MA283: Linear Algebra

70% Exam
30% Continuous Assessment (Homework)
10% Optional Project (Bonus)

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Contents

1 Review of Matrix Algebra

Matrix Addition

If a matrix has m rows and n columns, we say it is $m \times n$.**Two matrices can only be added if they have the same size.**. In this case, we just add the entries in each position.

The $m \times n$ **zero** matrix is a matrix with all entries equal to 0. It is the **Identity element** for matrix addition (adding it to any matrix does not change the matrix)

Matrix Multiplication by a Scalar

This simply means multiplying each entry of the matrix by the scalar. For example:

$$\alpha \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \alpha & 2\alpha \\ 3\alpha & 4\alpha \end{bmatrix}$$

Remark: Now that we have addition and scalar multiplication, we can subtract matrices ($A - B = A + (-1)B$), provided they are the same size.

Vector Space

With these operations of addition and scalar multiplication, the set of $m \times n$ matrices is a vector space. A **vector space** algebraic structure whose elements can be added, subtracted and multiplied by scalars.

Linear Combinations

Definition 1.1: Linear Combinations

Suppose $v_1, v_2, \ldots, v_k$ are elements that can be added together and multiplied by scalars.

A Linear Combination of $v_1, v_2, \ldots, v_k$ is an expression of the form:

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k$$

where $a_i \in \mathbb{R}$ are scalars, called **coefficients**.

Matrix-Vector Multiplication

Definition 1.2

Let A be a $m \times n$ matrix, and $\mathbf{v}$ be a column vector with n entries ($n \times 1$ matrix).

Then the matrix vector product Av is the column vector, with m entries, obtained by taking the linear combination of the columns of A with the entries of $\mathbf{v}$ as coefficients.

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix} = 7 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 9 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 41 \\ 33 \end{bmatrix}.$$

Remark: Av, if defined, has the same number of rows as A and the same number of columns as $\mathbf{v}$.

Matrix-Matrix Multiplication

Definition 1.3

Let A and B be matrices of size $m \times p$ and $p \times n$, respectively.Write $v_1, \ldots v_n$ for the columns of B. Then the product AB is the $m \times n$ matrices whose columns are $Av_1, \ldots, Av_n$.

The entry at row i and column j of the matrix A is given by A_{ij}. The entry in the i, j position of the product AB is the ith entry of the vector Av_j, where the vector v_j is the jth column of B. In other words, the entry in the i, j position of the product AB is given by:

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \ldots + A_{ip}B_{pj} = \sum_{k=1}^{p} A_{ik}B_{kj}$$

Definition 1.4

If A is $m \times p$ with rows $u_1, \ldots, u_m$ and B is $p \times n$ with columns $v_1, \ldots, v_n$, then the product AB is:

$$AB = \begin{bmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & \ldots & u_1 \cdot v_n \\ u_2 \cdot v_1 & u_2 \cdot v_2 & \ldots & u_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m \cdot v_1 & u_m \cdot v_2 & \ldots & u_m \cdot v_n \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} \quad AB = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

For matrices A and B, the products AB and BA are generally not equal, even if they are both defined and even if both have the same size.

Linear Transformations

Definition 1.5

Let m and n be positive integers.

A **linear transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$
- $T(\lambda \mathbf{u}) = \lambda T(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

Matrix of a Linear Transformation

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation:

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 7 \end{bmatrix}$$

Then for the vector in $\mathbb{R}^3$ with entries a, b, c:

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = aT \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + bT \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + cT \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -6 \\ -3 & 4 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Where the 2×3 matrix M_T is called the **standard matrix** of A. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be completely represented by an $m \times n$ matrix M_T.

Understanding the Matrix Representation

- The columns of matrix M_T are the images of the standard basis vectors $e_1, e_2, \ldots, e_n$ under T.
- For any vector $v \in \mathbb{R}^n$, we calculate $T(v)$ by multiplying: $M_T \cdot v$.
- Therefore, matrix-vector multiplication is simply evaluating a linear transformation.

Correspondence: Any $m \times n$ matrix A defines a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by: $T_A(v) = Av$. Linear transformations include rotations, reflections and scaling

Efficiency of Representation: A remarkable property of linear transformations is their information efficiency:

- To completely define $T : \mathbb{R}^n \to \mathbb{R}^m$, we need only mn values.
- These values are the coordinates of the n transformed basis vectors in $\mathbb{R}^m$.
- This differs fundamentally from general continuous functions $f : \mathbb{R} \to \mathbb{R}$, which cannot be fully determined by their values at finitely many points.

