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MP232: Applied Mathematics

60% Exam
40% Continuous Assessment (3 parts)

Contents

1 Prelim : The Exponential Function and Hyperbolic Functions

1.1 Exponential Function

Derivative

$$\frac{d}{dt}\left(e^{at}\right) = a\,e^{at}$$

Integral

$$\int e^{at}\,dt = \frac{1}{a}e^{at} + C$$

1.2 Hyperbolic Functions

Definitions:

$$\sinh(at) = \tfrac{e^{at}-e^{-at}}{2} \;\Big|\; \cosh(at) = \tfrac{e^{at}+e^{-at}}{2} \;\Big|\; \tanh(at) = \tfrac{\sinh(at)}{\cosh(at)}.$$

Derivatives

$$\tfrac{d}{dt}\big(\sinh(at)\big) = a\,\cosh(at), \;\Big|\; \tfrac{d}{dt}\big(\cosh(at)\big) = a\,\sinh(at), \;\Big|\; \tfrac{d}{dt}\big(\tanh(at)\big) = a\,\mathrm{sech}^2(at).$$

Integrals

$$\begin{aligned}
\int \sinh(at)\,dt &= \frac{1}{a}\cosh(at) + C \\
\int \cosh(at)\,dt &= \frac{1}{a}\sinh(at) + C, \\
\int \tanh(at)\,dt &= \frac{1}{a}\ln\big|\cosh(at)\big| + C.
\end{aligned}$$

Common Identities

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= 1, \\
\sinh(2x) &= 2\,\sinh x\,\cosh x, \\
\cosh(2x) &= \cosh^2 x + \sinh^2 x, \\
\tanh(2x) &= \frac{2\,\tanh x}{1+\tanh^2 x}.
\end{aligned}$$

1.3 Partial Fraction Decomposition

Unrepeated Linear Factors: A linear factor is of form $(ax+b)$

$$\frac{s+1}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

Repeated LinearFactors:

$$\frac{3}{(s+2)^2(s-3)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-3}$$

Unrepeated Quadratic Factors with complex roots: Where the discriminant (b^2-4ac) is negative (complex roots) but the factor is not repeated

$$\frac{3}{(s^2-s+1)(s+2)} = \frac{As+B}{s^2-s+1} + \frac{C}{s+2}$$

Repeated Quadratic Factors with complex roots:

$$\frac{1}{(s^2+1)^2(s-1)} = \frac{As+B}{(s^2+1)^2} + \frac{Cs+D}{s^2+1} + \frac{E}{s-1}$$

2 Intro to Laplace Transforms

2.1 What is a Laplace Transform?

Definition 2.1

The Laplace Transform, defined for $t \geq 0$, is given by

$$L\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st} dt$$

2.2 Common Laplace Transforms

Example 2.1: Find the Laplace Transform of $f(t) = 1$

We have:

$$L\{1\} = \int_0^\infty 1 \cdot e^{-st} dt = \lim_{R\to\infty} \int_0^R e^{-st} dt$$

This integral is equal to:

$$\int_0^R e^{-st} dt = \frac{e^{-st}}{-s}\bigg|_{t=0}^{t=R} = -\frac{1}{s}[e^{-sR} - 1] = \frac{1 - e^{-sR}}{s}$$

Taking the limit as $R \to \infty$ gives:

$$L\{1\} = \lim_{R\to\infty} \frac{1 - e^{-sR}}{s} = \frac{1}{s}$$

Example 2.2: Find the Laplace Transform of $f(t) = e^{2t}$

$$\begin{aligned}
L\{e^{2t}\} &= \int_0^\infty e^{2t} e^{-st}\, dt = \int_0^\infty e^{-(s-2)t}\, dt \\
&= \lim_{R\to\infty} \int_0^R e^{-(s-2)t}\, dt \\
&= \lim_{R\to\infty} \left[\frac{e^{-(s-2)t}}{-(s-2)}\right]_{t=0}^{t=R} \\
&= \lim_{R\to\infty} \left(\frac{e^{-(s-2)R} - e^0}{-(s-2)}\right) = \lim_{R\to\infty} \left(\frac{e^{-(s-2)R} - 1}{-(s-2)}\right) \\
&= \frac{1}{s-2} \quad (\text{since } e^{-(s-2)R} \to 0 \text{ as } R \to \infty \text{ provided } s > 2)
\end{aligned}$$

Example 2.3: Find the Laplace Transform of $f(t) = \cosh(at)$

We have:

$$\begin{aligned} L\{\cosh(at)\} &= L\left\{\frac{e^{at}+e^{-at}}{2}\right\} \quad \text{from the definition of } \cosh(at) \\ &= \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\} \quad \text{by linearity of the Laplace Transform} \\ &= \frac{1}{2}\left(\frac{1}{s-a}\right) + \frac{1}{2}\left(\frac{1}{s+a}\right) \end{aligned}$$

Hence:

$$L\{\cosh(at)\} = \frac{s}{s^2-a^2}$$

Noting that $\sinh(at) = (e^{at} - e^{-at})/2$, we can find that:

$$L\{\sinh(at)\} = \frac{a}{(s^2-a^2)}$$

Example 2.4: Find the Laplace Transform of $\cos(wt)$ and $\sin(wt)$ where w is a constant.

