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**MP232: Applied Mathematics**

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# 1 Prelim : The Exponential Function and Hyperbolic Functions

## 1.1 Exponential Function

**Derivative**

$$\frac{d}{dt}(e^{at}) = a e^{at}$$

**Integral**

$$\int e^{at} dt = \frac{1}{a} e^{at} + C$$

## 1.2 Hyperbolic Functions

**Definitions:**

$$\sinh(at) = \frac{e^{at} - e^{-at}}{2} \quad \bigg| \quad \cosh(at) = \frac{e^{at} + e^{-at}}{2} \quad \bigg| \quad \tanh(at) = \frac{\sinh(at)}{\cosh(at)}.$$

**Derivatives**

$$\frac{d}{dt}(\sinh(at)) = a \cosh(at), \quad \bigg| \quad \frac{d}{dt}(\cosh(at)) = a \sinh(at), \quad \bigg| \quad \frac{d}{dt}(\tanh(at)) = a \operatorname{sech}^2(at).$$

**Integrals**

$$\begin{aligned} \int \sinh(at) dt &= \frac{1}{a} \cosh(at) + C \\ \int \cosh(at) dt &= \frac{1}{a} \sinh(at) + C, \\ \int \tanh(at) dt &= \frac{1}{a} \ln|\cosh(at)| + C. \end{aligned}$$

**Common Identities**

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1, \\ \sinh(2x) &= 2 \sinh x \cosh x, \\ \cosh(2x) &= \cosh^2 x + \sinh^2 x, \\ \tanh(2x) &= \frac{2 \tanh x}{1 + \tanh^2 x}. \end{aligned}$$

## 1.3 Partial Fraction Decomposition

**Unrepeated Linear Factors:** A linear factor is of form  $(ax + b)$

$$\frac{s+1}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

**Repeated Linear Factors:**

$$\frac{3}{(s+2)^2(s-3)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-3}$$

**Unrepeated Quadratic Factors with complex roots:** Where the discriminant  $(b^2 - 4ac)$  is negative (complex roots) but the factor is not repeated

$$\frac{3}{(s^2 - s + 1)(s + 2)} = \frac{As + B}{s^2 - s + 1} + \frac{C}{s + 2}$$

**Repeated Quadratic Factors with complex roots:**

$$\frac{1}{(s^2 + 1)^2(s - 1)} = \frac{As + B}{(s^2 + 1)^2} + \frac{Cs + D}{s^2 + 1} + \frac{E}{s - 1}$$

## 2 Intro to Laplace Transforms

### 2.1 What is a Laplace Transform?

#### Definition

The Laplace Transform, defined for  $t \geq 0$ , is given by

$$L\{f(t)\}(s) = F(s) = \int_0^{\infty} e^{-st} dt$$

### 2.2 Common Laplace Transforms

#### Example Find the Laplace Transform of $f(t) = 1$

We have:

$$L\{1\} = \int_0^{\infty} 1 \cdot e^{-st} dt = \lim_{R \rightarrow \infty} \int_0^R e^{-st} dt$$

This integral is equal to:

$$\int_0^R e^{-st} dt = \left. \frac{e^{-st}}{-s} \right|_{t=0}^{t=R} = -\frac{1}{s} [e^{-sR} - 1] = \frac{1 - e^{-sR}}{s}$$

Taking the limit as  $R \rightarrow \infty$  gives:

$$L\{1\} = \lim_{R \rightarrow \infty} \frac{1 - e^{-sR}}{s} = \frac{1}{s}$$

#### Example Find the Laplace Transform of $f(t) = e^{2t}$

$$\begin{aligned} L\{e^{2t}\} &= \int_0^{\infty} e^{2t} e^{-st} dt = \int_0^{\infty} e^{-(s-2)t} dt \\ &= \lim_{R \rightarrow \infty} \int_0^R e^{-(s-2)t} dt \\ &= \lim_{R \rightarrow \infty} \left[ \frac{e^{-(s-2)t}}{-(s-2)} \right]_{t=0}^{t=R} \\ &= \lim_{R \rightarrow \infty} \left( \frac{e^{-(s-2)R} - e^0}{-(s-2)} \right) = \lim_{R \rightarrow \infty} \left( \frac{e^{-(s-2)R} - 1}{-(s-2)} \right) \\ &= \frac{1}{s-2} \quad (\text{since } e^{-(s-2)R} \rightarrow 0 \text{ as } R \rightarrow \infty \text{ provided } s > 2) \end{aligned}$$

#### Example Find the Laplace Transform of $f(t) = \cosh(at)$

We have:

$$\begin{aligned} L\{\cosh(at)\} &= L\left\{ \frac{e^{at} + e^{-at}}{2} \right\} \quad \text{from the definition of } \cosh(at) \\ &= \frac{1}{2} L\{e^{at}\} + \frac{1}{2} L\{e^{-at}\} \quad \text{by linearity of the Laplace Transform} \\ &= \frac{1}{2} \left( \frac{1}{s-a} \right) + \frac{1}{2} \left( \frac{1}{s+a} \right) \end{aligned}$$

Hence:

$$L\{\cosh(at)\} = \frac{s}{s^2 - a^2}$$

Noting that  $\sinh(at) = (e^{at} - e^{-at})/2$ , we can find that:

$$L\{\sinh(at)\} = \frac{a}{(s^2 - a^2)}$$

**Example Find the Laplace Transform of  $\cos(wt)$  and  $\sin(wt)$  where  $w$  is a constant.**

We first compute the Laplace Transform of  $e^{iwt}$  using its definition:

$$L\{e^{iwt}\} = \int_0^{\infty} e^{-st} e^{iwt} dt = \int_0^{\infty} e^{-(s-iw)t} dt = \frac{1}{s-iw}, \quad \text{for } \Re(s) > 0.$$

To express this in terms of real and imaginary parts, we multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{s-iw} = \frac{s+iw}{(s-iw)(s+iw)} = \frac{s+iw}{s^2+w^2}.$$

