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# MP232: Applied Mathematics

60% Exam
40% Continuous Assessment (3 parts)

## Contents

# 1 Prelim : The Exponential Function and Hyperbolic Functions

## 1.1 Exponential Function

**Derivative**

$$\frac{d}{dt}\left(e^{at}\right) = a\,e^{at}$$

**Integral**

$$\int e^{at}\,dt = \frac{1}{a}e^{at} + C$$

## 1.2 Hyperbolic Functions

**Definitions:**
$$\sinh(at) = \frac{e^{at}-e^{-at}}{2} \,\Big|\, \cosh(at) = \frac{e^{at}+e^{-at}}{2} \,\Big|\, \tanh(at) = \frac{\sinh(at)}{\cosh(at)}.$$

**Derivatives**
$$\frac{d}{dt}\left(\sinh(at)\right) = a\,\cosh(at),\,\Big|\,\frac{d}{dt}\left(\cosh(at)\right) = a\,\sinh(at),\,\Big|\,\frac{d}{dt}\left(\tanh(at)\right) = a\,\mathrm{sech}^2(at).$$

**Integrals**

$$\int \sinh(at)\,dt = \frac{1}{a}\cosh(at) + C$$

$$\int \cosh(at)\,dt = \frac{1}{a}\sinh(at) + C,$$

$$\int \tanh(at)\,dt = \frac{1}{a}\ln\left|\cosh(at)\right| + C.$$

**Common Identities**

$$\cosh^2 x - \sinh^2 x = 1,$$
$$\sinh(2x) = 2\,\sinh x\,\cosh x,$$
$$\cosh(2x) = \cosh^2 x + \sinh^2 x,$$
$$\tanh(2x) = \frac{2\,\tanh x}{1 + \tanh^2 x}.$$

## 1.3 Partial Fraction Decomposition

**Unrepeated Linear Factors**: A linear factor is of form $(ax + b)$

$$\frac{s+1}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

**Repeated LinearFactors**:

$$\frac{3}{(s+2)^2(s-3)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-3}$$

**Unrepeated Quadratic Factors with complex roots**: Where the discriminant $(b^2 - 4ac)$ is negative (complex roots) but the factor is not repeated

$$\frac{3}{(s^2 - s + 1)(s+2)} = \frac{As + B}{s^2 - s + 1} + \frac{C}{s+2}$$

**Repeated Quadratic Factors with complex roots**:

$$\frac{1}{(s^2 + 1)^2(s-1)} = \frac{As + B}{(s^2 + 1)^2} + \frac{Cs + D}{s^2 + 1} + \frac{E}{s-1}$$

# 2   Intro to Laplace Transforms

## 2.1   What is a Laplace Transform?

**Definition**

The Laplace Transform, defined for $t \geq 0$, is given by

$$L\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st} dt$$

## 2.2   Common Laplace Transforms

**Example Find the Laplace Transform of $f(t) = 1$**

We have:
$$L\{1\} = \int_0^\infty 1 \cdot e^{-st} dt = \lim_{R \to \infty} \int_0^R e^{-st} dt$$

This integral is equal to:
$$\int_0^R e^{-st} dt = \frac{e^{-st}}{-s} \bigg|_{t=0}^{t=R} = -\frac{1}{s}[e^{-sR} - 1] = \frac{1 - e^{-sR}}{s}$$

Taking the limit as $R \to \infty$ gives:
$$L\{1\} = \lim_{R \to \infty} \frac{1 - e^{-sR}}{s} = \frac{1}{s}$$

**Example  Find the Laplace Transform of $f(t) = e^{2t}$**

$$L\{e^{2t}\} = \int_0^\infty e^{2t} e^{-st} dt = \int_0^\infty e^{-(s-2)t} dt$$

$$= \lim_{R \to \infty} \int_0^R e^{-(s-2)t} dt$$

$$= \lim_{R \to \infty} \left[ \frac{e^{-(s-2)t}}{-(s-2)} \right]_{t=0}^{t=R}$$

$$= \lim_{R \to \infty} \left( \frac{e^{-(s-2)R} - e^0}{-(s-2)} \right) = \lim_{R \to \infty} \left( \frac{e^{-(s-2)R} - 1}{-(s-2)} \right)$$

$$= \frac{1}{s-2} \quad \text{(since } e^{-(s-2)R} \to 0 \text{ as } R \to \infty \text{ provided } s > 2)$$

**Example Find the Laplace Transform of $f(t) = \cosh(at)$**

We have:
$$L\{\cosh(at)\} = L\left\{ \frac{e^{at} + e^{-at}}{2} \right\} \quad \text{from the definition of } \cosh(at)$$

$$= \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\} \quad \text{by linearity of the Laplace Transform}$$

$$= \frac{1}{2}\left( \frac{1}{s-a} \right) + \frac{1}{2}\left( \frac{1}{s+a} \right)$$

Hence:
$$L\{\cosh(at)\} = \frac{s}{s^2 - a^2}$$

Noting that $\sinh(at) = (e^{at} - e^{-at})/2$, we can find that:
$$L\{\sinh(at)\} = \frac{a}{(s^2 - a^2)}$$