We first compute the Laplace Transform of e^{iwt} using its definition:

$$L\{e^{iwt}\} = \int_0^\infty e^{-st}e^{iwt}\,dt = \int_0^\infty e^{-(s-iw)t}\,dt = \frac{1}{s-iw}, \quad \text{for } \Re(s) > 0.$$

To express this in terms of real and imaginary parts, we multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{s-iw} = \frac{s+iw}{(s-iw)(s+iw)} = \frac{s+iw}{s^2+w^2}.$$

Since Euler's formula gives:

$$e^{iwt} = \cos(wt) + i\sin(wt),$$

the linearity of the Laplace Transform yields:

$$L\{e^{iwt}\} = L\{\cos(wt)\} + iL\{\sin(wt)\}.$$

Equating the two representations of $L\{e^{iwt}\}$, we have:

$$L\{\cos(wt)\} + iL\{\sin(wt)\} = \frac{s+iw}{s^2+w^2}.$$

Since the equality must hold for both the real and imaginary parts, we equate them separately:

$$L\{\cos(wt)\} = \frac{s}{s^2+w^2} \quad \text{and} \quad L\{\sin(wt)\} = \frac{w}{s^2+w^2}.$$

2.3 Linearity of the Laplace Transform

The Laplace Transform is a linear operator, i.e. for any constants a and b:

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

Proof:

$$\begin{aligned} L\{af(t)+bg(t)\} &= \int_0^\infty e^{-st}(af(t)+bg(t))\,dt \\ &= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt \\ &= aL\{f(t)\}+bL\{g(t)\} \end{aligned}$$

2.4 The First Shift Theorem

Theorem First Shift Theorem

If $f(t)$ has a Laplace Transform, $F(s)$, defined for $s>k$, then $e^{at}\ f(t)$ has a Laplace Transform, $F(s-a)$ defined for $s-a>k$ and is given by:

$$L\{e^{at}f(t)\} = F(s-a)$$

or, taking the inverse Laplace Transform of both sides:

$$e^{at}f(t) = L^{-1}\{F(s-a)\}$$

2.4.1 Examples

Example 2.5: Find the Laplace Transform of $e^{at}\cos(wt)$, where a, w are constants.

e know that $L\{\cos(wt)\} = \frac{s}{s^2+w^2}$, so by the First Shift Theorem:

$$\begin{aligned} L\{e^{at}\cos(wt)\} &= \frac{s-a}{(s-a)^2+w^2} \\ &= \frac{s-a}{s^2-2as+a^2+w^2} \end{aligned}$$

2.5 Existence of the Laplace Transform

Existence of a Laplace transform is not always guaranteed because we're integrating over an infinite integral. For a Laplace Transform to exist for a given s, then the integral must exist:

$$\int_0^\infty e^{-st}f(t)\,dt$$

Theorem Existence Theorem of Laplace Transforms

Suppose $f(t)$ is a piecewise continuous function on $[0,\infty)$. If $f(t)$ satisfies:

$$|f(t)| \le Me^{kt}\ (0 \le t \le \infty)$$

for some constants, M, k, then the Laplace Transform of $f(t)$ exists for $s>k$. In other words, the Laplace Transform of $f(t)$ exists if $f(t)$ is bounded by an exponential function.

Proof:

If $s>k$, then from the equation above, we have:

$$|F(s)| = \left|\int_0^\infty f(t)e^{-st}\ dt\right| \le \int_0^\infty |f(t)|e^{-st}\ dt \le \int_0^\infty Me^{(k-s)t}\ dt = \frac{M}{s-k}$$

3 Applications of Laplace Transforms

3.1 Integration by Parts

Starting with the product rule:

$$\frac{d}{dx}[uv] = u'v + uv',$$

we can express this in differential form as:

$$d(uv) = u\,dv + v\,du.$$

Integrate both sides with respect to x:

$$\int d(uv) = \int_a^b u\,dv + \int_a^b v\,du.$$

The Fundamental Theorem of Calculus tells us that the left-hand side is simply:

$$uv = \int_a^b u\,dv + \int_a^b v\,du.$$

Rearrange to solve for the desired integral:

$$\int_a^b u\,dv = uv - \int_a^b v\,du,$$

3.1.1 Examples

Example 3.1: Use integration by parts to find the Laplace of $f(t) = t$

$$L\{t\} = \int_0^\infty te^{-st}\ dt$$

We integrate by parts by setting:

$$u = t, \quad dv = e^{-st}, \quad du = 1, \quad v = -\frac{e^{-st}}{s}$$

Then intengrating by parts gives:

$$\begin{aligned} L\{t\} &= \left[-\frac{te^{-st}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\ dt \\ &= 0 + \frac{1}{s}\left[-\frac{e^{-st}}{s}\right]_0^\infty \end{aligned}$$

Hence:

$$L\{t\} = \frac{1}{s^2}$$

Example 3.2: Use integration by parts to find the Laplace of $f(t) = \cos(t)$

Let:

$$u = e^{-st}, \quad du = -se^{-st}, \quad dv = \cos(t), \quad v = \sin(t)$$

Then:

$$\int_0^\infty e^{-st}\cos(t)\,dt = \left[e^{-st}\sin(t)\right]_0^\infty + \int_0^\infty \sin(t)\cdot se^{-st}dt = 0 + s\int_0^\infty e^{-st}\sin(t)dt$$

Considering the sin part :

$$u = e^{-st}, \quad du = -se^{-st}, \quad dv = \sin(t), \quad v = -\cos(t)$$

$$\int_0^\infty e^{-st}\sin(t)\,dt = 1 - s\int_0^\infty e^{-st}\cos(t)\,dt$$

Substituting this back into the original integral gives:

$$\int_0^\infty e^{-st}\cos(t)\,dt = 1 - s\int_0^\infty e^{-st}\cos(t)\,dt = s - s^2\int_0^\infty e^{-st}\cos(t)\,dt$$

$$L\{cos(t)\} = \frac{s}{1+s^2}$$

3.2 Table of Laplace Transforms

$f(t)$	$L\{f(t)\}$
1	$\frac{1}{s}$, $s > 0$
t	$\frac{1}{s^2}$, $s > 0$
t^n, $n = 0, 1, 2, 3$	$\frac{n!}{s^{n+1}}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$, $s > a \geq 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}$, $s > a \geq 0$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}f(t)$	$F(s-a)$

