Robert Davidson
Complex Analysis Notes

60% Exam
40% Continuous Assessment (3 parts)

Contents

1 Preliminary **2**
- 1.1 The Complex Plane and the Four Quadrants . . . 2
- 1.2 Diagram of the Quadrants . . . 2
- 1.3 Adjusting Angles Based on Quadrants . . . 2

2 Foundations **3**
- 2.1 Intro to Complex Numbers . . . 3
- 2.2 Polar Form . . . 3
- 2.3 De Moivre's Theorem . . . 4
- 2.4 Roots of Unity . . . 4
- 2.5 Complex Roots . . . 5
- 2.6 Problem Sheet 1 . . . 5

3 Complex Functions **6**
- 3.1 Trigonemtric Functions . . . 6
- 3.2 Exponential Functions . . . 6
- 3.3 Complex Logarithms . . . 7
- 3.4 Complex Powers . . . 7

4 Geomtric Mappings and Transformations **8**
- 4.1 Mappings: . . . 8
  - 4.1.1 Example Mapping 1 . . . 8
  - 4.1.2 Example Mapping 2 . . . 9
- 4.2 Circle Preservation Theorem . . . 10
- 4.3 Prelim to Riemann Sphere . . . 11
  - 4.3.1 Euclidean Space and Compact Sets . . . 11
  - 4.3.2 Compactification of the Complex Plane . . . 11
- 4.4 Riemann Sphere . . . 11

5 Complex Analysis **12**
- 5.1 Mobius Transforms . . . 12
  - 5.1.1 Matrix Representation of Mobius Transforms . . . 13

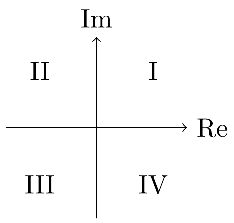
1 Preliminary

1.1 The Complex Plane and the Four Quadrants

The complex plane is a two-dimensional plane where the horizontal axis represents the real part and the vertical axis represents the imaginary part of a complex number. It is divided into four quadrants:

1. **Quadrant I** ($0^\circ < \theta < 90^\circ$): Both x and y are positive.
2. **Quadrant II** ($90^\circ < \theta < 180^\circ$): x is negative, y is positive.
3. **Quadrant III** ($180^\circ < \theta < 270^\circ$): Both x and y are negative.
4. **Quadrant IV** ($270^\circ < \theta < 360^\circ$): x is positive, y is negative.

1.2 Diagram of the Quadrants



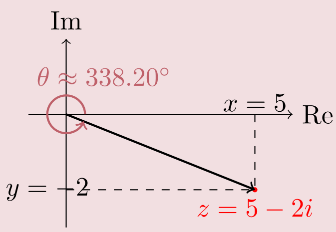
1.3 Adjusting Angles Based on Quadrants

To correctly determine θ, adjust the angle returned by $\tan^{-1}\left(\frac{y}{x}\right)$ according to the quadrant where z lies.

1. **Quadrant I** ($x > 0, y > 0$):
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
.
2. **Quadrant II** ($x < 0, y > 0$):
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$$
3. **Quadrant III** ($x < 0, y < 0$):
$$\theta = \pi + \tan^{-1}\left(\frac{y}{x}\right) - \pi$$
4. **Quadrant IV** ($x > 0, y < 0$):
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example: Let $z = 5 - 2i$

$x = 5$ and $y = -2$. Thus, we have: $\theta = 2\pi - \tan^{-1}(2/5) \approx 338.20^\circ$

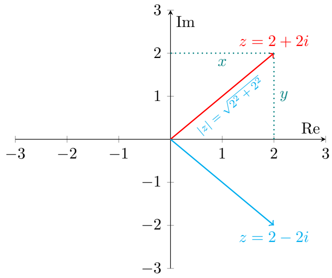


2 Foundations

2.1 Intro to Complex Numbers

Complex numbers can be written as the sum of a real and imaginary part:

$$z = x + iy$$



We denote the **complex conjugate** ($\overline{z}$) as:

$$\overline{z} = x - iy$$

Geometrically, $\overline{z}$ is the **reflection of z in the real axis**

With help from Pythagoras' we can now define the distance of z from the origin (**modulus**), that is the length of the vector pointing to z.

$$|z|^2 = x^2 + y^2 \Rightarrow |z| = \sqrt{x^2 + y^2}$$

We notice that:

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy - (iy)(iy) \\ &= x^2 - (i)^2(y^2) \\ &= x^2 - (-1)(y^2) \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

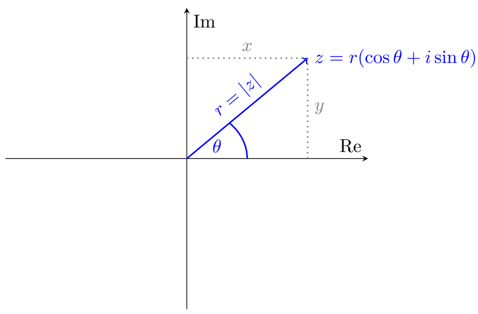
Thus, we have the distance of z from the origin as: $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ We refer to this as the **modulus** of z or the **absolute value** of z.

