

## **MA283: Linear Algebra**

70% Exam

30% Continuous Assessment (Homework)

10% Optional Project (Bonus)

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# 1 Review of Matrix Algebra

## Fields

- A field  $F$  is a set where addition, subtraction, multiplication and division (by nonzero elements) satisfy the usual algebraic properties. Common fields include  $\mathbb{R}$  and  $\mathbb{C}$
- We write  $\mathbb{F}^p$  for the vector space of all  $p$  vectors with entries in  $\mathbb{F}$ .
- We'll cheat and treat any ordered list of  $p$  elements of  $\mathbb{F}$  as an element of  $\mathbb{F}^p$ .
- For example, in  $\mathbb{R}^3$ , we might consider  $(1, 2, 3)$  as coordinates, a row vector, or a column vector with 3 real entries.

## Matrices Over a Field

- An  $m \times n$  matrix over a field  $\mathbb{F}$  is an array of  $m$  rows and  $n$  columns of elements from  $\mathbb{F}$ .
- When  $m = n$ , we write  $M_n(\mathbb{F})$ , otherwise we write  $M_{m \times n}(\mathbb{F})$ .

## Addition and Scalar Multiplication

- Two matrices of the same size  $m \times n$  can be added entrywise
- The  $m \times n$  matrix has all entries equal to zero and acts as the additive identity (adding it to any matrix does not change the matrix)
- Multiplying a matrix by a scalar means multiplying each entry by that scalar
- The set of all  $m \times n$  matrices over  $\mathbb{F}$  is a vector space over  $\mathbb{F}$

## Linear Combinations

- A linear combination of vectors  $v_1, v_2, \dots, v_k$  in a vector space  $V$  with coefficients  $a_1, a_2, \dots, a_k \in \mathbb{F}$  is defined as:

$$a_1 v_1 + a_2 v_2 + \dots + a_k v_k$$

- In particular, matrices themselves can be combined linearly, (e.g.  $2A - 3B$ )

## Row and Column Vectors

- A column vector is a matrix with one column
- A row vector is a matrix with one row

## Matrix-Vector Multiplication

- If  $A$  is  $m \times n$  matrix and  $v$  is an  $n$ -entry column vector, the product  $Av$  is defined by taking a linear combination of the columns of  $A$  with the entries of  $v$  as coefficients.
- The result  $Av$  is an  $m$ -entry column vector.
- For a row vector  $u$  with  $m$  entries, and an  $m \times n$  matrix  $A$  the product  $uA$  a row vector in  $\mathbb{F}^n$  formed by the linear combination of the rows of  $A$  with the entries of  $u$  as coefficients.

## Matrix-Matrix Multiplication

- If  $A$  is a  $m \times p$  and  $B$  is a  $p \times n$  matrix, the product  $AB$  is defined only when the inner dimensions match ( $p$ )
- To find each column of  $AB$ , multiply  $A$  with the corresponding column vector of  $B$ .
- In entrywise form:

$$(AB)_{ij} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \dots + A_{i,p}B_{p,j} = \sum_{k=1}^p A_{i,k}B_{k,j}$$

## Dot Product and Orthogonality

- For two  $p$ -entry vectors,  $u, v \in \mathbb{F}^p$ , their dot product is:

$$u \cdot v = \sum_{k=1}^p u_k v_k$$

- Vectors are **orthogonal** if their dot product is zero.
- If  $\mathbb{F} = \mathbb{R}$ , this means the vector are perpendicular.
- In matrix multiplication, the entry  $(AB)_{ij}$  can be viewed as the dot product of Row  $i$  with Column  $j$  of  $B$ .

## Matrices and Tables

Lets consider the table that gives the numbers of Maths  $M$ , Physics  $P$  and Chemistry  $C$  students in each of the 3 years of a course:

| Year | M  | P   | C  |
|------|----|-----|----|
| 2015 | 50 | 100 | 70 |
| 2016 | 60 | 80  | 80 |
| 2017 | 70 | 90  | 90 |

$$A = \begin{bmatrix} 50 & 100 & 70 \\ 60 & 80 & 80 \\ 80 & 70 & 70 \end{bmatrix}$$