3.3 Laplace Transforms of Derivatives

Theorem Laplace Transform of Derivatives

Suppose that $f(t)$ and $f'(t)$ are continous and that $|f(t)| \leq Me^{kt}, \forall t \geq 0$ and for constans M, k. Then the Laplace Transform of $f'(t)$ exists for $s > k$ and is given by:

$$L\left\{\frac{df}{dt}\right\} = sL\{f\} - f(0) \quad \text{for } s > k$$

We can easily extend this to higher order derivatives. Assume the Laplace Transform of $f^{(n)}(t)$ exists for $s > k$ and is given by:

$$L\left\{\frac{df^n}{dt^n}\right\} = s^n L\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0) \quad \text{for } s > k$$

3.3.1 Examples

Example 3.3: Find $L\{t^2\}$ using the fact $L\{s\} = 1/s$ for $s > 0$

$$L\{f''\} = s^2 L\{f\} - sf(0) - f'(0)$$

With $f(t) = t^2$. Since $f'(t) = 2t$, $f''(t) = 2$, $f'(0) = 0, f(0) = 0$, gives:

$$L\{2\} = s^2 L\{t^2\} - s \cdot 0 - 0$$

So that:

$$L\{t^2\} = \frac{L\{2\}}{s^2} = \frac{2}{s^3}$$

Example 3.4: Find $L\{\sin(t)\}$ and $L\{\cos(t)\}$

We again use ther equation:

$$L\{f''\} = s^2 L\{f\} - sf(0) - f'(0)$$

With $f(t) = \sin(t)$, $f'(t) = \cos(t)$, $f''(t) = -\sin(t)$, $\sin(0) = 0$, $\cos(0) = 1$. This gives:

$$L\{-\sin(t)\} = s^2 L\{\sin(t)\} - s \cdot 0 - 1$$

So that:

$$L\{\sin(t)\} = \frac{1}{s^2 + 1}$$

Similarly, we can find:

$$L\{\cos(t)\} = \frac{s}{s^2 + 1}$$

3.4 Solving Initial Value Problems

Consider an example from mechanics: A particle of mass $m > 0$ lies on rough table, attatched to a spring of stiffness $k > 0$. At any time $t > 0$, the mass is a distance $x(t)$ from the equillibrium position O, and $x(t)$ is much less than the length of the spring.
The mass is subject to a driving force $F_d(t)$, from Newtons second law, we have:

$$F_d(t) - kx - \gamma\frac{dx}{dt} = m\frac{dx^2}{dt^2}$$

Where $\gamma > 0$ is the **damping constant** and the term $\gamma\frac{dx}{dt}$ models the **fricition due to roughness** of the table, which oppposes direction of motion. The **restoring force** due to the spring is $-kx$; and always points towards O. The term $m\frac{dx^2}{dt^2}$ is the **acceleration of the mass**. We can rewrite this as:

$$F_d(t) = m\frac{dx^2}{dt^2} + \gamma\frac{dx}{dt} + kx$$

In order to solve this, we also need initial displacement $v_0 = x(0)$ and initial velocity $v_0 = \frac{dx}{dt}(0)$.

3.4.1 Examples

Example 3.5

$$\frac{dx^2}{dt^2} + 3\frac{dx}{dt} + 2x = 0, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1$$

1. Take Laplace of governing equation:

$$L\left\{\frac{dx^2}{dt^2}\right\} = s^2L\{x\} - sx(0) - x'(0) = s^2L\{x\} - 1$$

$$L\left\{\frac{dx}{dt}\right\} = sL\{x\} - x(0) = sLx$$

Hence:

$$s^2L\{x\} - 1 + 3sL\{x\} + 2L\{x\} = 0$$

This is known as the **subsidary equation**. Rearranging:

$$(s^2 + 3s + 2)L\{x\} = 1$$

2. Solve the subsidary equation:

$$L\{x\} = \frac{1}{s^2 + 3s + 2}$$

3. Find the inverse Laplace Transform:

$$x(t) = L^{-1}\left\{\frac{1}{s^2 + 3s + 2}\right\} = L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$$

Hence:

$$A(s+2) + B(s+1) = 1 \quad \rightarrow A = 1, B = -1$$

Thus:

$$x = L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-t} - e^{-2t}$$

4 Step Functions, Advanced Theorems, and Special Functions

4.1 Heaviside Step Function

Denote the Heaviside Step Function as $H(t)$, defined as:

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Clearly, for any constant a, we have:

$$H(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

4.1.1 Examples

Example 4.1: Express this function in terms of a Heaviside step function

$$f(t) = \begin{cases} 3 & \text{for } 0 \leq t < 2 \\ -1 & \text{for } t > 2 \end{cases}$$

We write the general form of $f(t)$ as:

$$f(t) = \alpha + \beta H(t-2)$$

Now, settings t to any value $\in [0, 2)$ gives:

$$f(t) = \alpha + (\beta)(0) = 0 \Rightarrow \alpha = 3$$

Setting t to any value greater than 2 gives:

$$f(t) = 3 + (\beta)(1) = -1 \Rightarrow \beta = -4$$

Thus, we have:

$$f(t) = 3 - 4H(t-2)$$

Example 4.2: Express this function in terms of a Heaviside step function

$$f(t) = \begin{cases} 3 & \text{for } 0 \leq t < 1 \\ 5 & \text{for } 1 < t < 3 \\ -2 & \text{for } t > 3 \end{cases}$$

