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MP232: Applied Mathematics

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Contents

1	Prelim : The Exponential Function and Hyperbolic Functions	2
1.1	Exponential Function	2
1.2	Hyperbolic Functions	2
1.3	Partial Fraction Decomposition	2
2	Intro to Laplace Transforms	3
2.1	What is a Laplace Transform?	3
2.2	Common Laplace Transforms	3
2.3	Linearity of the Laplace Transform	4
2.4	The First Shift Theorem	4
2.4.1	Examples	4
2.5	Existence of the Laplace Transform	5
3	Applications of Laplace Transforms	5
3.1	Integration by Parts	5
3.1.1	Examples	6
3.2	Table of Laplace Transforms	6
3.3	Laplace Transforms of Derivatives	7
3.3.1	Examples	7
3.4	Solving Initial Value Problems	8
3.4.1	Examples	8
4	Step Functions, Advanced Theorems, and Special Functions	9
4.1	Heaviside Step Function	9
4.1.1	Examples	9
4.2	The Second Shift Theorem	10
4.2.1	Examples	10
4.3	Practice Problems	10
4.4	The Dirac Delta Function	11
4.4.1	Examples	12
4.5	Differentiation of the Laplace Transform	13
4.6	The Convolution Function	13
5	Line Integrals	14
5.1	The Line Integral	14
5.1.1	Examples	15
5.2	Conervative Vector Fields	16
5.2.1	Examples	17

1 Prelim : The Exponential Function and Hyperbolic Functions

1.1 Exponential Function

Derivative

$$\frac{d}{dt}(e^{at}) = a e^{at}$$

Integral

$$\int e^{at} dt = \frac{1}{a} e^{at} + C$$

1.2 Hyperbolic Functions

Definitions:

$$\sinh(at) = \frac{e^{at} - e^{-at}}{2} \quad \bigg| \quad \cosh(at) = \frac{e^{at} + e^{-at}}{2} \quad \bigg| \quad \tanh(at) = \frac{\sinh(at)}{\cosh(at)}.$$

Derivatives

$$\frac{d}{dt}(\sinh(at)) = a \cosh(at), \quad \bigg| \quad \frac{d}{dt}(\cosh(at)) = a \sinh(at), \quad \bigg| \quad \frac{d}{dt}(\tanh(at)) = a \operatorname{sech}^2(at).$$

Integrals

$$\begin{aligned} \int \sinh(at) dt &= \frac{1}{a} \cosh(at) + C \\ \int \cosh(at) dt &= \frac{1}{a} \sinh(at) + C, \\ \int \tanh(at) dt &= \frac{1}{a} \ln|\cosh(at)| + C. \end{aligned}$$

Common Identities

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1, \\ \sinh(2x) &= 2 \sinh x \cosh x, \\ \cosh(2x) &= \cosh^2 x + \sinh^2 x, \\ \tanh(2x) &= \frac{2 \tanh x}{1 + \tanh^2 x}. \end{aligned}$$

1.3 Partial Fraction Decomposition

Unrepeated Linear Factors: A linear factor is of form $(ax + b)$

$$\frac{s+1}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

Repeated Linear Factors:

$$\frac{3}{(s+2)^2(s-3)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-3}$$

Unrepeated Quadratic Factors with complex roots: Where the discriminant $(b^2 - 4ac)$ is negative (complex roots) but the factor is not repeated

$$\frac{3}{(s^2 - s + 1)(s + 2)} = \frac{As + B}{s^2 - s + 1} + \frac{C}{s + 2}$$

Repeated Quadratic Factors with complex roots:

$$\frac{1}{(s^2 + 1)^2(s - 1)} = \frac{As + B}{(s^2 + 1)^2} + \frac{Cs + D}{s^2 + 1} + \frac{E}{s - 1}$$

2 Intro to Laplace Transforms

2.1 What is a Laplace Transform?

Definition

The Laplace Transform, defined for $t \geq 0$, is given by

$$L\{f(t)\}(s) = F(s) = \int_0^{\infty} e^{-st} dt$$

2.2 Common Laplace Transforms

Example Find the Laplace Transform of $f(t) = 1$

We have:

$$L\{1\} = \int_0^{\infty} 1 \cdot e^{-st} dt = \lim_{R \rightarrow \infty} \int_0^R e^{-st} dt$$

This integral is equal to:

$$\int_0^R e^{-st} dt = \left. \frac{e^{-st}}{-s} \right|_{t=0}^{t=R} = -\frac{1}{s} [e^{-sR} - 1] = \frac{1 - e^{-sR}}{s}$$

Taking the limit as $R \rightarrow \infty$ gives:

$$L\{1\} = \lim_{R \rightarrow \infty} \frac{1 - e^{-sR}}{s} = \frac{1}{s}$$

Example Find the Laplace Transform of $f(t) = e^{2t}$

$$\begin{aligned} L\{e^{2t}\} &= \int_0^{\infty} e^{2t} e^{-st} dt = \int_0^{\infty} e^{-(s-2)t} dt \\ &= \lim_{R \rightarrow \infty} \int_0^R e^{-(s-2)t} dt \\ &= \lim_{R \rightarrow \infty} \left[\frac{e^{-(s-2)t}}{-(s-2)} \right]_{t=0}^{t=R} \\ &= \lim_{R \rightarrow \infty} \left(\frac{e^{-(s-2)R} - e^0}{-(s-2)} \right) = \lim_{R \rightarrow \infty} \left(\frac{e^{-(s-2)R} - 1}{-(s-2)} \right) \\ &= \frac{1}{s-2} \quad (\text{since } e^{-(s-2)R} \rightarrow 0 \text{ as } R \rightarrow \infty \text{ provided } s > 2) \end{aligned}$$

