MA2287: Complex Analysis Exam Notes

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Contents

1 Question 1:

1.1 Sketch the region in the complex plane determined by the inequality

- $|z - 4| > 3|z + 4|$
- $\{z \in \mathbb{C} : |2z - 1| < 2|2z - i|\}$

1.2 Determine all solutions to roots of unity

- $z^6 - 1 = 0$ and factorize $x^6 - 1$ as a product of linear and quadratic factors
- $z^4 = -81i$ and find a polynomial $p(z)$ with complex coefficients with root w and $p(\overline{w}) \neq 0$

1.3 Determine and sketch the image under the mapping

- $w = e^z$, $\{z \in \mathbb{C} : \pi/4 \leq \text{Im}(z) \leq \pi/2\}$
- $w = \text{Log}(z)$, $\{z : |z| > 1, 0 \leq \text{Arg}(z) \leq \pi/2\}$

1.4 Find z where the function is 0

- $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

1.5 Calculate principal value Log(z)

- z $= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and prove e^z is the inverse function of Log(z)

1.6 Prove the following

- Define the complex conjugate ($\overline{w}$) and prove if w is a zero of a polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ then $\overline{w}$ is also a zero of $p(z)$
- Define the complex exponential function e^z and prove Eulers Formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

2 Question 2:

2.1 Determine image of the line

- $f(z) = \frac{1}{z}$ $\{z \in \mathbb{C} : \text{Re}(z) = 2\}$

  2023 Q2(a), 2021 Q2(b)
- $f(z) = \frac{1}{z}$ $\{z \in \mathbb{C} : \text{Re}(z) = 1\}$

  2022 Q2(a), 2018 Q2(a), 2017 Q2(a)

2.2 State and Use Cauchy-Riemann Equations

- State CRE, and use to prove $f(z) = \frac{1}{z}$ is holomoprhic on $\mathbb{C}\backslash\{0\}$

  2023 Q2(a)
- State CRE, and use to prove $f(z) = z^2$ is holomoprhic on $\mathbb{C}$

  2022 Q2(b)
- State CRE. Let $f = u + iv$ be holomoprhic on $\Omega \subset \mathbb{C}$. Prove ∇u and ∇v are perpendicular of equal length 2016 Q2(b)

2.3 Show that

- If $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$ then $f(x)$ is real for all $x \in \mathbb{R}$. And if in addition f is holomoprhic at $x \in \mathbb{R}$ then $f'(x)$ is real. 2023 Q2(c)
- Define that is meant for a function g to be harmonic. If $f = u + iv$ is holomoprhic on $\Omega \subset \mathbb{C}$, prove that $v(x, y)$ is a harmonic function, and that ∇u and ∇v are perpendicular of equal length. 2022 Q2(c), 2018 Q2(b)
- If $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$ then $f(x)$ is real for all $x \in \mathbb{R}$. And if in addition f is holomoprhic at 0 then the function $f'(0)$ is real. 2021 Q2(a), 2017 Q2(c)
- Let $f(z) = u + iv$ be holomoprhic on an open subset Ω of the complex plane and let $h(u, v)$ be a harmonic function of u and v on $f(\Omega)$. Prove that $g(x, y) = h(u(x, y), v(x, y))$ is harmonic on Ω (You may assume $\nabla u, \nabla v$ are equal length and perpendicular) 2021 Q2(c)
- Define what is meant for a function $f(z)$ to be holomoprhic at a point $z_0 \in \mathbb{C}$ and prove that $f(z) = z^2$ is holomoprhic and find its derivative there. Hence prove that the product uv is harmonic where $f = u + iv$ 2018 Q2(c)
- Define what is meant for a function $f(z)$ to be holomoprhic at a point $z_0 \in \mathbb{C}$ and prove that $f(z) = \frac{1}{z}$ is holomoprhic on $\mathbb{C}\backslash 0$ and find its derivative there (State any theorems used) 2017 Q2(b)
- Let $h(u, v)$ be a harmonic function of u, v on $f(\Omega)$ (See 2016 Q2(b)). Prove that $g(x, y) = h(u(x, y), v(x, y))$ is harmonic on Ω 2016 Q2(c)