Since Euler's formula gives:

$$e^{iwt} = \cos(wt) + i\sin(wt),$$

the linearity of the Laplace Transform yields:

$$L\{e^{iwt}\} = L\{\cos(wt)\} + iL\{\sin(wt)\}.$$

Equating the two representations of  $L\{e^{iwt}\}$ , we have:

$$L\{\cos(wt)\} + iL\{\sin(wt)\} = \frac{s+iw}{s^2+w^2}.$$

Since the equality must hold for both the real and imaginary parts, we equate them separately:

$$L\{\cos(wt)\} = \frac{s}{s^2+w^2} \quad \text{and} \quad L\{\sin(wt)\} = \frac{w}{s^2+w^2}.$$

**2.3 Linearity of the Laplace Transform**

The Laplace Transform is a linear operator, i.e. for any constants  $a$  and  $b$ :

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

**Proof**

$$\begin{aligned} L\{af(t) + bg(t)\} &= \int_0^{\infty} e^{-st} (af(t) + bg(t)) dt \\ &= a \int_0^{\infty} e^{-st} f(t) dt + b \int_0^{\infty} e^{-st} g(t) dt \\ &= aL\{f(t)\} + bL\{g(t)\} \end{aligned}$$

**2.4 The First Shift Theorem****Theorem First Shift Theorem**

If  $f(t)$  has a Laplace Transform,  $F(s)$ , defined for  $s > k$ , then  $e^{at} f(t)$  has a Laplace Transform,  $F(s-a)$  defined for  $s-a > k$  and is given by:

$$L\{e^{at} f(t)\} = F(s-a)$$

or, taking the inverse Laplace Transform of both sides:

$$e^{at} f(t) = L^{-1}\{F(s-a)\}$$

**2.4.1 Examples****Example Find the Laplace Transform of  $e^{at} \cos(wt)$ , where  $a, w$  are constants.**

We know that  $L\{\cos(wt)\} = \frac{s}{s^2+w^2}$ , so by the First Shift Theorem:

$$\begin{aligned} L\{e^{at} \cos(wt)\} &= \frac{s-a}{(s-a)^2+w^2} \\ &= \frac{s-a}{s^2-2as+a^2+w^2} \end{aligned}$$

## 2.5 Existence of the Laplace Transform

Existence of a Laplace transform is not always guaranteed because we're integrating over an infinite integral. For a Laplace Transform to exist for a given  $s$ , then the integral must exist:

$$\int_0^{\infty} e^{-st} f(t) dt$$

### Theorem Existence Theorem of Laplace Transforms

Suppose  $f(t)$  is a piecewise continuous function on  $[0, \infty)$ . If  $f(t)$  satisfies:

$$|f(t)| \leq M e^{kt} \quad (0 \leq t < \infty)$$

for some constants,  $M, k$ , then the Laplace Transform of  $f(t)$  exists for  $s > k$ . In other words, the Laplace Transform of  $f(t)$  exists if  $f(t)$  is bounded by an exponential function.

### Proof

If  $s > k$ , then from the equation above, we have:

$$|F(s)| = \left| \int_0^{\infty} f(t) e^{-st} dt \right| \leq \int_0^{\infty} |f(t)| e^{-st} dt \leq \int_0^{\infty} M e^{(k-s)t} dt = \frac{M}{s-k}$$

## 3 Applications of Laplace Transforms

### 3.1 Integration by Parts

Starting with the product rule:

$$\frac{d}{dx}[uv] = u'v + uv',$$

we can express this in differential form as:

$$d(uv) = u dv + v du.$$

Integrate both sides with respect to  $x$ :

$$\int d(uv) = \int_a^b u dv + \int_a^b v du.$$

The Fundamental Theorem of Calculus tells us that the left-hand side is simply:

$$uv = \int_a^b u dv + \int_a^b v du.$$

Rearrange to solve for the desired integral:

$$\int_a^b u dv = uv - \int_a^b v du,$$

### 3.1.1 Examples

**Example** Use integration by parts to find the Laplace of  $f(t) = t$

$$L\{t\} = \int_0^{\infty} t e^{-st} dt$$

We integrate by parts by setting:

$$u = t, \quad dv = e^{-st}, \quad du = 1, \quad v = -\frac{e^{-st}}{s}$$

Then integrating by parts gives:

$$\begin{aligned} L\{t\} &= \left[ -\frac{te^{-st}}{s} \right]_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt \\ &= 0 + \frac{1}{s} \left[ -\frac{e^{-st}}{s} \right]_0^{\infty} \end{aligned}$$

Hence:

$$L\{t\} = \frac{1}{s^2}$$

**Example** Use integration by parts to find the Laplace of  $f(t) = \cos(t)$

Let:

$$u = e^{-st}, \quad du = -se^{-st}, \quad dv = \cos(t), \quad v = \sin(t)$$

Then:

$$\int_0^{\infty} e^{-st} \cos(t) dt = \left[ e^{-st} \sin(t) \right]_0^{\infty} + \int_0^{\infty} \sin(t) \cdot se^{-st} dt = 0 + s \int_0^{\infty} e^{-st} \sin(t) dt$$

Considering the sin part :

$$u = e^{-st}, \quad du = -se^{-st}, \quad dv = \sin(t), \quad v = -\cos(t)$$

$$\int_0^{\infty} e^{-st} \sin(t) dt = 1 - s \int_0^{\infty} e^{-st} \cos(t) dt$$

Substituting this back into the original integral gives:

$$\int_0^{\infty} e^{-st} \cos(t) dt = 1 - s \int_0^{\infty} e^{-st} \cos(t) dt = s - s^2 \int_0^{\infty} e^{-st} \cos(t) dt$$

$$L\{\cos(t)\} = \frac{s}{1 + s^2}$$

## 3.2 Table of Laplace Transforms

$f(t)$	$L\{f(t)\}$
1	$\frac{1}{s}, s > 0$
$t$	$\frac{1}{s^2}, s > 0$
$t^n, n = 0, 1, 2, 3$	$\frac{n!}{s^{n+1}}, s > 0$
$e^{at}$	$\frac{1}{s-a}, s > a$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh(at)$	$\frac{s}{s^2 - a^2}, s > a \geq 0$
$\sinh(at)$	$\frac{a}{s^2 - a^2}, s > a \geq 0$
$e^{at} \cos(\omega t)$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$e^{at} \sin(\omega t)$	$\frac{\omega}{(s-a)^2 + \omega^2}$
$e^{at} f(t)$	$F(s-a)$

