

MA283: Linear Algebra

70% Exam

30% Continuous Assessment (Homework)

10% Optional Project (Bonus)

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Contents

1	Review of Matrix Algebra	3
2	Systems of linear equations	5
2.1	Linear equations and Solution Sets	5
2.2	Elementary Row Operations	5
2.2.1	REF and Strategy	6
2.2.2	Row Reduced Echelon Form	6
2.3	Leading variables and free variables	6
2.4	Consistent and Inconsistent Systems	6
2.5	Possible Outcomes when solving a system of equations	7
2.6	Elementary Row Operations as Matrix Transformations	7
2.6.1	Multiplying a Row by a Non-Zero Scalar	7
2.6.2	Switching Two Rows	7
2.6.3	Adding a Multiple of One Row to Another	8
2.7	EROs and Inverses	10
3	Spanning sets, bases and dimensions	11
3.1	Vector Spaces	11
3.2	Subspaces	11
3.3	Span of a set of vectors	12
3.4	Linear independence	13
3.4.1	Equivalent Definitions of Linear Independence	14

1 Review of Matrix Algebra

Fields

- A field F is a set where addition, subtraction, multiplication and division (by nonzero elements) satisfy the usual algebraic properties. Common fields include \mathbb{R} and \mathbb{C}
- We write \mathbb{F}^p for the vector space of all p vectors with entries in \mathbb{F} .
- We'll cheat and treat any ordered list of p elements of \mathbb{F} as an element of \mathbb{F}^p .
- For example, in \mathbb{R}^3 , we might consider $(1, 2, 3)$ as coordinates, a row vector, or a column vector with 3 real entries.

Matrices Over a Field

- An $m \times n$ matrix over a field \mathbb{F} is an array of m rows and n columns of elements from \mathbb{F} .
- When $m = n$, we write $M_n(\mathbb{F})$, otherwise we write $M_{m \times n}(\mathbb{F})$.

Addition and Scalar Multiplication

- Two matrices of the same size $m \times n$ can be added entrywise
- The $m \times n$ matrix has all entries equal to zero and acts as the additive identity (adding it to any matrix does not change the matrix)
- Multiplying a matrix by a scalar means multiplying each entry by that scalar
- The set of all $m \times n$ matrices over \mathbb{F} is a vector space over \mathbb{F}

Linear Combinations

- A linear combination of vectors v_1, v_2, \dots, v_k in a vector space V with coefficients $a_1, a_2, \dots, a_k \in \mathbb{F}$ is defined as:

$$a_1 v_1 + a_2 v_2 + \dots + a_k v_k$$

- In particular, matrices themselves can be combined linearly, (e.g. $2A - 3B$)

Row and Column Vectors

- A column vector is a matrix with one column
- A row vector is a matrix with one row

Matrix-Vector Multiplication

- If A is $m \times n$ matrix and v is an n -entry column vector, the product Av is defined by taking a linear combination of the columns of A with the entries of v as coefficients.
- The result Av is an m -entry column vector.
- For a row vector u with m entries, and an $m \times n$ matrix A the product uA a row vector in \mathbb{F}^n formed by the linear combination of the rows of A with the entries of u as coefficients.

Matrix-Matrix Multiplication

- If A is a $m \times p$ and B is a $p \times n$ matrix, the product AB is defined only when the inner dimensions match (p)
- To find each column of AB , multiply A with the corresponding column vector of B .
- In entrywise form:

$$(AB)_{ij} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \dots + A_{i,p}B_{p,j} = \sum_{k=1}^p A_{i,k}B_{k,j}$$

Dot Product and Orthogonality

- For two p -entry vectors, $u, v \in \mathbb{F}^p$, their dot product is:

$$u \cdot v = \sum_{k=1}^p u_k v_k$$

- Vectors are **orthogonal** if their dot product is zero.
- If $\mathbb{F} = \mathbb{R}$, this means the vector are perpendicular.
- In matrix multiplication, the entry $(AB)_{ij}$ can be viewed as the dot product of Row i with Column j of B .