We write the general form of $f(t)$ as:

$$f(t) = \alpha + \beta H(t-1) + \gamma H(t-3)$$

Setting t to any value $\in [0, 1)$ gives: $f(t) = \alpha + (\beta)(0) + (\gamma)(0) = 0 \Rightarrow \alpha = 3$
Setting t to any value $\in (1, 3)$ gives: $f(t) = 3 + (\beta)(2) + (\gamma)(0) = 5 \Rightarrow \beta = 2$
Setting t to any value greater than 3 gives: $f(t) = 3 + 2 + (\gamma)(1) = -7 \Rightarrow \gamma = -5$ Thus, we have:

$$f(t) = 3 + 2H(t-1) - 7H(t-3)$$

4.2 The Second Shift Theorem

Theorem

If $f(t)$ has the transform $F(s)$ $(s > k)$ then the shifted function,

$$\tilde{f}(t) = f(t-a)H(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

has the transform $e^{-as}F(s)$ $(s > k)$, that is:

$$L\{f(t-a)H(t-a)\} = e^{-as}F(s)$$

Proof:

$$L\{f(t-a)H(t-a)\} = \int_0^\infty e^{-st} f(t-a)H(t-a)\, dt = \int_a^\infty e^{-st} f(t-a)\, dt$$

We introduce a new integration variable $\tau = t - a$, we have

$$\int_0^\infty e^{-s(\tau+a)} f(\tau)\, d\tau = e^{-as} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau = e^{-as}F(s)$$

4.2.1 Examples

Example 4.3: Find the Laplace Transform of $H(t-a)$ for $a > 0$

$$L\{H(t-a)\} = L\{H(t-a)f(t-a)\} = e^{-as}F(s) = \frac{e^{-as}}{s}$$

4.3 Practice Problems

1. Use the First Shift Theorem $(L\{e^{at}f(t)\} = F(s-a))$ to find the Laplace transform of the following functions:

   (a) t^3e^{-3t} (b) $e^{-t}\cos(2t)$ (c) $e^{-4t}\cosh(5t)$ (d) $e^{-t}\sin^2(t)$
2. Use the First Shift Theorem $\left(L^{-1}\{F(s-a)\} = e^{at}f(t)\right)$ to find the inverse Laplace transform of the following functions:

   (a) $\dfrac{6s-4}{s^2-4s+20}$ (b) $\dfrac{3s+7}{s^2-2s-3}$ (c) $\dfrac{4s+12}{s^2+8s+16}$
3. Solve the following initial value problems using the method of Laplace transforms:

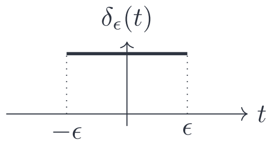
$$y'' + y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = 1;$$
$$y'' - y = t, \quad y(0) = 1, \quad y'(0) = 1.$$

4. Find the inverse Laplace transform of the following functions using the method of partial fractions:

   (a) $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}$. (b) $\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}$. (c) $\dfrac{3s+1}{(s-1)(s^2+1)}$. (d) $\dfrac{e^{-5s}}{(s^2+1)(s^2+2)}$.

4.4 The Dirac Delta Function

The **Dirac Delta Function** models extremely brief but intense forces like a hammer hitting a nail. It starts as a function δ_ε, that equals $\frac{1}{2\varepsilon}$ over the interval $t \in [-\varepsilon, \varepsilon]$ and 0 elsewhere.



$$\delta_\epsilon(t) = \begin{cases} \dfrac{1}{2\epsilon} & \text{if } t \in [-\epsilon, \epsilon], \\ 0 & \text{otherwise.} \end{cases}$$

This function creates a rectangular pulse with the following propertites:

$$\textbf{Height: } \frac{1}{2\varepsilon} \quad \textbf{Width: } 2\varepsilon \quad \textbf{Area: } 1 \text{ (always)}$$

As ε approaches 0, the function becomes infinitely tall and thin, but the area remains 1. This limit defines the Dirac Delta Function:

$$\delta(t) = \lim_{\varepsilon \to 0+} \{\delta_\varepsilon(t)\}$$

Properties of the Dirac Delta Function:

$$\delta(t) = 0 \text{ for } t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t)\, dt = f(t_0)$$

The Laplace Transform of the Dirac Delta Function is:

$$L\{\delta(t - t_0)\} = \int_0^\infty e^{-st} \delta(t - t_0)\, dt = \int_{-\infty}^{\infty} e^{-st} \delta(t - t_0)\, dt = e^{-st_0} \quad \text{for } \ t_0 > 0$$

4.4.1 Examples

Example 4.4: Solve the following initial value problem which governs the behavious of an RLC circuit

$$LQ'' + RQ' + \frac{Q}{C} = V_0\delta(t-a)$$
$$Q(0) = 0$$
$$Q'(0) = 0$$

Where a, L, R, C, V_0 are all positive constans and $4L > R^2C$.
Note that the applied voltage corresponds to an impulse of stength V_0 at $t = a$
We note that:

$$L\{Q''\} = s^2L\{Q\} - sQ(0) - Q'(0) = s^2L\{Q\}$$
$$L\{Q'\} = sL\{Q\} - Q(0) = sL\{Q\}$$
$$L\{\delta(t-a)\} = e^{-st_0} = e^{-as}$$

