

# **MA2287: Complex Analysis Exam Notes**

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## 1 Question 1:

### 1.1 Sketch the region in the complex plane determined by the inequality

- $|z - 4| > 3|z + 4|$  2023 Q1(a)
- $\{z \in \mathbb{C} : |2z - 1| < 2|2z - i|\}$  2022 Q1(a), 2021 Q1(d), 2017 Q1(a), 2016 Q1(a)

### 1.2 Determine all solutions to roots of unity

- $z^6 - 1 = 0$  and factorize  $x^6 - 1$  as a product of linear and quadratic factors 2023 Q1(b),
- $z^4 = -81i$  and find a polynomial  $p(z)$  with complex coefficients with root  $w$  and  $p(\bar{w}) \neq 0$  2022 Q1(b), 2018 Q1(b)
- $z^6 - 1 = 0$  and factorize  $x^6 - 1$  as a product of linear and quadratic factors 2021 Q1(c)
- $z^3 = 1 + i$ , let  $n \in \mathbb{N}$  and  $w \neq 1$  be an  $n$ -th root of unity. Prove  $1 + w + w^2 + \dots + w^{n-1} = 0$  2016 Q1(c)

### 1.3 Determine and sketch the image under the mapping

- $w = e^z$ ,  $\{z \in \mathbb{C} : \pi/4 \leq \text{Im}(z) \leq \pi/2\}$  2023 Q1(c), 2021 Q1(a), 2017 Q1(d)
- $w = \text{Log}(z)$ ,  $\{z : |z| > 1, 0 \leq \text{Arg}(z) \leq \pi/2\}$  2022 Q1(d), 2018 Q1(d), 2016 Q1(d)

### 1.4 Find $z$ where the function is 0

- $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$  2022 Q1(d)

### 1.5 Calculate principal value $\text{Log}(z)$

- $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$  and prove  $e^z$  is the inverse function of  $\text{Log}(z)$  2022 Q1(c), 2018 Q1(c), 2017 Q1(c)

### 1.6 Prove the following

- Define the complex conjugate ( $\bar{w}$ ) and prove if  $w$  is a zero of a polynomial  $p(z) = a_0 + a_1z + \dots + a_nz^n$  then  $\bar{w}$  is also a zero of  $p(z)$  2021 Q1(b), 2018 Q1(a), 2016 Q1(b)
- Define the complex exponential function  $e^z$  and prove Eulers Formula  $e^{i\theta} = \cos(\theta) + i\sin(\theta)$  2017 Q1(b)

## 2 Question 2:

### 2.1 Determine image of the line

- $f(z) = \frac{1}{z}$   $\{z \in \mathbb{C} : \text{Re}(z) = 2\}$  2023 Q2(a), 2021 Q2(a)
- $f(z) = \frac{1}{z}$   $\{z \in \mathbb{C} : \text{Re}(z) = 1\}$  2022 Q2(a), 2018 Q2(a)

### 2.2 State and Use Cauchy-Riemann Equations

- State CRE, and use to prove  $f(z) = \frac{1}{z}$  is holomorphic on  $\mathbb{C} \setminus \{0\}$  2023 Q2(a)
- State CRE, and use to prove  $f(z) = z^2$  is holomorphic on  $\mathbb{C}$  2022 Q2(b)

### 2.3 Show that

- If  $\overline{f(z)} = f(\bar{z})$  for all  $z \in \mathbb{C}$  then  $f(x)$  is real for all  $x \in \mathbb{R}$ . And if in addition  $f$  is holomorphic at  $x \in \mathbb{R}$  then  $f'(x)$  is real. 2023 Q2(c)
- Define that is meant for a function  $g$  to be harmonic. If  $f = u + iv$  is holomorphic on  $\Omega \subset \mathbb{C}$ , prove that  $v(x, y)$  is a harmonic function, and that  $\nabla u$  and  $\nabla v$  are perpendicular of equal length. 2022 Q2(c), 2018 Q2(b)
- If  $\overline{f(z)} = f(\bar{z})$  for all  $z \in \mathbb{C}$  then  $f(x)$  is real for all  $x \in \mathbb{R}$ . And if in addition  $f$  is holomorphic at 0 then  $f'(0)$  is real. 2021 Q2(a)
- Let  $f(z) = u + iv$  be holomorphic on an open subset  $\Omega$  of the complex plane and let  $h(u, v)$  be a harmonic function of  $u$  and  $v$  on  $f(\Omega)$ . Prove that  $g(x, y) = h(u(x, y), v(x, y))$  is harmonic on  $\Omega$  (You may assume  $\nabla u, \nabla v$  are equal length and perpendicular) 2021 Q2(c)

### 2.4 Find Mobius Transformation

- $T(z) : (-1, 1, \infty) \mapsto (-1, -i, 1)$  2023 Q1(d)
- $T(z) : (2, 1, -1) \mapsto (1, 0, \infty)$  2022 Q1(d)
- $T(z) : (-i, -1, 1) \mapsto (0, 1, \infty)$  and find the inverse Mobius Transformation 2021 Q1(d)