Letting $z = x + iy$ and $w = u + iv$, we see:

$$|z - w| = \sqrt{(x - u)^2 + (y - z^2)^2}$$

That is, $|z - w|$ is the distance between z and w in the complex plane.

2.2 Polar Form



Letting $r = |z| = \sqrt{x^2 + y^2}$, we can define x and y as:

$$\begin{aligned} \cos(\theta) &= \frac{x}{r} &\Rightarrow\quad x &= r\cos\theta, \\ \sin(\theta) &= \frac{y}{r} &\Rightarrow\quad y &= r\sin\theta. \end{aligned}$$

Now:

$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$

To find θ we usualy calculate $\tan^{-1}(y/x)$ and add/subtract π, when appropriate. Recalling $\tan^{-1}(y/x) \in (-\pi/2, \pi/2)$. We denote θ as as the **argument of z**, denoted as $\arg(z)$. Geometrically $\arg(z)$ represent the angle z makes with the positive real axis Thus, the pair (r, arg(z)) is called the **polar coordinates of z**. We introduce the idea that $\arg(z)$ is a version of $\mathrm{Arg}(z)$ that can take multiple values outside of $\mathrm{Arg}(z)$'s bounds, $(-\pi, \pi)$, more precisely:

$$\arg(z) = \mathrm{Arg}(z) + 2n\pi, \quad n \in \mathbb{Z}$$

Example: Find $\text{Arg}(i)$ and $\arg(i)$

Since $i = 0 + 1i$, we have $x = 0$ and $y = 1$.
Using $\tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \arg(z) = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$
Therefore:

$$\text{Arg}(i) = \frac{\pi}{2} \quad \text{and} \quad \arg(i) = \frac{\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}$$

2.3 De Moivre's Theorem

Theorem: Let $z_1, z_2 \in \mathbb{C}$, be nonzero numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then:

$$\begin{aligned} z_1 z_2 &= r_1 r_2[(cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

Thus, we have:

$$\begin{aligned} |z_1 z_2| &= |z_1||z_2| \\ \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \end{aligned}$$

Corollary: De Moivre's Theorem

Let $n \in \mathbb{Z}$, and $z = |z|(\cos\theta + i\sin\theta)$, then:

$$z^n = |z|^n = [\cos(n\theta) + i\sin(n\theta)]$$

2.4 Roots of Unity

Roots of unity are solutions to $z^n = 1$, where z is a complex number on the unit circle.
Eulers formula states that $e^{i\alpha} = \cos\alpha + i\sin\alpha$.
Given $z = x + iy$, then:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Since z lies on the unit circle, we know $R = 1$, thus we have

$$z = e^{i\theta}$$

Also, we can rewrite 1 as:

$$\begin{aligned} 1 &= 1 + 0i = \cos(0) + i\sin(0) \\ &= \cos(2\pi) + i\sin(2\pi) = \cos(2\pi k) + i\sin(2\pi k) \quad \text{(Periodic with} 2\pi \text{ k multiples don't change the result)} \\ &= e^{i2\pi k} \quad \text{where } k \in \mathbb{Z} \quad \text{(By Eulers Formula)} \end{aligned}$$

So we have, $z^n = e^{n(i\theta)}$:

$$\begin{aligned} e^{in\theta} &= e^{i2\pi k} \\ in\theta &= i2\pi k \\ n\theta &= 2\pi k \\ \theta &= \frac{2\pi k}{n} \end{aligned}$$

So θ is the angle corresponding to the n-th roots of unity. Using eulers formula again, the solutions are given as:

$$z_k = e_{i\theta} = e^{i(\frac{2\pi k}{n})} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

2.5 Complex Roots

Recall, square roots can be written as $4^{1/2} = \sqrt{4} = 2$, thus, we can write the n-th root as $x^{1/n}$.
What if we wanted to find the n-th root of a complex number?
Consider $f(z) = z^{1/n}$, where $n \in \mathbb{Z}$. To solve this, we aim to find some w such that $w^n = z$.

$$z = R[\cos(\theta) + i\sin(\theta)] \quad \text{and} \quad w = r[\cos(\phi) + i\sin(\phi)]$$