Thus:

$$L\{LQ'' + RQ' + \frac{Q}{C} = V_0\delta(t-a)\} = Ls^2L\{Q\} + RsL\{Q\} + \frac{1}{C}L\{Q\} = V_0e^{-as}$$

Grouping terms:

$$L\{Q\}(Ls^2 + Rs + \frac{1}{C}) = V_0e^{-as}$$

Hence:

$$L\{Q\} = V_0e^{-as} \cdot \frac{1}{Ls^2 + Rs + \frac{1}{C}}$$

Removing the L from the denominator gives:

$$\begin{aligned} L\{Q\} &= \frac{V_0}{L}e^{-as} \cdot \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \\ &= \frac{V_0}{L} \cdot \frac{e^{-as}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \end{aligned}$$

We notice that:

$$\begin{aligned} (s + \frac{R}{2L})^2 &= s^2 + s\frac{2R}{2L} + \frac{R^2}{4L^2} \\ &= s^2 + \frac{R}{L}s + \frac{R^2}{4L^2} \end{aligned}$$

So that:

$$L\{Q\} = \frac{V_0}{L}\frac{e^{-as}}{(s + \frac{R}{2L})^2 - \frac{R^2}{4L^2} + \frac{1}{LC}}$$

Rewriting with $\alpha = \frac{R}{2L}$ and $\beta = \frac{1}{LC} - \frac{R^2}{4L^2}$

$$L\{Q\} = \frac{V_0}{L}\frac{e^{-as}}{(s+\alpha)^2 + \beta}$$

We also note that:

$$L\{\sin(\beta t)\} = \frac{\beta}{s^2 + B^2} \xRightarrow{\text{First Shift Theorem}} L\{e^{-as}\sin(\beta t)\} = \frac{B}{(s+a)^2 + \beta^2}$$

Or,

$$L^{-1}\left\{\frac{B}{(s+a)^2 + \beta^2}\right\} = e^{-at}\sin(\beta t)$$

We can also write:

$$L^{-1}\{F(s)\} = f(t)$$

Notice that:

$$f(t-a) = e^{-a(t-a)}\sin(\beta[t-a])$$

Applying the Second Shift Theorem gives:

$$L^{-1}\{e^{-a}F(s)\} = f(t-a)H(t-a)$$

$$L^{-1}\left\{e^{-as}\frac{\beta}{(s+a)^2 + \beta^2}\right\} = e^{-a(t-a)}\sin(\beta[t-a])H(t-a)$$

Thus:

$$\begin{aligned} Q(t) &= \frac{V_0}{L\beta}e^{-a(t-a)}\sin(\beta[t-a])H(t-a) \\ &= \begin{cases} 0 & \text{for } 0 \le t < a \\ \frac{V_0}{L\beta}e^{-a(t-a)}\sin(\beta[t-a])H(t-a) & \text{for } t > a \end{cases} \end{aligned}$$

4.5 Differentiation of the Laplace Transform

Suppose $f(t), t \geq 0$ satisfies the conditions of the existence theorem so that its Laplace Transform $(F(s))$ exists for some $s > k$. Then:

$$F'(s) = \frac{d}{ds}\left\{\int_0^\infty e^{-st} f(t)\, dt\right\} = \int_0^\infty \frac{\partial}{\partial s}\{e^{-st} f(t)\, dt\}$$

We are allowed to bring the derivative inside the integral provided the conditions of the existence theorem are satisfied, hence:

$$F'(s) = -\int_0^\infty e^{-st}\{tf(t)\}\, dt = -L\{tf(t)\}$$

so that,

$$L\{tf(t)\} = -F'(s)$$

We can sometimes use this to calculate transforms and inverse transforms. For example:

$$L\{t\} = L\{t \cdot 1\} = -\frac{d}{ds}L\{1\} = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

4.6 The Convolution Function

Let $f(t), g(t)$ be two functions. Define the Convolution function

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau$$

Where τ is integrated over the interval $[0, t]$. The Convolution is:

$$\begin{aligned} \textbf{Commutative:} & f \star g = g \star f \\ \textbf{Associative:} & f \star (g \star h) = (f \star g) \star h \\ \textbf{Distruibutive:} & f \star (g + h) = f \star g + f \star h \\ \textbf{Multiplication by 0:} & = f \star (ag) = a(f \star g) \end{aligned}$$

Theorem

Let $f(t)$ and $g(t)$ have Laplace Transforms $F(s)$ and $G(s)$ respectively defined for $s > k \geq 04$. Then

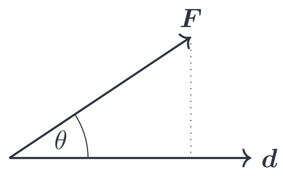
$$L\{f \star g\} = F(s)G(s), \quad s > k$$

Proof:

Write $F(s) = \int_0^\infty e^{-s\sigma} f(\sigma)\, d\sigma$ and $G(s) = \int_0^\infty e^{-s\tau} g(\tau)\, d\tau$. Then:

$$\begin{aligned} F(s)G(s) &= \left\{\int_0^\infty e^{-s\sigma} f(\sigma)\, d\sigma\right\}\left\{\int_0^\infty e^{-s\tau} g(\tau)\, d\tau\right\} \\ &= \int_0^\infty e^{-s\tau} g(\tau)\left\{\int_0^\infty e^{-s\sigma} f(\sigma)\, d\sigma\right\} d\tau \\ &= \int_0^\infty g(\tau)\left\{\int_0^\infty e^{-s(\sigma+\tau)} f(\sigma)\, d\sigma\right\} d\tau. \end{aligned}$$