> **Example** Find the Laplace Transform of $\cos(wt)$ and $\sin(wt)$ where $w$ is a constant.
>
> We first compute the Laplace Transform of $e^{iwt}$ using its definition:
>
> $$L\{e^{iwt}\} = \int_0^\infty e^{-st}e^{iwt}\, dt = \int_0^\infty e^{-(s-iw)t}\, dt = \frac{1}{s-iw}, \quad \text{for } \Re(s) > 0.$$
>
> To express this in terms of real and imaginary parts, we multiply the numerator and denominator by the complex conjugate of the denominator:
>
> $$\frac{1}{s-iw} = \frac{s+iw}{(s-iw)(s+iw)} = \frac{s+iw}{s^2+w^2}.$$
>
> Since Euler's formula gives:
>
> $$e^{iwt} = \cos(wt) + i\sin(wt),$$
>
> the linearity of the Laplace Transform yields:
>
> $$L\{e^{iwt}\} = L\{\cos(wt)\} + iL\{\sin(wt)\}.$$
>
> Equating the two representations of $L\{e^{iwt}\}$, we have:
>
> $$L\{\cos(wt)\} + iL\{\sin(wt)\} = \frac{s+iw}{s^2+w^2}.$$
>
> Since the equality must hold for both the real and imaginary parts, we equate them separately:
>
> $$L\{\cos(wt)\} = \frac{s}{s^2+w^2} \quad \text{and} \quad L\{\sin(wt)\} = \frac{w}{s^2+w^2}.$$

## 2.3  Linearity of the Laplace Transform

The Laplace Transform is a linear operator, i.e. for any constants $a$ and $b$:

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

> **Proof**
>
> $$
> \begin{aligned}
> L\{af(t) + bg(t)\} &= \int_0^\infty e^{-st}(af(t) + bg(t))\, dt \\
> &= a\int_0^\infty e^{-st}f(t)\, dt + b\int_0^\infty e^{-st}g(t)\, dt \\
> &= aL\{f(t)\} + bL\{g(t)\}
> \end{aligned}
> $$

## 2.4  The First Shift Theorem

> **Theorem First Shift Theorem**
>
> If $f(t)$ has a Laplace Transform, $F(s)$, defined for $s > k$, then $e^{at}\,f(t)$ has a Laplace Transform, $F(s-a)$ defined for $s - a > k$ and is given by:
>
> $$L\{e^{at}f(t)\} = F(s-a)$$
>
> or, taking the inverse Laplace Transform of both sides:
>
> $$e^{at}f(t) = L^{-1}\{F(s-a)\}$$

### 2.4.1  Examples

> **Example** Find the Laplace Transform of $e^{at}\cos(wt)$, where $a, w$ are constants.
>
> We know that $L\{\cos(wt)\} = \frac{s}{s^2+w^2}$, so by the First Shift Theorem:
>
> $$
> \begin{aligned}
> L\{e^{at}\cos(wt)\} &= \frac{s-a}{(s-a)^2+w^2} \\
> &= \frac{s-a}{s^2-2as+a^2+w^2}
> \end{aligned}
> $$

## 2.5 Existence of the Laplace Transform

Existence of a Laplace transform is not always guaranteed because we're integrating over an infinite integral. For a Laplace Transform to exist for a given $s$, then the integral must exist:

$$\int_0^\infty e^{-st} f(t)\, dt$$

**Theorem Existence Theorem of Laplace Transforms**

Suppose $f(t)$ is a piecewise continuous function on $[0, \infty)$. If $f(t)$ satisfies:

$$|f(t)| \le Me^{kt}\ (0 \le t \le \infty)$$

for some constants, $M, k$, then the Laplace Transform of $f(t)$ exists for $s > k$. In other words, the Laplace Transform of $f(t)$ exists if $f(t)$ is bounded by an exponential function.

**Proof**

If $s > k$, then from the equation above, we have:

$$|F(s)| = \left| \int_0^\infty f(t)e^{-st}\, dt \right| \le \int_0^\infty |f(t)|e^{-st}\, dt \le \int_0^\infty Me^{(k-s)t}\, dt = \frac{M}{s-k}$$

# 3 Applications of Laplace Transforms

## 3.1 Integration by Parts

Starting with the product rule:

$$\frac{d}{dx}[uv] = u'v + uv',$$

we can express this in differential form as:

$$d(uv) = u\, dv + v\, du.$$

Integrate both sides with respect to $x$:

$$\int d(uv) = \int_a^b u\, dv + \int_a^b v\, du.$$

The Fundamental Theorem of Calculus tells us that the left-hand side is simply:

$$uv = \int_a^b u\, dv + \int_a^b v\, du.$$

Rearrange to solve for the desired integral:

$$\int_a^b u\, dv = uv - \int_a^b v\, du,$$

### 3.1.1 Examples

**Example Use integration by parts to find the Laplace of $f(t) = t$**

$$L\{t\} = \int_0^\infty t e^{-st}\, dt$$

We integrate by parts by setting:

$$u = t, \quad dv = e^{-st}, \quad du = 1, \quad v = -\frac{e^{-st}}{s}$$

Then intengrating by parts gives:

$$L\{t\} = \left[-\frac{t e^{-st}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= 0 + \frac{1}{s}\left[-\frac{e^{-st}}{s}\right]_0^\infty$$

Hence:

$$L\{t\} = \frac{1}{s^2}$$

**Example Use integration by parts to find the Laplace of $f(t) = \cos(t)$**

Let:
$$u = e^{-st}, \quad du = -s e^{-st}, \quad dv = \cos(t), \quad v = \sin(t)$$

Then:
$$\int_0^\infty e^{-st}\cos(t)\, dt = \left[e^{-st}\sin(t)\right]_0^\infty + \int_0^\infty \sin(t)\cdot s e^{-st}\, dt = 0 + s\int_0^\infty e^{-st}\sin(t)\, dt$$

