

# **MA2287: Complex Analysis Exam Notes**

Robert Davidson

## Contents

<b>1</b>	<b>Question 1:</b>	<b>3</b>
1.1	Sketch the region in the complex plane determined by the inequality . . . . .	3
1.2	Determine all solutions to roots of unity . . . . .	3
1.3	Determine and sketch the image under the mapping . . . . .	3
1.4	Find $z$ where the function is 0 . . . . .	3
1.5	Calculate principal value $\text{Log}(z)$ . . . . .	3
1.6	Prove the following . . . . .	3
<b>2</b>	<b>Question 2:</b>	<b>5</b>
2.1	Determine image of the line . . . . .	5
2.2	State and Use Cauchy-Riemann Equations . . . . .	5
2.3	Show that . . . . .	5
2.4	Find Mobius Transformation . . . . .	5

## Prelims

### Completing a Square

#### Definition

Given a quadratic expression  $x^2 + bx$ , we can complete the square by following these steps:

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

## 1 Question 1:

### 1.1 Sketch the region in the complex plane determined by the inequality

- $|z - 4| > 3|z + 4|$  2023 Q1(a)
- $\{z \in \mathbb{C} : |2z - 1| < 2|2z - i|\}$  2022 Q1(a), 2021 Q1(d), 2017 Q1(a), 2016 Q1(a)

### 1.2 Determine all solutions to roots of unity

- $z^6 - 1 = 0$  and factorize  $x^6 - 1$  as a product of linear and quadratic factors 2023 Q1(b),
- $z^4 = -81i$  and find a polynomial  $p(z)$  with complex coefficients with root  $w$  and  $p(\bar{w}) \neq 0$  2022 Q1(b), 2018 Q1(b)
- $z^6 - 1 = 0$  and factorize  $x^6 - 1$  as a product of linear and quadratic factors 2021 Q1(c)
- $z^3 = 1 + i$ , let  $n \in \mathbb{N}$  and  $w \neq 1$  be an  $n$ -th root of unity. Prove  $1 + w + w^2 + \dots + w^{n-1} = 0$  2016 Q1(c)

### 1.3 Determine and sketch the image under the mapping

- $w = e^z, \{z \in \mathbb{C} : \pi/4 \leq \text{Im}(z) \leq \pi/2\}$  2023 Q1(c), 2021 Q1(a), 2017 Q1(d)
- $w = \text{Log}(z), \{z : |z| > 1, 0 \leq \text{Arg}(z) \leq \pi/2\}$  2022 Q1(d), 2018 Q1(d), 2016 Q1(d)

### 1.4 Find $z$ where the function is 0

- $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$  2022 Q1(d)

### 1.5 Calculate principal value $\text{Log}(z)$

- $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$  and prove  $e^z$  is the inverse function of  $\text{Log}(z)$  2022 Q1(c), 2018 Q1(c), 2017 Q1(c)

### 1.6 Prove the following

- Define the complex conjugate ( $\bar{w}$ ) and prove if  $w$  is a zero of a polynomial  $p(z) = a_0 + a_1z + \dots + a_nz^n$  then  $\bar{w}$  is also a zero of  $p(z)$  2021 Q1(b), 2018 Q1(a), 2016 Q1(b)
- Define the complex exponential function  $e^z$  and prove Eulers Formula  $e^{i\theta} = \cos(\theta) + i\sin(\theta)$  2017 Q1(b)

**Example 2023 Q1(a)****Given**  $|z - 4| > 3|z + 4|$ **Write**  $z = x + iy$ 

$$\begin{aligned}
 |x + iy - 4| &> 3|x + iy + 4| \\
 |(x - 4) + iy| &> 3|(x + 4) + iy| \\
 \sqrt{(x - 4)^2 + y^2} &> 3\sqrt{(x + 4)^2 + y^2}
 \end{aligned}$$

**Square both sides**

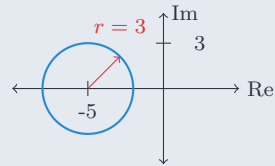
$$\begin{aligned}
 (x - 4)^2 + y^2 &> 9((x + 4)^2 + y^2) \\
 (x^2 - 8x + 16 + y^2) &> 9x^2 + 72x + 144 + 9y^2 \\
 x^2 - 8x + 16 + y^2 - 9x^2 - 72x - 144 - 9y^2 &> 0 && \text{Moving all terms to one side} \\
 -8x^2 - 80x - 8y^2 - 128 &> 0 && \text{Simplify} \\
 x^2 + 10x + y^2 - 16 &< 0 && \text{Dividing by -8 and reversing inequality}
 \end{aligned}$$