5 Line Integrals



Consider a mass which undergoes a displacement, $\boldsymbol{d}$, under a constant force $\boldsymbol{F}$. Define the work, $\boldsymbol{W}$, done by $\boldsymbol{F}$ to be the magnitude of the force multiplied by the distance moved in the direction of the force. Inspecting the diagram, we see that work done $\boldsymbol{W}$ is given by the dot product of $\boldsymbol{F}$ and $\boldsymbol{d}$:

$$W = |F| \cdot |d| \cdot \cos(\theta) = \boldsymbol{F} \cdot \boldsymbol{d}$$

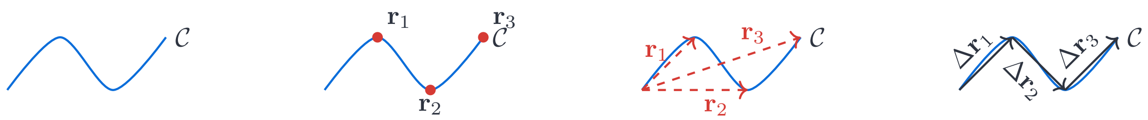
Now, lets suppose F is not constant:

$$F = F(x, y, z) = F(r) = r(x, y, z)$$

Suppose further, that F acts for a time $t_1 \leq t \leq t_2$ and the path of the object in this time interval is given by a curve $\mathcal{C}$ defined by:

$$\mathbf{r} = (x(t), y(t), z(t)) \quad t \in [t_1, t_2]$$

But how do we calculate the work done by F along $\mathcal{C}$?



As seen as the diagram above, we can divide $\mathcal{C}$ into a a large number $N-1$ of small segment of $\Delta\mathbf{r}_i$ and approximate the work done by F along $\mathcal{C}$ by the sum of the work done along each segment:

$$W \approx \sum_{i=1}^{N-1} F(\mathbf{r}_i) \cdot \Delta\mathbf{r}_i$$

5.1 The Line Integral

Taking the limit $N \to \infty$

$$W = \lim_{N\to\infty} \left\{ \sum_{i=1}^{N-1} F(\mathbf{r}_i) \cdot \Delta\mathbf{r}_i \right\}$$

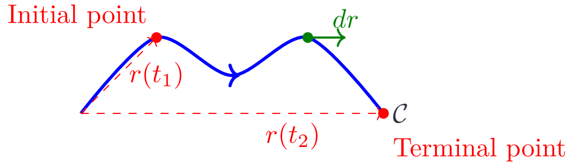
This limit is called the **line integral** of F along $\mathcal{C}$ and is denoted by $\int_{\mathcal{C}} F(r) \cdot d\mathbf{r}$, that is:

$$\int_{\mathcal{C}} F(r) \cdot dr = \lim_{N\to\infty} \left\{ \sum_{i=1}^{N-1} F(r_i) \cdot \Delta r_i \right\}$$

Since $r = r(t)$ we can calculate the line integral as:

$$\int_{\mathcal{C}} F(r) \cdot dr = \int_{t_1}^{t_2} F(r(t)) \cdot \frac{dr}{dt}\, dt$$

In general, t, may be any variable that parametrizes (traces out) the curve $\mathcal{C}$. Then $dr = \frac{dr}{dt}\, dt$ is the tangent vector to $\mathcal{C}$ at the point $r(t)$. We call $\mathcal{C}$ the **path of integration** and $r(t_1)$ the initial point, $r(t_2)$ the **terminal point**. $\mathcal{C}$ is now **oriented** from $r(t_1)$ to $r(t_2)$.
The direction for $r(t_1) \to r(t_2)$, in which t increases, is called the **positive direction** of $\mathcal{C}$, we indicate this by an arrow on $\mathcal{C}$.



If $r(t_1) = r(t_2)$ then $\mathcal{C}$ is a **closed curve** and the line integral is denoted by:

$$\oint_{\mathcal{C}} F(r) \cdot dr$$

The line integral of F alonged a closed curve $\mathcal{C}$ is called the **circulation** of F around $\mathcal{C}$.

5.1.1 Examples

Example 5.1: For a time period $0 \leq t \leq 1$, a particle moves along a trajectory defined by $\mathcal{C} = x = t, y = t,$

We have:

$$\begin{aligned}\mathbf{r} &= (t, t, 2t^2) \\ \frac{d\mathbf{r}}{dt} &= (1, 1, 4t) \\ F(\mathbf{r}) &= (t, t, 2t^2)\end{aligned}$$

The work done is:

$$\begin{aligned}\int_{\mathcal{C}} F(r) \cdot dr &= \int_0^1 (t, t, 2t^2) \cdot (1, 1, 4t)\, dt \\ &= \int_0^1 (t + t + 8t^3)\, dt \\ &= \int_0^1 (2t + 8t^3)\, dt \\ &= \left[t^2 + 2t^4\right]_0^1 \\ &= 1 + 2 \\ &= 3\end{aligned}$$

Example 5.2: Evaluate $\int_{\mathcal{C}} F(r) \cdot dr$ for $F(r) = (y + z, x + z, x + y)$, where $\mathcal{C}$ is the line segment joinint $A : (-$