Considering the sin part :

$$u = e^{-st}, \quad du = -s e^{-st}, \quad dv = \sin(t), \quad v = -\cos(t)$$

$$\int_0^\infty e^{-st}\sin(t)\, dt = 1 - s\int_0^\infty e^{-st}\cos(t)\, dt$$

Substituting this back into the original integral gives:

$$\int_0^\infty e^{-st}\cos(t)\, dt = 1 - s\int_0^\infty e^{-st}\cos(t)\, dt = s - s^2\int_0^\infty e^{-st}\cos(t)\, dt$$

$$L\{cos(t)\} = \frac{s}{1 + s^2}$$

## 3.2 Table of Laplace Transforms

| $f(t)$ | $L\{f(t)\}$ |
|---|---|
| 1 | $\frac{1}{s}$, $s > 0$ |
| $t$ | $\frac{1}{s^2}$, $s > 0$ |
| $t^n$, $n = 0, 1, 2, 3$ | $\frac{n!}{s^{n+1}}$, $s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}$, $s > a$ |
| $\cos(\omega t)$ | $\frac{s}{s^2+\omega^2}$ |
| $\sin(\omega t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cosh(at)$ | $\frac{s}{s^2-a^2}$, $s > a \geq 0$ |
| $\sinh(at)$ | $\frac{a}{s^2-a^2}$, $s > a \geq 0$ |
| $e^{at}\cos(\omega t)$ | $\frac{s-a}{(s-a)^2+\omega^2}$ |
| $e^{at}\sin(\omega t)$ | $\frac{\omega}{(s-a)^2+\omega^2}$ |
| $e^{at}f(t)$ | $F(s - a)$ |

## 3.3 Laplace Transforms of Derivatives

**Theorem Laplace Transform of Derivatives**

Suppose that $f(t)$ and $f'(t)$ are continous and that $|f(t)| \leq Me^{kt}, \forall t \geq 0$ and for constans $M, k$. Then the Laplace Transform of $f'(t)$ exists for $s > k$ and is given by:

$$L\left\{\frac{df}{dt}\right\} = sL\{f\} - f(0) \quad \text{for } s > k$$

We can easily extend this to higher order derivatives. Assume the Laplace Transform of $f^{(n)}(t)$ exists for $s > k$ and is given by:

$$L\left\{\frac{df^n}{dt^n}\right\} = s^n L\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0) \quad \text{for } s > k$$

### 3.3.1 Examples

**Example Find $L\{t^2\}$ using the fact $L\{s\} = 1/s$ for $s > 0$**

$$L\{f''\} = s^2 L\{f\} - sf(0) - f'(0)$$

With $f(t) = t^2$. Since $f'(t) = 2t$, $f''(t) = 2$, $f'(0) = 0$, $f(0) = 0$, gives:

$$L\{2\} = s^2 L\{t^2\} - s \cdot 0 - 0$$

So that:

$$L\{t^2\} = \frac{L\{2\}}{s^2} = \frac{2}{s^3}$$

**Example Find $L\{\sin(t)\}$ and $L\{\cos(t)\}$**

We again use ther equation:

$$L\{f''\} = s^2 L\{f\} - sf(0) - f'(0)$$

With $f(t) = \sin(t)$, $f'(t) = \cos(t)$, $f''(t) = -\sin(t)$, $\sin(0) = 0$, $\cos(0) = 1$. This gives:

$$L\{-\sin(t)\} = s^2 L\{\sin(t)\} - s \cdot 0 - 1$$

So that:

$$L\{\sin(t)\} = \frac{1}{s^2 + 1}$$

Similarly, we can find:

$$L\{\cos(t)\} = \frac{s}{s^2 + 1}$$

## 3.4 Solving Initial Value Problems

Consider an example from mechanics: A particle of mass $m > 0$ lies on rough table, attatched to a spring of stiffness $k > 0$. At any time $t > 0$, the mass is a distance $x(t)$ from the equillibrium position $O$, and $x(t)$ is much less than the length of the spring.

The mass is subject to a driving force $F_d(t)$, from Newtons second law, we have:

$$F_d(t) - kx - \gamma \frac{dx}{dt} = m \frac{dx^2}{dt^2}$$

Where $\gamma > 0$ is the **damping constant** and the term $\gamma \frac{dx}{dt}$ models the **fricition due to roughness** of the table, which oppposes direction of motion. The **restoring force** due to the spring is $-kx$; and always points towards $O$. The term $m \frac{dx^2}{dt^2}$ is the **acceleration of the mass**. We can rewrite this as:

$$F_d(t) = m \frac{dx^2}{dt^2} + \gamma \frac{dx}{dt} + kx$$

In order to solve this, we also need initial displacement $v_0 = x(0)$ and initial velocity $v_0 = \frac{dx}{dt}(0)$.

### 3.4.1 Examples

**Example**

$$\frac{dx^2}{dt^2} + 3\frac{dx}{dt} + 2x = 0, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1$$

1. **Take Laplace of governing equation:**

$$L\left\{\frac{dx^2}{dt^2}\right\} = s^2 L\{x\} - sx(0) - x'(0) = s^2 L\{x\} - 1$$

$$L\left\{\frac{dx}{dt}\right\} = sL\{x\} - x(0) = sLx$$

Hence:

$$s^2 L\{x\} - 1 + 3sL\{x\} + 2L\{x\} = 0$$

This is known as the **subsidary equation**. Rearranging:

$$(s^2 + 3s + 2)L\{x\} = 1$$

2. **Solve the subsidary equation:**

$$L\{x\} = \frac{1}{s^2 + 3s + 2}$$

3. **Find the inverse Laplace Transform:**

$$x(t) = L^{-1}\left\{\frac{1}{s^2 + 3s + 2}\right\} = L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$$