Matrices and Tables

Lets consider the table that gives the numbers of Maths M , Physics P and Chemistry C students in each of the 3 years of a course:

Year	M	P	C
2015	50	100	70
2016	60	80	80
2017	70	90	90

$$A = \begin{bmatrix} 50 & 100 & 70 \\ 60 & 80 & 80 \\ 80 & 70 & 70 \end{bmatrix}$$

Each student of M, P, C must also take a course in X and Y . We can represent the credits they earn as a matrix:

Subject	X	Y
M	10	0
P	15	15
C	20	10

$$B = \begin{bmatrix} 10 & 0 \\ 15 & 15 \\ 20 & 10 \end{bmatrix}$$

The total number of credits earned each year can be found by the matrix product AB :

$$AB = \begin{bmatrix} 50 & 100 & 70 \\ 60 & 80 & 80 \\ 80 & 70 & 70 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 15 & 15 \\ 20 & 10 \end{bmatrix} = \begin{bmatrix} 50 \cdot 10 + 100 \cdot 15 + 70 \cdot 20 & 50 \cdot 0 + 100 \cdot 15 + 70 \cdot 10 \\ 60 \cdot 10 + 80 \cdot 15 + 80 \cdot 20 & 60 \cdot 0 + 80 \cdot 15 + 80 \cdot 10 \\ 80 \cdot 10 + 70 \cdot 15 + 70 \cdot 20 & 80 \cdot 0 + 70 \cdot 15 + 70 \cdot 10 \end{bmatrix}$$

We can represent the result as a table:

Year	X credits	Y credits
2015	3400	2200
2016	3400	2000
2017	3250	1750

$$A = \begin{bmatrix} 3400 & 2200 \\ 3400 & 2000 \\ 3250 & 1750 \end{bmatrix}$$

Linear Transformations

Let m and n be positive integers, A linear transformation T from \mathbb{R}^n to \mathbb{R}^m , denoted $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, is a function that satisfies the following properties:

- $T(u + v) = T(u) + T(v)$
- $T(\lambda u) = \lambda T(u)$

$\forall u, v \in \mathbb{R}^n$ and scalars $\lambda \in \mathbb{R}$

When $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, if we know T applied to the three standard basis vectors of \mathbb{R}^3 :

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we can form a 2×3 matrix A whose columns are exactly these image, then $T(v) = Av$ for any column vector $v \in \mathbb{R}^3$.

Composition of Linear Transformations

- If $T : \mathbb{R}^n \rightarrow \mathbb{R}^p$ and $S : \mathbb{R}^p \rightarrow \mathbb{R}^m$, then the composition $(S \circ T)(v) = S(T(v))$
- If T is represented by a $p \times n$ matrix A and S by an $m \times p$ matrix B then the composition $S \circ T$ is represented by the matrix product BA .
- Also $(AB)C = A(BC)$
- Composing transformations is only possible if the codomain of the first transformation matches the domain of the second transformation, that is:

$$A \in M_{m \times n} \quad B \in M_{p \times m} \quad \Rightarrow \quad AB \in M_{m \times n}$$

The $n \times n$ Identity Matrix

I_n has 1s on the main diagonal:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The identity matrix is the **neutral element** for multiplication:

$$A \cdot I_n = A \quad \text{and} \quad I_n \cdot B = B \quad (\text{where } A \text{ has } n \text{ columns and } B \text{ has } n \text{ rows})$$

This is interpreted as the **identity transformation** on \mathbb{R}^n , so composing with it has no effect on any linear map.

Invertible (Non-Singular) Matrices

A square $n \times n$ matrix A has an inverse A^{-1} if there exists another $n \times n$ matrix such that:

$$AB = I_n \quad \text{and} \quad BA = I_n$$

If A has an inverse, we say it is **unique**; there cannot be two different inverses for the same matrix.

Not all matrices are invertible. A key fact (explained later) is that:

$$A \text{ is invertible} \Leftrightarrow \text{the determinant} \neq 0$$

Transpose of a Matrix

For a $m \times n$ matrix A , the transpose A^T is the $n \times m$ matrix obtained by turning the rows of A into the columns of A^T :

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

2 Systems of linear equations

2.1 Linear equations and Solution Sets

A linear equation in the variables x and y is an equation of the form

$$2x + y = 3$$

If we replace x and y with some numbers, the statement **becomes true or false**.

Definition 2.1: Solution to a linear equation

A pair, $(x_0, y_0) \in \mathbb{R}$, is a solution to a linear equation if setting $x = x_0$ and $y = y_0$ **makes the equation true**.