We can parametrize any straight line in three dimensions by writing:

$$x = c_1 t + c_2 \quad y = c_3 t + c_4 \quad z = c_5 t + c_6$$

where the c_i are constants.
We can take $t = 0$ to correspond with A and $t = 1$ to correspond with B.
Hence setting $t = 0$ gives:

$$c_2 = -2 \quad c_4 = 2 \quad c_6 = -3$$

Setting $t = 1$ gives: $c_1 + c_2 = 2 \quad c_3 + c_4 = 4 \quad c_5 + c_6 = 6$ so that:

$$c_1 = 4 \quad c_3 = 2 \quad c_5 = 9$$

Hence, the required line is given by:

$$x = 4t - 2 \quad y = 2t + 2 \quad z = 9t - 3$$

We have:

$$\begin{aligned}\int_{\mathcal{C}} F(r) \cdot dr &= \int_0^1 (11t - 1, 13t - 5, 6t) \cdot \frac{d}{dt}(4t - 2, 2t + 2, 9t - 3)\, dt \\ &= \int_0^1 (11t - 1, 13t - 5, 6t) \cdot (4, 2, 9)\, dt \\ &= \int_0^1 (44t - 4 + 26t - 10 + 54t)\, dt \\ &= \int_0^1 (124t - 12)\, dt \\ &= \left[62t^2 - 14t\right]_0^1 \\ &= 48\end{aligned}$$

Example 5.3: Find the circulation of the vector $F = (y, -x, 0)$ around the unit circle $\mathcal{C} = x^2 + y^2 = 1$, $z = 0$

We can parametrize the unit circle by setting x, y and z to:

$$x = \cos(t) \quad y = \sin(t) \quad z = 0$$

We can write the position vector as:

$$\mathbf{r} = (\cos(t), \sin(t), 0)$$

The tangent vector is:

$$\frac{d\mathbf{r}}{dt} = (-\sin(t), \cos(t), 0)$$

The force vector is:

$$F(\mathbf{r}) = (\sin(t), -\cos(t), 0)$$

The circulation is:

$$\begin{align} \oint_{\mathcal{C}} F(r) \cdot dr &= \int_0^{2\pi} (\sin(t), -\cos(t), 0) \cdot (-\sin(t), \cos(t), 0)\ dt \tag{1} \\ &= \int_0^{2\pi} -\sin^2(t) - \cos^2(t)\ dt \tag{2} \\ &= -\int_0^{2\pi} (\sin^2(t) + \cos^2(t))\ dt \tag{3} \\ &= -\int_0^{2\pi} 1\ dt \tag{4} \\ &= -[t]_0^{2\pi} \tag{5} \\ &= -2\pi \tag{6} \end{align}$$

5.2 Convervative Vector Fields

A vector field F is called **conservative** if the line integral of F along any closed curve $\mathcal{C}$ is zero, that is:

$$\oint_{\mathcal{C}} F(r) \cdot dr = 0$$

An equivalent definition is that F is conservative if the line integral of F depends only on the end points of the curce, not on the path taken, so that:

$$\int_{\mathcal{C}_1} F(r) \cdot dr = \int_{\mathcal{C}_2} F(r) \cdot dr$$

Where $\mathcal{C}_1$ and $\mathcal{C}_2$ are two curves with the same initial and terminal points but different paths.

Path $\mathcal{C}_1$ Path $\mathcal{C}_2$

Consider two curves, $\mathcal{C}_1$ and $\mathcal{C}_2$, that start at A and end at B. Let $\mathcal{C}$ be the closed curve that starts at A follows the curve $\mathcal{C}_1$ and then follows $\mathcal{C}_2$ in the reverse direction to B. Then:

$$\begin{aligned} \oint_{\mathcal{C}} F(r) \cdot dr &= \int_{A\mathcal{C}_1}^{B} F(r) \cdot dr + \int_{B\mathcal{C}_2}^{A} F(r) \cdot dr \\ &= \int_{A\mathcal{C}_1}^{B} F(r) \cdot dr - \int_{A\mathcal{C}_2}^{B} F(r) \cdot dr = 0 \end{aligned}$$

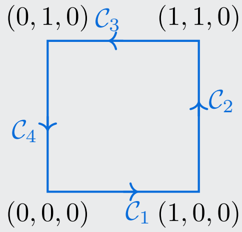
Thus:

$$\oint_{\mathcal{C}} F(r) \cdot dr = 0 \Rightarrow \int_{A\mathcal{C}_1}^{B} F(r) \cdot dr = \int_{A\mathcal{C}_2}^{B} F(r) \cdot dr$$

5.2.1 Examples

Example 5.4: By considering the line integral of $F = (y, x^2 - x, 0)$ around the square C in the x, y plane c

Split $\mathcal{C}$ into four segments, $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ and calculate the line integral of F along each segment.



We have:

$$\int_{\mathcal{C}_1} F(r) \cdot dr = \int_0^1 (0, t^2 - t, 0) \cdot (1, 0, 0)\ dt = 0$$
$$\int_{\mathcal{C}_2} F(r) \cdot dr = \int_0^1 (t, 1 - t, 0) \cdot (0, 1, 0)\ dt = \int_0^1 (1 - t)\ dt = [t - \frac{t^2}{2}]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$
$$\int_{\mathcal{C}_3} F(r) \cdot dr = \int_0^1 (1, (1 - t)^2 - (1 - t), 0) \cdot (-1, 0, 0)\ dt = \int_0^1 -1\ dt = -1$$
$$\int_{\mathcal{C}_4} F(r) \cdot dr = \int_0^1 (1 - t, 0, 0) \cdot (0, -1, 0)\ dt = 0$$

Hence:

$$\oint_{\mathcal{C}} F(r) \cdot dr = \int_{\mathcal{C}_1} F(r) \cdot dr + \int_{\mathcal{C}_2} F(r) \cdot dr + \int_{\mathcal{C}_3} F(r) \cdot dr + \int_{\mathcal{C}_4} F(r) \cdot dr = 0 + \frac{1}{2} + (-1) + 0 = -\frac{1}{2} \neq 0$$

Thus, F is not conservative.