Hence:

$$A(s+2) + B(s+1) = 1 \quad \rightarrow A = 1, B = -1$$

Thus:

$$x = L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-t} - e^{-2t}$$

# 4 Step Functions, Advanced Theorems, and Special Functions

## 4.1 Heaviside Step Function

Denote the Heaviside Step Function as $H(t)$, defined as:

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Clearly, for any constant $a$, we have:

$$H(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

### 4.1.1 Examples

**Example Express this function in terms of a Heaviside step function**

$$f(t) = \begin{cases} 3 & \text{for } 0 \leq t < 2 \\ -1 & \text{for } t > 2 \end{cases}$$

We write the general form of $f(t)$ as:

$$f(t) = \alpha + \beta H(t - 2)$$

Now, settings $t$ to any value $\in [0, 2)$ gives:

$$f(t) = \alpha + (\beta)(0) = 0 \Rightarrow \alpha = 3$$

Setting $t$ to any value greater than 2 gives:

$$f(t) = 3 + (\beta)(1) = -1 \Rightarrow \beta = -4$$

Thus, we have:

$$f(t) = 3 - 4H(t - 2)$$

**Example Express this function in terms of a Heaviside step function**

$$f(t) = \begin{cases} 3 & \text{for } 0 \leq t < 1 \\ 5 & \text{for } 1 < t < 3 \\ -2 & \text{for } t > 3 \end{cases}$$

We write the general form of $f(t)$ as:

$$f(t) = \alpha + \beta H(t - 1) + \gamma H(t - 3)$$

Setting $t$ to any value $\in [0, 1)$ gives: $f(t) = \alpha + (\beta)(0) + (\gamma)(0) = 0 \Rightarrow \alpha = 3$
Setting $t$ to any value $\in (1, 3)$ gives: $f(t) = 3 + (\beta)(2) + (\gamma)(0) = 5 \Rightarrow \beta = 2$
Setting $t$ to any value greater than 3 gives: $f(t) = 3 + 2 + (\gamma)(1) = -7 \Rightarrow \gamma = -5$ Thus, we have:

$$f(t) = 3 + 2H(t - 1) - 7H(t - 3)$$

## 4.2 The Second Shift Theorem

### 4.2.1 Examples

## 4.3 Practice Problems

1. Use the First Shift Theorem $(L\{e^{at}f(t)\} = F(s-a))$ to find the Laplace transform of the following functions:

   (a) $t^3 e^{-3t}$ (b) $e^{-t}\cos(2t)$ (c) $e^{-4t}\cosh(5t)$ (d) $e^{-t}\sin^2(t)$

2. Use the First Shift Theorem $(L^{-1}\{F(s-a)\} = e^{at}f(t))$ to find the inverse Laplace transform of the following functions:

   (a) $\dfrac{6s-4}{s^2-4s+20}$ (b) $\dfrac{3s+7}{s^2-2s-3}$ (c) $\dfrac{4s+12}{s^2+8s+16}$

3. Solve the following initial value problems using the method of Laplace transforms:

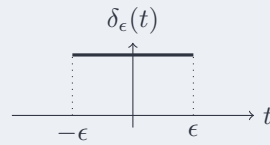$$y'' + y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = 1;$$
$$y'' - y = t, \quad y(0) = 1, \quad y'(0) = 1.$$

4. Find the inverse Laplace transform of the following functions using the method of partial fractions:

   (a) $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}$. (b) $\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}$. (c) $\dfrac{3s+1}{(s-1)(s^2+1)}$. (d) $\dfrac{e^{-5s}}{(s^2+1)(s^2+2)}$.

## 4.4 The Dirac Delta Function

The **Dirac Delta Function** models extremely brief but intense forces like a hammer hitting a nail. It starts as a function $\delta_\varepsilon$, that equals $\frac{1}{2\varepsilon}$ over the interval $t \in [-\varepsilon, \varepsilon]$ and 0 elsewhere.

$$\delta_\epsilon(t)$$



$$\delta_\epsilon(t) = \begin{cases} \dfrac{1}{2\epsilon} & \text{if } t \in [-\epsilon, \epsilon], \\ 0 & \text{otherwise.} \end{cases}$$

This function creates a rectangular pulse with the following propertites:

$$\textbf{Height: } \frac{1}{2\varepsilon} \quad \textbf{Width: } 2\varepsilon \quad \textbf{Area: } 1 \text{ (always)}$$

As $\varepsilon$ approaches 0, the function becomes infinitely tall and thin, but the area remains 1. This limit defines the Dirac Delta Function:

$$\delta(t) = \lim_{\varepsilon \to 0+} \{\delta_\varepsilon(t)\}$$

**Properties of the Dirac Delta Function:**

$$\delta(t) = 0 \text{ for } t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$
$$\int_{-\infty}^{\infty} \delta(t - t_0)f(t)\, dt = f(t_0)$$

The Laplace Transform of the Dirac Delta Function is:

$$L\{\delta(t - t_0)\} = \int_0^\infty e^{-st}\delta(t - t_0)\, dt = \int_{-\infty}^{\infty} e^{-st}\delta(t - t_0)\, dt = e^{-st_0} \quad \text{for } t_0 > 0$$

### 4.4.1   Examples

**Example : Solve the following initial value problem which governs the behavious of an RLC circuit**

$$LQ'' + RQ' + \frac{Q}{C} = V_0 \delta(t - a)$$
$$Q(0) = 0$$
$$Q'(0) = 0$$

Where $a, L, R, C, V_0$ are all positive constans and $4L > R^2 C$.
Note that the applied voltage corresponds to an impulse of stength $V_0$ at $t = a$
We note that:

$$L\{Q''\} = s^2 L\{Q\} - sQ(0) - Q'(0) = s^2 L\{Q\}$$
$$L\{Q'\} = sL\{Q\} - Q(0) = sL\{Q\}$$
$$L\{\delta(t - a)\} = e^{-st_0} = e^{-as}$$