Each student of  $M, P, C$  must also take a course in  $X$  and  $Y$ . We can represent the credits they earn as a matrix:

| Subject | X  | Y  |
|---------|----|----|
| M       | 10 | 0  |
| P       | 15 | 15 |
| C       | 20 | 10 |

$$B = \begin{bmatrix} 10 & 0 \\ 15 & 15 \\ 20 & 10 \end{bmatrix}$$

The total number of credits earned each year can be found by the matrix product  $AB$ :

$$AB = \begin{bmatrix} 50 & 100 & 70 \\ 60 & 80 & 80 \\ 80 & 70 & 70 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 15 & 15 \\ 20 & 10 \end{bmatrix} = \begin{bmatrix} 50 \cdot 10 + 100 \cdot 15 + 70 \cdot 20 & 50 \cdot 0 + 100 \cdot 15 + 70 \cdot 10 \\ 60 \cdot 10 + 80 \cdot 15 + 80 \cdot 20 & 60 \cdot 0 + 80 \cdot 15 + 80 \cdot 10 \\ 80 \cdot 10 + 70 \cdot 15 + 70 \cdot 20 & 80 \cdot 0 + 70 \cdot 15 + 70 \cdot 10 \end{bmatrix}$$

We can represent the result as a table:

| Year | X credits | Y credits |
|------|-----------|-----------|
| 2015 | 3400      | 2200      |
| 2016 | 3400      | 2000      |
| 2017 | 3250      | 1750      |

$$A = \begin{bmatrix} 3400 & 2200 \\ 3400 & 2000 \\ 3250 & 1750 \end{bmatrix}$$

## Linear Transformations

Let  $m$  and  $n$  be positive integers, A linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ , denoted  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , is a function that satisfies the following properties:

- $T(u + v) = T(u) + T(v)$
- $T(\lambda u) = \lambda T(u)$

$\forall u, v \in \mathbb{R}^n$  and scalars  $\lambda \in \mathbb{R}$

When  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ , if we know  $T$  applied to the three standard basis vectors of  $\mathbb{R}^3$ :

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we can form a  $2 \times 3$  matrix  $A$  whose columns are exactly these image, then  $T(v) = Av$  for any column vector  $v \in \mathbb{R}^3$ .

## Composition of Linear Transformations

- If  $T : \mathbb{R}^n \rightarrow \mathbb{R}^p$  and  $S : \mathbb{R}^p \rightarrow \mathbb{R}^m$ , then the composition  $(S \circ T)(v) = S(T(v))$
- If  $T$  is represented by a  $p \times n$  matrix  $A$  and  $S$  by an  $m \times p$  matrix  $B$  then the composition  $S \circ T$  is represented by the matrix product  $BA$ .
- Also  $(AB)C = A(BC)$
- Composing transformations is only possible if the codomain of the first transformation matches the domain of the second transformation, that is:

$$A \in M_{m \times n} \quad B \in M_{p \times m} \quad \Rightarrow \quad AB \in M_{m \times n}$$

## The $n \times n$ Identity Matrix

$I_n$  has 1s on the main diagonal:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The identity matrix is the **neutral element** for multiplication:

$$A \cdot I_n = A \quad \text{and} \quad I_n \cdot B = B \quad (\text{where } A \text{ has } n \text{ columns and } B \text{ has } n \text{ rows})$$

This is interpreted as the **identity transformation** on  $\mathbb{R}^n$ , so composing with it has no effect on any linear map.

## Invertible (Non-Singular) Matrices

A square  $n \times n$  matrix  $A$  has an inverse  $A^{-1}$  if there exists another  $n \times n$  matrix such that:

$$AB = I_n \quad \text{and} \quad BA = I_n$$

If  $A$  has an inverse, we say it is **unique**; there cannot be two different inverses for the same matrix.

Not all matrices are invertible. A key fact (explained later) is that:

$$A \text{ is invertible} \Leftrightarrow \text{the determinant} \neq 0$$

## Transpose of a Matrix

For a  $m \times n$  matrix  $A$ , the transpose  $A^T$  is the  $n \times m$  matrix obtained by turning the rows of  $A$  into the columns of  $A^T$ :

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

## 2 Systems of linear equations

### 2.1 Linear equations and Solution Sets

A linear equation in the variables  $x$  and  $y$  is an equation of the form

$$2x + y = 3$$

If we replace  $x$  and  $y$  with some numbers, the statement **becomes true or false**.

#### Definition 2.1: Solution to a linear equation

A pair,  $(x_0, y_0) \in \mathbb{R}$ , is a solution to a linear equation if setting  $x = x_0$  and  $y = y_0$  **makes the equation true**.

#### Definition 2.2: Solution set

The **solution set** is the set of all solutions to a linear equation.

$$a_1X_1 + a_2X_2 + \dots + a_nX_n = b \quad \text{where } a_i, b \in \mathbb{R}$$

is an **affine hyperplane** in  $\mathbb{R}^n$ ; geometrically resembles a copy of  $\mathbb{R}^{n-1}$  inside  $\mathbb{R}^n$ .