**Focus on x and complete the square**

$$\begin{aligned}
 x + bx &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \Rightarrow x^2 + 10x = (x + 5)^2 - 25 && \text{Complete the square} \\
 (x + 5)^2 - 25 + y^2 + 16 &< 0 && \text{Substitute back into inequality} \\
 (x + 5)^2 + y^2 + 9 &< 0 && \text{Simplify} \\
 (x + 5)^2 + y^2 &< -9 && \text{Subtract 9}
 \end{aligned}$$

**Recall the equation of a circle**

$$(x - a)^2 + (y - b)^2 = r^2 \Rightarrow (x + 5)^2 + y^2 < -9$$

**Therefore the region is a circle with radius 3 and center at (-5, 0)****Example 2022 Q1(a), 2021 Q1(d), 2017 Q1(a), 2016 Q1(a)****Given**  $\{z \in \mathbb{C} : |2z - 1| < 2|2z - i|\}$ **Write**  $z = x + iy$

## 2 Question 2:

### 2.1 Determine image of the line

- $f(z) = \frac{1}{z}$   $\{z \in \mathbb{C} : \operatorname{Re}(z) = 2\}$  2023 Q2(a), 2021 Q2(b)
- $f(z) = \frac{1}{z}$   $\{z \in \mathbb{C} : \operatorname{Re}(z) = 1\}$  2022 Q2(a), 2018 Q2(a), 2017 Q2(a)

### 2.2 State and Use Cauchy-Riemann Equations

- State CRE, and use to prove  $f(z) = \frac{1}{z}$  is holomorphic on  $\mathbb{C} \setminus \{0\}$  2023 Q2(a)
- State CRE, and use to prove  $f(z) = z^2$  is holomorphic on  $\mathbb{C}$  2022 Q2(b)
- State CRE. Let  $f = u + iv$  be holomorphic on  $\Omega \subset \mathbb{C}$ . Prove  $\nabla u$  and  $\nabla v$  are perpendicular of equal length 2016 Q2(b)

### 2.3 Show that

- If  $\overline{f(z)} = f(\bar{z})$  for all  $z \in \mathbb{C}$  then  $f(x)$  is real for all  $x \in \mathbb{R}$ . And if in addition  $f$  is holomorphic at  $x \in \mathbb{R}$  then  $f'(x)$  is real. 2023 Q2(c)
- Define that is meant for a function  $g$  to be harmonic. If  $f = u + iv$  is holomorphic on  $\Omega \subset \mathbb{C}$ , prove that  $v(x, y)$  is a harmonic function, and that  $\nabla u$  and  $\nabla v$  are perpendicular of equal length. 2022 Q2(c), 2018 Q2(b)
- If  $\overline{f(z)} = f(\bar{z})$  for all  $z \in \mathbb{C}$  then  $f(x)$  is real for all  $x \in \mathbb{R}$ . And if in addition  $f$  is holomorphic at 0 then the function  $f'(0)$  is real. 2021 Q2(a), 2017 Q2(c)
- Let  $f(z) = u + iv$  be holomorphic on an open subset  $\Omega$  of the complex plane and let  $h(u, v)$  be a harmonic function of  $u$  and  $v$  on  $f(\Omega)$ . Prove that  $g(x, y) = h(u(x, y), v(x, y))$  is harmonic on  $\Omega$  (You may assume  $\nabla u, \nabla v$  are equal length and perpendicular) 2021 Q2(c)
- Define what is meant for a function  $f(z)$  to be holomorphic at a point  $z_0 \in \mathbb{C}$  and prove that  $f(z) = z^2$  is holomorphic and find its derivative there. Hence prove that the product  $uv$  is harmonic where  $f = u + iv$  2018 Q2(c)
- Define what is meant for a function  $f(z)$  to be holomorphic at a point  $z_0 \in \mathbb{C}$  and prove that  $f(z) = \frac{1}{z}$  is holomorphic on  $\mathbb{C} \setminus \{0\}$  and find its derivative there (State any theorems used) 2017 Q2(b)
- Let  $h(u, v)$  be a harmonic function of  $u, v$  on  $f(\Omega)$  (See 2016 Q2(b)). Prove that  $g(x, y) = h(u(x, y), v(x, y))$  is harmonic on  $\Omega$  2016 Q2(c)

### 2.4 Find Mobius Transformation

- $T(z) : (-1, 1, \infty) \mapsto (-1, -i, 1)$  2023 Q2(d)
- $T(z) : (2, 1, -1) \mapsto (1, 0, \infty)$  2022 Q2(d)
- $T(z) : (-i, -1, 1) \mapsto (1, 0, \infty)$  and find the inverse Mobius Transformation 2021 Q2(d)
- $T(z) : (-i, -1, i) \mapsto (1, 0, \infty)$  and find the inverse Mobius Transformation 2018 Q2(d), 2017 Q2(d)
- $T(z) : (-1, \frac{1}{2}, 2) \mapsto (1, 0, \infty)$  and find the inverse Mobius Transformation 2016 Q2(d)