2.4 Find Mobius Transformation

- $T(z) : (-1, 1, \infty) \mapsto (-1, -i, 1)$

  2023 Q2(d)
- $T(z) : (2, 1, -1) \mapsto (1, 0, \infty)$

  2022 Q2(d)
- $T(z) : (-i, -1, 1) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation

  2021 Q2(d)
- $T(z) : (-i, -1, i) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation

  2018 Q2(d), 2017 Q2(d)
- $T(z) : (-1, \frac{1}{2}, 2) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation

  2016 Q2(d)

3 Worked Examples - Q1

Example 3.1: 2023 Q1(a)

Given $|z - 4| > 3|z + 4|$
Write $z = x + iy$

$$|x + iy - 4| > 3|x + iy + 4|$$
$$|(x - 4) + iy| > 3|(x + 4) + iy|$$
$$\sqrt{(x - 4)^2 + y^2} > 3\sqrt{(x + 4)^2 + y^2}$$

Square both sides

$$(x - 4)^2 + y^2 > 9((x + 4)^2 + y^2)$$
$$(x^2 - 8x + 16 + y^2) > 9x^2 + 72x + 144 + 9y^2$$
$$x^2 - 8x + 16 + y^2 - 9x^2 - 72x - 144 - 9y^2 > 0 \qquad \textit{Moving all terms to one side}$$
$$-8x^2 - 80x - 8y^2 - 128 > 0 \qquad \textit{Simplify}$$
$$x^2 + 10x + y^2 - 16 < 0 \qquad \textit{Dividing by -8 and reversing inequality}$$
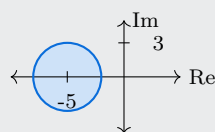
Focus on x and complete the square

$$x + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \Rightarrow x^2 + 10x = (x + 5)^2 - 25 \qquad \textit{Complete the square}$$
$$(x + 5)^2 - 25 + y^2 + 16 < 0 \qquad \textit{Substitute back into inequality}$$
$$(x + 5)^2 + y^2 + 9 < 0 \qquad \textit{Simplify}$$
$$(x + 5)^2 + y^2 < -9 \qquad \textit{Subtract 9}$$

Recall the eqation of a circle

$$(x - a)^2 + (y - b)^2 = r^2 \Rightarrow (x + 5)^2 + y^2 < -9$$

Therefore the region is all the points inside circle
with radius 3 and center at (-5, 0)

Given $\{z \in \mathbb{C} : |2z - 1| < 2|2z - i|\}$
Write $z = x + iy$

$$|2x + i2y - 1| < 2|2x + i2y - i|$$
$$|(2x - 1) + i2y| < 2|2x + i(2y - 1)|$$
$$\sqrt{(2x - 1)^2 + 4y^2} < 2\sqrt{4x^2 + (2y - 1)^2}$$

$$(2x - 1)^2 + 4y^2 < 4[4x^2 + (2y - 1)^2] \qquad \textit{Square both sides}$$
$$4x^2 - 4x + 1 + 4y^2 < 16x^2 + 16y^2 - 16y + 4 \qquad \textit{Expand}$$
$$-12x^2 - 4x - 12y^2 + 16y - 3 < 0 \qquad \textit{Move all terms to one side}$$
$$12x^2 + 4x + 12y^2 - 16y + 3 > 0 \qquad \textit{Multiply by -1 and reverse inequality}$$
$$x^2 + \frac{1}{3}x + y^2 - \frac{4}{3}y + \frac{1}{4} > 0 \qquad \textit{Divide by 12}$$

Complete square for x

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \Rightarrow x^2 + \frac{1}{3}x = \left(x + \frac{1}{6}\right)^2 - \left(\frac{1}{36}\right)$$

Complete square for y

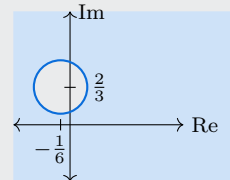$$y^2 + by = \left(y - \frac{2}{3}\right)^2 - \left(\frac{4}{9}\right)$$

Substitute back into inequality

$$\left(x + \frac{1}{6}\right)^2 - \left(\frac{1}{36}\right) + \left(y - \frac{2}{3}\right)^2 - \left(\frac{4}{9}\right) + \frac{1}{4} > 0 \qquad \textit{Substitute back into inequality}$$
$$\left(x + \frac{1}{6}\right)^2 + \left(y - \frac{2}{3}\right)^2 > \frac{2}{9} \qquad \textit{Simplify and move constant across}$$