Definition 2.2: Solution set

The **solution set** is the set of all solutions to a linear equation.

$$a_1X_1 + a_2X_2 + \dots + a_nX_n = b \quad \text{where } a_i, b \in \mathbb{R}$$

is an **affine hyperplane** in \mathbb{R}^n ; geometrically resembles a copy of \mathbb{R}^{n-1} inside \mathbb{R}^n .

2.2 Elementary Row Operations

To solve a system of linear equations we associate an **augmented matrix** to the system of equations. For example:

$$\begin{array}{rrrrrr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array} \right]$$

To solve, we can perform the following **Elementary Row Operations (EROs)**:

1. Multiply a row by a non-zero constant.
2. Add a multiple of one row to another row.
3. Swap two rows.

The goal of these operations is to transform the augmented matrix into **row echelon form (REF)** or **reduced row echelon form (RREF)**.

2.2.1 REF and Strategy

We say a matrix is in **row echelon form** (REF) if:

- The first non zero entry in each row is a 1 (called the **leading 1**).
- If a column has a leading 1, then all entries below it are 0.
- The leading 1 in each row is to the right of the leading 1 in the previous row.
- All rows of 0s are at the bottom of the matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Example of REF

We have produced a new system of equations. This is easily solved by back substitution.

Concept 2.1: Strategy for Obtaining REF

- Get a 1 as the top left entry
- Use this 1 to clear the entries below it
- Move to the next column and repeat
- Continue until all leading 1s are in place
- Use back substitution to solve the system

2.2.2 Row Reduced Echelon Form

A matrix is in **reduced row echelon form** (RREF) if:

- It is in REF
- The leading 1 in each row is the only non-zero entry in its column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Example of RREF

2.3 Leading variables and free variables

We'll start by an example:

$$\begin{array}{rrrrrrr} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{array} \right]$$

Solving this system of equations, we get:

$$\text{RREF: } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \Rightarrow \begin{array}{l} x_1 + 2x_4 = 4 \\ x_2 - x_4 = 2 \\ x_3 + x_4 = 2 \end{array} \Rightarrow \begin{array}{l} x_1 = 4 - 2x_4 \\ x_2 = 2 + x_4 \\ x_3 = 2 - x_4 \end{array}$$

This RREF tells us how the **leading variables** (x_1, x_2, x_3) depend on the **free variable** (x_4). The free variable can take any value in \mathbb{R} . We write the solution set as:

$$x_1 = 4 - 2t, \quad x_2 = 2 + t, \quad x_3 = 2 - t, \quad x_4 = t \quad \text{where } t \in \mathbb{R}$$

$$(x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t); \quad t \in \mathbb{R}$$

Definition 2.3: Leading and Free Variables

- **Leading variable** : A variable whose columns in the RREF contain a leading 1
- **Free variable** : A variable whose columns in the RREF do not contain a leading 1

2.4 Consistent and Inconsistent Systems

Consider the following system of equations:

$$\begin{array}{rrrrr} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (\text{REF})$$

We can see the last row of the REF is:

$$0x + 0y + 0z = 1$$

This equation clearly has no solution, and hence the system has no solutions. We say the system is **inconsistent**. Alternatively, we say the system is **consistent** if it has at least one solution.

2.5 Possible Outcomes when solving a system of equations

- The system may be **inconsistent** (no solutions) - i.e:

$$[0 \ 0 \ \dots \ 0 \ | \ a] \quad a \neq 0$$

- The system may be **consistent** which occurs if:

- **Unique Solutions** each column (aside from the rightmost) contains a single leading 1. - i.e:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

- **Infinitely many solutions** at least one variable does not appear as a leading 1 in any row, making it a free variable - i.e:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

2.6 Elementary Row Operations as Matrix Transformations

Elementary row operations may be interpreted as **matrix multiplication**. To see this, first we introduce the **Identity matrix**:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The I_m Identity matrix is an $m \times m$ matrix with 1s on the diagonal and 0s elsewhere. We also introduce the $E_{i,j}$ matrix which has 1 in the (i,j) position and 0s elsewhere. For example:

$$E_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then:

$$I_3 + 4E_{1,2} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing a row operation on A is the same as multiplying A by an appropriate matrix E on the left. These matrices are called **elementary matrices**. They are **always invertible**, and **their inverses are also elementary matrices**. The statement:

"every matrix can be reduced to RREF through EROs"

is equivalent to saying that

"for every matrix A with m rows, there exists a $m \times m$ matrix B which is a product of elementary matrices such that BA is in RREF."

2.6.1 Multiplying a Row by a Non-Zero Scalar

When multiplying row i of matrix A by a scalar $\alpha \neq 0$, we can use the matrix:

$$I_m + (\alpha - 1)E_{i,i}$$

This works because it modifies only the (i,i) entry of the identity matrix to be α while keeping all other entries unchanged. When multiplied with A , it scales row i by α and leaves all other rows intact.

Example: If $\alpha = 5$ and $i = 2$, then:

$$I_3 + 4E_{2,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad (I_3 + 4E_{2,2})A = \begin{bmatrix} 1 & 2 & 3 \\ 20 & 25 & 30 \\ 7 & 8 & 9 \end{bmatrix}$$

2.6.2 Switching Two Rows

To swap rows i and k , we use:

$$S = I_m + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k}$$

This works by:

- Removing the 1's at positions (i,i) and (k,k) from the identity matrix
- Adding 1's at positions (i,k) and (k,i)

Example: Swapping rows 1 and 3:

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad SA = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

2.6.3 Adding a Multiple of One Row to Another

To replace row k with row $k + \alpha \times$ row i , use:

$$I_m + \alpha E_{k,i}$$

This adds α times row i to row k while leaving all other rows unchanged because:

- For any row $j \neq k$, the corresponding row in this matrix is just the standard basis row
- Row k becomes the sum of the standard basis row k plus α times the standard basis row i

Example: Adding 3 times row 1 to row 2:

$$I_3 + 3E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad (I_3 + 3E_{2,1})A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \\ 7 & 8 & 9 \end{bmatrix}$$

Example 2.1

Write the inverse of an elementary matrix and show it is an elementary matrix.

Multiplying a row by a nonzero scalar:

- **Operation:** Multiply row i by $\alpha \neq 0$.
- **Elementary Matrix:** $E = I_m + (\alpha - 1)E_{i,i}$
- **Inverse:** To reverse the operation, multiply row i by $1/\alpha$. Hence, the inverse is

$$E^{-1} = I_m + \left(\frac{1}{\alpha} - 1\right) E_{i,i} = I_m + \frac{1 - \alpha}{\alpha} E_{i,i}.$$

Swapping two rows:

- **Operation:** Swap rows i and k .
- **Elementary Matrix:** $S = I_m - E_{i,i} - E_{k,k} + E_{i,k} + E_{k,i}$
- **Inverse:** Since swapping the same two rows twice returns them to their original positions,

$$S^{-1} = S.$$

Adding a multiple of one row to another:

- **Operation:** Add α times row i to row k .
- **Elementary Matrix:** $E = I_m + \alpha E_{k,i}$
- **Inverse:** To undo the operation, subtract α times row i from row k . Therefore,

$$E^{-1} = I_m - \alpha E_{k,i}.$$

Example 2.2

Prove that every invertible matrix in $M_n(\mathbb{R})$ is a product of elementary matrices.

Let A be an invertible matrix in $M_n(\mathbb{R})$. Since A is invertible, we can use Gaussian elimination to transform A into the identity matrix I_n .

Let E_1, E_2, \dots, E_k be the elementary matrices corresponding to the row operations used in the elimination process. Then, we have:

Multiplying a row by a scalar: $I_n + (\alpha - 1)E_{i,i}$

Swapping two rows: $I_n + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k}$

Adding a multiple of one row to another: $I_n + \alpha E_{k,i}$

Applying these in sequence to A gives:

$$E_k \cdots E_2 E_1 A = I_n$$

Since $E_k \cdots E_2 E_1 = I_n$, we can multiply both sides by $(E_k \cdots E_2 E_1)^{-1}$ on the left to obtain:

$$A = (E_k \cdots E_2 E_1)^{-1} I_n = (E_k \cdots E_2 E_1)^{-1}$$

Using the property:

$$(E_k \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1},$$

we can express A as a product of elementary matrices:

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Since each E_i is an elementary matrix, its inverse is also an elementary matrix. Therefore, A can be expressed as a product of elementary matrices.