Example Find the Laplace Transform of $f(t) = \cosh(at)$

We have:

$$\begin{aligned} L\{\cosh(at)\} &= L\left\{ \frac{e^{at} + e^{-at}}{2} \right\} \quad \text{from the definition of } \cosh(at) \\ &= \frac{1}{2} L\{e^{at}\} + \frac{1}{2} L\{e^{-at}\} \quad \text{by linearity of the Laplace Transform} \\ &= \frac{1}{2} \left(\frac{1}{s-a} \right) + \frac{1}{2} \left(\frac{1}{s+a} \right) \end{aligned}$$

Hence:

$$L\{\cosh(at)\} = \frac{s}{s^2 - a^2}$$

Noting that $\sinh(at) = (e^{at} - e^{-at})/2$, we can find that:

$$L\{\sinh(at)\} = \frac{a}{(s^2 - a^2)}$$

Example Find the Laplace Transform of $\cos(wt)$ and $\sin(wt)$ where w is a constant.

We first compute the Laplace Transform of e^{iwt} using its definition:

$$L\{e^{iwt}\} = \int_0^{\infty} e^{-st} e^{iwt} dt = \int_0^{\infty} e^{-(s-iw)t} dt = \frac{1}{s-iw}, \quad \text{for } \Re(s) > 0.$$

To express this in terms of real and imaginary parts, we multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{s-iw} = \frac{s+iw}{(s-iw)(s+iw)} = \frac{s+iw}{s^2+w^2}.$$

Since Euler's formula gives:

$$e^{iwt} = \cos(wt) + i\sin(wt),$$

the linearity of the Laplace Transform yields:

$$L\{e^{iwt}\} = L\{\cos(wt)\} + iL\{\sin(wt)\}.$$

Equating the two representations of $L\{e^{iwt}\}$, we have:

$$L\{\cos(wt)\} + iL\{\sin(wt)\} = \frac{s+iw}{s^2+w^2}.$$

Since the equality must hold for both the real and imaginary parts, we equate them separately:

$$L\{\cos(wt)\} = \frac{s}{s^2+w^2} \quad \text{and} \quad L\{\sin(wt)\} = \frac{w}{s^2+w^2}.$$

2.3 Linearity of the Laplace Transform

The Laplace Transform is a linear operator, i.e. for any constants a and b :

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

Proof

$$\begin{aligned} L\{af(t) + bg(t)\} &= \int_0^{\infty} e^{-st} (af(t) + bg(t)) dt \\ &= a \int_0^{\infty} e^{-st} f(t) dt + b \int_0^{\infty} e^{-st} g(t) dt \\ &= aL\{f(t)\} + bL\{g(t)\} \end{aligned}$$

2.4 The First Shift Theorem**Theorem First Shift Theorem**

If $f(t)$ has a Laplace Transform, $F(s)$, defined for $s > k$, then $e^{at} f(t)$ has a Laplace Transform, $F(s-a)$ defined for $s-a > k$ and is given by:

$$L\{e^{at} f(t)\} = F(s-a)$$

or, taking the inverse Laplace Transform of both sides:

$$e^{at} f(t) = L^{-1}\{F(s-a)\}$$

2.4.1 Examples**Example Find the Laplace Transform of $e^{at} \cos(wt)$, where a, w are constants.**

We know that $L\{\cos(wt)\} = \frac{s}{s^2+w^2}$, so by the First Shift Theorem:

$$\begin{aligned} L\{e^{at} \cos(wt)\} &= \frac{s-a}{(s-a)^2+w^2} \\ &= \frac{s-a}{s^2-2as+a^2+w^2} \end{aligned}$$

2.5 Existence of the Laplace Transform

Existence of a Laplace transform is not always guaranteed because we're integrating over an infinite integral. For a Laplace Transform to exist for a given s , then the integral must exist:

$$\int_0^{\infty} e^{-st} f(t) dt$$

Theorem Existence Theorem of Laplace Transforms

Suppose $f(t)$ is a piecewise continuous function on $[0, \infty)$. If $f(t)$ satisfies:

$$|f(t)| \leq M e^{kt} \quad (0 \leq t < \infty)$$

for some constants, M, k , then the Laplace Transform of $f(t)$ exists for $s > k$. In other words, the Laplace Transform of $f(t)$ exists if $f(t)$ is bounded by an exponential function.