From De Moivre's Theorem, we have:

$$w^n = r^n[\cos(n\phi) + i\sin(n\phi)] = R[\cos(\theta) + i\sin(\theta)]$$

We see:

$$r^n = R \to r = \sqrt[n]{R} = R^{1/2}$$
$$n\phi = \theta = \theta + 2\pi k \to \phi = \frac{\theta}{n} + \frac{2\pi k}{n}$$

Note that since sin and cos are periodic with 2π, the addition of $2\pi k$ doesn't change the result.
So we have:

$$z^{1/n} = R^{1/n}[\cos\phi + i\sin\phi] \quad \text{with} \quad \phi = \frac{\theta + 2k\pi}{n}, \quad k \in (0, 1, 2, \ldots, n-1)$$

Note that we reserve the notation $\sqrt[n]{z}$ to denote the **principal root**, defined when $k = 0$.

Example: Find the cube roots of $z = -1 + i$

$R = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$
We know z is in the second quadrant, so must adjust θ accordingly:

$$\theta = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

We have $k = 0, 1, 2$ for the cube roots.
Thus, the cubic roots are:

$$w_k = \sqrt[3]{2}\left[\cos\left(\frac{\theta + 2\pi k}{3}\right) + i\sin\left(\frac{\theta + 2\pi k}{3}\right)\right]$$

2.6 Problem Sheet 1

1. Simplify the following (write in form a + ib)

   (a) $3\left(\frac{1+i}{1-i}\right)^2 - 2\left(\frac{1-i}{1+i}\right)^3$

3 Complex Functions

3.1 Trigonemtric Functions

Recall:

$$\text{cosine is an even function} \Rightarrow \cos(-\theta) = \cos(\theta)$$
$$\text{sine is an odd function} \Rightarrow \sin(-\theta) = -\sin(\theta)$$

Also recall Eulers formula states $e^{iz} = \cos(z) + i\sin(z)$ also that:

$$\begin{aligned} e^{-iz} &= \cos(-z) + i\sin(-z) \\ &= \cos(z) - i\sin(z) \end{aligned}$$

If we add these expressions, we get an expression for $\cos(z)$:

$$\begin{aligned} e^{iz} + e^{-iz} &= (\cos(z) + i\sin(z)) + (\cos(z) - i\sin(z)) \\ e^{iz} + e^{-iz} &= 2\cos(z) \Rightarrow \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \end{aligned}$$

If we subtract the expressions, we get an expression for $\sin(z)$:

$$\begin{aligned} e^{iz} - e^{-iz} &= (\cos(z) + i\sin(z)) - (\cos(z) - i\sin(z)) \\ e^{iz} - e^{-iz} &= 2i\sin(z) \Rightarrow \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

We can now also derive $\tan(z)$ and $\cot(z)$:

$$\begin{aligned} \tan(z) &= \frac{\sin(z)}{\cos(z)} = \frac{\frac{e^{iz}-e^{-iz}}{2i}}{\frac{e^{iz}+e^{-iz}}{2}} = i\frac{e^{iz} + e^{-iz}}{e^{iz} + e^{-iz}} \\ \cot(z) &= \frac{\cos(z)}{\sin(z)} = \frac{\frac{e^{iz}+e^{-iz}}{2}}{\frac{e^{iz}-e^{-iz}}{2i}} = -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \end{aligned}$$

Proposition. Let $z, z_1, z_2 \in \mathbb{C}$

$$\begin{aligned} &(i) \quad \sin(z + 2\pi) = \sin(z) \quad \text{and} \quad \cos(z + 2\pi) = \cos(z) \\ &(ii) \quad \cos^2(z) + \sin^2(z) = 1 \\ &(iii) \quad \sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2) \end{aligned}$$

3.2 Exponential Functions

Recall the **Taylor Series** for e^x, that is: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
We can now define the exponential function for complex numbers as:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}$$

Recall also, that $z = rei\theta = e^{i\theta}$ it then follows:

$$z = e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right)}_{\cos\theta} + i\underbrace{\left(1 - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\right)}_{\cos\theta} = \cos(\theta) + i\sin(\theta)$$

3.3 Complex Logarithms

Recall the log rule: $\log(e^x) = x$. Also recall we defined $\theta = \text{Arg}(z)$ with $\arg(z) = \text{Arg}(z) + 2\pi k$. Lastly, recall the polar form of z:

$$z = |z|(\cos(\theta) + i\sin(\theta)) = e^{i\theta} = |z|e^{i\text{Arg}(z)} = e^{\ln|z| + i\text{Arg}z}$$