2.7 EROs and Inverses

Elementary Row Operations can be used to find the inverse of a square matrix. Consider a square matrix $A \in M_n(\mathbb{F})$ (that is, an $n \times n$ matrix over a field \mathbb{F}). If A is invertible, let

$$A^{-1} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}$$

be its inverse, where each \mathbf{v}_i is the i th column of A^{-1} . By definition of the matrix inverse, we have

$$A A^{-1} = A [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] = [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n] = I_n,$$

the $n \times n$ identity matrix. This implies that

$$A\mathbf{v}_i = \mathbf{e}_i, \quad \text{for each } i = 1, 2, \dots, n,$$

where \mathbf{e}_i is the i th column of I_n (which has a 1 in the i th row and 0 everywhere else). In other words, each column \mathbf{v}_i of A^{-1} is the unique solution to the linear system

$$A\mathbf{v}_i = \mathbf{e}_i.$$

To find A^{-1} effectively, we form the augmented matrix $[A \mid I_n]$ and apply EROs to transform A into I_n . When this is achieved, the augmented portion becomes A^{-1} . Thus, we have

$$\text{RREF}([A \mid I_n]) = [I_n \mid A^{-1}].$$

Example 2.3

Find A^{-1} if $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$.

We form a 3×6 matrix $A' = [A \mid I_3]$:

$$A' = \begin{bmatrix} 3 & 4 & -1 & | & 1 & 0 & 0 \\ 1 & 0 & 3 & | & 0 & 1 & 0 \\ 2 & 5 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

We apply the following EROs to A' :

- $R_1 \leftrightarrow R_2$
- $R_2 \rightarrow R_2 - 3R_1$
- $R_3 \rightarrow R_3 - 2R_1$
- $R_3 \rightarrow R_3 + R - 2$
- $R_3 \leftrightarrow R_2$
- $R_3 \rightarrow R_3 - 4R_2$
- $R_3 \times (-\frac{1}{10})$
- $R_1 \rightarrow R_1 - 3R_3$

To obtain:

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & | & -1 & 1 & 1 \\ 0 & 0 & 1 & | & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

That is:

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

It is easily checked that $AA^{-1} = I_3$.

3 Spanning sets, bases and dimensions

3.1 Vector Spaces

A **vector space** \mathbf{V} over \mathbb{F} is a non empty set of objects equipped with an addition operation and whose elements can be multiplied by scalars in \mathbb{F} , subject to the following axioms:

1. $u + v = v + u$, $\forall u, v \in \mathbf{V}$
2. $(u + v) + w = u + (v + w)$, $\forall u, v, w \in \mathbf{V}$
3. $\exists 0_{\mathbf{V}}$, so that $0_{\mathbf{V}} + v = v$, $\forall v \in \mathbf{V}$
4. $\exists -v \in \mathbf{V}$, so that $v + (-v) = 0_{\mathbf{V}}$, $\forall v \in \mathbf{V}$
5. $\alpha(\beta v) = \alpha\beta(v)$, $\forall \alpha, \beta \in \mathbb{F}, v \in \mathbf{V}$
6. $(\alpha + \beta)v = \alpha v + \beta v$, $\forall \alpha, \beta \in \mathbb{F}, v \in \mathbf{V}$
7. $1v = v$, $\forall v \in \mathbf{V}$

In the definitions axioms above, the field \mathbb{F} can be replaced with any other field, such as \mathbb{R} or \mathbb{C} .

Examples of vector spaces over \mathbb{R}

- The space $M_{m \times n}(\mathbb{R})$ of $m \times n$ with real entries.
- The space of all polynomials with real coefficients
- The set of complex numbers is a vector space over \mathbb{R} .

Consider the space \mathbf{V} consisting of all **symmetric** 2×2 matrices in $M_2(\mathbb{R})$ with **trace zero**.

- **Trace zero** means that the sum of the diagonal elements is zero.
- **Symmetric** means that the matrix is equal to its transpose.

So a matrix of trace zero has the form:

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \quad \text{where } a, b \in \mathbb{R}$$

Since it takes two real number to specify an element of \mathbf{V} , this is another example of a 2-dimensional vector.