Thus:

$$L\{LQ'' + RQ' + \frac{Q}{C} = V_0 \delta(t - a)\} = Ls^2 L\{Q\} + RsL\{Q\} + \frac{1}{C}L\{Q\} = V_0 e^{-as}$$

Grouping terms:

$$L\{Q\}(Ls^2 + Rs + \frac{1}{C}) = V_0 e^{-as}$$

Hence:

$$L\{Q\} = V_0 e^{-as} \cdot \frac{1}{Ls^2 + Rs + \frac{1}{C}}$$

Removing the $L$ from the denominator gives:

$$L\{Q\} = \frac{V_0}{L} e^{-as} \cdot \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$
$$= \frac{V_0}{L} \cdot \frac{e^{-as}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

We notice that:

$$(s + \frac{R}{2L})^2 = s^2 + s\frac{2R}{2L} + \frac{R^2}{4L^2}$$
$$= s^2 + \frac{R}{L}s + \frac{R^2}{4L^2}$$

So that:

$$L\{Q\} = \frac{V_0}{L} \frac{e^{-as}}{(s + \frac{R}{2L})^2 - \frac{R^2}{4L^2} + \frac{1}{LC}}$$

Rewriting with $\alpha = \frac{R}{2L}$ and $\beta = \frac{1}{LC} - \frac{R^2}{4L^2}$

$$L\{Q\} = \frac{V_0}{L} \frac{e^{-as}}{(s + \alpha)^2 + \beta}$$

We also note that:

$$L\{\sin(\beta t)\} = \frac{\beta}{s^2 + B^2} \xrightarrow{\text{First Shift Theorem}} L\{e^{-as}\sin(\beta t)\} = \frac{B}{(s + a)^2 + \beta^2}$$

Or,

$$L^{-1}\left\{\frac{B}{(s + a)^2 + \beta^2}\right\} = e^{-at}\sin(\beta t)$$

We can also write:

$$L^{-1}\{F(s)\} = f(t)$$

Notice that:

$$f(t - a) = e^{-a(t-a)}\sin(\beta[t - a])$$

Applying the Second Shift Theorem gives:

$$L^{-1}\{e^{-a}F(s)\} = f(t - a)H(t - a)$$
$$L^{-1}\left\{e^{-as}\frac{\beta}{(s + a)^2 + \beta^2}\right\} = e^{-a(t-a)}\sin(\beta[t - a])H(t - a)$$

Thus:

$$Q(t) = \frac{V_0}{L\beta} e^{-a(t-a)}\sin(\beta[t - a])H(t - a)$$
$$= \begin{cases} 0 & \text{for } 0 \le t < a \\ \frac{V_0}{L\beta} e^{-a(t-a)}\sin(\beta[t - a])H(t - a) & \text{for } t > a \end{cases}$$

## 4.5 Differentiation of the Laplace Transform

Suppose $f(t), t \geq 0$ satisfies the conditions of the existence theorem so that its Laplace Transform $(F(s))$ exists for some $s > k$. Then:

$$F'(s) = \frac{d}{ds}\left\{\int_0^\infty e^{-st}f(t)\,dt\right\} = \int_0^\infty \frac{\partial}{\partial s}\{e^{-st}f(t)\,dt\}$$

We are allowed to bring the derivative inside the integral provided the conditions of the existence theorem are satisfied, hence:

$$F'(s) = -\int_0^\infty e^{-st}\{tf(t)\}\,dt = -L\{tf(t)\}$$

so that,

$$L\{tf(t)\} = -F'(s)$$

We can sometimes use this to calculate transforms and inverse transforms. For example:

$$L\{t\} = L\{t \cdot 1\} = -\frac{d}{ds}L\{1\} = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

## 4.6 The Convolution Function

Let $f(t), g(t)$ be two functions. Define the Convolution function

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

Where $\tau$ is integrated over the interval $[0, t]$. The Convolution is:

$$\textbf{Commutative:} f \star g = g \star f$$
$$\textbf{Associative:} f \star (g \star h) = (f \star g) \star h$$
$$\textbf{Distruibutive:} f \star (g + h) = f \star g + f \star h$$
$$\textbf{Multiplication by 0:} = f \star (ag) = a(f \star g)$$

---

**Theorem**

Let $f(t)$ and $g(t)$ have Laplace Transforms $F(s)$ and $G(s)$ respectively defined for $s > k \geq 04$. Then

$$L\{f \star g\} = F(s)G(s), \quad s > k$$

---

**Proof**

Write $F(s) = \int_0^\infty e^{-s\sigma}f(\sigma)\,d\sigma$ and $G(s) = \int_0^\infty e^{-s\tau}g(\tau)\,d\tau$. Then:

$$F(s)G(s) = \left\{\int_0^\infty e^{-s\sigma}f(\sigma)\,d\sigma\right\}\left\{\int_0^\infty e^{-s\tau}g(\tau)\,d\tau\right\}$$
$$= \int_0^\infty e^{-s\tau}g(\tau)\left\{\int_0^\infty e^{-s\sigma}f(\sigma)\,d\sigma\right\}d\tau$$
$$= \int_0^\infty g(\tau)\left\{\int_0^\infty e^{-s(\sigma+\tau)}f(\sigma)\,d\sigma\right\}d\tau.$$