We can now define the **Logarithm of a Complex Number**:

$$\begin{aligned} \text{Log}(z) &= \log\left(e^{\ln|z|+i\text{Arg}z}\right) & &= \ln|z| + i\,\text{Arg}(z) \\ \log(z) &= \ln|z| + i\arg z & &= \ln|z| + i(\text{Arg}(z) + 2\pi k) \end{aligned}$$

Note: Denote $\text{Log}(z)$ as the **principal branch** of the complex logarithm and denote $\log(z)$ as any branch with $k \neq 0$.
We can also write the **Complex logarithm** as:

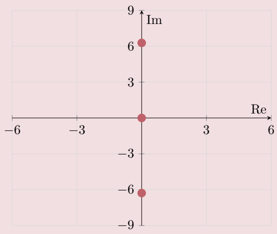
$$\begin{aligned} \log(z) &= \ln|z| + i\arg(z) \\ &= \ln|z| + i(\text{Arg}(z) + 2k\pi) \\ &= \ln|z| + i\text{Arg}(z) + 2k\pi i \end{aligned}$$

Example: Find the log of $z = 1 + 0i$

- $z = 1 + 0i = 1 \Rightarrow |z| = 1$
- $\text{Arg}(z) = \tan^{-1} 0/1 = 0$

Thus, we have:

$$\begin{aligned} \log(1) &= \ln|1| + i(\text{Arg}(z) + 2k\pi) \\ &= 0 + i(0 + 2k\pi) \\ &= 2k\pi i \quad \text{where } k \in \mathbb{Z} \end{aligned}$$



3.4 Complex Powers

Recall the Logarithm Rule: $\log(a^b) = b\log(a)$. We want to define z^α, in such a way that $\log(z^\alpha) = \alpha\log(z)$. That is the **Complex Power** is defined as:

$$z^\alpha = e^{\alpha\log(z)} = e^{\alpha(\text{Log}(z)+2k\pi i)} \quad \text{for } k \in \mathbb{Z}$$

So that we have:

$$\begin{aligned} \log(z^\alpha) &= \log(e^{\alpha(\text{Log}(z)+2k\pi i)}) \\ &= \alpha(\text{Log}(z) + 2k\pi i) \\ &= \alpha\log(z) \end{aligned}$$

As example, consider $z = 1 + 0i$:

$$\begin{aligned} 1^\alpha &= e^{\alpha(Log(1)+2k\pi i)} \\ &= e^{2k\alpha\pi i} \end{aligned}$$

If $\alpha \in \mathbb{Z}$ $(1, 2, 3, \dots)$

$$1^\alpha = \left(e^{2k\pi i}\right)^\alpha = (\cos(2\pi k) + \sin(2\pi k))^\alpha = 1^\alpha = 1$$

If $\alpha = \frac{m}{n} \in \mathbb{Q}$, then 1^α is the set of all n-th roots of unity:

$$1^\alpha = e^{\frac{2k\pi im}{n}} = \cos\left(\frac{2\pi km}{n}\right) + i\sin\left(\frac{2\pi km}{n}\right)\cos\left(\frac{2\pi r}{n}\right) + i\sin\left(\frac{2\pi r}{n}\right)$$

If $\alpha = i$ then we see:

$$1^\alpha = 1^i = e^{2k\pi i\cdot i} = e^{-2k\pi}$$

4 Geomtric Mappings and Transformations

4.1 Mappings:

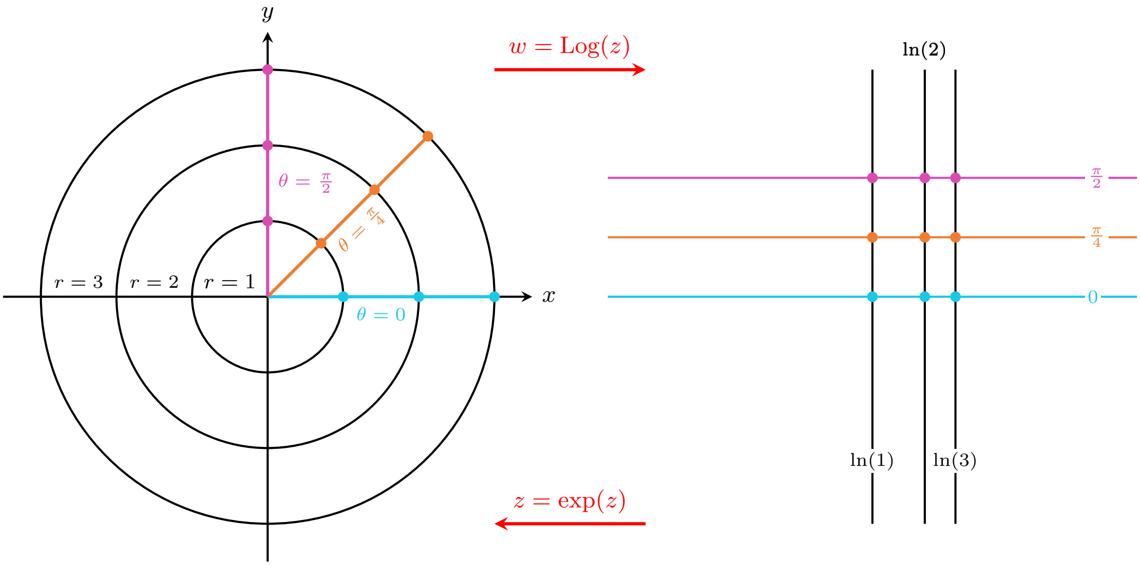
Recall we defined the principal branch as

