MA283: Linear Algebra

70% Exam
30% Continuous Assessment (Homework)
10% Optional Project (Bonus)

Robert Davidson

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1 Review of Matrix Algebra

Matrix Addition

If a matrix has m rows and n columns, we say it is $m \times n$.**Two matrices can only be added if they have the same size.**. In this case, we just add the entries in each position.

The $m \times n$ **zero** matrix is a matrix with all entries equal to 0. It is the **Identity element** for matrix addition (adding it to any matrix does not change the matrix)

Matrix Multiplication by a Scalar

This simply means multiplying each entry of the matrix by the scalar. For example:

$$\alpha \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \alpha & 2\alpha \\ 3\alpha & 4\alpha \end{bmatrix}$$

Remark: Now that we have addition and scalar multiplication, we can subtract matrices ($A - B = A + (-1)B$), provided they are the same size.

Vector Space

With these operations of addition and scalar multiplication, the set of $m \times n$ matrices is a vector space. A **vector space** algebraic structure whose elements can be added, subtracted and multiplied by scalars.

Linear Combinations

Definition 1.1: Linear Combinations

Suppose $v_1, v_2, \ldots, v_k$ are elements that can be added together and multiplied by scalars.

A Linear Combination of $v_1, v_2, \ldots, v_k$ is an expression of the form:

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k$$

where $a_i \in \mathbb{R}$ are scalars, called **coefficients**.

Matrix-Vector Multiplication

Definition 1.2

Let A be a $m \times n$ matrix, and $\mathbf{v}$ be a column vector with n entries ($n \times 1$ matrix).

Then the matrix vector product Av is the column vector, with m entries, obtained by taking the linear combination of the columns of A with the entries of $\mathbf{v}$ as coefficients.

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix} = 7 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 9 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 41 \\ 33 \end{bmatrix}.$$

Remark: Av, if defined, has the same number of rows as A and the same number of columns as $\mathbf{v}$.

Matrix-Matrix Multiplication

Definition 1.3

Let A and B be matrices of size $m \times p$ and $p \times n$, respectively.Write $v_1, \ldots v_n$ for the columns of B. Then the product AB is the $m \times n$ matrices whose columns are $Av_1, \ldots, Av_n$.

The entry at row i and column j of the matrix A is given by A_{ij}. The entry in the i, j position of the product AB is the ith entry of the vector Av_j, where the vector v_j is the jth column of B. In other words, the entry in the i, j position of the product AB is given by:

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \ldots + A_{ip}B_{pj} = \sum_{k=1}^{p} A_{ik}B_{kj}$$

Definition 1.4

If A is $m \times p$ with rows $u_1, \ldots, u_m$ and B is $p \times n$ with columns $v_1, \ldots, v_n$, then the product AB is:

$$AB = \begin{bmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & \ldots & u_1 \cdot v_n \\ u_2 \cdot v_1 & u_2 \cdot v_2 & \ldots & u_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m \cdot v_1 & u_m \cdot v_2 & \ldots & u_m \cdot v_n \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} \quad AB = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

For matrices A and B, the products AB and BA are generally not equal, even if they are both defined and even if both have the same size.

Linear Transformations

Definition 1.5

Let m and n be positive integers.

A **linear transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$
- $T(\lambda \mathbf{u}) = \lambda T(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

Matrix of a Linear Transformation

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation:

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 7 \end{bmatrix}$$

Then for the vector in $\mathbb{R}^3$ with entries a, b, c:

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = aT \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + bT \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + cT \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -6 \\ -3 & 4 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Where the 2×3 matrix M_T is called the **standard matrix** of A. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be completely represented by an $m \times n$ matrix M_T.

Understanding the Matrix Representation

- The columns of matrix M_T are the images of the standard basis vectors $e_1, e_2, \ldots, e_n$ under T.
- For any vector $v \in \mathbb{R}^n$, we calculate $T(v)$ by multiplying: $M_T \cdot v$.
- Therefore, matrix-vector multiplication is simply evaluating a linear transformation.

Correspondence: Any $m \times n$ matrix A defines a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by: $T_A(v) = Av$. Linear transformations include rotations, reflections and scaling

Efficiency of Representation: A remarkable property of linear transformations is their information efficiency:

- To completely define $T : \mathbb{R}^n \to \mathbb{R}^m$, we need only mn values.
- These values are the coordinates of the n transformed basis vectors in $\mathbb{R}^m$.
- This differs fundamentally from general continuous functions $f : \mathbb{R} \to \mathbb{R}$, which cannot be fully determined by their values at finitely many points.

