the population mean, and σ is the population standard deviation.

The **Central Limit Theorem** states that the sampling distribution of the sample mean $\bar{X}$ (the distribution of all sample means) approaches a normal distribution as the sample size n increases, **regardless of the shape of the population distribution.**

This means that for large enough sample sizes, we can use the normal distribution to make inferences about the population mean μ.

Practically, many apply the rule of thumb $n \geq 30$ to treat $\bar{X}$ as normally distributed.

Definition 1.8: Unbiased Estimators

We say a statistic T is an **unbiased estimator** of a population parameter θ, if $E(T) = \theta$.

For example, the sample mean $\bar{X}$ is an unbiased estimator of the population mean μ because $E(\bar{X}) = \mu$.

The sample standard deviation s (using Bessel's correction, dividing by multiplying by $\frac{1}{n-1}$ rather than $\frac{1}{N}$) is an unbiased estimator of the population standard deviation σ.

1.1.3 Example

Example 1.1: Weekly rent

If a population mean rent is $\mu = 225$, with $\sigma = 25$ for a population sample size $n = 30$, the sample distribution of the sample mean is approximately:

$$\bar{X} \sim N\left(225, \frac{25^2}{30}\right)$$

This lets us compute probabilities for specific sample mean ranges using the normal distribution (e.g. $P(\bar{X} < 220)$).

1.2 Part 2 - Confidence Intervals

1.2.1 Recap

A **sample statistic** (e.g. the sample mean $\bar{X}$) varies from one sample to another. Understanding this variation (and quantifying it via the standard error) is crucial for knowing how precise (or imprecise) an estimate really is.

If we have a large sample size n from a population with mean μ and standard deviation σ, then our sample distribution of the sample mean $\bar{X}$ is approximately normal:

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

In practice, for $n \geq 30$, $\bar{X}$ can be treated as normally distributed even if the original population is not strictly normal.

1.2.2 Confidence Intervals

Concept 1.3: Why confidence intervals?

Why do we need confidence intervals, instead of a single point estimate, like the sample mean $\bar{X}$?

A confidence interval provides a range of plausible values for the population parameter (e.g. μ) based on the sample data.

Analogy: Using a single point estimate is like trying to catch a fish wih a spear; your aim may not be perfect. Using a confidence interval is like using a net; we have a better chance of "catching" (capturing) the true population parameter.

Definition 1.9: Confidence Interval

$$\bar{X} \pm (\text{critical value}) \times SE(\bar{X})$$

where $SE(\bar{X}) = \frac{\sigma}{\sqrt{n}}$ is the standard error of the sample mean, $\pm$ is the margin of error.

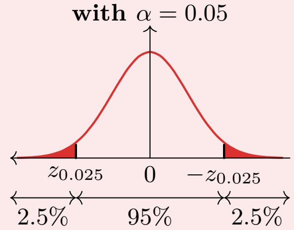
Interpretation:
If we repeat the sampling process many times and construct confidence intervals from each sample, then approximately $100 \times (1 - \alpha)\%$ of those intervals will contain the true population parameter μ.
In other words, you do not say *"there is a 95% chance that μ lies in my interval"*. Rather we say, **"on repeated sampling 95% of such intervals will contain the true population mean μ."**

Definition 1.10: Critical Values

The **critical value** is a z-score that corresponds to the desired confidence level.

For example, for a 95% confidence level, the critical value is $Z_{\alpha/2} = 1.96$ (where $\alpha = 0.05$). This means that 95% of the area under the normal curve lies within 1.96 standard deviations of the mean.

Standard Normal Distribution (z)
with $\alpha = 0.05$



Example 1.2: Find crtitical value for the 95% CI

For a confidence interval of 95%, we want to find the z-score that leaves 2.5% in each tail of the normal distribution.
We want to find the z-value where the cumulative area)from the left up to that $z - score$) is $1 - 0.025 = 0.975$.
We look in the z-tables for the value closest to 0.975 and read the row and column headers to find the z-value.
The z-value is 1.96.

1.2.3 CI with large n, σ uknown but replaced with s

Form of the 95% CI (Z **based)**:
For a large sample ($n \geq 30$) or when the population standard deviation σ is known **or** reasonably approximated, the confidence interval for μ is:

$$\bar{X} \pm Z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}}$$

where $Z_{\alpha/2}$ is the z-score for the chosen confidence level (e.g. $Z_{0.025} = 1.96$ for a 95% CI). If σ is not known but n is large, we replace σ with the sample standard deviation s: