Complex Analysis

Exams:
70% Exam
30% Continuous Assessment (Homework)
10% Optional Project (Bonus)

Contents

1 Week 1: Systems of Linear Equations

1.1 Intro to Systems of Linear Equations

We call linear equations because each variable is raised to the first power. Products of variables, squares, square roots, etc., are not linear. A solution to a system of linear equations is an assignment of numerical values to each variables. Systems can have multiple solutions.

1.2 Augmented Matrices and Element row operations

$$\begin{aligned} x + 2y - z &= 5, \\ 3x + y - 2z &= 9, \\ -x + 4y + 2z &= 0 \end{aligned} \quad \Longleftrightarrow \quad \left(\begin{array}{rrr|r} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array}\right)$$

To solve system of linear equations, we work with this augmented matrix, applying three types of operations to convert to a simpler form. These operations include:

1. Adding scalar multiple of one row to another
2. Multiplying all entries of a row by same non-zero scalar
3. Swapping two rows.

Why does this work? Every ERO changes changes the system, but the new system has exactly the same solutions as the original.

Example

$$\boxed{\begin{aligned} x_1 + 3x_2 + 5x_3 - 9x_4 &= 5, \\ 3x_1 - x_2 - 5x_3 + 13x_4 &= 5, \\ 2x_1 - 3x_2 - 8x_3 + 18x_4 &= 1. \end{aligned}}$$

$$\left(\begin{array}{rrrr|r} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{array}\right) \xrightarrow[R2 \to R2 - 3R1]{R3 \to R3 - 2R1} \left(\begin{array}{rrrr|r} 1 & 3 & 5 & -9 & 5 \\ 0 & -10 & -20 & 40 & -10 \\ 0 & -9 & -18 & 36 & -9 \end{array}\right)$$

$$\xrightarrow{R2 \times \left(-\frac{1}{10}\right)} \left(\begin{array}{rrrr|r} 1 & 3 & 5 & -9 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & -9 & -18 & 36 & -9 \end{array}\right) \xrightarrow{R3 \to R3 + 9R2} \left(\begin{array}{rrrr|r} 1 & 3 & 5 & -9 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R1 \to R1 - 3R2} \left(\begin{array}{rrrr|r} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

1.3 How to read the solution from RREF

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{lccccccccc} \text{Equation 1:} & x_1 & + & 0 & - & x_3 & + & 3x_4 & = & 2 \\ \text{Equation 2:} & 0 & + & x_2 & + & 2x_3 & - & 4x_4 & = & 1 \\ \text{Equation 3:} & 0 & + & 0 & + & 0 & + & 0 & = & 0 \end{array}$$

That is:

$$x_1 = 2 + x_3 - 3x_4$$

$$x_2 = 1 - 2x_3 + 4x_4$$

Equation 3 has no content, so we can ignore it.

A solution of the system must satisfy all equations. We write s and t for the values of x_3 and x_4 in a solution of the system. The General Solution is written as:

$$(x_1, x_2, x_3, x_4) = (2 + s - 3t, 1 - 2s + 4t, s, t), \quad s, t \in \mathbb{R}$$

The general solution is a description of all the infinite solutions to the system. We can assign any values to s and t to get a solution. For example, if we set $s = 0$ and $t = 0$, we get the particular solution $(2, 1, 0, 0)$.

1.4 Vocabulary and Definitions

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{array}\right] \xrightarrow[\text{Elementary Row Operations}]{\text{Gauss-Jordan Elimination}} \left[\begin{array}{cccc|c} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{array}\right]$$

The second matrix is in reduced row echelon form (RREF). The RREF is a matrix where:

1. Every row that is not all zeros has a leading 1.
2. Every column thats containing a leady 1 has a 0 in every other entry.
3. The leading 1 go left to right as we move down the rows
4. Any rows that are all zero are at the bottom of the matrix.

Remark: A matrix is in row echelon form (REF) if it satisfies the first three conditions above.

Example: This matrix is in REF, not RREF.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 4 & 5 \\ 0 & 1 & 5 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right]$$

1.5 Leading Variables and Free Variables

$$\begin{aligned} x_1 + 3x_2 + 5x_3 - 9x_4 &= 50 \\ 3x_1 - x_2 - 5x_3 + 13x_4 &= 5 \\ 2x_1 - 3x_2 - 8x_3 + 18x_4 &= 1. \end{aligned} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Leading 1's in the RREF occur in the columns of the varables x_1 and x_2. The columns of x_3 and x_4 do not contain leading 1's.

Definition

Any variables who columns in the RREF **do not contain leading 1's are called free variables.**
Any variables who columns in the RREF **contain leading 1's are called leading variables.**

1.6 How to write the solution

1. Give independant parameters to the free variables.
2. Read from the RREF, how the correspodning values of the leading variables depend on s and t.
$$(x_1, x_2, x_3, x_4) = (2 + s - 3t, 1 - 2s + 4t, s, t), \quad s, t \in \mathbb{R}$$
3. **Note:** we can also write:
$$(x_1, x_2, x_3, x_4) = (2, 1, 0, 0) + s(1, -2, 1, 0) + t(-3, 4, 0, 1), \quad s, t \in \mathbb{R}$$