Matrix multiplication is composition

Suppose that $T : \mathbb{R}^n \to \mathbb{R}^p$ and $S : \mathbb{R}^p \to \mathbb{R}^m$ are linear transformations. Then the composition $S \circ T : \mathbb{R}^n \to \mathbb{R}^m$ is also a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined for $\mathbf{v} \in \mathbb{R}^n$ by:

$$S \circ T(\mathbf{v}) = S(T(\mathbf{v}))$$

To see how that the $m \times n$ matrix $M_{S \circ T}$ depends on the matrix $M_S(m \times p)$ and $M_T(p \times n)$ we look at the definition of $M_{S \circ T}$:

- The first column has coordinates $S \circ T(e_1) = S(T(e_1))$
- $T(e_1)$ is first column of M_T
- Then $S(T(e_1))$ is the matrix-vector product $M_S \cdot M_T(e_1)$
- Same for all other columns $\Longrightarrow M_{S \circ T} = M_S \cdot M_T$

Thus, we conclude **matrix multiplication is composition of linear transformations**.

2 Systems of linear equations

2.1 Linear equations and Solution Sets

A linear equation in the variables x and y is an equation of the form

$$2x + y = 3$$

If we replace x and y with some numbers, the statement **becomes true or false**.

Definition 2.1: Solution to a linear equation

A pair, $(x_0, y_0) \in \mathbb{R}$, is a solution to an linear equation if setting $x = x_0$ and $y = y_0$ **makes the equation true.**

Definition 2.2: Solution set

The **solution set** is the set of all solutions to a linear equation.

$$a_1X_1 + a_2X_2 + \ldots + a_nX_n = b \quad \text{where } a_i, b \in \mathbb{R}$$

is an **affine hyperplane** in $\mathbb{R}^n$; geometrically resembles a copy of $\mathbb{R}^{n-1}$ inside $\mathbb{R}^n$.

2.1.1 Interpreting Linear Systems as Matrix Equations

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 0 \end{bmatrix}$$

2.2 Elementary Row Operations

To solve a system of linear equations we associate an **augmented matrix** to the system of equations. For example:

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array}\right]$$

To solve, we can perform the following **Elementary Row Operations (EROs)**:

1. Multiply a row by a non-zero constant.
2. Add a multiple of one row to another row.
3. Swap two rows.

The goal of these operations is to transform the augmented matrix into **row echelon form** (REF) or **reduced row echelon form** (RREF).

2.2.1 REF and Strategy

We say a matrix is in **row echelon form** (REF) if:

- The first non zero entry in each row is a 1 (called the **leading 1**).
- If a column has a leading 1, then all entries below it are 0.
- The leading 1 in each row is to the right of the leading 1 in the previous row.
- All rows of 0s are at the bottom of the matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Example of REF

We have produced a new system of equations. This is easily solved by back substitution.

Concept 2.1: Stategy for Obtaining REF

- Get a 1 as the top left entry
- Use this 1 to clear the entries below it
- Move to the next column and repeat
- Continue until all leading 1s are in place
- Use back substitution to solve the system

2.2.2 Row Reduced Echelon Form

A matrix is in **reduced row echelon form** (RREF) if:

- It is in REF
- The leading 1 in each row is the only non-zero entry in its column.

$$\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

Example of RREF

2.3 Leading variables and free variables

We'll start by an example:

$$\begin{array}{rcrcrcrcl} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 \end{array} \Rightarrow \begin{bmatrix} 1 & -1 & -1 & 2 & | & 0 \\ 2 & 1 & -1 & 2 & | & 8 \\ 1 & -3 & 2 & 7 & | & 2 \end{bmatrix}$$

Solving this system of equations, we get:

$$\text{RREF:} \quad \begin{bmatrix} 1 & 0 & 0 & 2 & | & 4 \\ 0 & 1 & 0 & -1 & | & 2 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix} \Rightarrow \begin{array}{rcrcl} x_1 & + & 2x_4 & = & 4 \\ x_2 & - & x_4 & = & 2 \\ x_3 & + & x_4 & = & 2 \end{array} \Rightarrow \begin{array}{rcl} x_1 & = & 4-2x_4 \\ x_2 & = & 2+x_4 \\ x_3 & = & 2-x_4 \end{array}$$

This RREF tells us how the **leading variables** (x_1, x_2, x_3) depend on the **free variable** (x_4). The free variable can take any value in $\mathbb{R}$. We write the solution set as:

$$x_1 = 4-2t, \quad x_2 = 2+t, \quad x_3 = 2-t, \quad x_4 = t \quad \text{where } t \in \mathbb{R}$$

$$(x_1, x_2, x_3, x_4) = (4-2t, 2+t, 2-t, t); \quad t \in \mathbb{R}$$

Definition 2.3: Leading and Free Variables

- **Leading variable** : A variable whose columns in the RREF contain a leading 1
- **Free variable** : A variable whose columns in the RREF do not contain a leading 1