Proof

If $s > k$, then from the equation above, we have:

$$|F(s)| = \left| \int_0^{\infty} f(t) e^{-st} dt \right| \leq \int_0^{\infty} |f(t)| e^{-st} dt \leq \int_0^{\infty} M e^{(k-s)t} dt = \frac{M}{s-k}$$

3 Applications of Laplace Transforms

3.1 Integration by Parts

Starting with the product rule:

$$\frac{d}{dx}[uv] = u'v + uv',$$

we can express this in differential form as:

$$d(uv) = u dv + v du.$$

Integrate both sides with respect to x :

$$\int d(uv) = \int_a^b u dv + \int_a^b v du.$$

The Fundamental Theorem of Calculus tells us that the left-hand side is simply:

$$uv = \int_a^b u dv + \int_a^b v du.$$

Rearrange to solve for the desired integral:

$$\int_a^b u dv = uv - \int_a^b v du,$$

3.1.1 Examples

Example Use integration by parts to find the Laplace of $f(t) = t$

$$L\{t\} = \int_0^{\infty} te^{-st} dt$$

We integrate by parts by setting:

$$u = t, \quad dv = e^{-st}, \quad du = 1, \quad v = -\frac{e^{-st}}{s}$$

Then integrating by parts gives:

$$\begin{aligned} L\{t\} &= \left[-\frac{te^{-st}}{s} \right]_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt \\ &= 0 + \frac{1}{s} \left[-\frac{e^{-st}}{s} \right]_0^{\infty} \end{aligned}$$

Hence:

$$L\{t\} = \frac{1}{s^2}$$

Example Use integration by parts to find the Laplace of $f(t) = \cos(t)$

Let:

$$u = e^{-st}, \quad du = -se^{-st}, \quad dv = \cos(t), \quad v = \sin(t)$$

Then:

$$\int_0^{\infty} e^{-st} \cos(t) dt = \left[e^{-st} \sin(t) \right]_0^{\infty} + \int_0^{\infty} \sin(t) \cdot se^{-st} dt = 0 + s \int_0^{\infty} e^{-st} \sin(t) dt$$

Considering the sin part :

$$u = e^{-st}, \quad du = -se^{-st}, \quad dv = \sin(t), \quad v = -\cos(t)$$

$$\int_0^{\infty} e^{-st} \sin(t) dt = 1 - s \int_0^{\infty} e^{-st} \cos(t) dt$$

Substituting this back into the original integral gives:

$$\int_0^{\infty} e^{-st} \cos(t) dt = 1 - s \int_0^{\infty} e^{-st} \cos(t) dt = s - s^2 \int_0^{\infty} e^{-st} \cos(t) dt$$

$$L\{\cos(t)\} = \frac{s}{1 + s^2}$$

3.2 Table of Laplace Transforms

$f(t)$	$L\{f(t)\}$
1	$\frac{1}{s}, s > 0$
t	$\frac{1}{s^2}, s > 0$
$t^n, n = 0, 1, 2, 3$	$\frac{n!}{s^{n+1}}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh(at)$	$\frac{s}{s^2 - a^2}, s > a \geq 0$
$\sinh(at)$	$\frac{a}{s^2 - a^2}, s > a \geq 0$
$e^{at} \cos(\omega t)$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$e^{at} \sin(\omega t)$	$\frac{\omega}{(s-a)^2 + \omega^2}$
$e^{at} f(t)$	$F(s-a)$

3.3 Laplace Transforms of Derivatives

Theorem Laplace Transform of Derivatives

Suppose that $f(t)$ and $f'(t)$ are continuous and that $|f(t)| \leq Me^{kt}$, $\forall t \geq 0$ and for constants M, k . Then the Laplace Transform of $f'(t)$ exists for $s > k$ and is given by:

$$L\left\{\frac{df}{dt}\right\} = sL\{f\} - f(0) \quad \text{for } s > k$$

We can easily extend this to higher order derivatives. Assume the Laplace Transform of $f^{(n)}(t)$ exists for $s > k$ and is given by:

$$L\left\{\frac{d^n f}{dt^n}\right\} = s^n L\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0) \quad \text{for } s > k$$

3.3.1 Examples

Example Find $L\{t^2\}$ using the fact $L\{s\} = 1/s$ for $s > 0$

$$L\{f''\} = s^2 L\{f\} - sf(0) - f'(0)$$

With $f(t) = t^2$. Since $f'(t) = 2t$, $f''(t) = 2$, $f'(0) = 0$, $f(0) = 0$, gives:

$$L\{2\} = s^2 L\{t^2\} - s \cdot 0 - 0$$

So that:

$$L\{t^2\} = \frac{L\{2\}}{s^2} = \frac{2}{s^3}$$

Example Find $L\{\sin(t)\}$ and $L\{\cos(t)\}$

We again use the equation:

$$L\{f''\} = s^2 L\{f\} - sf(0) - f'(0)$$

With $f(t) = \sin(t)$, $f'(t) = \cos(t)$, $f''(t) = -\sin(t)$, $\sin(0) = 0$, $\cos(0) = 1$. This gives:

$$L\{-\sin(t)\} = s^2 L\{\sin(t)\} - s \cdot 0 - 1$$

So that:

$$L\{\sin(t)\} = \frac{1}{s^2 + 1}$$

Similarly, we can find:

$$L\{\cos(t)\} = \frac{s}{s^2 + 1}$$

3.4 Solving Initial Value Problems

Consider an example from mechanics: A particle of mass $m > 0$ lies on rough table, attached to a spring of stiffness $k > 0$. At any time $t > 0$, the mass is a distance $x(t)$ from the equilibrium position O , and $x(t)$ is much less than the length of the spring.