$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$$

So, when we take the principal branch of the logarithm, we see that it maps to the complex number $w = u + iv$ where $u = \ln|z|$ and $v = \mathrm{Arg}(z)$.
In essence. Log maps $\mathbb{C}$ to the horizontal strip:

$$\{w = u + iv : -\pi < v \leq \pi\}$$

Diagram of the Logarithm Mapping:



4.1.1 Example Mapping 1

Let $f(z) = z^3$, we see that:

Using exponential rules and polar representation:

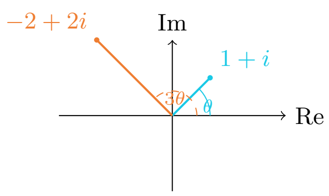
$$\begin{aligned} z &= |z|e^{i\theta} \\ z^3 &= \left(|z|e^{i\theta}\right)^3 \\ &= |z|^3 e^{i3\theta} \\ &= |z|^3 \Big(\cos(3\theta) + i\sin(3\theta)\Big) \end{aligned}$$

Alternatively, using our definition for complex powers:

$$\begin{aligned} z^3 &= e^{3\log(z)} \\ &= e^{3\left(\ln|z| + i\,\mathrm{Arg}(z)\right)} \\ &= e^{3\ln|z| + i3\,\mathrm{Arg}(z)} \\ &= |z|^3 e^{i3\,\mathrm{Arg}(z)} \\ &= |z|^3\Big(\cos(3\theta) + i\sin(3\theta)\Big) \end{aligned}$$

Letting $z = 1 + 1i$, we see: $\theta = \tan^{-1}\left(\frac{1}{1}\right) = 45° = \frac{\pi}{4}$, and $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Thus, we have:

$$\begin{aligned} z^3 &= |z|^3 \cdot [\cos(3\theta) + i\sin(3\theta)] \\ &= (\sqrt{2})^3 \cdot \left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right] \\ &= -2\sqrt{2} + i2\sqrt{2} \end{aligned}$$



In essence, the mapping $f(z) = z^3$ rotates the complex number z by 3θ and scales it by $|z|^3$. We can imagine this, for the complex numbers with $|z| = 1$, and $0 < \theta \leq \frac{\pi}{2}$, as an arc of radius 1, from the angle $0 \to 90°$, mapped to an arc of radius 8, from the angles $0 \to 270°$.