### 3.3 Laplace Transforms of Derivatives

#### Theorem Laplace Transform of Derivatives

Suppose that  $f(t)$  and  $f'(t)$  are continuous and that  $|f(t)| \leq Me^{kt}$ ,  $\forall t \geq 0$  and for constants  $M, k$ . Then the Laplace Transform of  $f'(t)$  exists for  $s > k$  and is given by:

$$L\left\{\frac{df}{dt}\right\} = sL\{f\} - f(0) \quad \text{for } s > k$$

We can easily extend this to higher order derivatives. Assume the Laplace Transform of  $f^{(n)}(t)$  exists for  $s > k$  and is given by:

$$L\left\{\frac{d^n f}{dt^n}\right\} = s^n L\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0) \quad \text{for } s > k$$

#### 3.3.1 Examples

##### Example Find $L\{t^2\}$ using the fact $L\{s\} = 1/s$ for $s > 0$

$$L\{f''\} = s^2 L\{f\} - sf(0) - f'(0)$$

With  $f(t) = t^2$ . Since  $f'(t) = 2t$ ,  $f''(t) = 2$ ,  $f'(0) = 0$ ,  $f(0) = 0$ , gives:

$$L\{2\} = s^2 L\{t^2\} - s \cdot 0 - 0$$

So that:

$$L\{t^2\} = \frac{L\{2\}}{s^2} = \frac{2}{s^3}$$

##### Example Find $L\{\sin(t)\}$ and $L\{\cos(t)\}$

We again use the equation:

$$L\{f''\} = s^2 L\{f\} - sf(0) - f'(0)$$

With  $f(t) = \sin(t)$ ,  $f'(t) = \cos(t)$ ,  $f''(t) = -\sin(t)$ ,  $\sin(0) = 0$ ,  $\cos(0) = 1$ . This gives:

$$L\{-\sin(t)\} = s^2 L\{\sin(t)\} - s \cdot 0 - 1$$

So that:

$$L\{\sin(t)\} = \frac{1}{s^2 + 1}$$

Similarly, we can find:

$$L\{\cos(t)\} = \frac{s}{s^2 + 1}$$

### 3.4 Solving Initial Value Problems

Consider an example from mechanics: A particle of mass  $m > 0$  lies on rough table, attached to a spring of stiffness  $k > 0$ . At any time  $t > 0$ , the mass is a distance  $x(t)$  from the equilibrium position  $O$ , and  $x(t)$  is much less than the length of the spring.

The mass is subject to a driving force  $F_d(t)$ , from Newton's second law, we have:

$$F_d(t) - kx - \gamma \frac{dx}{dt} = m \frac{dx^2}{dt^2}$$

Where  $\gamma > 0$  is the **damping constant** and the term  $\gamma \frac{dx}{dt}$  models the **friction due to roughness** of the table, which opposes direction of motion. The **restoring force** due to the spring is  $-kx$ ; and always points towards  $O$ . The term  $m \frac{dx^2}{dt^2}$  is the **acceleration of the mass**. We can rewrite this as:

$$F_d(t) = m \frac{dx^2}{dt^2} + \gamma \frac{dx}{dt} + kx$$

In order to solve this, we also need initial displacement  $v_0 = x(0)$  and initial velocity  $v_0 = \frac{dx}{dt}(0)$ .

#### 3.4.1 Examples

##### Example

$$\frac{dx^2}{dt^2} + 3 \frac{dx}{dt} + 2x = 0, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1$$

1. Take Laplace of governing equation:

$$L \left\{ \frac{dx^2}{dt^2} \right\} = s^2 L\{x\} - sx(0) - x'(0) = s^2 L\{x\} - 1$$

$$L \left\{ \frac{dx}{dt} \right\} = sL\{x\} - x(0) = sLx$$

Hence:

$$s^2 L\{x\} - 1 + 3sL\{x\} + 2L\{x\} = 0$$

This is known as the **subsidiary equation**. Rearranging:

$$(s^2 + 3s + 2)L\{x\} = 1$$

2. Solve the subsidiary equation:

$$L\{x\} = \frac{1}{s^2 + 3s + 2}$$

3. Find the inverse Laplace Transform:

$$x(t) = L^{-1} \left\{ \frac{1}{s^2 + 3s + 2} \right\} = L^{-1} \left\{ \frac{1}{(s+1)(s+2)} \right\}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$$

Hence:

$$A(s+2) + B(s+1) = 1 \rightarrow A = 1, B = -1$$

Thus:

$$x = L^{-1} \left\{ \frac{1}{s+1} - \frac{1}{s+2} \right\} = L^{-1} \left\{ \frac{1}{s+1} \right\} - L^{-1} \left\{ \frac{1}{s+2} \right\} = e^{-t} - e^{-2t}$$



## 4 Step Functions, Advanced Theorems, and Special Functions

### 4.1 Heaviside Step Function

Denote the Heaviside Step Function as  $H(t)$ , defined as:

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Clearly, for any constant  $a$ , we have:

$$H(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

#### 4.1.1 Examples

**Example Express this function in terms of a Heaviside step function**

$$f(t) = \begin{cases} 3 & \text{for } 0 \leq t < 2 \\ -1 & \text{for } t > 2 \end{cases}$$

We write the general form of  $f(t)$  as:

$$f(t) = \alpha + \beta H(t - 2)$$

Now, settings  $t$  to any value  $\in [0, 2)$  gives:

$$f(t) = \alpha + (\beta)(0) = 0 \Rightarrow \alpha = 3$$

Setting  $t$  to any value greater than 2 gives:

$$f(t) = 3 + (\beta)(1) = -1 \Rightarrow \beta = -4$$

Thus, we have:

$$f(t) = 3 - 4H(t - 2)$$

**Example Express this function in terms of a Heaviside step function**

$$f(t) = \begin{cases} 3 & \text{for } 0 \leq t < 1 \\ 5 & \text{for } 1 < t < 3 \\ -2 & \text{for } t > 3 \end{cases}$$