Recall the eqation of a circle

$$(x - a)^2 + (y - b)^2 = r^2 \Rightarrow \left(x + \frac{1}{6}\right)^2 + \left(y - \frac{2}{3}\right)^2 < \frac{2}{9}$$

**Therefore the region is all the points OUTSIDE
the circle with radius $\frac{\sqrt{2}}{3}$ and center at $(-\frac{1}{6}, \frac{2}{3})$**

Example 3.3: Determine all solutions to $z^6 - 1 = 0$ and factor $x^6 - 1$ as a product of linear and quadratic factors

Given $z^6 - 1 = 0$
Write $z = e^{i\theta}$ and $1 = e^{i2\pi k}$ for $k \in \mathbb{Z}$

$$z^6 - 1 = 0$$
$$e^{i6\theta} - e^{i2\pi k} = 0$$
$$e^{i6\theta} = e^{i2\pi k}$$
$$6\theta = 2\pi k$$
$$\theta = \frac{\pi k}{3}$$

Therefore the solutions are

$$z = e^{i\theta} = e^{i\frac{\pi k}{3}} = \cos\left(\frac{\pi k}{3}\right) + i\sin\left(\frac{\pi k}{3}\right) \quad \text{for} \quad k = 0, 1, 2, 3, 4, 5$$

$k = 0 : w_0 = \cos(0) + i\sin(0) = 1 + i0$
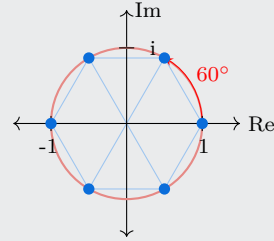
$k = 1 : w_1 = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k = 2 : w_2 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k = 3 : w_3 = \cos(\pi) + i\sin(\pi) = -1$

$k = 4 : w_4 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$k = 5 : w_5 = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

We can write:
$$x^6 - 1 = (x - w_0)(x - w_1)(x - w_2)(x - w_3)(x - w_4)(x - w_5)$$

Rewriting to group complex conjugates
$$x^6 - 1 = (z - w_0)(z - w_3) \cdot (z - w_1)(z - w_5) \cdot (z - w_2)(z - w_4)$$

Note that
$$(w - z)(w - \overline{z}) = w^2 - w\overline{z} - zw + z\overline{z}$$
$$= w^2 - 2(\overline{z} + z) + 1$$

We recall that
$$z = x + iy = e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
$$\overline{z} = x - iy = e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

Then
$$\overline{z} + z = \cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta)$$
$$= 2\cos(\theta)$$

Thus
$$(w - z)(w - \overline{z}) = w^2 - 2\cos(\theta) + 1$$

We see that $-\frac{5\pi}{3} = \frac{\pi}{3} - 2\pi$, thus:

$(z - w_1)(z - w_5) = (z - e^{i\frac{\pi}{3}})(z - e^{i\frac{5\pi}{3}})$

$(z - w_1)(z - w_5) = z^2 - 2\cos\left(\frac{\pi}{3}\right) + 1$

$(z - w_1)(z - w_5) = z^2 + z + 1$

We see that $-\frac{4\pi}{3} = \frac{\pi}{3} - \pi$, thus:

$(z - w_2)(z - w_4) = (z - e^{i\frac{2\pi}{3}})(z - e^{i\frac{4\pi}{3}})$

$(z - w_2)(z - w_4) = z^2 - 2\cos\left(\frac{2\pi}{3}\right) + 1$

$(z - w_2)(z - w_4) = z^2 - z + 1$

Therefore
$$x^6 - 1 = (x + 1)(x - 1)(x^2 + x + 1)(x^2 - x + 1)$$

Given $z^4 = -81i$, we want to find $z^{4(\frac{1}{4})} = w$
Recall:
$$z^{1/n} = R^{1/n}[\cos\phi + i\sin\phi] \quad \text{with } \phi = \frac{\theta + 2k\pi}{n}, \ k \in (0, 1, 2, \ldots, n-1) \quad \text{and } R = |z|$$