4.1.2 Example Mapping 2

We wish to find the image of the line $x = 1$ under

$$f(z) = \frac{1}{z}, \quad z = x + iy, \quad w = u + iv.$$

For $z = x + iy$ we have

$$w = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2},$$

so that

$$u = \frac{x}{x^2+y^2}, \quad v = -\frac{y}{x^2+y^2}.$$

Setting $x = 1$ yields

$$u = \frac{1}{1+y^2}, \quad v = -\frac{y}{1+y^2}.$$

Since

$$|w|^2 = u^2 + v^2 = \frac{1}{1+y^2} = u,$$

it follows that

$$u^2 + v^2 = u \quad \Longrightarrow \quad u^2 - u + v^2 = 0.$$

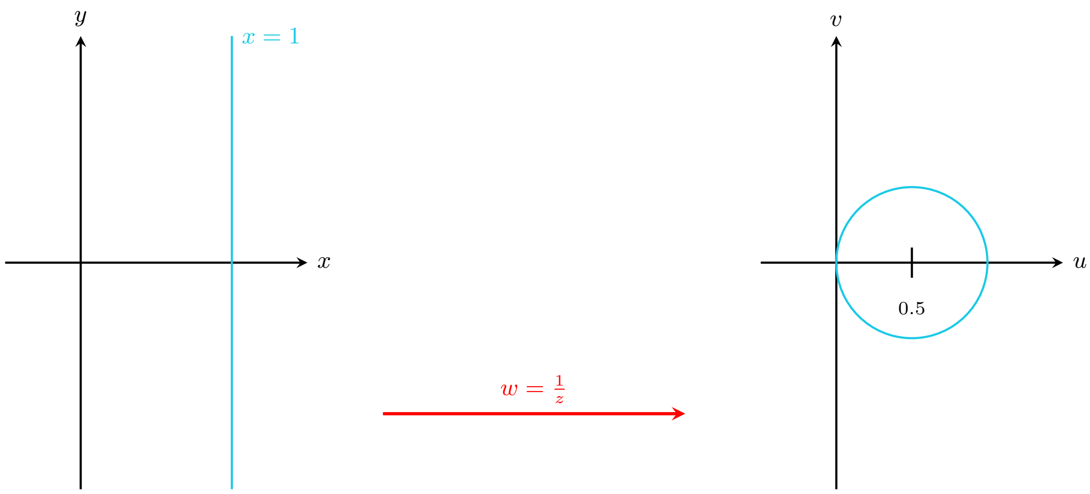
Completing the square in u by adding and subtracting $\frac{1}{4}$:

$$u^2 - u + \frac{1}{4} + v^2 = \frac{1}{4} \quad \Longrightarrow \quad \left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}.$$

Thus, the image of $x = 1$ is the circle

$$\boxed{\left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}},$$

centered at $\left(\frac{1}{2}, 0\right)$ with radius $\frac{1}{2}$



In general, $f(z) = \frac{1}{z}$ maps circle and lines to circles and lines, respectively.

4.2 Circle Preservation Theorem

Consider the equation:

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

We can we that if $A \neq 0$, then we can divide by A:

$$x^2 + y^2 + \frac{B}{A}x + \frac{C}{A}y + \frac{D}{A} = 0$$

Completing the square yields:

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{B^2 + C^2 - 4AD}{4A^2}\right)$$

Thus, if $A \neq 0$, we have a circle with center (-B/2A, -C/2A) and radius $\sqrt{\frac{B^2+C^2-4AD}{4A^2}}$.
If $A = 0$, then the equation represents a line:

$$Bx + Cy + D = 0$$

If $D = 0$, the circle or line contains 0:

$$Bx + Cy + D \mid_{(0,0)} = D = 0$$

Why is This Important?

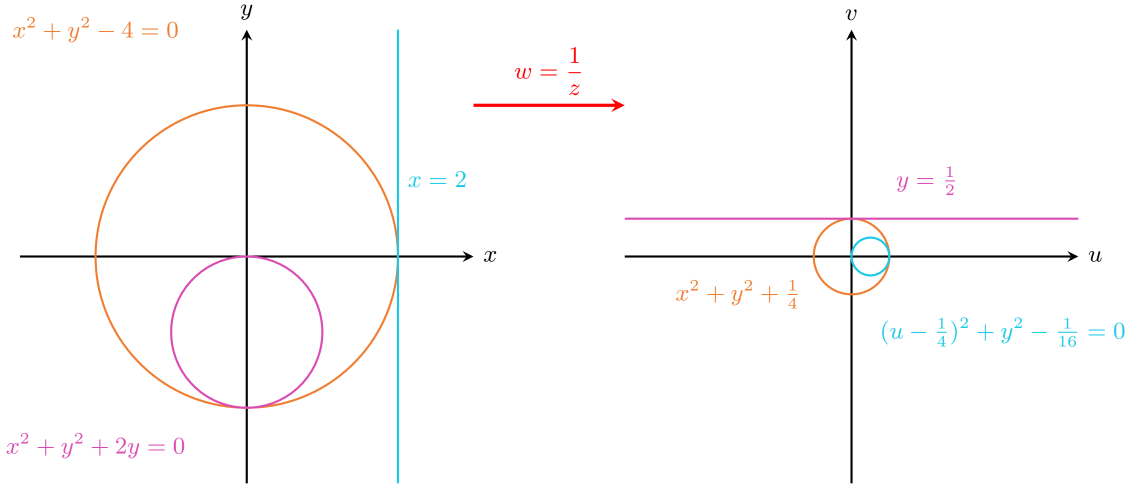
Under the inversion $f(z) = \frac{1}{z}$ with $z = x + iy$ and $w = u + iv$, one can show that the general equation

$$A(x^2 + y^2) + Bx + Cy + D = 0 \quad \underrightarrow{\text{maps to}} \quad D(u^2 + v^2) + Bu - Cv + A = 0.$$

In this transformed equation:

- If the original set does not contain the origin image is a circle.
- If the original set does contain the origin then the equation becomes linear:
- If the original set is a line (with $A = 0$), if it does not pass through the origin, its inversion is a circle that passes through the origin.