We write the general form of  $f(t)$  as:

$$f(t) = \alpha + \beta H(t - 1) + \gamma H(t - 3)$$

Setting  $t$  to any value  $\in [0, 1)$  gives:  $f(t) = \alpha + (\beta)(0) + (\gamma)(0) = 0 \Rightarrow \alpha = 3$

Setting  $t$  to any value  $\in (1, 3)$  gives:  $f(t) = 3 + (\beta)(2) + (\gamma)(0) = 5 \Rightarrow \beta = 2$

Setting  $t$  to any value greater than 3 gives:  $f(t) = 3 + 2 + (\gamma)(1) = -7 \Rightarrow \gamma = -5$  Thus, we have:

$$f(t) = 3 + 2H(t - 1) - 5H(t - 3)$$

## 4.2 The Second Shift Theorem

### Theorem

If  $f(t)$  has the transform  $F(s)$  ( $s > k$ ) then the shifted function,

$$\tilde{f}(t) = f(t-a)H(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

has the transform  $e^{-as}F(s)$  ( $s > k$ ), that is:

$$L\{f(t-a)H(t-a)\} = e^{-as}F(s)$$

### Proof

$$L\{f(t-a)H(t-a)\} = \int_0^\infty e^{-st} f(t-a)H(t-a) dt = \int_a^\infty e^{-st} f(t-a) dt$$

We introduce a new integration variable  $\tau = t - a$ , we have

$$\int_0^\infty e^{-s(\tau+a)} f(\tau) d\tau = e^{-as} \int_0^\infty e^{-s\tau} f(\tau) d\tau = e^{-as}F(s)$$

### 4.2.1 Examples

**Example :** Find the Laplace Transform of  $H(t-a)$  for  $a > 0$

$$L\{H(t-a)\} = L\{H(t-a)f(t-a)\} = e^{-as}F(s) = \frac{e^{-as}}{s}$$

## 4.3 Practice Problems

- Use the First Shift Theorem ( $L\{e^{at}f(t)\} = F(s-a)$ ) to find the Laplace transform of the following functions:

(a)  $t^3e^{-3t}$  (b)  $e^{-t}\cos(2t)$  (c)  $e^{-4t}\cosh(5t)$  (d)  $e^{-t}\sin^2(t)$

- Use the First Shift Theorem ( $L^{-1}\{F(s-a)\} = e^{at}f(t)$ ) to find the inverse Laplace transform of the following functions:

(a)  $\frac{6s-4}{s^2-4s+20}$  (b)  $\frac{3s+7}{s^2-2s-3}$  (c)  $\frac{4s+12}{s^2+8s+16}$

- Solve the following initial value problems using the method of Laplace transforms:

$$y'' + y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = 1;$$

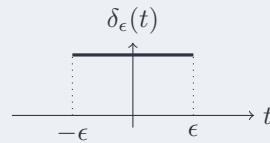
$$y'' - y = t, \quad y(0) = 1, \quad y'(0) = 1.$$

- Find the inverse Laplace transform of the following functions using the method of partial fractions:

(a)  $\frac{2s^2-4}{(s+1)(s-2)(s-3)}$  (b)  $\frac{5s^2-15s-11}{(s+1)(s-2)^3}$  (c)  $\frac{3s+1}{(s-1)(s^2+1)}$  (d)  $\frac{e^{-5s}}{(s^2+1)(s^2+2)}$

## 4.4 The Dirac Delta Function

The **Dirac Delta Function** models extremely brief but intense forces like a hammer hitting a nail. It starts as a function  $\delta_\epsilon$ , that equals  $\frac{1}{2\epsilon}$  over the interval  $t \in [-\epsilon, \epsilon]$  and 0 elsewhere.



$$\delta_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon} & \text{if } t \in [-\epsilon, \epsilon], \\ 0 & \text{otherwise.} \end{cases}$$

This function creates a rectangular pulse with the following properties:

$$\text{Height: } \frac{1}{2\epsilon} \quad \text{Width: } 2\epsilon \quad \text{Area: } 1 \text{ (always)}$$

As  $\epsilon$  approaches 0, the function becomes infinitely tall and thin, but the area remains 1. This limit defines the Dirac Delta Function:

$$\delta(t) = \lim_{\epsilon \rightarrow 0^+} \{\delta_\epsilon(t)\}$$

**Properties of the Dirac Delta Function:**

$$\begin{aligned} \delta(t) &= 0 \text{ for } t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t) dt &= 1 \\ \int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt &= f(t_0) \end{aligned}$$

The Laplace Transform of the Dirac Delta Function is:

$$L\{\delta(t - t_0)\} = \int_0^{\infty} e^{-st} \delta(t - t_0) dt = \int_{-\infty}^{\infty} e^{-st} \delta(t - t_0) dt = e^{-st_0} \quad \text{for } t_0 > 0$$

### 4.4.1 Examples

**Example :** Solve the following initial value problem which governs the behaviour of an RLC circuit

$$\begin{aligned} LQ'' + RQ' + \frac{Q}{C} &= V_0\delta(t-a) \\ Q(0) &= 0 \\ Q'(0) &= 0 \end{aligned}$$

Where  $a, L, R, C, V_0$  are all positive constants and  $4L > R^2C$ .  
Note that the applied voltage corresponds to an impulse of strength  $V_0$  at  $t = a$   
We note that:

$$\begin{aligned} L\{Q''\} &= s^2L\{Q\} - sQ(0) - Q'(0) = s^2L\{Q\} \\ L\{Q'\} &= sL\{Q\} - Q(0) = sL\{Q\} \\ L\{\delta(t-a)\} &= e^{-st_0} = e^{-as} \end{aligned}$$