# 5    Line Integrals

Consider a mass which undergoes a displacement, $\boldsymbol{d}$, under a constant force $\boldsymbol{F}$. Define the work, $\boldsymbol{W}$, done by $\boldsymbol{F}$ to be the magnitude of the force multiplied by the distance moved in the direction of the force. Inspecting the diagram, we see that work done $\boldsymbol{W}$ is given by the dot product of $\boldsymbol{F}$ and $\boldsymbol{d}$:

$$W = |F| \cdot |d| \cdot \cos(\theta) = \boldsymbol{F} \cdot \boldsymbol{d}$$
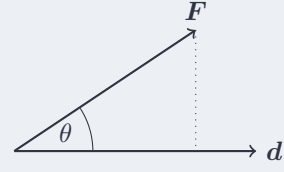
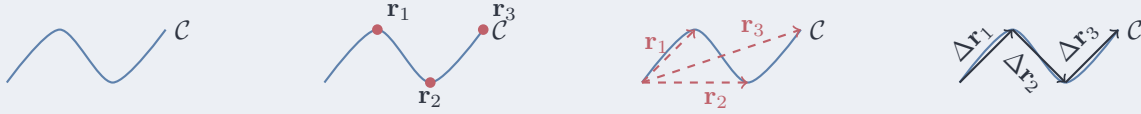Now, lets suppose $F$ is not constant:

$$F = F(x, y, z) = F(r) = r(x, y, z)$$

Suppose further, that $F$ acts for a time $t_1 \leq t \leq t_2$ and the path of the object in this time interval is given by a curve $\mathcal{C}$ defined by:

$$\mathbf{r} = (x(t), y(t), z(t)) \quad t \in [t_1, t_2]$$

But how do we calculate the work done by $F$ along $\mathcal{C}$?

As seen as the diagram above, we can divide $\mathcal{C}$ into a a large number $N-1$ of small segment of $\Delta \mathbf{r}_i$ and approximate the work done by $F$ along $\mathcal{C}$ by the sum of the work done along each segment:

$$W \approx \sum_{i=1}^{N-1} F(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i$$

## 5.1    The Line Integral

Taking the limit $N \to \infty$

$$W = \lim_{N \to \infty} \left\{ \sum_{i=1}^{N-1} F(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i \right\}$$

This limit is called the **line integral** of $F$ along $\mathcal{C}$ and is denoted by $\int_{\mathcal{C}} F(r) \cdot d\mathbf{r}$, that is:

$$\int_{\mathcal{C}} F(r) \cdot dr = \lim_{N \to \infty} \left\{ \sum_{i=1}^{N-1} F(r_i) \cdot \Delta r_i \right\}$$

Since $r = r(t)$ we can calculate the line integral as:

$$\int_{\mathcal{C}} F(r) \cdot dr = \int_{t_1}^{t_2} F(r(t)) \cdot \frac{dr}{dt} \, dt$$

In general, $t$, may be any variable that parametrizes (traces out) the curve $\mathcal{C}$. Then $dr = \frac{dr}{dt} \, dt$ is the tangent vector to $\mathcal{C}$ at the point $r(t)$. We call $\mathcal{C}$ the **path of integration** and $r(t_1)$ the initial point, $r(t_2)$ the **terminal point**. $\mathcal{C}$ is now **oriented** from $r(t_1)$ to $r(t_2)$.

The direction for $r(t_1) \to r(t_2)$, in which $t$ increases, is called the **positive direction** of $\mathcal{C}$, we indicate this by an arrow on $\mathcal{C}$.
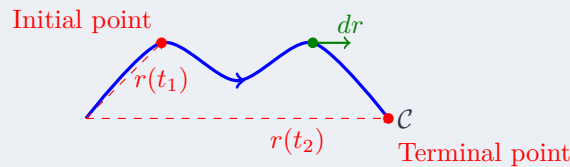
If $r(t_1) = r(t_2)$ then $\mathcal{C}$ is a **closed curve** and the line integral is denoted by:

$$\oint_{\mathcal{C}} F(r) \cdot dr$$

The line integral of $F$ alonged a closed curve $\mathcal{C}$ is called the **circulation** of $F$ around $\mathcal{C}$.

### 5.1.1 Examples

**Example : For a time period $0 \leq t \leq 1$, a particle moves along a trajectory defined by $\mathcal{C} = x = t, y = t, z = 2t^2$, a force $F(r) = (y, x, z)$ acts. Calculate work done.**

We have:

$$\mathbf{r} = (t, t, 2t^2)$$

$$\frac{d\mathbf{r}}{dt} = (1, 1, 4t)$$

$$F(\mathbf{r}) = (t, t, 2t^2)$$

The work done is:

$$\int_{\mathcal{C}} F(r) \cdot dr = \int_0^1 (t, t, 2t^2) \cdot (1, 1, 4t) \, dt$$

$$= \int_0^1 (t + t + 8t^3) \, dt$$

$$= \int_0^1 (2t + 8t^3) \, dt$$

$$= \left[ t^2 + 2t^4 \right]_0^1$$

$$= 1 + 2$$

$$= 3$$

**Example Evaluate $\int_{\mathcal{C}} F(r) \cdot dr$ for $F(r) = (y + z, x + z, x + y)$, where $\mathcal{C}$ is the line segment joinint $A : (-2, 2, -3) \rightarrow B(2, 4, 6)$**