The mass is subject to a driving force $F_d(t)$, from Newton's second law, we have:

$$F_d(t) - kx - \gamma \frac{dx}{dt} = m \frac{dx^2}{dt^2}$$

Where $\gamma > 0$ is the **damping constant** and the term $\gamma \frac{dx}{dt}$ models the **friction due to roughness** of the table, which opposes direction of motion. The **restoring force** due to the spring is $-kx$; and always points towards O . The term $m \frac{dx^2}{dt^2}$ is the **acceleration of the mass**. We can rewrite this as:

$$F_d(t) = m \frac{dx^2}{dt^2} + \gamma \frac{dx}{dt} + kx$$

In order to solve this, we also need initial displacement $v_0 = x(0)$ and initial velocity $v_0 = \frac{dx}{dt}(0)$.

3.4.1 Examples

Example

$$\frac{dx^2}{dt^2} + 3 \frac{dx}{dt} + 2x = 0, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1$$

1. Take Laplace of governing equation:

$$L \left\{ \frac{dx^2}{dt^2} \right\} = s^2 L\{x\} - sx(0) - x'(0) = s^2 L\{x\} - 1$$

$$L \left\{ \frac{dx}{dt} \right\} = sL\{x\} - x(0) = sLx$$

Hence:

$$s^2 L\{x\} - 1 + 3sL\{x\} + 2L\{x\} = 0$$

This is known as the **subsidiary equation**. Rearranging:

$$(s^2 + 3s + 2)L\{x\} = 1$$

2. Solve the subsidiary equation:

$$L\{x\} = \frac{1}{s^2 + 3s + 2}$$

3. Find the inverse Laplace Transform:

$$x(t) = L^{-1} \left\{ \frac{1}{s^2 + 3s + 2} \right\} = L^{-1} \left\{ \frac{1}{(s+1)(s+2)} \right\}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$$

Hence:

$$A(s+2) + B(s+1) = 1 \quad \rightarrow A = 1, B = -1$$

Thus:

$$x = L^{-1} \left\{ \frac{1}{s+1} - \frac{1}{s+2} \right\} = L^{-1} \left\{ \frac{1}{s+1} \right\} - L^{-1} \left\{ \frac{1}{s+2} \right\} = e^{-t} - e^{-2t}$$

4 Step Functions, Advanced Theorems, and Special Functions

4.1 Heaviside Step Function

Denote the Heaviside Step Function as $H(t)$, defined as:

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Clearly, for any constant a , we have:

$$H(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

4.1.1 Examples

Example Express this function in terms of a Heaviside step function

$$f(t) = \begin{cases} 3 & \text{for } 0 \leq t < 2 \\ -1 & \text{for } t > 2 \end{cases}$$

We write the general form of $f(t)$ as:

$$f(t) = \alpha + \beta H(t - 2)$$

Now, setting t to any value $\in [0, 2)$ gives:

$$f(t) = \alpha + (\beta)(0) = 0 \Rightarrow \alpha = 3$$

Setting t to any value greater than 2 gives:

$$f(t) = 3 + (\beta)(1) = -1 \Rightarrow \beta = -4$$

Thus, we have:

$$f(t) = 3 - 4H(t - 2)$$

Example Express this function in terms of a Heaviside step function

$$f(t) = \begin{cases} 3 & \text{for } 0 \leq t < 1 \\ 5 & \text{for } 1 < t < 3 \\ -2 & \text{for } t > 3 \end{cases}$$

We write the general form of $f(t)$ as:

$$f(t) = \alpha + \beta H(t - 1) + \gamma H(t - 3)$$

Setting t to any value $\in [0, 1)$ gives: $f(t) = \alpha + (\beta)(0) + (\gamma)(0) = 0 \Rightarrow \alpha = 3$

Setting t to any value $\in (1, 3)$ gives: $f(t) = 3 + (\beta)(2) + (\gamma)(0) = 5 \Rightarrow \beta = 2$

Setting t to any value greater than 3 gives: $f(t) = 3 + 2 + (\gamma)(1) = -7 \Rightarrow \gamma = -5$ Thus, we have:

$$f(t) = 3 + 2H(t - 1) - 5H(t - 3)$$

4.2 The Second Shift Theorem

Theorem

If $f(t)$ has the transform $F(s)$ ($s > k$) then the shifted function,

$$\tilde{f}(t) = f(t-a)H(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

has the transform $e^{-as}F(s)$ ($s > k$), that is:

$$L\{f(t-a)H(t-a)\} = e^{-as}F(s)$$

Proof

$$L\{f(t-a)H(t-a)\} = \int_0^\infty e^{-st} f(t-a)H(t-a) dt = \int_a^\infty e^{-st} f(t-a) dt$$

We introduce a new integration variable $\tau = t - a$, we have

$$\int_0^\infty e^{-s(\tau+a)} f(\tau) d\tau = e^{-as} \int_0^\infty e^{-s\tau} f(\tau) d\tau = e^{-as}F(s)$$