Thus:

$$L\{LQ'' + RQ' + \frac{Q}{C}\} = V_0\delta(t-a) = Ls^2L\{Q\} + RsL\{Q\} + \frac{1}{C}L\{Q\} = V_0e^{-as}$$

Grouping terms:

$$L\{Q\}(Ls^2 + Rs + \frac{1}{C}) = V_0e^{-as}$$

Hence:

$$L\{Q\} = V_0e^{-as} \cdot \frac{1}{Ls^2 + Rs + \frac{1}{C}}$$

Removing the  $L$  from the denominator gives:

$$\begin{aligned} L\{Q\} &= \frac{V_0}{L}e^{-as} \cdot \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \\ &= \frac{V_0}{L} \cdot \frac{e^{-as}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \end{aligned}$$

We notice that:

$$\begin{aligned} (s + \frac{R}{2L})^2 &= s^2 + s\frac{2R}{2L} + \frac{R^2}{4L^2} \\ &= s^2 + \frac{R}{L}s + \frac{R^2}{4L^2} \end{aligned}$$

So that:

$$L\{Q\} = \frac{V_0}{L} \frac{e^{-as}}{(s + \frac{R}{2L})^2 - \frac{R^2}{4L^2} + \frac{1}{LC}}$$

Rewriting with  $\alpha = \frac{R}{2L}$  and  $\beta = \frac{1}{LC} - \frac{R^2}{4L^2}$

$$L\{Q\} = \frac{V_0}{L} \frac{e^{-as}}{(s + \alpha)^2 + \beta}$$

We also note that:

$$L\{\sin(\beta t)\} = \frac{\beta}{s^2 + \beta^2} \xrightarrow{\text{First Shift Theorem}} L\{e^{-as}\sin(\beta t)\} = \frac{B}{(s + a)^2 + \beta^2}$$

Or,

$$L^{-1}\left\{\frac{B}{(s + a)^2 + \beta^2}\right\} = e^{-at}\sin(\beta t)$$

We can also write:

$$L^{-1}\{F(s)\} = f(t)$$

Notice that:

$$f(t-a) = e^{-a(t-a)}\sin(\beta[t-a])$$

Applying the Second Shift Theorem gives:

$$\begin{aligned} L^{-1}\{e^{-a}F(s)\} &= f(t-a)H(t-a) \\ L^{-1}\left\{e^{-as}\frac{\beta}{(s + a)^2 + \beta^2}\right\} &= e^{-a(t-a)}\sin(\beta[t-a])H(t-a) \end{aligned}$$

Thus:

$$\begin{aligned} Q(t) &= \frac{V_0}{L\beta}e^{-a(t-a)}\sin(\beta[t-a])H(t-a) \\ &= \begin{cases} 0 & \text{for } 0 \leq t < a \\ \frac{V_0}{L\beta}e^{-a(t-a)}\sin(\beta[t-a])H(t-a) & \text{for } t > a \end{cases} \end{aligned}$$

## 4.5 Differentiation of the Laplace Transform

Suppose  $f(t), t \geq 0$  satisfies the conditions of the existence theorem so that its Laplace Transform ( $F(s)$ ) exists for some  $s > k$ . Then:

$$F'(s) = \frac{d}{ds} \left\{ \int_0^\infty e^{-st} f(t) dt \right\} = \int_0^\infty \frac{\partial}{\partial s} \{ e^{-st} f(t) \} dt$$

We are allowed to bring the derivative inside the integral provided the conditions of the existence theorem are satisfied, hence:

$$F'(s) = - \int_0^\infty e^{-st} \{ t f(t) \} dt = -L\{ t f(t) \}$$

so that,

$$L\{ t f(t) \} = -F'(s)$$

We can sometimes use this to calculate transforms and inverse transforms. For example:

$$L\{ t \} = L\{ t \cdot 1 \} = -\frac{d}{ds} L\{ 1 \} = -\frac{d}{ds} \left( \frac{1}{s} \right) = \frac{1}{s^2}$$

## 4.6 The Convolution Function

Let  $f(t), g(t)$  be two functions. Define the Convolution function

$$(f \star g)(t) = \int_0^t f(t - \tau) g(\tau) d\tau$$

Where  $\tau$  is integrated over the interval  $[0, t]$ . The Convolution is:

$$\textbf{Commutative: } f \star g = g \star f$$

$$\textbf{Associative: } f \star (g \star h) = (f \star g) \star h$$

$$\textbf{Distributive: } f \star (g + h) = f \star g + f \star h$$

$$\textbf{Multiplication by 0: } f \star (ag) = a(f \star g)$$

### Theorem

Let  $f(t)$  and  $g(t)$  have Laplace Transforms  $F(s)$  and  $G(s)$  respectively defined for  $s > k \geq 0$ . Then

$$L\{ f \star g \} = F(s)G(s), \quad s > k$$

### Proof

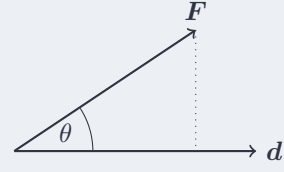
Write  $F(s) = \int_0^\infty e^{-s\sigma} f(\sigma) d\sigma$  and  $G(s) = \int_0^\infty e^{-s\tau} g(\tau) d\tau$ . Then:

$$\begin{aligned} F(s)G(s) &= \left\{ \int_0^\infty e^{-s\sigma} f(\sigma) d\sigma \right\} \left\{ \int_0^\infty e^{-s\tau} g(\tau) d\tau \right\} \\ &= \int_0^\infty e^{-s\tau} g(\tau) \left\{ \int_0^\infty e^{-s\sigma} f(\sigma) d\sigma \right\} d\tau \\ &= \int_0^\infty g(\tau) \left\{ \int_0^\infty e^{-s(\sigma+\tau)} f(\sigma) d\sigma \right\} d\tau. \end{aligned}$$