Examples Illustrating the Inversion Effects



4.3 Prelim to Riemann Sphere

Our goal is to define the **Riemann Sphere**, which is the complex plane $\mathbb{C}$, together with an extra point at infinity. In essence The Riemann sphere is a way to "wrap up" the entire complex plane into a compact, closed surface that is **homeomorphic** (toplogically equivalent) to the sphere S^2 and the connection between them is made via the **stereographic projection**.

4.3.1 Euclidean Space and Compact Sets

Euclidean space, denoted as $\mathbb{R}^n$, is the collection of all points in n-dimensional space, where each point is described by n real numbers. In Euclidean spaces (such as the real line $\mathbb{R}$ or the plane $\mathbb{R}^2$), a set is **compact** if it is both: **Closed** (contains all its limit points), and **Bounded** (contained within a finite region).

Examples of Compact Sets:

The closed interval $[0, 1] \subset \mathbb{R}^1$,

A closed disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$

Examples of Non-Compact Sets:

The open interval $(0, 1) \subset \mathbb{R}^1$ (not closed),

The entire real line $\mathbb{R}$ (not bounded)

4.3.2 Compactification of the Complex Plane

The complex plane $\mathbb{C}$ is not compact - it streches out infinitely in all directions. By adding a single point at infinity, we "close" the plane, turning it into a compact set. This new space, is **homeomorphic** (a one-to-one mapping that is continuous in both directions or toplogically equivalent) to to the Riemann Sphere . We define the new space as:

$$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

4.4 Riemann Sphere

Define $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then $\tilde{\mathbb{C}} \overset{1:1}{\longleftrightarrow} S^2\{X = (x, y, z) : x^2 + y^2 + z^2 = 1\}$ (*homeomorphic*) via the sterographic projection, denoted St, defined as follows:

1. Projection from $S^2 \to \tilde{\mathbb{C}}$:
For a point $(x, y, z) \in S^2$, with $z \neq 1$ (the point is not the north pole) the projection is defined as:

$$St(x, y, z) = \frac{1}{1 - x_3}(x_1, x_2) \quad \text{for } z \neq 1$$

This takes a point on the sphere and maps it to a point in the complex plane.

2. Projection from $\tilde{\mathbb{C}} \to S^2$:
For a point $z \in \mathbb{C}$, the inverse projection is defined as:

$$St^{-1}(z) = \frac{1}{|z|^2 + 1}\langle 2\text{Re}(z), 2\text{Im}(z), |z|^2 - 1\rangle$$

This takes a complex number, z, written in terms of its real (Re(z)) and imaginary (Im(z)) parts, and maps it to the sphere

3. Mapping the North Pole:
The projection leaves out the north pole from projection onto $\mathbb{C}$

$$St(N) = \infty \quad \text{and} \quad St^{-1}(\infty) = N \quad \text{where } N = \langle 0, 0, 1\rangle$$

The north pole is mapped to the point at infinity, and vice versa.

5 Complex Analysis

5.1 Mobius Transforms

Recall: The complex plane $\mathbb{C}$ can be throught as points $(x, y) \in \mathbb{R}^2$, but we usually label a point as $z = x + iy$. We can extend $\mathbb{C}$ by adding a point at infinity, the resulting set is called the **Riemann Sphere** $\tilde{\mathbb{C}}$. Visually, we can imaigine wrapping the complex plane onto the surface of a sphere, where ∞ is the north pole of the sphere.

Now, letting a, b, c, d be complex numbers (i.e. $a = x_a + iy_a$), we define a Mobius Transform as a function $T : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$:

$$T(z) = \frac{az + b}{cz + d}$$

where $ad - bc \neq 0$ (that is the determinant $\neq 0 \to$ matrix is invertible).
These functions occur on the Riemann Sphere, because we need to define that happens when $cz + d = 0$ and when $z = \infty$:

$$\text{If } c \neq 0 : \quad T(\infty) = \frac{a}{c} \quad \text{and} \quad T\left(-\frac{d}{c}\right) = \infty$$

$$\text{If } c = 0 : \quad T(z) = \frac{az + b}{d} \quad \text{and} \quad T(\infty) = \infty$$

Mobius transforms can be uniquely determined by its action on three distinct points. For example, we'll find a mobius transform that maps three points $\{z_1, z_2, z_3\}$ to $\{1, 0, \infty\}$

1. We want $T(z_2) = 0 : az_2 + b = 0 \Rightarrow b = -az_2$, then $T(z)$ becomes:

$$T(z) = \frac{az + b}{cz + d} = \frac{az - az_2}{cz + d} = \frac{a(z - z_2)}{cz + d}$$