4.2.1 Examples

Example : Find the Laplace Transform of $H(t-a)$ for $a > 0$

$$L\{H(t-a)\} = L\{H(t-a)f(t-a)\} = e^{-as}F(s) = \frac{e^{-as}}{s}$$

4.3 Practice Problems

1. Use the First Shift Theorem ($L\{e^{at}f(t)\} = F(s-a)$) to find the Laplace transform of the following functions:

(a) t^3e^{-3t} (b) $e^{-t}\cos(2t)$ (c) $e^{-4t}\cosh(5t)$ (d) $e^{-t}\sin^2(t)$

2. Use the First Shift Theorem ($L^{-1}\{F(s-a)\} = e^{at}f(t)$) to find the inverse Laplace transform of the following functions:

(a) $\frac{6s-4}{s^2-4s+20}$ (b) $\frac{3s+7}{s^2-2s-3}$ (c) $\frac{4s+12}{s^2+8s+16}$

3. Solve the following initial value problems using the method of Laplace transforms:

$$y'' + y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = 1;$$

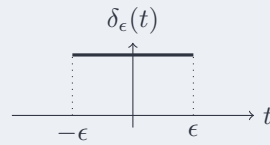
$$y'' - y = t, \quad y(0) = 1, \quad y'(0) = 1.$$

4. Find the inverse Laplace transform of the following functions using the method of partial fractions:

(a) $\frac{2s^2-4}{(s+1)(s-2)(s-3)}$. (b) $\frac{5s^2-15s-11}{(s+1)(s-2)^3}$. (c) $\frac{3s+1}{(s-1)(s^2+1)}$. (d) $\frac{e^{-5s}}{(s^2+1)(s^2+2)}$.

4.4 The Dirac Delta Function

The **Dirac Delta Function** models extremely brief but intense forces like a hammer hitting a nail. It starts as a function δ_ϵ , that equals $\frac{1}{2\epsilon}$ over the interval $t \in [-\epsilon, \epsilon]$ and 0 elsewhere.



$$\delta_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon} & \text{if } t \in [-\epsilon, \epsilon], \\ 0 & \text{otherwise.} \end{cases}$$

This function creates a rectangular pulse with the following properties:

$$\text{Height: } \frac{1}{2\epsilon} \quad \text{Width: } 2\epsilon \quad \text{Area: } 1 \text{ (always)}$$

As ϵ approaches 0, the function becomes infinitely tall and thin, but the area remains 1. This limit defines the Dirac Delta Function:

$$\delta(t) = \lim_{\epsilon \rightarrow 0^+} \{\delta_\epsilon(t)\}$$

Properties of the Dirac Delta Function:

$$\begin{aligned} \delta(t) &= 0 \text{ for } t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t) dt &= 1 \\ \int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt &= f(t_0) \end{aligned}$$

The Laplace Transform of the Dirac Delta Function is:

$$L\{\delta(t - t_0)\} = \int_0^{\infty} e^{-st} \delta(t - t_0) dt = \int_{-\infty}^{\infty} e^{-st} \delta(t - t_0) dt = e^{-st_0} \quad \text{for } t_0 > 0$$

4.4.1 Examples

Example : Solve the following initial value problem which governs the behaviour of an RLC circuit

$$\begin{aligned} LQ'' + RQ' + \frac{Q}{C} &= V_0\delta(t-a) \\ Q(0) &= 0 \\ Q'(0) &= 0 \end{aligned}$$

Where a, L, R, C, V_0 are all positive constants and $4L > R^2C$.
Note that the applied voltage corresponds to an impulse of strength V_0 at $t = a$
We note that:

$$\begin{aligned} L\{Q''\} &= s^2L\{Q\} - sQ(0) - Q'(0) = s^2L\{Q\} \\ L\{Q'\} &= sL\{Q\} - Q(0) = sL\{Q\} \\ L\{\delta(t-a)\} &= e^{-st_0} = e^{-as} \end{aligned}$$

Thus:

$$L\{LQ'' + RQ' + \frac{Q}{C}\} = V_0\delta(t-a) = Ls^2L\{Q\} + RsL\{Q\} + \frac{1}{C}L\{Q\} = V_0e^{-as}$$

Grouping terms:

$$L\{Q\}(Ls^2 + Rs + \frac{1}{C}) = V_0e^{-as}$$

Hence:

$$L\{Q\} = V_0e^{-as} \cdot \frac{1}{Ls^2 + Rs + \frac{1}{C}}$$

Removing the L from the denominator gives:

$$\begin{aligned} L\{Q\} &= \frac{V_0}{L}e^{-as} \cdot \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \\ &= \frac{V_0}{L} \cdot \frac{e^{-as}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \end{aligned}$$

We notice that:

$$\begin{aligned} (s + \frac{R}{2L})^2 &= s^2 + s\frac{2R}{2L} + \frac{R^2}{4L^2} \\ &= s^2 + \frac{R}{L}s + \frac{R^2}{4L^2} \end{aligned}$$

So that:

$$L\{Q\} = \frac{V_0}{L} \frac{e^{-as}}{(s + \frac{R}{2L})^2 - \frac{R^2}{4L^2} + \frac{1}{LC}}$$