## 5 Line Integrals

Consider a mass which undergoes a displacement,  $\mathbf{d}$ , under a constant force  $\mathbf{F}$ . Define the work,  $\mathbf{W}$ , done by  $\mathbf{F}$  to be the magnitude of the force multiplied by the distance moved in the direction of the force. Inspecting the diagram, we see that work done  $\mathbf{W}$  is given by the dot product of  $\mathbf{F}$  and  $\mathbf{d}$ :

$$W = |\mathbf{F}| \cdot |\mathbf{d}| \cdot \cos(\theta) = \mathbf{F} \cdot \mathbf{d}$$



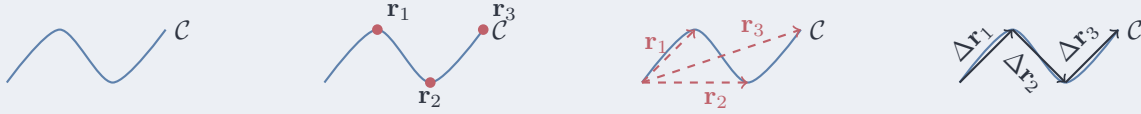
Now, let's suppose  $F$  is not constant:

$$F = F(x, y, z) = F(r) = r(x, y, z)$$

Suppose further, that  $F$  acts for a time  $t_1 \leq t \leq t_2$  and the path of the object in this time interval is given by a curve  $\mathcal{C}$  defined by:

$$\mathbf{r} = (x(t), y(t), z(t)) \quad t \in [t_1, t_2]$$

But how do we calculate the work done by  $F$  along  $\mathcal{C}$ ?



As seen as the diagram above, we can divide  $\mathcal{C}$  into a large number  $N - 1$  of small segments of  $\Delta \mathbf{r}_i$  and approximate the work done by  $F$  along  $\mathcal{C}$  by the sum of the work done along each segment:

$$W \approx \sum_{i=1}^{N-1} F(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i$$

### 5.1 The Line Integral

Taking the limit  $N \rightarrow \infty$

$$W = \lim_{N \rightarrow \infty} \left\{ \sum_{i=1}^{N-1} F(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i \right\}$$

This limit is called the **line integral** of  $F$  along  $\mathcal{C}$  and is denoted by  $\int_{\mathcal{C}} F(r) \cdot d\mathbf{r}$ , that is:

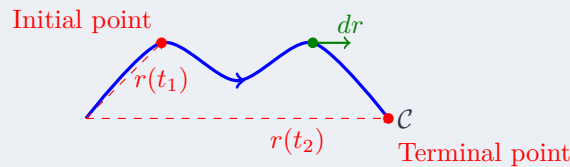
$$\int_{\mathcal{C}} F(r) \cdot d\mathbf{r} = \lim_{N \rightarrow \infty} \left\{ \sum_{i=1}^{N-1} F(r_i) \cdot \Delta r_i \right\}$$

Since  $r = r(t)$  we can calculate the line integral as:

$$\int_{\mathcal{C}} F(r) \cdot d\mathbf{r} = \int_{t_1}^{t_2} F(r(t)) \cdot \frac{d\mathbf{r}}{dt} dt$$

In general,  $t$ , may be any variable that parametrizes (traces out) the curve  $\mathcal{C}$ . Then  $d\mathbf{r} = \frac{d\mathbf{r}}{dt} dt$  is the tangent vector to  $\mathcal{C}$  at the point  $r(t)$ . We call  $\mathcal{C}$  the **path of integration** and  $r(t_1)$  the initial point,  $r(t_2)$  the **terminal point**.  $\mathcal{C}$  is now **oriented** from  $r(t_1)$  to  $r(t_2)$ .

The direction for  $r(t_1) \rightarrow r(t_2)$ , in which  $t$  increases, is called the **positive direction** of  $\mathcal{C}$ , we indicate this by an arrow on  $\mathcal{C}$ .



If  $r(t_1) = r(t_2)$  then  $\mathcal{C}$  is a **closed curve** and the line integral is denoted by:

$$\oint_{\mathcal{C}} F(r) \cdot d\mathbf{r}$$

The line integral of  $F$  along a closed curve  $\mathcal{C}$  is called the **circulation** of  $F$  around  $\mathcal{C}$ .

### 5.1.1 Examples

**Example :** For a time period  $0 \leq t \leq 1$ , a particle moves along a trajectory defined by  $\mathcal{C} = x = t, y = t, z = 2t^2$ , a force  $F(r) = (y, x, z)$  acts. Calculate work done.

We have:

$$\begin{aligned}\mathbf{r} &= (t, t, 2t^2) \\ \frac{d\mathbf{r}}{dt} &= (1, 1, 4t) \\ F(\mathbf{r}) &= (t, t, 2t^2)\end{aligned}$$

The work done is:

$$\begin{aligned}\int_{\mathcal{C}} F(r) \cdot dr &= \int_0^1 (t, t, 2t^2) \cdot (1, 1, 4t) dt \\ &= \int_0^1 (t + t + 8t^3) dt \\ &= \int_0^1 (2t + 8t^3) dt \\ &= [t^2 + 2t^4]_0^1 \\ &= 1 + 2 \\ &= 3\end{aligned}$$

**Example Evaluate  $\int_{\mathcal{C}} F(r) \cdot dr$  for  $F(r) = (y + z, x + z, x + y)$ , where  $\mathcal{C}$  is the line segment jointint  $A : (-2, 2, -3) \rightarrow B(2, 4, 6)$**

We can parametrize any straight line in three dimensions by writing:

$$x = c_1 t + c_2 \quad y = c_3 t + c_4 \quad z = c_5 t + c_6$$

where the  $c_i$  are constants.

We can take  $t = 0$  to correspond with  $A$  and  $t = 1$  to correspond with  $B$ .