2. We want $T(z_3) = \infty : cz_3 + d = 0 \Rightarrow d = -cz_3$, then $T(z)$ becomes:

$$T(z) = \frac{a(z - z_2)}{c(z - z_3)}$$

3. We want $T(z_1) = 1$, then $T(z_1)$ becomes:

$$T(z_1) = \frac{a(z_1 - z_2)}{c(z_1 - z_3)} = 1 \Rightarrow \frac{a}{c} = \frac{z_1 - z_3}{z_1 - z_2}$$

Finally, we see that $T(z)$ is:

$$T(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z - z_2}{z - z_3}$$

We can now solve problems, such as : Find the Mobius Transform that maps the 3 points $z_1 = -i, z_2 = -1, z_3 = 1$ to $1, 0, \infty$

$$T(z) = \frac{-1 - 1}{i + 1} \cdot \frac{z + 1}{z - 1} = (-i)\frac{z + 1}{z - 1} = \frac{-iz - i}{z - 1}$$

5.1.1 Matrix Representation of Mobius Transforms

We associate a 2×2 matrix M to a Mobius Transform $T(z)$:

$$T(z) = \frac{az+b}{cz+d} \longleftrightarrow M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Note that: $kM \longleftrightarrow T(z)$ for any $k \in \mathbb{C}, k \neq 0$.

We can also define the **inverse map** T^{-1} as the Mobius transform:

$$T^{-1} \longleftrightarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We can also define the **composition** of two Mobius Transforms, if $T_1(z) = \frac{az+b}{cz+d}$ with matrix M and $T_2(z) = \frac{ez+f}{gz+g}$ with matrix M_2, then:

$$T \circ T_2 \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Putting it all together, we can map any three points to any other three point:

Three-Point Theorem for Möbius Transformations

If $T \longleftrightarrow M : (z_1, z_2, z_3) \mapsto (1, 0, \infty)$ and if $T_2 \longleftrightarrow M_2 : (z_1', z_2', z_3') \mapsto (1, 0, \infty)$ then:

$$T^{-1} \circ T_2 \longleftrightarrow M^{-1} \cdot : (z_1, z_2, z_3)f \mapsto (z_1', z_2', z'3)$$

This can be visualized like so:

$$\begin{tikzcd}
z_1', z_2', z_3' \arrow[rr, "T^{-1} \circ T'"] \arrow[dr, "T_2 \mapsto M_2"'] & & z_1, z_2, z_3 \arrow[dl, "T \mapsto M"] \\
& 1, 0, \infty &
\end{tikzcd}$$

Note that, M, M_2 and $T^{-1} \circ T_2$ have matrices: Three-Point Theorem for Mobius Transformations

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad M_2 = \begin{bmatrix} e & f \\ g & g \end{bmatrix}, \quad T^{-1} \circ T_2 = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Example: Find a Mobius transformation, $T : (0, -i, -1) \mapsto (i, 1, 0)$

If we can find a map $T_1 : (0, -i, 1) \mapsto (1, 0, \infty)$ and a map $T_2 : (1, -i, -1) \mapsto (i, 1, 0)$. Then, by the Theorem above, we can find a T such that: $T : (0, -i, -1) \mapsto (i, 1, 0)$ Recall, we define a general transform T, that takes 3 points $(z_1, z_2, z_3) \mapsto (1, 0, \infty)$

$$T(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z - z_2}{z - z_3}$$

T_1 becomes:

$$\begin{aligned} T_1(z) &= \frac{0+1}{0+i} \cdot \frac{z+i}{z+1} \\ &= \frac{1}{i} \cdot \frac{z+i}{z+1} \\ &= \frac{z+1}{iz+i} \\ &\Rightarrow \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \end{aligned}$$

T_2 becomes:

$$\begin{aligned} T_2(z) &= \frac{i-0}{i-1} \cdot \frac{z-1}{z-0} \\ &= \frac{i}{i-1} \cdot \frac{z-1}{z} \\ &= \frac{iz-i}{(i-1)z} \\ &\Rightarrow \begin{bmatrix} i & -i \\ i-1 & 1 \end{bmatrix} \end{aligned}$$

Thus, T is:

$$T = T_2^{-1} \circ T_1 \leftrightarrow \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} i & -i \\ i-1 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} 0 & i \\ 1 & i-1 \end{bmatrix} = \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} i & -i \\ i-1 & 1 \end{bmatrix} = \begin{bmatrix} 0(1)+(i)(i) & (0)(i)+(i)(i) \\ (1-i)(1)+(i)(i) & (1-i)(i)+(i)(i) \end{bmatrix} = \begin{bmatrix} i^2 & i^2 \\ -i & i \end{bmatrix}$$

$$T(z) = -\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \longleftrightarrow = -i\frac{z+1}{z-1}$$