### 2.2 Elementary Row Operations

To solve a system of linear equations we associate an **augmented matrix** to the system of equations. For example:

$$\begin{array}{rrrrrr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \Rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array} \right]$$

To solve, we can perform the following **Elementary Row Operations (EROs)**:

1. Multiply a row by a non-zero constant.
2. Add a multiple of one row to another row.
3. Swap two rows.

The goal of these operations is to transform the augmented matrix into **row echelon form (REF)** or **reduced row echelon form (RREF)**.

## 2.2.1 REF and Strategy

We say a matrix is in **row echelon form** (REF) if:

- The first non zero entry in each row is a 1 (called the **leading 1**).
- If a column has a leading 1, then all entries below it are 0.
- The leading 1 in each row is to the right of the leading 1 in the previous row.
- All rows of 0s are at the bottom of the matrix.

$$\left[ \begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

*Example of REF*

We have produced a new system of equations. This is easily solved by back substitution.

### Concept 2.1: Strategy for Obtaining REF

- Get a 1 as the top left entry
- Use this 1 to clear the entries below it
- Move to the next column and repeat
- Continue until all leading 1s are in place
- Use back substitution to solve the system

## 2.2.2 Row Reduced Echelon Form

A matrix is in **reduced row echelon form** (RREF) if:

- It is in REF
- The leading 1 in each row is the only non-zero entry in its column.

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

*Example of RREF*

## 2.3 Leading variables and free variables

We'll start by an example:

$$\begin{array}{rrrrrrrrcl} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 & \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 & \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 & \end{array} \Rightarrow \left[ \begin{array}{cccc|c} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{array} \right]$$

Solving this system of equations, we get:

$$\text{RREF: } \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \Rightarrow \begin{array}{lcl} x_1 + 2x_4 & = & 4 \\ x_2 - x_4 & = & 2 \\ x_3 + x_4 & = & 2 \end{array} \Rightarrow \begin{array}{lcl} x_1 & = & 4 - 2x_4 \\ x_2 & = & 2 + x_4 \\ x_3 & = & 2 - x_4 \end{array}$$

This RREF tells us how the **leading variables** ( $x_1, x_2, x_3$ ) depend on the **free variable** ( $x_4$ ). The free variable can take any value in  $\mathbb{R}$ . We write the solution set as:

$$x_1 = 4 - 2t, \quad x_2 = 2 + t, \quad x_3 = 2 - t, \quad x_4 = t \quad \text{where } t \in \mathbb{R}$$

$$(x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t); \quad t \in \mathbb{R}$$

### Definition 2.3: Leading and Free Variables

- **Leading variable** : A variable whose columns in the RREF contain a leading 1
- **Free variable** : A variable whose columns in the RREF do not contain a leading 1

## 2.4 Consistent and Inconsistent Systems

Consider the following system of equations:

$$\begin{array}{rrrrrrcl} 3x & + & 2y & - & 5z & = & 4 & \\ x & + & y & - & 2z & = & 1 & \\ 5x & + & 3y & - & 8z & = & 6 & \end{array} \Rightarrow \left[ \begin{array}{ccc|c} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{array} \right] \Rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (\text{REF})$$

We can see the last row of the REF is:

$$0x + 0y + 0z = 1$$

This equation clearly has no solution, and hence the system has no solutions. We say the system is **inconsistent**. Alternatively, we say the system is **consistent** if it has at least one solution.

## 2.5 Possible Outcomes when solving a system of equations

- The system may be **inconsistent** (no solutions) - i.e:

$$[0 \ 0 \ \dots \ 0 \mid a] \quad a \neq 0$$

- The system may be **consistent** which occurs if:

- **Unique Solutions** each column (aside from the rightmost) contains a single leading 1. - i.e:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

- **Infinitely many solutions** at least one variable does not appear as a leading 1 in any row, making it a free variable - i.e:

$$\left[ \begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

## 2.6 Elementary Row Operations as Matrix Transformations

Elementary row operations may be interpreted as **matrix multiplication**. To see this, first we introduce the **Identity matrix**:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The  $I_m$  Identity matrix is an  $m \times m$  matrix with 1s on the diagonal and 0s elsewhere. We also introduce the  $E_{i,j}$  matrix which has 1 in the  $(i,j)$  position and 0s elsewhere. For example:

$$E_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then:

$$I_3 + 4E_{1,2} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing a row operation on  $A$  is the same as multiplying  $A$  by an appropriate matrix  $E$  on the left. These matrices are called **elementary matrices**. They are **always invertible**, and **their inverses are also elementary matrices**. The statement:

*"every matrix can be reduced to RREF through EROs"*

is equivalent to saying that

*"for every matrix  $A$  with  $m$  rows, there exists a  $m \times m$  matrix  $B$  which is a product of elementary matrices such that  $BA$  is in RREF."*

### 2.6.1 Multiplying a Row by a Non-Zero Scalar

When multiplying row  $i$  of matrix  $A$  by a scalar  $\alpha \neq 0$ , we can use the matrix:

$$I_m + (\alpha - 1)E_{i,i}$$

This works because it modifies only the  $(i,i)$  entry of the identity matrix to be  $\alpha$  while keeping all other entries unchanged. When multiplied with  $A$ , it scales row  $i$  by  $\alpha$  and leaves all other rows intact.

**Example:** If  $\alpha = 5$  and  $i = 2$ , then:

$$I_3 + 4E_{2,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad (I_3 + 4E_{2,2})A = \begin{bmatrix} 1 & 2 & 3 \\ 20 & 25 & 30 \\ 7 & 8 & 9 \end{bmatrix}$$

### 2.6.2 Switching Two Rows

To swap rows  $i$  and  $k$ , we use:

$$S = I_m + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k}$$

This works by:

- Removing the 1's at positions  $(i,i)$  and  $(k,k)$  from the identity matrix
- Adding 1's at positions  $(i,k)$  and  $(k,i)$

**Example:** Swapping rows 1 and 3:

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad SA = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

### 2.6.3 Adding a Multiple of One Row to Another

To replace row  $k$  with row  $k + \alpha \times$  row  $i$ , use:

$$I_m + \alpha E_{k,i}$$

This adds  $\alpha$  times row  $i$  to row  $k$  while leaving all other rows unchanged because:

- For any row  $j \neq k$ , the corresponding row in this matrix is just the standard basis row
- Row  $k$  becomes the sum of the standard basis row  $k$  plus  $\alpha$  times the standard basis row  $i$

**Example:** Adding 3 times row 1 to row 2:

$$I_3 + 3E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad (I_3 + 3E_{2,1})A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \\ 7 & 8 & 9 \end{bmatrix}$$

#### Example 2.1

Write the inverse of an elementary matrix and show it is an elementary matrix.

**Multiplying a row by a nonzero scalar:**

- **Operation:** Multiply row  $i$  by  $\alpha \neq 0$ .
- **Elementary Matrix:**  $E = I_m + (\alpha - 1)E_{i,i}$
- **Inverse:** To reverse the operation, multiply row  $i$  by  $1/\alpha$ . Hence, the inverse is

$$E^{-1} = I_m + \left(\frac{1}{\alpha} - 1\right) E_{i,i} = I_m + \frac{1 - \alpha}{\alpha} E_{i,i}.$$

**Swapping two rows:**

- **Operation:** Swap rows  $i$  and  $k$ .
- **Elementary Matrix:**  $S = I_m - E_{i,i} - E_{k,k} + E_{i,k} + E_{k,i}$
- **Inverse:** Since swapping the same two rows twice returns them to their original positions,

$$S^{-1} = S.$$

**Adding a multiple of one row to another:**

- **Operation:** Add  $\alpha$  times row  $i$  to row  $k$ .
- **Elementary Matrix:**  $E = I_m + \alpha E_{k,i}$
- **Inverse:** To undo the operation, subtract  $\alpha$  times row  $i$  from row  $k$ . Therefore,

$$E^{-1} = I_m - \alpha E_{k,i}.$$



### Example 2.2

Prove that every invertible matrix in  $M_n(\mathbb{R})$  is a product of elementary matrices.

Let  $A$  be an invertible matrix in  $M_n(\mathbb{R})$ . Since  $A$  is invertible, we can use Gaussian elimination to transform  $A$  into the identity matrix  $I_n$ .

Let  $E_1, E_2, \dots, E_k$  be the elementary matrices corresponding to the row operations used in the elimination process. Then, we have:

Multiplying a row by a scalar:  $I_n + (\alpha - 1)E_{i,i}$

Swapping two rows:  $I_n + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k}$

Adding a multiple of one row to another:  $I_n + \alpha E_{k,i}$

Applying these in sequence to  $A$  gives:

$$E_k \cdots E_2 E_1 A = I_n$$

Since  $E_k \cdots E_2 E_1 = I_n$ , we can multiply both sides by  $(E_k \cdots E_2 E_1)^{-1}$  on the left to obtain:

$$A = (E_k \cdots E_2 E_1)^{-1} I_n = (E_k \cdots E_2 E_1)^{-1}$$

Using the property:

$$(E_k \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1},$$

we can express  $A$  as a product of elementary matrices:

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Since each  $E_i$  is an elementary matrix, its inverse is also an elementary matrix. Therefore,  $A$  can be expressed as a product of elementary matrices.