Rewriting with $\alpha = \frac{R}{2L}$ and $\beta = \frac{1}{LC} - \frac{R^2}{4L^2}$

$$L\{Q\} = \frac{V_0}{L} \frac{e^{-as}}{(s + \alpha)^2 + \beta}$$

We also note that:

$$L\{\sin(\beta t)\} = \frac{\beta}{s^2 + \beta^2} \xrightarrow{\text{First Shift Theorem}} L\{e^{-as}\sin(\beta t)\} = \frac{B}{(s + a)^2 + \beta^2}$$

Or,

$$L^{-1}\left\{\frac{B}{(s + a)^2 + \beta^2}\right\} = e^{-at}\sin(\beta t)$$

We can also write:

$$L^{-1}\{F(s)\} = f(t)$$

Notice that:

$$f(t-a) = e^{-a(t-a)}\sin(\beta[t-a])$$

Applying the Second Shift Theorem gives:

$$\begin{aligned} L^{-1}\{e^{-a}F(s)\} &= f(t-a)H(t-a) \\ L^{-1}\left\{e^{-as}\frac{\beta}{(s + a)^2 + \beta^2}\right\} &= e^{-a(t-a)}\sin(\beta[t-a])H(t-a) \end{aligned}$$

Thus:

$$\begin{aligned} Q(t) &= \frac{V_0}{L\beta}e^{-a(t-a)}\sin(\beta[t-a])H(t-a) \\ &= \begin{cases} 0 & \text{for } 0 \leq t < a \\ \frac{V_0}{L\beta}e^{-a(t-a)}\sin(\beta[t-a])H(t-a) & \text{for } t > a \end{cases} \end{aligned}$$

4.5 Differentiation of the Laplace Transform

Suppose $f(t), t \geq 0$ satisfies the conditions of the existence theorem so that its Laplace Transform ($F(s)$) exists for some $s > k$. Then:

$$F'(s) = \frac{d}{ds} \left\{ \int_0^\infty e^{-st} f(t) dt \right\} = \int_0^\infty \frac{\partial}{\partial s} \{ e^{-st} f(t) \} dt$$

We are allowed to bring the derivative inside the integral provided the conditions of the existence theorem are satisfied, hence:

$$F'(s) = - \int_0^\infty e^{-st} \{ t f(t) \} dt = -L\{ t f(t) \}$$

so that,

$$L\{ t f(t) \} = -F'(s)$$

We can sometimes use this to calculate transforms and inverse transforms. For example:

$$L\{ t \} = L\{ t \cdot 1 \} = -\frac{d}{ds} L\{ 1 \} = -\frac{d}{ds} \left(\frac{1}{s} \right) = \frac{1}{s^2}$$

4.6 The Convolution Function

Let $f(t), g(t)$ be two functions. Define the Convolution function

$$(f \star g)(t) = \int_0^t f(t - \tau) g(\tau) d\tau$$

Where τ is integrated over the interval $[0, t]$. The Convolution is:

$$\textbf{Commutative: } f \star g = g \star f$$

$$\textbf{Associative: } f \star (g \star h) = (f \star g) \star h$$

$$\textbf{Distributive: } f \star (g + h) = f \star g + f \star h$$

$$\textbf{Multiplication by 0: } f \star (ag) = a(f \star g)$$

Theorem

Let $f(t)$ and $g(t)$ have Laplace Transforms $F(s)$ and $G(s)$ respectively defined for $s > k \geq 0$. Then

$$L\{ f \star g \} = F(s)G(s), \quad s > k$$

Proof

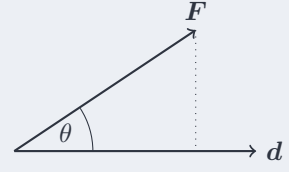
Write $F(s) = \int_0^\infty e^{-s\sigma} f(\sigma) d\sigma$ and $G(s) = \int_0^\infty e^{-s\tau} g(\tau) d\tau$. Then:

$$\begin{aligned} F(s)G(s) &= \left\{ \int_0^\infty e^{-s\sigma} f(\sigma) d\sigma \right\} \left\{ \int_0^\infty e^{-s\tau} g(\tau) d\tau \right\} \\ &= \int_0^\infty e^{-s\tau} g(\tau) \left\{ \int_0^\infty e^{-s\sigma} f(\sigma) d\sigma \right\} d\tau \\ &= \int_0^\infty g(\tau) \left\{ \int_0^\infty e^{-s(\sigma+\tau)} f(\sigma) d\sigma \right\} d\tau. \end{aligned}$$

5 Line Integrals

Consider a mass which undergoes a displacement, \mathbf{d} , under a constant force \mathbf{F} . Define the work, \mathbf{W} , done by \mathbf{F} to be the magnitude of the force multiplied by the distance moved in the direction of the force. Inspecting the diagram, we see that work done \mathbf{W} is given by the dot product of \mathbf{F} and \mathbf{d} :

$$W = |\mathbf{F}| \cdot |\mathbf{d}| \cdot \cos(\theta) = \mathbf{F} \cdot \mathbf{d}$$



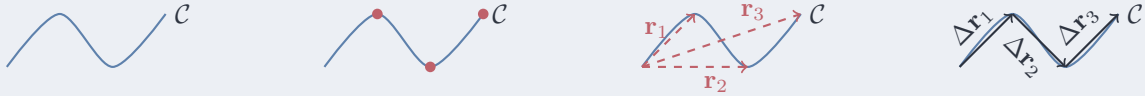
Now, let's suppose F is not constant:

$$F = F(x, y, z) = F(r) = r(x, y, z)$$

Suppose further, that F acts for a time $t_1 \leq t \leq t_2$ and the path of the object in this time interval is given by a curve \mathcal{C} defined by:

$$\mathbf{r} = (x(t), y(t), z(t)) \quad t \in [t_1, t_2]$$

But how do we calculate the work done by F along \mathcal{C} ?