Thus

$$R = |81i| = \sqrt{0^2 + 81^2} = 81$$

$$\theta = -\frac{\pi}{2}$$

$$\phi = \frac{\theta + 2k\pi}{n} = \frac{-\frac{\pi}{2} + 2k\pi}{4} = \frac{-\pi}{8} + \frac{k\pi}{2}$$
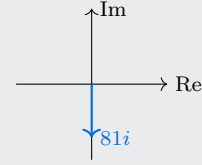
Therefore

$$w_k = 81^{1/4}\left[\cos\left(\frac{-\pi}{8} + \frac{k\pi}{2}\right) + i\sin\left(\frac{-\pi}{8} + \frac{k\pi}{2}\right)\right] \quad k \in (0, 1, 2, 3)$$

$$w_0 = 3\left[\cos\left(\frac{-\pi}{8}\right) + i\sin\left(\frac{-\pi}{8}\right)\right] \approx 2.77 - 1.155i$$

$$w_1 = 3\left[\cos\left(-\frac{\pi}{8} + \frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{8} + \frac{\pi}{2}\right)\right] \approx 1.155 + 2.77i$$

$$w_2 = 3\left[\cos\left(-\frac{\pi}{8} + \pi\right) + i\sin\left(-\frac{\pi}{8} + \pi\right)\right] \approx -1.55 + 2.77i$$

$$w_3 = 3\left[\cos\left(-\frac{\pi}{8} + \frac{3\pi}{2}\right) + i\sin\left(-\frac{\pi}{8} + \frac{3\pi}{2}\right)\right] \approx -2.77 - 1.55i$$

Part 2:
Given $p(z)$ with complex coefficients has root w and $p(\overline{w}) \neq 0$
In other words, we want $p(w) = 0$ and $p(\overline{w}) \neq 0$

Using the most simple polynomial, $p(z) = z - w$ and letting $w = 3e^{i\frac{-\pi}{8}}$ we have

$$p(z) = z - 3e^{i\frac{-\pi}{8}}$$

$$p(w) = w - w$$
$$= 3e^{i\frac{-\pi}{8}} - 3e^{i\frac{-\pi}{8}}$$
$$= 0$$

$$p(\overline{w}) = \overline{w} - 3e^{i\frac{-\pi}{8}}$$
$$= 3e^{-i\frac{\pi}{8}} - 3e^{i\frac{-\pi}{8}}$$
$$= 3\left[\cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right) - \left(\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right)\right]$$
$$= 3\left[\cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right) - \cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right)\right]$$
$$= 3\left[-2i\sin\left(\frac{\pi}{8}\right)\right]$$
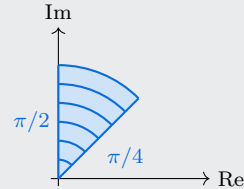$$= -6i\sin\left(\frac{\pi}{8}\right)$$
$$\approx -2.3i \neq 0$$

Example 3.5: Determine and sketch the image under the map $w = e^z$, $\{z \in \mathbb{C} : \pi/4 \leq \mathbf{Im}(z) \leq \pi/2\}$

$$w = e^z = e^{x+iy}$$
$$= e^x e^{iy}$$
$$= e^x[\cos(y) + i\sin(y)]$$

Recall the polar form of a complex number $z = |z|[\cos(\theta) + i\sin(\theta)]$
We see, e^x acts as the radius, and is always positive, and $[\cos(y) + i\sin(y)]$ acts draws out a section of the unit circle, thus the mapping $w = e^z$ sends the set to:

$$\left\{ w \in \mathbb{C} : |w| > 0, \frac{\pi}{4} \leq \arg(w) \leq \frac{\pi}{2} \right\}$$
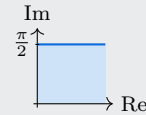

Example 3.6: Determine and sketch the region $w = \mathbf{Log}(z)$, $\{z : |z| > 1, 0 \leq \mathbf{Arg}(z) \leq \pi/2\}$

$$w = \mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z) = u + iv$$

Note that $|z| > 1$ implies $\ln|z| > 0$ Thus:

$$\left\{ w = u + iv \in \mathbb{C} : u > 0, 0 \leq v \leq \frac{\pi}{2} \right\}$$

Example 3.7: Find where the function is 0 : $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