6 Gradient, Divergence and Curl

6.1 Gradient

Definition 6.1

The gradient of a differentiable function $\phi(x, y, z)$ is the vector field:

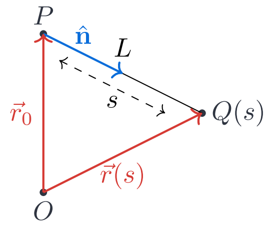
$$\text{grad}(\phi) = \nabla\phi = \frac{d\phi}{dx}\hat{i} + \frac{d\phi}{dy}\hat{j} + \frac{d\phi}{dz}\hat{k}$$

(a) Contour Plot with Gradient Vectors

(b) 3D Surface Plot

Figure 1: Illustration of the function $\phi(x, y) = x^2 + y^2$ with its gradient vectors and corresponding 3D surface.

6.1.1 The Directional Derivative



Consider a differentiable function $\phi(x, y, z)$ at a point P, with a position vector $\vec{r}_0$. We want to find the rate of change of ϕ as we move from P in the direction of $\hat{\mathbf{n}}$.
Consider the line segment L through P in the direction of $\hat{\mathbf{n}}$
Let $Q(s)$ be the point on L a distance s from P
Define the **Directional Derivative** of $\phi(x, y, z)$ in the direction of $\hat{\mathbf{n}}$ by:

$$D_{\hat{\mathbf{n}}}\phi = \lim_{s \to 0} \frac{\phi(Q(s)) - \phi(P)}{s}$$

That is, as s gets smaller and smaller we get a better and better approximation of With L described by $r(s) = x(s)\hat{\imath} + y(s)\hat{\jmath} + z(s)\hat{k}$, we have:

$$\begin{aligned} D_{\hat{\mathbf{n}}}\phi &= \lim_{s \to 0} \frac{\phi(r(s)) - \phi(r(0))}{s} \\ &= \lim_{s \to 0} \frac{\phi(x(s), y(s), z(s)) - \phi(x(0), y(0), z(0))}{s} \\ &= \frac{d\phi}{ds} \end{aligned}$$

Writing $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ and using the chain rule:

$$\begin{aligned} D_{\hat{\mathbf{n}}}\phi = \frac{d\phi}{ds} &= \frac{d\phi}{dx}\frac{dx}{ds} + \frac{d\phi}{dy}\frac{dy}{ds} + \frac{d\phi}{dz}\frac{dz}{ds} \\ &= \frac{d\phi}{dx}n_1 + \frac{d\phi}{dy}n_2 + \frac{d\phi}{dz}n_3 \qquad (7) \end{aligned}$$

and so with $|\hat{\mathbf{n}}| = 1$, we have:

$$D_{\hat{\mathbf{n}}}\phi = \hat{\mathbf{n}} \cdot \nabla\phi$$

Example 6.1: Find the directional derivative of $\phi(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point $P :: (2, 1, 3)$ in the di

First we calculate the gradient of ϕ:

$$\nabla\phi = \frac{d}{dx}(2x^2)\hat{\imath} + \frac{d}{dy}(3y^2)\hat{\jmath} + \frac{d}{dz}(z^2)\hat{k} = 4x\hat{\imath} + 6y\hat{\jmath} + 2z\hat{k}$$

At $P : (2, 1, 3)$, the gradient is evaluated as

$$\nabla\phi = 4(2)\hat{\imath} + 6(1)\hat{\jmath} + 2(3)\hat{k} = 8\hat{\imath} + 6\hat{\jmath} + 6\hat{k}$$

Since $\vec{a}$ is not a unit vector, i.e. $|\vec{a}| \neq 1$, we need to normalize it to get a unit vector $\hat{\mathbf{n}}$:

$$\hat{\mathbf{n}} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{\imath} - 2\hat{k}}{\sqrt{1 + 4}} = \frac{\hat{\imath} - 2\hat{k}}{\sqrt{5}}$$

The directional derivative is then:

$$D_{\hat{\mathbf{n}}}\phi\mid_P = \hat{\mathbf{n}} \cdot \nabla\phi\mid_P = \frac{\hat{\imath} - 2\hat{k} \cdot (8\hat{\imath} + 6\hat{\jmath} + 6\hat{k})}{\sqrt{5}} = \frac{8 - 12}{\sqrt{5}} = -\frac{4}{\sqrt{5}}$$

Note: The fact the directional derivative is negative indicates that ϕ decreases in the direction of $\vec{a}$.

6.1.2 Meaning of $\nabla\phi$

Theorem

Let $\phi(x, y, z)$ be a scalar function for which $\nabla\phi$ exists.
Then if $\nabla\phi\mid_P \neq 0$ at a point P, then $\nabla\phi\mid_P$ is the direction of maximum increase of ϕ at P.

Proof:

From the definition of the dot product, we know that:

$$D_{\hat{\mathbf{n}}}\phi = \hat{\mathbf{n}} \cdot \nabla\phi = |\hat{\mathbf{n}}||\nabla\phi| \cos(\gamma) = |\nabla\phi| \cos(\gamma)$$

where γ is the angle between $\hat{\mathbf{n}}$ and $\nabla\phi$.
For $\nabla\phi\mid_P \neq 0$ then $D_{\hat{\mathbf{n}}}\phi$ is maximized when $\cos(\gamma) = 1$, i.e. when $\gamma = 0$. That is, when $\hat{\mathbf{n}}$ is parallel to $\nabla\phi$.

Remark 1: We note that $-\nabla\phi\mid_P$ is the direction of maximum decrease of ϕ at P.
Remark 2: If $\nabla\phi\mid_P = 0$, then, P is a critical point of ϕ.

6.2 Gradient as Normal to a Surface