As seen in the diagram above, we can divide \mathcal{C} into a large number $N - 1$ of small segments $\Delta \mathbf{r}_i$ and approximate the work done by F along \mathcal{C} by the sum of the work done along each segment:

$$W \approx \sum_{i=1}^{N-1} F(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i$$

5.1 The Line Integral

Taking the limit $N \rightarrow \infty$

$$W = \lim_{N \rightarrow \infty} \left\{ \sum_{i=1}^{N-1} F(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i \right\}$$

This limit is called the **line integral** of F along \mathcal{C} and is denoted by $\int_{\mathcal{C}} F(r) \cdot d\mathbf{r}$, that is:

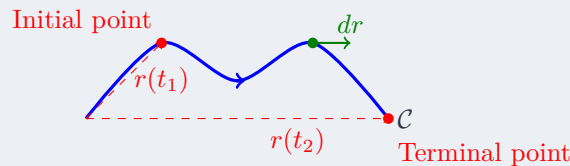
$$\int_{\mathcal{C}} F(r) \cdot d\mathbf{r} = \lim_{N \rightarrow \infty} \left\{ \sum_{i=1}^{N-1} F(r_i) \cdot \Delta r_i \right\}$$

Since $r = r(t)$ we can calculate the line integral as:

$$\int_{\mathcal{C}} F(r) \cdot d\mathbf{r} = \int_{t_1}^{t_2} F(r(t)) \cdot \frac{d\mathbf{r}}{dt} dt$$

In general, t , may be any variable that parametrizes (traces out) the curve \mathcal{C} . Then $d\mathbf{r} = \frac{d\mathbf{r}}{dt} dt$ is the tangent vector to \mathcal{C} at the point $r(t)$. We call \mathcal{C} the **path of integration** and $r(t_1)$ the initial point, $r(t_2)$ the **terminal point**. \mathcal{C} is now **oriented** from $r(t_1)$ to $r(t_2)$.

The direction for $r(t_1) \rightarrow r(t_2)$, in which t increases, is called the **positive direction** of \mathcal{C} , we indicate this by an arrow on \mathcal{C} .



If $r(t_1) = r(t_2)$ then \mathcal{C} is a **closed curve** and the line integral is denoted by:

$$\oint_{\mathcal{C}} F(r) \cdot d\mathbf{r}$$

The line integral of F along a closed curve \mathcal{C} is called the **circulation** of F around \mathcal{C} .

5.1.1 Examples

Example : For a time period $0 \leq t \leq 1$, a particle moves along a trajectory defined by $\mathcal{C} = x = t, y = t, z = 2t^2$, a force $F(r) = (y, x, z)$ acts. Calculate work done.

We have:

$$\begin{aligned}\mathbf{r} &= (t, t, 2t^2) \\ \frac{d\mathbf{r}}{dt} &= (1, 1, 4t) \\ F(\mathbf{r}) &= (t, t, 2t^2)\end{aligned}$$

The work done is:

$$\begin{aligned}\int_{\mathcal{C}} F(r) \cdot dr &= \int_0^1 (t, t, 2t^2) \cdot (1, 1, 4t) dt \\ &= \int_0^1 (t + t + 8t^3) dt \\ &= \int_0^1 (2t + 8t^3) dt \\ &= [t^2 + 2t^4]_0^1 \\ &= 1 + 2 \\ &= 3\end{aligned}$$

Example Evaluate $\int_{\mathcal{C}} F(r) \cdot dr$ for $F(r) = (y + z, x + z, x + y)$, where \mathcal{C} is the line segment jointint $A : (-2, 2, -3) \rightarrow B(2, 4, 6)$

We can parametrize any straight line in three dimensions by writing:

$$x = c_1 t + c_2 \quad y = c_3 t + c_4 \quad z = c_5 t + c_6$$

where the c_i are constants.

We can take $t = 0$ to correspond with A and $t = 1$ to correspond with B .

