# Complex Analysis

Exams:

60% Exam

40% Continuous Assessment

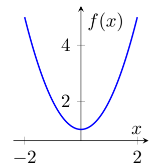
## Contents

# 1 Week 1: Introduction to Complex Numbers

## 1.1 Quadratics with Complex Roots

Everybody knows that, for coefficients $a, b, c \in \mathbb{R}$, the quadratic $ax^2 + bx + c = 0$ has real values solutions given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{if } b^2 - 4ac \geq 0$$




but if $b^2 - 4ac < 0$, then we need the roots of negative numbers, and thus the solutions are complex numbers.
For example, the, the plot of $x^2 + 1 = 0$, below implies imaginary soltuons, since there are no real $x$-values that make y=0

## 1.2 Real valued solutions of a cubic

Oddly enough, complex numbers are needed to find real-valued solutions of a cubic equation.

**Definition**

For $p, q \in \mathbb{R}$,

$$x^3 = px + q,$$

has the solution, by Cardano's formula:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

**Example**

Consider $x^3 = 15x + 4$, staring at this long enough, one could guess that $x = 4$ is a solution, and then factor out $(x - 4)$ to get a quadratic, but that's not the point.
By Cardano's Formula, with $p = 15$ and $q = 4$, we get:

$$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{121}}$$

Setting $i = \sqrt{-1}$, thus $\sqrt{-121} = 11i$
And noticing that:

$$\begin{aligned}
(2+i)^3 &= 2^3 + 3 \cdot 2^2 \cdot i + 3 \cdot 2 \cdot i^2 + i^3 \\
&= 8 + 12i - 6 - i \\
&= 2 + 11i \\
\text{Thus} \quad (2+1)^3 = 2 + 11i \quad &\text{and} \quad (2-1)^3 = 2 - 11i
\end{aligned}$$

Thus, the solution is:

$$\begin{aligned}
&= \sqrt[3]{(2+i)^3} + \sqrt[3]{(2-i)^3} \\
&= 2 + i + 2 - i \\
&= 4
\end{aligned}$$

## 1.3 Definition of Complex Numbers

**Definition**

The set of complex numbers is defined as:

$$\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\}$$

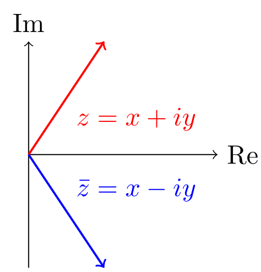
where $a$ is the real part and $yi$ is the imaginary part, and $i^2 = -1$

## 1.4 Attributes of Complex Numbers

Given a complex number of the form: $z = x + yi$, we have:

- $\text{Re}(z) = x$ is the real part of $z$
- $\text{Im}(z) = y$ is the imaginary part of $z$
- $\bar{z} = x - yi$ is the complex conjugate of $z$
- $|z| = \sqrt{x^2 + y^2}$ is the modulus of $z$

Note that: $z\bar{z} = x^2 + y^2 = |z|^2 \in \mathbb{R}$
Also note how this formula is used in the computation of the inverse of $z : z^{-1} = \dfrac{\overline{z}}{|z|^2}$
The geometric meaning of these attributes can be seen below.




## 1.5 Polar Form of a Complex Number

Given a non zero complex number $z = x + iy$:

1. **Magnitude**: $\mid z \mid = \sqrt{x^2 + y^2}$
2. **Argument**: If you think of $(x, y)$ as a point in the plane, $\theta$, is the angle the vector from the origin makes with the positive $x - axis$
   - In polar form, we write: $z = |z|(\cos\theta + i\sin\theta)$
   - **Principle Argument** (Arg z): This is the "main" angle $\theta$ chosen to lie in $(-\pi, \pi]$
   - Because angles can differ by full turns $(2\pi)$, the general argument of $z$ can be written as:

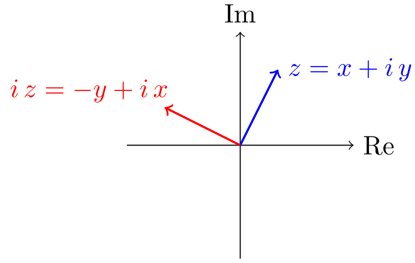
$$\arg(z) = \text{Arg}(z) = 2\pi n, \quad n \in \mathbb{Z}$$

**Why angles are "multi-valued"** A direction in the plane can be expressed by infintetly many angles differing by whole circle $(2\pi)$. The "principal" angle is just a standard choice in $(-\pi, \pi]$

## 1.6 Multplication by $i$ and Geometric Interpretation

Multiplying a complex number $z$ by $i$ correspond to a rotation by $\frac{\pi}{2}$ (90°) in the complex plane.. Example:

$$\begin{aligned} z &= x + iy \\ iz &= i(x + iy) \\ &= ix - y \\ &= -y + ix \end{aligned}$$