3.2 Subspaces

Definition 3.1: Vector Subspaces

Let \mathbf{V} be a vector space over a field \mathbb{F} . A subset \mathbf{U} is a **subspace** of \mathbf{V} if \mathbf{U} is itself a vector space over \mathbb{F} , under the addition and scalar multiplication operations defined on \mathbf{V} .

Two things need to be checked to confirm that $U \subseteq V$ is a subspace:

1. \mathbf{U} is **closed** under the addition in \mathbf{V} , i.e. $u_1 + u_2 \in \mathbf{U}$ for all $u_1, u_2 \in \mathbf{U}$.
2. \mathbf{U} is **closed** under scalar multiplication, i.e. $\alpha u \in \mathbf{U}$, whenever $u \in \mathbf{U}$ and $\alpha \in \mathbb{F}$.

Examples of subspaces

1. Let $\mathbb{Q}[x]$ be the set of all polynomials with rational coefficients. Let $P_2 \subseteq \mathbb{Q}[x]$ be the set of all polynomials of degree at most 2. This means $P_2 = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{Q}\}$. Then P_2 is a vector subspace of $\mathbb{Q}[x]$. If $f(x)$ and $g(x)$ are rational polynomials of degree at most 2, then also is $f(x) + g(x)$ and $\alpha f(x)$, where $\alpha \in \mathbb{Q}$.
2. The set \mathbb{C} is a vector space over the set of real numbers. Within \mathbb{C} , the subset \mathbb{R} is an example of a vector subspace over \mathbb{R} . An example of a subset of \mathbb{C} that is not a real vector subset is the unit circle S in the complex plane- this is the set of complex numbers of modulus 1, it consists of all complex numbers of the form $a+bi$, where $a^2 + b^2 = 1$. This is closed neither under addition nor multiplication by real scalars.
3. The Cartesian plane (\mathbb{R}^2) is a real vector space. Within \mathbb{R}^2 , let $U = \{(a, b) : a > 0, b > 0\}$. Then \mathbf{U} is closed under addition and under multiplication by positive scalars. It is not a vector subspace of \mathbb{R}^2 , because it is not closed under multiplication by negative scalars.
4. Let v be a fixed non-zero vector $\in \mathbb{R}^3$ and let $v^\perp = \{u \in \mathbb{R}^3 : u^T v = 0\}$. Then v^\perp is not empty since $0 \in v^\perp$. Suppose $u_1, u_2 \in v^\perp$. If $u \in v^\perp$ and $\alpha \in \mathbb{R}$, then $(\alpha u)^T v = \alpha u^T v = \alpha 0 = 0$. Hence v^\perp is closed under scalar multiplication. Thus v^\perp is a vector subspace of \mathbb{R}^3 . Note that v^\perp is not all \mathbb{R}^3 , since $v \notin v^\perp$.

3.3 Span of a set of vectors

Definition 3.2: Span

Let \mathbf{V} be a vector space over a field \mathbb{F} , and let S be a non empty subset of \mathbf{V} .

The \mathbb{F} -linear span, commonly called the **span** of S , denoted $\langle S \rangle$, is the set of all \mathbb{F} -linear combinations of the elements of $S \in \mathbf{V}$.

If $S = \mathbf{V}$, then S is called a spanning set of \mathbf{V} ; meaning that every element of \mathbf{V} is a linear combination of the elements of S .

For a subset S of a \mathbb{F} -vector space \mathbf{V} , the sum of any two linear combinations of S is an element of S , and any scalar multiple of a linear combination of S is also an element of S ; hence the following lemma:

Lemma 3.1

For any subset, S , of a vector space, \mathbf{V} , the span, $\langle S \rangle$, is a subspace of \mathbf{V} .

Examples

- **Polynomials over \mathbb{Q}**

$\mathbb{Q}[x]$ is the set of all polynomials with rational coefficients, and $P_2 \subset \mathbb{Q}[x]$ consists of polynomials of degree at most 2. If $S = \{x^2 + 1, x + 1\}$, then

$$\langle S \rangle = \{a(x^2 + 1) + b(x + 1) : a, b \in \mathbb{Q}\}.$$

All members of $\langle S \rangle$ are degree- ≤ 2 polynomials with constant term equal to the sum of the x - and x^2 -coefficients. For instance, $x^2 + 2x + 3 \in \langle S \rangle$ but $x^2 + 2x + 4 \notin \langle S \rangle$. Since $\langle S \rangle$ does not include all degree- ≤ 2 polynomials in P_2 , S is not a spanning set for P_2 over \mathbb{Q} .