Hence setting  $t = 0$  gives:

$$c_2 = -2 \quad c_4 = 2 \quad c_6 = -3$$

Setting  $t = 1$  gives:  $c_1 + c_2 = 2 \quad c_3 + c_4 = 4 \quad c_5 + c_6 = 6$  so that:

$$c_1 = 4 \quad c_3 = 2 \quad c_5 = 9$$

Hence, the required line is given by:

$$x = 4t - 2 \quad y = 2t + 2 \quad z = 9t - 3$$

We have:

$$\begin{aligned}\int_{\mathcal{C}} F(r) \cdot dr &= \int_0^1 (11t - 1, 13t - 5, 6t) \cdot \frac{d}{dt}(4t - 2, 2t + 2, 9t - 3) dt \\ &= \int_0^1 (11t - 1, 13t - 5, 6t) \cdot (4, 2, 9) dt \\ &= \int_0^1 (44t - 4 + 26t - 10 + 54t) dt \\ &= \int_0^1 (124t - 12) dt \\ &= [62t^2 - 12t]_0^1 \\ &= 48\end{aligned}$$

**Example** Find the circulation of the vector  $F = (y, -x, 0)$  around the unit circle  $\mathcal{C} = x^2 + y^2 = 1, z = 0$ , taken in an anti-clockwise direction

We can parametrize the unit circle by setting  $x, y$  and  $z$  to:

$$x = \cos(t) \quad y = \sin(t) \quad z = 0$$

We can write the position vector as:

$$\mathbf{r} = (\cos(t), \sin(t), 0)$$

The tangent vector is:

$$\frac{d\mathbf{r}}{dt} = (-\sin(t), \cos(t), 0)$$

The force vector is:

$$F(\mathbf{r}) = (\sin(t), -\cos(t), 0)$$

The circulation is:

$$\oint_{\mathcal{C}} F(r) \cdot dr = \int_0^{2\pi} (\sin(t), -\cos(t), 0) \cdot (-\sin(t), \cos(t), 0) dt \quad (1)$$

$$= \int_0^{2\pi} -\sin^2(t) - \cos^2(t) dt \quad (2)$$

$$= - \int_0^{2\pi} (\sin^2(t) + \cos^2(t)) dt \quad (3)$$

$$= - \int_0^{2\pi} 1 dt \quad (4)$$

$$= - [t]_0^{2\pi} \quad (5)$$

$$= -2\pi \quad (6)$$

## 5.2 Conservative Vector Fields

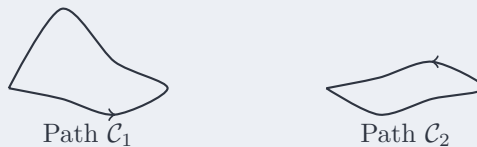
A vector field  $F$  is called **conservative** if the line integral of  $F$  along any closed curve  $\mathcal{C}$  is zero, that is:

$$\oint_{\mathcal{C}} F(r) \cdot dr = 0$$

An equivalent definition is that  $F$  is conservative if the line integral of  $F$  depends only on the end points of the curve, not on the path taken, so that:

$$\int_{\mathcal{C}_1} F(r) \cdot dr = \int_{\mathcal{C}_2} F(r) \cdot dr$$

Where  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are two curves with the same initial and terminal points but different paths.



Consider two curves,  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , that start at  $A$  and end at  $B$ . Let  $\mathcal{C}$  be the closed curve that starts at  $A$  follows the curve  $\mathcal{C}_1$  and then follows  $\mathcal{C}_2$  in the reverse direction to  $B$ . Then:

$$\begin{aligned} \oint_{\mathcal{C}} F(r) \cdot dr &= \int_{AC_1}^B F(r) \cdot dr + \int_{BC_2}^A F(r) \cdot dr \\ &= \int_{AC_1}^B F(r) \cdot dr - \int_{AC_2}^B F(r) \cdot dr = 0 \end{aligned}$$

Thus:

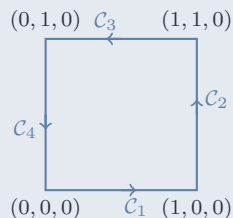
$$\oint_{\mathcal{C}} F(r) \cdot dr = 0 \Rightarrow \int_{AC_1}^B F(r) \cdot dr = \int_{AC_2}^B F(r) \cdot dr$$



### 5.2.1 Examples

**Example** By considering the line integral of  $F = (y, x^2 - x, 0)$  around the square  $C$  in the  $x, y$  plane connecting for point  $(0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 1, 0)$ , show that  $F$  cannot be conservative.

Split  $C$  into four segments,  $C_1, C_2, C_3, C_4$  and calculate the line integral of  $F$  along each segment.



We have:

$$\begin{aligned} \int_{C_1} F(r) \cdot dr &= \int_0^1 (0, t^2 - t, 0) \cdot (1, 0, 0) dt = 0 \\ \int_{C_2} F(r) \cdot dr &= \int_0^1 (t, 1 - t, 0) \cdot (0, 1, 0) dt = \int_0^1 (1 - t) dt = [t - \frac{t^2}{2}]_0^1 = 1 - \frac{1}{2} = \frac{1}{2} \\ \int_{C_3} F(r) \cdot dr &= \int_0^1 (1, (1 - t)^2 - (1 - t), 0) \cdot (-1, 0, 0) dt = \int_0^1 -1 dt = -1 \\ \int_{C_4} F(r) \cdot dr &= \int_0^1 (1 - t, 0, 0) \cdot (0, -1, 0) dt = 0 \end{aligned}$$

Hence:

$$\oint_C F(r) \cdot dr = \int_{C_1} F(r) \cdot dr + \int_{C_2} F(r) \cdot dr + \int_{C_3} F(r) \cdot dr + \int_{C_4} F(r) \cdot dr = 0 + \frac{1}{2} + (-1) + 0 = -\frac{1}{2} \neq 0$$

Thus,  $F$  is not conservative.