We can parametrize any straight line in three dimensions by writing:

$$x = c_1 t + c_2 \quad y = c_3 t + c_4 \quad z = c_5 t + c_6$$

where the $c_i$ are constants.
We can take $t = 0$ to correspond with $A$ and $t = 1$ to correspond with $B$.
Hence setting $t = 0$ gives:

$$c_2 = -2 \quad c_4 = 2 \quad c_6 = -3$$

Setting $t = 1$ gives: $c_1 + c_2 = 2 \quad c_3 + c_4 = 4 \quad c_5 + c_6 = 6$ so that:

$$c_1 = 4 \quad c_3 = 2 \quad c_5 = 9$$

Hence, the required line is given by:

$$x = 4t - 2 \quad y = 2t + 2 \quad z = 9t - 3$$

We have:

$$\int_{\mathcal{C}} F(r) \cdot dr = \int_0^1 (11t - 1, 13t - 5, 6t) \cdot \frac{d}{dt}(4t - 2, 2t + 2, 9t - 3) \, dt$$

$$= \int_0^1 (11t - 1, 13t - 5, 6t) \cdot (4, 2, 9) \, dt$$

$$= \int_0^1 (44t - 4 + 26t - 10 + 54t) \, dt$$

$$= \int_0^1 (124t - 12) \, dt$$

$$= \left[ 62t^2 - 14t \right]_0^1$$

$$= 48$$

**Example Find the circulation of the vector $F = (y, -x, 0)$ around the unit circle $\mathcal{C} = x^2 + y^2 = 1$, $z = 0$, taken in an anti-clockwise direction**

We can parametrize the unit circle by setting $x, y$ and $z$ to:

$$x = \cos(t) \quad y = \sin(t) \quad z = 0$$

We can write the position vector as:

$$\mathbf{r} = (\cos(t), \sin(t), 0)$$

The tangent vector is:

$$\frac{d\mathbf{r}}{dt} = (-\sin(t), \cos(t), 0)$$

The force vector is:

$$F(\mathbf{r}) = (\sin(t), -\cos(t), 0)$$

The circulation is:

$$\oint_{\mathcal{C}} F(r) \cdot dr = \int_0^{2\pi} (\sin(t), -\cos(t), 0) \cdot (-\sin(t), \cos(t), 0)\ dt \tag{1}$$

$$= \int_0^{2\pi} -\sin^2(t) - \cos^2(t)\ dt \tag{2}$$

$$= -\int_0^{2\pi} (\sin^2(t) + \cos^2(t))\ dt \tag{3}$$

$$= -\int_0^{2\pi} 1\ dt \tag{4}$$

$$= -[t]_0^{2\pi} \tag{5}$$

$$= -2\pi \tag{6}$$

## 5.2 Convervative Vector Fields

A vector field $F$ is called **conservative** if the line integral of $F$ along any closed curve $\mathcal{C}$ is zero, that is:

$$\oint_{\mathcal{C}} F(r) \cdot dr = 0$$

An equivalent definition is that $F$ is conservative if the line integral of $F$ depends only on the end points of the curce, not on the path taken, so that:

$$\int_{\mathcal{C}_1} F(r) \cdot dr = \int_{\mathcal{C}_2} F(r) \cdot dr$$

Where $\mathcal{C}_1$ and $\mathcal{C}_2$ are two curves with the same initial and terminal points but different paths.



Path $\mathcal{C}_1$        Path $\mathcal{C}_2$
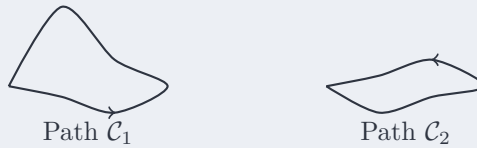
Consider two curves, $\mathcal{C}_1$ and $\mathcal{C}_2$, that start at $A$ and end at $B$. Let $\mathcal{C}$ be the closed curve that starts at $A$ follows the curve $\mathcal{C}_1$ and then follows $\mathcal{C}_2$ in the reverse direction to $B$. Then:

$$\oint_{\mathcal{C}} F(r) \cdot dr = \int_{A\mathcal{C}_1}^{B} F(r) \cdot dr + \int_{B\mathcal{C}_2}^{A} F(r) \cdot dr$$

$$= \int_{A\mathcal{C}_1}^{B} F(r) \cdot dr - \int_{A\mathcal{C}_2}^{B} F(r) \cdot dr = 0$$

Thus:

$$\oint_{\mathcal{C}} F(r) \cdot dr = 0 \Rightarrow \int_{A\mathcal{C}_1}^{B} F(r) \cdot dr = \int_{A\mathcal{C}_2}^{B} F(r) \cdot dr$$

### 5.2.1 Examples

**Example** By considering the line integral of $F = (y, x^2 - x, 0)$ around the square $C$ in the $x, y$ plane connecting for point $(0,0,0), (1,0,0), (1,1,0), (0,1,0)$, show that $F$ cannot be conservative.