## 1.7 Products of Complex Numbers and De Moivre's Theorem

**Theorem**

Let $z, z' \in \mathbb{C}\backslash\{0\}$, then:

- $|zz'| = |z||z'|$ and $\left|\dfrac{z}{z'}\right| = \dfrac{|z|}{|z'|}$
- $\mathrm{Arg}(zz') = \arg(z) + \arg(z')$ and $\arg\left(\dfrac{z}{z'}\right) = \arg(z)$ - $\arg(z')$

**Proof**

Let $x - x + iy$ and $x' + iy'$, thus:

$$z = |z|(\cos\theta + i\sin\theta) \quad \text{and} \quad z' = |z'|(\cos\theta' + i\sin\theta')$$

Then:

$$\begin{aligned} |zz'| &= |z||z'|(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta') \\ &= |z||z'|\left[(\cos\theta\cos\theta' - \sin\theta\sin\theta' + i(\cos\theta\sin\theta' + \sin\theta\cos\theta'))\right] \\ &= |z||z'|\left[\cos(\theta + \theta') + i\sin(\theta + \theta')\right] \end{aligned}$$

Thus, $|zz'| = |z||z'|$ and $\arg(zz') = \theta + \theta' = \arg(z) + \arg(z')$
Note that: $|z||z'|$ acts as a stretch/ shrink, $\cos(\theta + \theta')$ and $\sin(\theta + \theta')$ act as a rotation.

**Corollary**

**De Moivre's Thereom:** If $z = |z|(\cos\theta + i\sin\theta)$, then:

$$z^n = |z|^n \cos(n\theta) + isin(\theta)$$

De Moivre's Thereom is extemely useful for raising a complex number to inger powers and expressing $nt$th roots of unity (as a special case where $r = 1$ and $z^n = 1$)

## 1.8 Roots of Unity

A **root of unity** is a complex number $z$, such that:

$$z^n = 1 \quad \text{for some integer} n \geq 1$$

Geometrically, roots of unity lie on the unit circle in the complex plane.
Specifically:

**Definition**

The $n$th roots of unity are given by:

$$z_k = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right) \quad \text{for } k = 0, 1, 2, \ldots, n-1$$

**Example**

Recalling $z = |z| [\cos\theta + i \sin\theta]$
Letting $z^4 = 1$, then we see:

$$z^4 = \cos(4\theta) + i \sin(4\theta) = 1 + 0i$$

We see, $z$ is one of $w_k = \cos\left(\frac{2\pi k}{4}\right) + i \sin\left(\frac{2\pi k}{4}\right) = 1, i, -1, i$
And that the 4 roots of unity form a square in the unit circle, and their sum is 0.

# 2 Week 2: Functions of a Complex Variable

We study functions $f : \mathbb{C} \to \mathbb{C}$, that map complex numbers to complex numbers, like $f(z) = z^2 + z - 6$.
**Recall from Week 1:**

$$z = x + iy = R(\cos\theta + i\sin\theta)$$

where $R^2 = z^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, but $\frac{y}{x} = \frac{-y}{-x}$, that is, tan has period $\pi$ only.

## 2.1 Complex Roots

For $n \in \mathbb{N}$, consider the function $f(z) = z^{1/n}$.
Given $z = R(\cos\theta + i\sin\theta) \neq 0$, find all complex numbers $w = r(\cos\phi + i\sin\phi)$ such that $w^n = z$.
By De moivre's Theorem (applied to $w$), $r^n = R$ and $n\phi = \theta + 2k\pi$ for some $k \in \mathbb{Z}$.

$$r = \sqrt[n]{R} > 0 \quad \text{and} \quad \phi = \theta/n + 2k\pi/n$$

Note how $\phi = \frac{\theta+2k\pi}{n}$ takes exactly $n$ disctinct values for $k = 0, 1, 2, \ldots, n-1$. Thus, every complex number $z = R(\cos\theta + i\sin\theta)$ has exactly $n$ distinct $n$th roots.
We reserve the notation $\sqrt[n]{z}$ for the principal root, which is the one with $k = 0$.

**Example**

**Find the cube roots of z = -1 + 1**
Here, $R = \sqrt{2}$ and $\theta = \frac{3\pi}{4}$. Hence, the 3 cubic roots of $z = -1 + i$ are:

$$w_k = \sqrt[6]{2} \cdot \left[\cos\left(\frac{\pi}{4} + 23k\pi\right) + i\sin\left(\frac{\pi}{4} + \frac{2}{3}k\pi\right)\right]$$

Yielding: $w_0 = \sqrt[6]{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, $w_1 = \sqrt[6]{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$, $w_2 = \sqrt[6]{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$