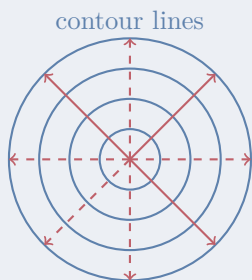
## 6 Gradient, Divergence and Curl

### 6.1 Gradient

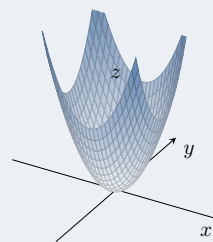
#### Definition

The gradient of a differentiable function  $\phi(x, y, z)$  is the vector field:

$$\text{grad}(\phi) = \nabla \phi = \frac{d\phi}{dx} \hat{i} + \frac{d\phi}{dy} \hat{j} + \frac{d\phi}{dz} \hat{k}$$



(a) Contour Plot with Gradient Vectors

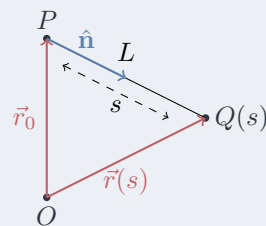


(b) 3D Surface Plot

Figure 1: Illustration of the function  $\phi(x, y) = x^2 + y^2$  with its gradient vectors and corresponding 3D surface.

### 6.1.1 The Directional Derivative

Consider a differentiable function  $\phi(x, y, z)$  at a point  $P$ , with a position vector  $\vec{r}_0$ . We want to find the rate of change of  $\phi$  as we move from  $P$  in the direction of  $\hat{\mathbf{n}}$ . Consider the line segment  $L$  through  $P$  in the direction of  $\hat{\mathbf{n}}$ . Let  $Q(s)$  be the point on  $L$  a distance  $s$  from  $P$ . Define the **Directional Derivative** of  $\phi(x, y, z)$  in the direction of  $\hat{\mathbf{n}}$  by:



$$D_{\hat{\mathbf{n}}}\phi = \lim_{s \rightarrow 0} \frac{\phi(Q(s)) - \phi(P)}{s}$$

That is, as  $s$  gets smaller and smaller we get a better and better approximation of  $\phi$  at  $P$ . With  $L$  described by  $\vec{r}(s) = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$ , we have:

$$\begin{aligned} D_{\hat{\mathbf{n}}}\phi &= \lim_{s \rightarrow 0} \frac{\phi(\vec{r}(s)) - \phi(\vec{r}(0))}{s} \\ &= \lim_{s \rightarrow 0} \frac{\phi(x(s), y(s), z(s)) - \phi(x(0), y(0), z(0))}{s} \\ &= \frac{d\phi}{ds} \end{aligned}$$

Writing  $\hat{\mathbf{n}} = (n_1, n_2, n_3)$  and using the chain rule:

$$\begin{aligned} D_{\hat{\mathbf{n}}}\phi &= \frac{d\phi}{ds} = \frac{d\phi}{dx} \frac{dx}{ds} + \frac{d\phi}{dy} \frac{dy}{ds} + \frac{d\phi}{dz} \frac{dz}{ds} \\ &= \frac{d\phi}{dx} n_1 + \frac{d\phi}{dy} n_2 + \frac{d\phi}{dz} n_3 \end{aligned} \quad (7)$$

and so with  $|\hat{\mathbf{n}}| = 1$ , we have:

$$D_{\hat{\mathbf{n}}}\phi = \hat{\mathbf{n}} \cdot \nabla \phi$$

**Example** Find the directional derivative of  $\phi(x, y, z) = 2x^2 + 3y^2 + z^2$  at the point  $P :: (2, 1, 3)$  in the direction of the vector  $\vec{a} = \hat{i} - 2\hat{k}$

First we calculate the gradient of  $\phi$ :

$$\nabla \phi = \frac{d}{dx}(2x^2)\hat{i} + \frac{d}{dy}(3y^2)\hat{j} + \frac{d}{dz}(z^2)\hat{k} = 4x\hat{i} + 6y\hat{j} + 2z\hat{k}$$

At  $P : (2, 1, 3)$ , the gradient is evaluated as

$$\nabla \phi = 4(2)\hat{i} + 6(1)\hat{j} + 2(3)\hat{k} = 8\hat{i} + 6\hat{j} + 6\hat{k}$$

Since  $\vec{a}$  is not a unit vector, i.e.  $|\vec{a}| \neq 1$ , we need to normalize it to get a unit vector  $\hat{\mathbf{n}}$ :

$$\hat{\mathbf{n}} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} - 2\hat{k}}{\sqrt{1+4}} = \frac{\hat{i} - 2\hat{k}}{\sqrt{5}}$$

The directional derivative is then:

$$D_{\hat{\mathbf{n}}}\phi|_P = \hat{\mathbf{n}} \cdot \nabla \phi|_P = \frac{\hat{i} - 2\hat{k} \cdot (8\hat{i} + 6\hat{j} + 6\hat{k})}{\sqrt{5}} = \frac{8 - 12}{\sqrt{5}} = -\frac{4}{\sqrt{5}}$$

*Note:* The fact the directional derivative is negative indicates that  $\phi$  decreases in the direction of  $\vec{a}$ .

### 6.1.2 Meaning of $\nabla\phi$

#### Theorem

Let  $\phi(x, y, z)$  be a scalar function for which  $\nabla\phi$  exists.  
Then if  $\nabla\phi|_P \neq 0$  at a point  $P$ , then  $\nabla\phi|_P$  is the direction of maximum increase of  $\phi$  at  $P$ .

#### Proof

From the definition of the dot product, we know that:

$$D_{\hat{\mathbf{n}}}\phi = \hat{\mathbf{n}} \cdot \nabla\phi = |\hat{\mathbf{n}}||\nabla\phi|\cos(\gamma) = |\nabla\phi|\cos(\gamma)$$

where  $\gamma$  is the angle between  $\hat{\mathbf{n}}$  and  $\nabla\phi$ .

For  $\nabla\phi|_P \neq 0$  then  $D_{\hat{\mathbf{n}}}\phi$  is maximized when  $\cos(\gamma) = 1$ , i.e. when  $\gamma = 0$ . That is, when  $\hat{\mathbf{n}}$  is parallel to  $\nabla\phi$ .

**Remark 1:** We note that  $-\nabla\phi|_P$  is the direction of maximum decrease of  $\phi$  at  $P$ .

**Remark 2:** If  $\nabla\phi|_P = 0$ , then,  $P$  is a critical point of  $\phi$ .

## 6.2 Gradient as Normal to a Surface