We want $cos(z) = \frac{e^{iz} + e^{-iz}}{2} = 0$, some basic algebra gives us:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = 0$$
$$e^{iz} + e^{-iz} = 0$$
$$e^{iz} = -e^{-iz}$$
$$e^{iz} \cdot e^{iz} = -e^{-iz} \cdot e^{iz} \qquad\qquad \textit{Multiply both sides by } e^{iz}$$
$$e^{2iz} = -e^0 \qquad\qquad\qquad e^a \cdot e^b = e^{a+b}$$
$$e^{2iz} = -1$$

Recall:
$$-1 = cos(\pi + 2\pi k) + i\sin(\pi + 2\pi k) = e^{i(\pi + 2\pi k)}$$

Thus
$$e^{2iz} = e^{i(\pi + 2\pi k)}$$
$$2iz = i(\pi + 2\pi k) \qquad\qquad \textit{Taking the natrual log of both sides}$$
$$2z = \pi + 2\pi k \qquad\qquad\qquad \textit{Divide by 2}$$
$$z = \frac{\pi}{2} + \pi k \qquad\qquad\qquad \textit{Divide by i}$$

Therefore, the zeros of $\cos(z)$ are:

$$z = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}$$

Example 3.8: Calculate the principal value $\text{Log}(z)$ of $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and prove e^z is the inverse function of $\text{Log}(z)$

Part 1.
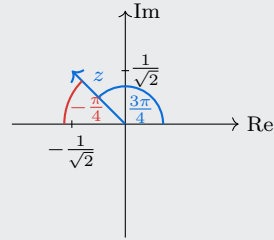**Given** $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$:

$$\ln|z| = \sqrt{\left(1/\sqrt{2}\right)^2 + \left(1/\sqrt{2}\right)^2} = 1$$

and

$$\begin{aligned} \text{Arg}(z) &= \tan^{-1}(-1) \\ &= -\tan^{-1}(1) \\ &= -\frac{\pi}{4} \Rightarrow \frac{3\pi}{4} \end{aligned}$$

Therefore

$$\text{Log}(z) = \ln|z| + i\text{Arg}(z) = i\frac{3\pi}{4}$$

Part 2: We need to show that (a) $e^{\textbf{Log(z)}} = z$ and (b) $\textbf{Log}(e^z) = z$
(a) Let $z = |z|e^{i\theta}, |z| > 0$ and $\theta = \text{Arg}(z)$

$$\begin{aligned} \text{Log}(z) &= \ln|z| + i\theta \\ e^{\text{Log}(z)} &= e^{\ln|z|+i\theta} && \textit{Exponentiate both sides} \\ &= e^{\ln|z|} \cdot e^{i\theta} && \textit{Exponentiation rules} \\ &= |z| \cdot e^{i\theta} \\ e^{\text{Log}(z)} &= z \end{aligned}$$

(b) Let $z = x + iy, y \in [-\pi, \pi]$

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x \cdot e^{iy} \\ \text{Log}(e^z) &= \ln|e^x \cdot e^{iy}| && \textit{Take log of both sides} \\ &= \ln|e^x| + \ln|e^{iy}| && \log(a \cdot b) = \log(a) + \log(b) \\ &= x + iy \\ \text{Log}(e^z) &= z \end{aligned}$$

Example 3.9: Define the complex conjugate ($\overline{w}$) and prove if w is a zero of a polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ then $\overline{w}$

Definition: For a complex number $w = a + bi$ the complex conjugate of w is defined as $\overline{w} = a - bi$ (with $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$)
This has several properties:

$$\overline{z + w} = \overline{z} + \overline{w}$$
$$\overline{zw} = \overline{z} \cdot \overline{w}$$
$$\overline{(w^n)} = (\overline{w})^n$$

Proof: If w is zero of a polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ then $p(\overline{w}) = 0$

$$\begin{aligned} \text{Assume} \quad & p(w) = a_0 + a_1 w + \ldots + a_n w^n = 0 \\ \text{Take the conjugate of both sides} \quad & \overline{p(w)} = \overline{0} = 0 \\ \text{Evalute} \quad & p(\overline{w}) = a_0 + a_1\overline{w} + \ldots + a_n\overline{w}^n \\ & = a_0 + a_1\overline{w} + \ldots + a_n\overline{w^n} \\ & = \overline{a_0} + \overline{a_1 w} + \ldots + \overline{a_n w^n} \\ & = \overline{a_0 + a_1 w + \ldots + a_n w^n} \\ & = \overline{p(w)} = 0 \end{aligned}$$

