Mathematical Methods II

Exams:
70% Exam
30% Continuous Assessment (3 parts)

Contents

1 Week 1: Intro to Laplace Transforms

1.1 Preliminary: Exponential Functions

Recall the following facts:

1. $e^t = \exp(t) = 1 + \frac{t^1}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{t^i}{i!}$.
2. $e^0 = 1$.
3. As $t \to \infty$, $e^t \to \infty$; as $t \to -\infty$, $e^t \to 0$.
4. $\frac{d}{dt} e^t = e^t$, and $\frac{d}{dt} e^{st} = s\, e^{st}$.
5. $\displaystyle\int e^t \, dt = e^t + C$, and $\displaystyle\int e^{st} \, dt = \frac{1}{s} e^{st} + C$.
6. $e^{t_1} \cdot e^{t_2} = e^{\,t_1 + t_2}$.

1.2 Laplace Transforms

Definition

Consider a function $f(t)$ for $t > 0$.

We define the Laplace transform of $f(t)$ as

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \, dt.$$

Note

We can also write $\mathcal{L}\{f(t)\}$ as $F(s)$.

Alternatively,

$$\mathcal{L}\{f(t)\} = \lim_{R \to \infty} \int_0^{R} e^{-st} f(t) \, dt.$$

Recalling that

$$\int_0^1 st^2 \, dt = s\left[\frac{t^3}{3}\right]_0^1 = \frac{s}{3}$$

we see that $\mathcal{L}\{f(t)\}$ is just a function of s.

1.3 Laplace Transforms of Common Functions

Given the function $f(t)$, its Laplace transform is denoted as:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Note

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

At $t = 0$, $e^{-st} = 1$ and as $t \to \infty$, $e^{-st} = 0$, so

$$\mathcal{L}\{1\} = \left(0 - \frac{1}{-s}\right) = \frac{1}{s} \quad s > 0$$

Note

$$\begin{aligned}
\mathcal{L}\{e^{kt}\} &= \int_0^\infty e^{-st} e^{kt}\, dt \\
&= \int_0^\infty e^{(k-s)t}\, dt \\
&= \left[\frac{e^{(k-s)t}}{k-s}\right]_0^\infty
\end{aligned}$$

As $t \to \infty$, $e^{(k-s)t} = 0$, and at $t = 0$, $e^{(k-s)t} = 1$.
Applying the limits:

$$\begin{aligned}
\left[\frac{e^{(k-s)t}}{k-s}\right]_0^\infty &= \frac{0}{k-s} - \frac{1}{k-s} \\
&= -\frac{1}{k-s} \\
&= \frac{1}{s-k} \quad \text{for } s > k
\end{aligned}$$

Thus,

$$\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}, \quad s > k$$

Note

$$\begin{aligned}
\mathcal{L}\{af(t) + bg(t)\} &= \int_0^\infty e^{-st}\Big(af(t) + bg(t)\Big)\, dt \\
&= a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt \\
&= a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}.
\end{aligned}$$

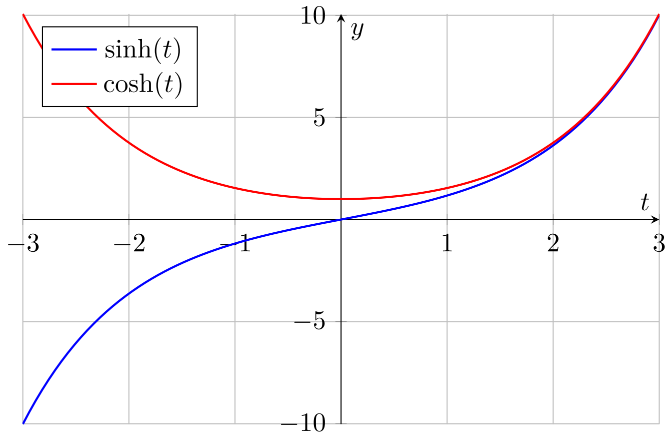
2 Week 2 : Laplace Transforms of Hyperbolic Functions

Last week we saw:

- $\mathcal{L}\{1\} = \dfrac{1}{s}$
- $\mathcal{L}\{e^{kt}\} = \dfrac{1}{s-k}$
- $\mathcal{L}\{af(t) + bg(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$

2.1 Laplace Transforms of Hyperbolic Functions

Recall:



- $\cosh(t) = \frac{e^t + e^{-t}}{2}$.
- $\sinh(t) = \frac{e^t - e^{-t}}{2}$.
- $\tanh(t) = \frac{\sinh(t)}{\cosh(t)}$.

Example

Find the Laplace Transofrm of $f(t) = \cosh(5), t > 0$

$$\begin{aligned}
\mathcal{L}\{\cosh(5t)\} &= L\left\{\frac{e^{5t} + e^{-5t}}{2}\right\} \\
&= \frac{1}{2}\left(L\{e^{5t}\} + L\{e^{-5t}\}\right) \\
&= \frac{1}{2}\left(\frac{1}{s-5} + \frac{1}{s+5}\right) \\
&= \frac{1}{2}\left(\frac{s+5+s-5}{s^2-25}\right) \\
&= \frac{1}{s^2-25}
\end{aligned}$$

Similarly: The Laplace Transofrm of $\cosh at$ is:

$$\frac{s}{s^2 - a^2}$$

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Example

Find the Laplace Transorm of $\cos(\omega t)$ and $i\sin(wt)$, where w is a const. and $t > 0$.
Recall **De Moivre's Theorem**:

$$e^{iwt} = \cos(wt) + i\sin(wt)$$

Here: $\mathcal{L}\{e^{iwt}\} = \frac{1}{s-iw}$, with $k = iw$

2.2 First Shift Theorem

Theorem

If $f(t)$ has a Laplace Transform $F(s)$, defined for $s > k$, then $e^{at}f(t)$ has transform $F(s-a)$, defined for $s - a > k$, that is:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

or, taking the inverse on both sides:

$$e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$$

Proof

From the definition of the Laplace Transform:

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}[e^{at}f(t)]\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$$

We see, if $\mathcal{L}\{f(t)\}$ exists for $s > k$, then $\mathcal{L}\{e^{at}f(t)\}$ exists for $s > k + a$.

Example

Find the Laplace Transform of $e^{at}\cos(\omega t)$ We recalll that:

$$\mathcal{L}\{\cos(\omega t)\} = F(s), \quad \text{where } F(S) = \frac{s}{s^2+\omega^2}$$

Hence, using the first shift theorem above, we have:

$$\mathcal{L}\{e^{at}\cos(\omega t)\} = F(s-a) = \frac{s-a}{(s-a)^2+\omega^2}$$