Hence setting $t = 0$ gives:

$$c_2 = -2 \quad c_4 = 2 \quad c_6 = -3$$

Setting $t = 1$ gives: $c_1 + c_2 = 2 \quad c_3 + c_4 = 4 \quad c_5 + c_6 = 6$ so that:

$$c_1 = 4 \quad c_3 = 2 \quad c_5 = 9$$

Hence, the required line is given by:

$$x = 4t - 2 \quad y = 2t + 2 \quad z = 9t - 3$$

We have:

$$\begin{aligned}\int_{\mathcal{C}} F(r) \cdot dr &= \int_0^1 (11t - 1, 13t - 5, 6t) \cdot \frac{d}{dt}(4t - 2, 2t + 2, 9t - 3) dt \\ &= \int_0^1 (11t - 1, 13t - 5, 6t) \cdot (4, 2, 9) dt \\ &= \int_0^1 (44t - 4 + 26t - 10 + 54t) dt \\ &= \int_0^1 (124t - 12) dt \\ &= [62t^2 - 12t]_0^1 \\ &= 48\end{aligned}$$

Example Find the circulation of the vector $F = (y, -x, 0)$ around the unit circle $\mathcal{C} = x^2 + y^2 = 1, z = 0$, taken in an anti-clockwise direction

We can parametrize the unit circle by setting x, y and z to:

$$x = \cos(t) \quad y = \sin(t) \quad z = 0$$

We can write the position vector as:

$$\mathbf{r} = (\cos(t), \sin(t), 0)$$

The tangent vector is:

$$\frac{d\mathbf{r}}{dt} = (-\sin(t), \cos(t), 0)$$

The force vector is:

$$F(\mathbf{r}) = (\sin(t), -\cos(t), 0)$$

The circulation is:

$$\oint_{\mathcal{C}} F(r) \cdot dr = \int_0^{2\pi} (\sin(t), -\cos(t), 0) \cdot (-\sin(t), \cos(t), 0) dt \quad (1)$$

$$= \int_0^{2\pi} -\sin^2(t) - \cos^2(t) dt \quad (2)$$

$$= - \int_0^{2\pi} (\sin^2(t) + \cos^2(t)) dt \quad (3)$$

$$= - \int_0^{2\pi} 1 dt \quad (4)$$

$$= -[t]_0^{2\pi} \quad (5)$$

$$= -2\pi \quad (6)$$

5.2 Conservative Vector Fields

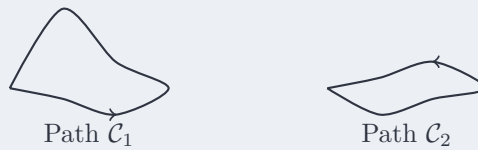
A vector field F is called **conservative** if the line integral of F along any closed curve \mathcal{C} is zero, that is:

$$\oint_{\mathcal{C}} F(r) \cdot dr = 0$$

An equivalent definition is that F is conservative if the line integral of F depends only on the end points of the curve, not on the path taken, so that:

$$\int_{\mathcal{C}_1} F(r) \cdot dr = \int_{\mathcal{C}_2} F(r) \cdot dr$$

Where \mathcal{C}_1 and \mathcal{C}_2 are two curves with the same initial and terminal points but different paths.



Consider two curves, \mathcal{C}_1 and \mathcal{C}_2 , that start at A and end at B . Let \mathcal{C} be the closed curve that starts at A follows the curve \mathcal{C}_1 and then follows \mathcal{C}_2 in the reverse direction to B . Then:

$$\begin{aligned} \oint_{\mathcal{C}} F(r) \cdot dr &= \int_{AC_1}^B F(r) \cdot dr + \int_{BC_2}^A F(r) \cdot dr \\ &= \int_{AC_1}^B F(r) \cdot dr - \int_{AC_2}^B F(r) \cdot dr = 0 \end{aligned}$$

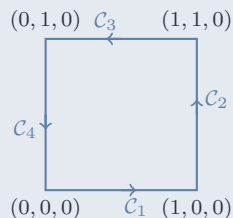
Thus:

$$\oint_{\mathcal{C}} F(r) \cdot dr = 0 \Rightarrow \int_{AC_1}^B F(r) \cdot dr = \int_{AC_2}^B F(r) \cdot dr$$

5.2.1 Examples

Example By considering the line integral of $F = (y, x^2 - x, 0)$ around the square C in the x, y plane connecting for point $(0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 1, 0)$, show that F cannot be conservative.

Split C into four segments, C_1, C_2, C_3, C_4 and calculate the line integral of F along each segment.



We have:

$$\begin{aligned}\int_{C_1} F(r) \cdot dr &= \int_0^1 (0, t^2 - t, 0) \cdot (1, 0, 0) dt = 0 \\ \int_{C_2} F(r) \cdot dr &= \int_0^1 (t, 1 - t, 0) \cdot (0, 1, 0) dt = \int_0^1 (1 - t) dt = [t - \frac{t^2}{2}]_0^1 = 1 - \frac{1}{2} = \frac{1}{2} \\ \int_{C_3} F(r) \cdot dr &= \int_0^1 (1, (1 - t)^2 - (1 - t), 0) \cdot (-1, 0, 0) dt = \int_0^1 -1 dt = -1 \\ \int_{C_4} F(r) \cdot dr &= \int_0^1 (1 - t, 0, 0) \cdot (0, -1, 0) dt = 0\end{aligned}$$

Hence:

$$\oint_C F(r) \cdot dr = \int_{C_1} F(r) \cdot dr + \int_{C_2} F(r) \cdot dr + \int_{C_3} F(r) \cdot dr + \int_{C_4} F(r) \cdot dr = 0 + \frac{1}{2} + (-1) + 0 = -\frac{1}{2} \neq 0$$

Thus, F is not conservative.