Split $C$ into four segments, $C_1, C_2, C_3, C_4$ and calculate the line integral of $F$ along each segment.
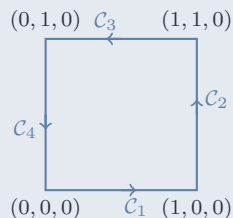


We have:

$$\int_{C_1} F(r) \cdot dr = \int_0^1 (0, t^2 - t, 0) \cdot (1, 0, 0) \, dt = 0$$

$$\int_{C_2} F(r) \cdot dr = \int_0^1 (t, 1 - t, 0) \cdot (0, 1, 0) \, dt = \int_0^1 (1 - t) \, dt = [t - \frac{t^2}{2}]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\int_{C_3} F(r) \cdot dr = \int_0^1 (1, (1 - t)^2 - (1 - t), 0) \cdot (-1, 0, 0) \, dt = \int_0^1 -1 \, dt = -1$$

$$\int_{C_4} F(r) \cdot dr = \int_0^1 (1 - t, 0, 0) \cdot (0, -1, 0) \, dt = 0$$

Hence:

$$\oint_C F(r) \cdot dr = \int_{C_1} F(r) \cdot dr + \int_{C_2} F(r) \cdot dr + \int_{C_3} F(r) \cdot dr + \int_{C_4} F(r) \cdot dr = 0 + \frac{1}{2} + (-1) + 0 = -\frac{1}{2} \neq 0$$
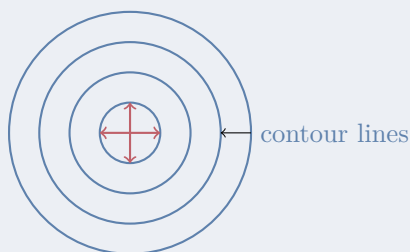
Thus, $F$ is not conservative.

# 6 Gradient, Divergence and Curl

## 6.1 Gradient
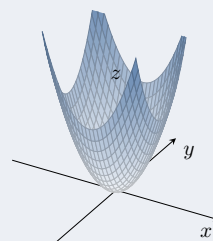
**Definition**

The gradient of a differentiable function $\phi(x, y, z)$ is the vector field:

$$\mathrm{grad}(\phi) = \nabla \phi = \frac{d\phi}{dx}\hat{i} + \frac{d\phi}{dy}\hat{j} + \frac{d\phi}{dz}\hat{k}$$



(a) Contour Plot with Gradient Vectors          (b) 3D Surface Plot

Figure 1: Illustration of the function $\phi(x, y) = x^2 + y^2$ with its gradient vectors and corresponding 3D surface.

### 6.1.1   The Directional Derivative

Consider a differentiable function $\phi(x, y, z)$ at a point $P$, with a position vector $\vec{r}_0$.
We want to find the rate of change of $\phi$ as we move from $P$ in the direction of $\hat{\mathbf{n}}$.
Consider the line segment $L$ through $P$ in the direction of $\hat{\mathbf{n}}$
Let $Q(s)$ be the point on $L$ a distance $s$ from $P$
Define the **Directional Derivative** of $\phi(x, y, z)$ in the direction of $\hat{\mathbf{n}}$ by:
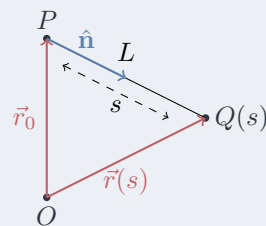
$$D_{\hat{\mathbf{n}}}\phi = \lim_{s \to 0} \frac{\phi(Q(s)) - \phi(P)}{s}$$

That is, as $s$ gets smaller and smaller we get a better and better approximation of With $L$ described by
$r(s) = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$, we have:

$$
\begin{aligned}
D_{\hat{\mathbf{n}}}\phi &= \lim_{s \to 0} \frac{\phi(r(s)) - \phi(r(0))}{s} \\
&= \lim_{s \to 0} \frac{\phi(x(s), y(s), z(s)) - \phi(x(0), y(0), z(0))}{s} \\
&= \frac{d\phi}{ds}
\end{aligned}
$$

Writing $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ and using the chain rule:

$$
\begin{aligned}
D_{\hat{\mathbf{n}}}\phi = \frac{d\phi}{ds} &= \frac{d\phi}{dx}\frac{dx}{ds} + \frac{d\phi}{dy}\frac{dy}{ds} + \frac{d\phi}{dz}\frac{dz}{ds} \\
&= \frac{d\phi}{dx}n_1 + \frac{d\phi}{dy}n_2 + \frac{d\phi}{dz}n_3
\end{aligned}
\tag{7}
$$

and so with $|\hat{\mathbf{n}}| = 1$, we have:

$$D_{\hat{\mathbf{n}}}\phi = \hat{\mathbf{n}} \cdot \nabla\phi$$

---

**Example Find the directional derivative of $\phi(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point $P :: (2, 1, 3)$ in the direction of the vector $\vec{a} = \hat{i} - 2\hat{k}$**

First we calculate the gradient of $\phi$:

$$\nabla\phi = \frac{d}{dx}(2x^2)\hat{i} + \frac{d}{dy}(3y^2)\hat{j} + \frac{d}{dz}(z^2)\hat{k} = 4x\hat{i} + 6y\hat{j} + 2z\hat{k}$$

At $P : (2, 1, 3)$, the gradient is evaluated as

$$\nabla\phi = 4(2)\hat{i} + 6(1)\hat{j} + 2(3)\hat{k} = 8\hat{i} + 6\hat{j} + 6\hat{k}$$

Since $\vec{a}$ is not a unit vector, i.e. $|\vec{a}| \neq 1$, we need to normalize it to get a unit vector $\hat{\mathbf{n}}$:

$$\hat{\mathbf{n}} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} - 2\hat{k}}{\sqrt{1+4}} = \frac{\hat{i} - 2\hat{k}}{\sqrt{5}}$$

The directional derivative is then:

$$D_{\hat{\mathbf{n}}}\phi \mid_P = \hat{\mathbf{n}} \cdot \nabla\phi \mid_P = \frac{\hat{i} - 2\hat{k} \cdot (8\hat{i} + 6\hat{j} + 6\hat{k})}{\sqrt{5}} = \frac{8 - 12}{\sqrt{5}} = -\frac{4}{\sqrt{5}}$$

*Note:* The fact the directional derivative is negative indicates that $\phi$ decreases in the direction of $\vec{a}$.