## 2.7 EROs and Inverses

Elementary Row Operations can be used to find the inverse of a square matrix. Consider a square matrix  $A \in M_n(\mathbb{F})$  (that is, an  $n \times n$  matrix over a field  $\mathbb{F}$ ). If  $A$  is invertible, let

$$A^{-1} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}$$

be its inverse, where each  $\mathbf{v}_i$  is the  $i$ th column of  $A^{-1}$ . By definition of the matrix inverse, we have

$$A A^{-1} = A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} = I_n,$$

the  $n \times n$  identity matrix. This implies that

$$A\mathbf{v}_i = \mathbf{e}_i, \quad \text{for each } i = 1, 2, \dots, n,$$

where  $\mathbf{e}_i$  is the  $i$ th column of  $I_n$  (which has a 1 in the  $i$ th row and 0 everywhere else). In other words, each column  $\mathbf{v}_i$  of  $A^{-1}$  is the unique solution to the linear system

$$A\mathbf{v}_i = \mathbf{e}_i.$$

To find  $A^{-1}$  effectively, we form the augmented matrix  $[A \mid I_n]$  and apply EROs to transform  $A$  into  $I_n$ . When this is achieved, the augmented portion becomes  $A^{-1}$ . Thus, we have

$$\text{RREF}([A \mid I_n]) = [I_n \mid A^{-1}].$$

### Example 2.3

Find  $A^{-1}$  if  $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$ .

We form a  $3 \times 6$  matrix  $A' = [A \mid I_3]$ :

$$A' = \begin{bmatrix} 3 & 4 & -1 & | & 1 & 0 & 0 \\ 1 & 0 & 3 & | & 0 & 1 & 0 \\ 2 & 5 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

We apply the following EROs to  $A'$ :

- $R_1 \leftrightarrow R_2$
- $R_2 \rightarrow R_2 - 3R_1$
- $R_3 \rightarrow R_3 - 2R_1$
- $R_3 \rightarrow R_3 + R - 2$
- $R_3 \leftrightarrow R_2$
- $R_3 \rightarrow R_3 - 4R_2$
- $R_3 \times (-\frac{1}{10})$
- $R_1 \rightarrow R_1 - 3R_3$

To obtain:

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & | & -1 & 1 & 1 \\ 0 & 0 & 1 & | & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

That is:

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

It is easily checked that  $AA^{-1} = I_3$ .

### 3 Spanning sets, bases and dimensions

A **vector space**  $\mathbf{V}$  over  $\mathbb{F}$  is a non empty set of objects equipped with an addition operation and whose elements can be multiplied by scalars in  $\mathbb{F}$ , subject to the following axioms:

1.  $u + v = v + u, \quad \forall u, v \in \mathbf{V}$
2.  $(u + v) + w = u + (v + w), \quad \forall u, v, w \in \mathbf{V}$
3.  $\exists 0_{\mathbf{V}}$ , so that  $0_v + v = v, \quad \forall v \in \mathbf{V}$
4.  $\exists -v \in \mathbf{V}$ , so that  $v + (-v) = 0_{\mathbf{V}}, \quad \forall v \in \mathbf{V}$
5.  $\alpha(\beta v) = \alpha\beta(v), \quad \forall \alpha, \beta \in \mathbb{F}, v \in \mathbf{V}$
6.  $(\alpha + \beta)v = \alpha v + \beta v, \quad \forall \alpha, \beta \in \mathbb{F}, v \in \mathbf{V}$
7.  $1v = v, \quad \forall v \in \mathbf{V}$

In the definitions axioms above, the field  $\mathbb{F}$  can be replaced with any other field, such as  $\mathbb{R}$  or  $\mathbb{C}$ .

#### Examples of vector spaces over $\mathbb{R}$

- The space  $M_{m \times n}(\mathbb{R})$  of  $m \times n$  with real entries.
- The space of all polynomials with real coefficients
- The set of complex numbers is a vector space over  $\mathbb{R}$ .

Another example of a 2-dimensional vector space is the space  $\mathbf{V}$  consisting of all **symmetric**  $2 \times 2$  matrices in  $M_2(\mathbb{R})$  with **trace zero**.

- **Trace 0** means that the sum of the diagonal elements is zero.
- **Symmetric** means that the matrix is equal to its transpose.

So a matrix of trace zero has the form:

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \quad \text{where } a, b \in \mathbb{R}$$