MA283: Linear Algebra

70% Exam
30% Continuous Assessment (Homework)
10% Optional Project (Bonus)

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Contents

1 Systems of linear equations

1.1 Linear equations and Solution Sets

A linear equation in the variables x and y is an equation of the form

$$2x + y = 3$$

If we replace x and y with some numbers, the statement **becomes true or false**.

Definition 1.1: Solution to a linear equation

A pair, $(x_0, y_0) \in \mathbb{R}$, is a solution to an linear equation if setting $x = x_0$ and $y = y_0$ **makes the equation true.**

Definition 1.2: Solution set

The **solution set** is the set of all solutions to a linear equation.

$$a_1X_1 + a_2X_2 + \ldots + a_nX_n = b \quad \text{where } a_i, b \in \mathbb{R}$$

is an **affine hyperplane** in $\mathbb{R}^n$; geometrically resembles a copy of $\mathbb{R}^{n-1}$ inside $\mathbb{R}^n$.

1.2 Elementary Row Operations

To solve a system of linear equations we associate an **augmented matrix** to the system of equations. For example:

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array}\right]$$

To solve, we can perform the following **Elementary Row Operations (EROs)**:

1. Multiply a row by a non-zero constant.
2. Add a multiple of one row to another row.
3. Swap two rows.

The goal of these operations is to transform the augmented matrix into **row echelon form** (REF) or **reduced row echelon form** (RREF).

1.2.1 REF and Strategy

We say a matrix is in **row echelon form** (REF) if:

- The first non zero entry in each row is a 1 (called the **leading 1**).
- If a column has a leading 1, then all entries below it are 0.
- The leading 1 in each row is to the right of the leading 1 in the previous row.
- All rows of 0s are at the bottom of the matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Example of REF

We have produced a new system of equations. This is easily solved by back substitution.

Concept 1.1: Stategy for Obtaining REF

- Get a 1 as the top left entry
- Use this 1 to clear the entries below it
- Move to the next column and repeat
- Continue until all leading 1s are in place
- Use back substitution to solve the system

1.2.2 Row Reduced Echelon Form

A matrix is in **reduced row echelon form** (RREF) if:

- It is in REF
- The leading 1 in each row is the only non-zero entry in its column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Example of RREF

1.3 Leading variables and free variables

We'll start by an example:

$$\begin{array}{ccccccccc} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 \end{array} \quad \Rightarrow \quad \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{array}\right]$$

Solving this system of equations, we get:

$$\text{RREF:} \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \quad \Rightarrow \quad \begin{array}{ccccc} x_1 & + & 2x_4 & = & 4 \\ x_2 & - & x_4 & = & 2 \\ x_3 & + & x_4 & = & 2 \end{array} \quad \Rightarrow \quad \begin{array}{ccc} x_1 & = & 4-2x_4 \\ x_2 & = & 2+x_4 \\ x_3 & = & 2-x_4 \end{array}$$

This RREF tells us how the **leading variables** (x_1, x_2, x_3) depend on the **free variable** (x_4). The free variable can take any value in $\mathbb{R}$. We write the solution set as:

$$x_1 = 4-2t, \quad x_2 = 2+t, \quad x_3 = 2-t, \quad x_4 = t \quad \text{where } t \in \mathbb{R}$$

$$(x_1, x_2, x_3, x_4) = (4-2t, 2+t, 2-t, t); \quad t \in \mathbb{R}$$

Definition 1.3: Leading and Free Variables

- **Leading variable** : A variable whose columns in the RREF contain a leading 1
- **Free variable** : A variable whose columns in the RREF do not contain a leading 1

1.4 Consistent and Inconsistent Systems

Consider the following system of equations:

$$\begin{array}{ccccccc} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \text{(REF)}$$

We can see the last row of the REF is:

$$0x + 0y + 0z = 1$$

This equation clearly has no solution, and hence the system has no solutions. We say the system is **inconsistent**. Alternatively, we say the system is **consistent** if it has at least one solution.

1.5 Possible Outcomes when solving a system of equations

- The system may be **inconsistent** (no solutions) - i.e:

$$[0\,0\,\ldots\,0 \mid a] \quad a \neq 0$$

- The system may be **consistent** which occurs if:
  - **Unique Solutions** each column (aside from the rightmost) contains a single leading 1. - i.e:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

  - **Infinitely many solutions** at least one variable does not appear as a leading 1 in any row, making it a free variable - i.e:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

1.6 Gaussian Elimination and Matrix Algebra

Elementary row operations may be interpreted as **matrix multiplication**. To see this, first we introduce the **Identity matrix:**

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The I_m Identity matrix is an $m \times m$ matrix with 1s on the diagonal and 0s elsewhere. We also introduce the $E_{i,j}$ matrix which has 1 in the (i, j) position and 0s elsewhere. For example:

$$E_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then:

$$I_3 + 4E_{1,2} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing a row operation on A is the same as multiplying A by an appropriate matrix E on the left

Multiplying a row by a non zero scalar

- When you multiply row i of A, by some scalar $\alpha \neq 0$, all other rows remain unchanged, and the ith row is scaled by α.
- The Identity matrix I_m leaved A unchanged if we multiply $I_m A$
- To scale row i by α,you want the (i, i)-entry of this matrix to be α, so the ith row of A is scaled by α.
- Hence $I_m + (\alpha - 1)E_{i,i}$ causes the ith row of A to be scaled by α.

That is what the matrix

$$I_m + (\alpha - 1)E_{i,i}$$

does to A: Observe that I_m has all diagonal entries $= 1$. Adding $(\alpha - 1)E_{i,i}$ makes that diagonal entry become α and all other diagonal entries remain 1. Example:

$$I_3 + 4E2, 2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$