Thus, since we assumed $p(w) = 0$:

$$p(\overline{w}) = \overline{p(w)} = 0$$

Example 3.10: Define the complex exponential function e^z and prove Eulers Foruma $e^{i\theta} = \cos(\theta) + i\sin\theta$

Defition : For any $z \in mathbbC$, e^z is defined by its power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

The series converges for all $z \in \mathbb{C}$ and has the following properties:

$$e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$$

$$e^z \cdot e^{-z} = 1$$

Proof of Eulers Formula

$$\text{Eulers Formula} \quad e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} & \textit{Substitute } z = i\theta \\
&= \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} & \textit{Split into even and odd powers} \\
&= \sum_{n=0}^{\infty} \frac{(i)^{2n}(\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i)^{2n+1}(\theta)^{2n+1}}{(2n+1)!} & \textit{Seperate powers}
\end{aligned}$$

We note that:

$$i^{2n} = (i^2)^n = (-1)^n$$
$$i^{2n+1} = i \cdot i^{2n} = i(-1)^n$$

Thus:

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(-1)^n(\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i(-1)^n(\theta)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(\theta)^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n(\theta)^{2n+1}}{(2n+1)!} \\
&:= \cos(\theta) + i\sin(\theta)
\end{aligned}$$

4   Worked Examples - Q2

Example 4.1: Determine the image of the line $f(z) = \frac{1}{z}$ $\{z \in \mathbb{C} : \mathbf{Re}(z) = 2\}$

Let:

$$w = f(z) = \frac{1}{z}$$

Then we have:

$$w = \frac{1}{z}, \quad \text{so that } zw = 1 \quad \Rightarrow \quad z = \frac{1}{w},$$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

We note that: $\mathrm{Re}(z) = \frac{u}{u^2+v^2} = a$

$$u = a(u^2 + v^2) \quad \Rightarrow \quad \frac{1}{a}u = u^2 + v^2 \quad \Rightarrow u^2 - \frac{1}{a}u + v^2 = 0$$
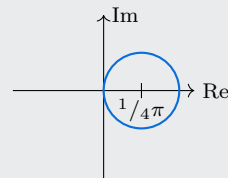
Completing the square in u:

$$\left(u^2 - \frac{1}{2a}\right)^2 - \frac{1}{4a^2} + av^2 = 0 \Rightarrow \left(u^2 - \frac{1}{2a}\right)^2 + av^2 = \frac{1}{4a^2}$$

Letting $a = 2$:

$$\left(u^2 - \frac{1}{4}\right)^2 + 4v^2 = \frac{1}{16}$$

Thus the image is sphere with radius $\frac{1}{4}$ and centre $(\frac{1}{4}, 0)$

Example 4.2: Determine the image of the line $f(z) = \frac{1}{z}$ $\{z \in \mathbb{C} : \mathbf{Re}(z) = 1\}$

Let:
$$w = f(z) = \frac{1}{z}$$

Then we have:
$$w = \frac{1}{z}, \quad \text{so that } zw = 1 \quad \Rightarrow \quad z = \frac{1}{w},$$

$$z = \frac{1}{u + iv} = \frac{u - iv}{(u + iv)(u - iv)} = \frac{u - iv}{u^2 + v^2} = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

We note that: $\mathrm{Re}(z) = \frac{u}{u^2 + v^2} = a$

$$u = a(u^2 + v^2) \quad \Rightarrow \quad \frac{1}{a}u = u^2 + v^2 \quad \Rightarrow u^2 - \frac{1}{a}u + v^2 = 0$$

Completing the square in u:

$$\left(u^2 - \frac{1}{2a}\right)^2 - \frac{1}{4a^2} + av^2 = 0 \Rightarrow \left(u^2 - \frac{1}{2a}\right)^2 + av^2 = \frac{1}{4a^2}$$

Letting $a = 1$:

$$\left(u^2 - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}$$

Thus the image is sphere with radius $\frac{1}{2}$ **and centre**
$(\frac{1}{2}, 0)$