- **Column vectors in \mathbb{R}^2**

Let

$$S = \{(3, 1), (2, 1), (1, -1)\}.$$

Any vector $(a, b) \in \mathbb{R}^2$ can be written as a linear combination of these three vectors in more than one way. However, $(1, -1)$ itself is a linear combination of $(3, 1)$ and $(2, 1)$, so it is not necessary to span \mathbb{R}^2 . Hence S has redundant elements and is not a minimal spanning set of \mathbb{R}^2 .

The second example above motivates the following lemma:

Lemma 3.2

Suppose that $S_1 \subset S$, where $S \subseteq \mathbf{V}$, then

$$\langle S_1 \rangle \subseteq \langle S \rangle$$

if and only if every element of $S \setminus S_1$ is a linear combination of the elements of S_1 .

Definition 3.3

- **Finite dimensional:** A vector space that has a finite spanning set
- **Infinite dimensional:** A vector space that has an infinite spanning set

Example of infinite dimensional vector space

- The vector space $\mathbb{R}[x]$ of all polynomials with real coefficients is infinite dimensional. To see this let S be a finite subset of $\mathbb{R}[x]$ and let x^k be the highest power of x in S . Then $x^{k+1} \notin \langle S \rangle$ since x^{k+1} cannot be expressed as a linear combination of the elements of S .
- The set of \mathbb{R} is infinite dimensional as a vector space over the field, \mathbb{Q} , of rational numbers.

3.4 Linear independence

Definition 3.4

Let $S \subseteq \mathbf{V}$ with at least two elements.

Then S is linearly independent if **no element of S can be expressed as a linear combination of the other elements of S .**

Equivalently, if no element of S belongs to the span of the other elements of S .

It follows, a subset consisting of a single element is linearly independent if and only if that element is non-zero. The definition above takes a lot of work to check for large sets, the following definition is often more useful:

Definition 3.5

Let S be a non-empty subset of \mathbf{V} .

Then S is **linearly independent** if the only linear combination of the elements of S that equals zero is to take all the coefficients to be zero.

Equivalence of the two definitions

Let $S = \{v_1, \dots, v_k\}$ and suppose $v_1 \in \langle v_2, \dots, v_k \rangle$. Then:

$$v_1 = \alpha_2 v_2 + \dots + \alpha_k v_k \quad \Rightarrow \quad 0 = -v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$$

is an expression for the zero vector as a linear combination of elements of S , whose coefficients are not all zero. On the other hand suppose:

$$0 = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

where not all $c_i = 0$ Then:

$$v_1 = -\frac{c_2}{c_1} v_2 - \dots - \frac{c_k}{c_1} v_k \quad \Rightarrow \quad v_1 \in \langle v_2, \dots, v_k \rangle$$

Example 3.1

In \mathbb{R}^3 , let $S = \{[1, 2, -1], [-2, 3, 2], [-3, 8, 3]\}$. Show that S is linearly independent.

To determine if S is linearly independent, we need to investigate whether the system of equations has solutions other than $(x, y, z) = (0, 0, 0)$:

$$x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + z \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 2 & 3 & 8 & 0 \\ -1 & 2 & 3 & 0 \end{array} \right]$$

Reducing it to its RREF we get:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{array}{l} x + t = 0 \\ y + 2t = 0 \\ z + t = 0 \end{array} \Rightarrow (x, y, z) = (-t, -2t, t)$$

Setting $t = 1$ gives:

$$-1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, each of the three elements of S is a linear combination of the other two. So S is not linearly independent (we say it is linearly dependent).

3.4.1 Equivalent Definitions of Linear Independence

Theorem

Let S be a subset of a vector space \mathbf{V} . The following are equivalent:

1. S is linearly independent **if S is a minimal spanning set of its linear span** - no proper subset of S spans the same subspace of \mathbf{V} that S does.
2. S is linearly independent if every element of $\langle S \rangle$ has a **unique expression as a linear combination of elements of S** .
3. S is linearly independent if every element of $\langle S \rangle$ has unique coordinates in terms of the elements of S .