2.4 Consistent and Inconsistent Systems

Consider the following system of equations:

$$\begin{array}{rcrcrcl} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array} \Rightarrow \begin{bmatrix} 3 & 2 & -5 & | & 4 \\ 1 & 1 & -2 & | & 1 \\ 5 & 3 & -8 & | & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -2 & | & 1 \\ 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & | & 1 \end{bmatrix} \quad \text{(REF)}$$

We can see the last row of the REF is:

$$0x + 0y + 0z = 1$$

This equation clearly has no solution, and hence the system has no solutions. We say the system is **inconsistent**. Alternatively, we say the system is **consistent** if it has at least one solution.

2.5 Possible Outcomes when solving a system of equations

- The system may be **inconsistent** (no solutions) - i.e:

$$[0\ 0\ \dots\ 0 \mid a] \quad a \neq 0$$

- The system may be **consistent** which occurs if:
  - **Unique Solutions** each column (aside from the rightmost) contains a single leading 1. - i.e:

$$\begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$$

  - **Infinitely many solutions** at least one variable does not appear as a leading 1 in any row, making it a free variable - i.e:

$$\begin{bmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 0 & 1 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

2.6 Elementary Row Operations as Matrix Transformations

Elementary row operations may be interpreted as **matrix multiplication**. To see this, first we introduce the **Identity matrix:**

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The I_m Identity matrix is an $m \times m$ matrix with 1s on the diagonal and 0s elsewhere. We also introduce the $E_{i,j}$ matrix which has 1 in the (i, j) position and 0s elsewhere. For example:

$$E_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then:

$$I_3 + 4E_{1,2} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing a row operation on A is the same as multiplying A by an appropriate matrix E on the left. These matrices are called **elementary matrices**. They are **always invertible,** and **their inverses are also elementary matrices**. The statement:

"every matrix can be reduced to RREF through EROs"

is equivalent to saying that

"for every matrix A with m rows, there exists a $m \times m$ matrix B which is a product of elementary matrices such that BA is in RREF."

2.6.1 Multiplying a Row by a Non-Zero Scalar

When multiplying row i of matrix A by a scalar $\alpha \neq 0$, we can use the matrix:

$$I_m + (\alpha - 1)E_{i,i}$$

This works because it modifies only the (i, i) entry of the identity matrix to be α while keeping all other entries unchanged. When multiplied with A, it scales row i by α and leaves all other rows intact.
Example: If $\alpha = 5$ and $i = 2$, then:

$$I_3 + 4E_{2,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad (I_3 + 4E_{2,2})A = \begin{bmatrix} 1 & 2 & 3 \\ 20 & 25 & 30 \\ 7 & 8 & 9 \end{bmatrix}$$

2.6.2 Switching Two Rows

To swap rows i and k, we use:

$$S = I_m + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k}$$

This works by:

- Removing the 1's at positions (i, i) and (k, k) from the identity matrix
- Adding 1's at positions (i, k) and (k, i)

Example: Swapping rows 1 and 3:

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad SA = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

2.6.3 Adding a Multiple of One Row to Another

To replace row k with row $k + \alpha\times$ row i, use:

$$I_m + \alpha E_{k,i}$$

This adds α times row i to row k while leaving all other rows unchanged because:

- For any row $j \neq k$, the corresponding row in this matrix is just the standard basis row
- Row k becomes the sum of the standard basis row k plus α times the standard basis row i

Example: Adding 3 times row 1 to row 2:

$$I_3 + 3E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad (I_3 + 3E_{2,1})A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \\ 7 & 8 & 9 \end{bmatrix}$$

Example 2.1

Write the inverse of an elementary matrix and show it is an elementary matrix.

Multiplying a row by a nonzero scalar:

- **Operation:** Multiply row i by $\alpha \neq 0$.
- **Elementary Matrix:** $E = I_m + (\alpha - 1)E_{i,i}$
- **Inverse:** To reverse the operation, multiply row i by $1\backslash\alpha$. Hence, the inverse is

$$E^{-1} = I_m + \left(\frac{1}{\alpha} - 1\right) E_{i,i} = \; I_m + \frac{1-\alpha}{\alpha} E_{i,i}.$$

Swapping two rows:

- **Operation:** Swap rows i and k.
- **Elementary Matrix:** $S = I_m - E_{i,i} - E_{k,k} + E_{i,k} + E_{k,i}$
- **Inverse:** Since swapping the same two rows twice returns them to their original positions,

$$S^{-1} = S.$$

Adding a multiple of one row to another:

- **Operation:** Add α times row i to row k.
- **Elementary Matrix:** $E = I_m + \alpha\, E_{k,i}$
- **Inverse:** To undo the operation, subtract α times row i from row k. Therefore,

$$E^{-1} = I_m - \alpha\, E_{k,i}.$$

Example 2.2

Prove that every invertible matrix in $M_n(\mathbb{R})$ is a product of elementary matrices.

Let A be an invertible matrix in $M_n(\mathbb{R})$. Since A is invertible, we can use Gaussian elimination to transform A into the identity matrix I_n.

Let $E_1, E_2, \ldots, E_k$ be the elementary matrices corresponding to the row operations used in the elimination process. Then, we have:

$$\begin{aligned} \text{Multiplying a row by a scalar:} &\quad I_n + (\alpha - 1)E_{i,i} \\ \text{Swapping two rows:} &\quad I_n + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k} \\ \text{Adding a multiple of one row to another:} &\quad I_n + \alpha\, E_{k,i} \end{aligned}$$

Applying these in sequence to A gives:

$$E_k \cdots E_2 E_1 A = I_n$$

Since $E_k \cdots E_2 E_1 = I_n$, we can multiply both sides by $(E_k \cdots E_2 E_1)^{-1}$ on the left to obtain:

$$A = (E_k \cdots E_2 E_1)^{-1} I_n = (E_k \cdots E_2 E_1)^{-1}$$

Using the property:

$$(E_k \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1},$$

we can express A as a product of elementary matrices:

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Since each E_i is an elementary matrix, its inverse is also an elementary matrix. Therefore, A can be expressed as a product of elementary matrices.

2.7 EROs and Inverses

Elementary Row Operations can be used to find the inverse of a square matrix. Consider a square matrix $A \in M_n(\mathbb{F})$ (that is, an $n \times n$ matrix over a field $\mathbb{F}$). If A is invertible, let

$$A^{-1} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

be its inverse, where each $\mathbf{v}_i$ is the ith column of A^{-1}. By definition of the matrix inverse, we have

$$A\,A^{-1} = A\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} = I_n,$$

the $n \times n$ identity matrix. This implies that

$$A\mathbf{v}_i = \mathbf{e}_i, \quad \text{for each } i = 1, 2, \ldots, n,$$

where $\mathbf{e}_i$ is the ith column of I_n (which has a 1 in the ith row and 0 everywhere else). In other words, each column $\mathbf{v}_i$ of A^{-1} is the unique solution to the linear system

$$A\mathbf{v}_i = \mathbf{e}_i.$$

To find A^{-1} effectively, we form the augmented matrix $[A \mid I_n]$ and apply EROs to trasnform A into I_n. When this is achieved, the augmented portion becomes A^{-1}. Thus, we have

$$\text{RREF}\big([\,A \mid I_n\,]\big) = \big[\,I_n \mid A^{-1}\big].$$

Example 2.3

Find A^{-1} if $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$.

We form a 3×6 matrix $A' = [A \mid I_3]$:

$$A' = \begin{bmatrix} 3 & 4 & -1 & | & 1 & 0 & 0 \\ 1 & 0 & 3 & | & 0 & 1 & 0 \\ 2 & 5 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

We apply the following EROs to A':

- $R_1 \leftrightarrow R_2$
- $R_2 \to R_2 - 3R_1$
- $R_3 \to R_3 - 2R_1$
- $R_3 \to R_3 + R - 2$
- $R_3 \leftrightarrow R_2$
- $R_3 \to R_3 - 4R_2$
- $R_3 \times (-\frac{1}{10})$
- $R_1 \to R_1 - 3R_3$

To obtain:

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & | & -1 & 1 & 1 \\ 0 & 0 & 1 & | & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

That is:

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

It is easily checked that $AA^{-1} = I_3$.

3 Vector Spaces and Subspace Structure

3.1 The Image and Kernel of a Linear Transformation

$T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation defined with:

$$M_T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix}$$

The **image** of T is the subset of $\mathbb{R}^3$ consisting of all elements $T(\mathbf{v}), \mathbf{v} \in \mathbb{R}^3$. This is the set of all vectors of the form:

$$a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

In matrix terms, this is the **column space** of M_T.
The **kernel** of T is the set of all vectors $\mathbf{v} \in \mathbb{R}^3$ such that $T(\mathbf{v}) = \mathbf{0}$. This is the set of all column vectors, whose entries, a, b, c satisfies:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The kernel is a line and the image is a plane

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 & 0 & | & 0 \\ 0 & -5 & 5 & | & 0 \\ 1 & 1 & 1 & | & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The kernel (or nullspace) is $\{-2t, t, t\}, t \in \mathbb{R}$, which is a line in $\mathbb{R}^3$.