Matrix multiplication is composition

Suppose that $T : \mathbb{R}^n \to \mathbb{R}^p$ and $S : \mathbb{R}^p \to \mathbb{R}^m$ are linear transformations. Then the composition $S \circ T : \mathbb{R}^n \to \mathbb{R}^m$ is also a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined for $\mathbf{v} \in \mathbb{R}^n$ by:

$$S \circ T(\mathbf{v}) = S(T(\mathbf{v}))$$

To see how that the $m \times n$ matrix $M_{S \circ T}$ depends on the matrix $M_S(m \times p)$ and $M_T(p \times n)$ we look at the definition of $M_{S \circ T}$:

- The first column has coordinates $S \circ T(e_1) = S(T(e_1))$
- $T(e_1)$ is first column of M_T
- Then $S(T(e_1))$ is the matrix-vector product $M_S \cdot M_T(e_1)$
- Same for all other columns $\implies M_{S\circ T} = M_S \cdot M_T$

Thus, we conclude **matrix multiplication is composition of linear transformations**.

2 Systems of linear equations

2.1 Linear equations and Solution Sets

A linear equation in the variables x and y is an equation of the form

$$2x + y = 3$$

If we replace x and y with some numbers, the statement **becomes true or false**.

Definition 2.1: Solution to a linear equation

A pair, $(x_0, y_0) \in \mathbb{R}$, is a solution to an linear equation if setting $x = x_0$ and $y = y_0$ **makes the equation true.**

Definition 2.2: Solution set

The **solution set** is the set of all solutions to a linear equation.

$$a_1X_1 + a_2X_2 + \ldots + a_nX_n = b \quad \text{where } a_i, b \in \mathbb{R}$$

is an **affine hyperplane** in $\mathbb{R}^n$; geometrically resembles a copy of $\mathbb{R}^{n-1}$ inside $\mathbb{R}^n$.

2.1.1 Interpreting Linear Systems as Matrix Equations

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 0 \end{bmatrix}$$

2.2 Elementary Row Operations

To solve a system of linear equations we associate an **augmented matrix** to the system of equations. For example:

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array}\right]$$

To solve, we can perform the following **Elementary Row Operations (EROs)**:

1. Multiply a row by a non-zero constant.
2. Add a multiple of one row to another row.
3. Swap two rows.

The goal of these operations is to transform the augmented matrix into **row echelon form** (REF) or **reduced row echelon form** (RREF).

2.2.1 REF and Strategy

We say a matrix is in **row echelon form** (REF) if:

- The first non zero entry in each row is a 1 (called the **leading 1**).
- If a column has a leading 1, then all entries below it are 0.
- The leading 1 in each row is to the right of the leading 1 in the previous row.
- All rows of 0s are at the bottom of the matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Example of REF

We have produced a new system of equations. This is easily solved by back substitution.

Concept 2.1: Stategy for Obtaining REF

- Get a 1 as the top left entry
- Use this 1 to clear the entries below it
- Move to the next column and repeat
- Continue until all leading 1s are in place
- Use back substitution to solve the system

2.2.2 Row Reduced Echelon Form

A matrix is in **reduced row echelon form** (RREF) if:

- It is in REF
- The leading 1 in each row is the only non-zero entry in its column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Example of RREF

2.3 Leading variables and free variables

We'll start by an example:

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{array}\right]$$

Solving this system of equations, we get:

$$\text{RREF: } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{array}{rcrcr} x_1 & + & 2x_4 & = & 4 \\ x_2 & - & x_4 & = & 2 \\ x_3 & + & x_4 & = & 2 \end{array} \Rightarrow \begin{array}{rcl} x_1 & = & 4 - 2x_4 \\ x_2 & = & 2 + x_4 \\ x_3 & = & 2 - x_4 \end{array}$$

This RREF tells us how the **leading variables** (x_1, x_2, x_3) depend on the **free variable** (x_4). The free variable can take any value in $\mathbb{R}$. We write the solution set as:

$$x_1 = 4 - 2t, \quad x_2 = 2 + t, \quad x_3 = 2 - t, \quad x_4 = t \quad \text{where } t \in \mathbb{R}$$

$$(x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t); \quad t \in \mathbb{R}$$

Definition 2.3: Leading and Free Variables

- **Leading variable** : A variable whose columns in the RREF contain a leading 1
- **Free variable** : A variable whose columns in the RREF do not contain a leading 1

2.4 Consistent and Inconsistent Systems

Consider the following system of equations:

$$\begin{array}{rcrcrcr} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \text{ (REF)}$$

We can see the last row of the REF is:

$$0x + 0y + 0z = 1$$

This equation clearly has no solution, and hence the system has no solutions. We say the system is **inconsistent**. Alternatively, we say the system is **consistent** if it has at least one solution.

2.5 Possible Outcomes when solving a system of equations

- The system may be **inconsistent** (no solutions) - i.e:

$$[0\ 0\ \ldots\ 0 \mid a] \quad a \neq 0$$

- The system may be **consistent** which occurs if:
  - **Unique Solutions** each column (aside from the rightmost) contains a single leading 1. - i.e:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

  - **Infinitely many solutions** at least one variable does not appear as a leading 1 in any row, making it a free variable - i.e:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2.6 Elementary Row Operations as Matrix Transformations

Elementary row operations may be interpreted as **matrix multiplication**. To see this, first we introduce the **Identity matrix:**

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The I_m Identity matrix is an $m \times m$ matrix with 1s on the diagonal and 0s elsewhere. We also introduce the $E_{i,j}$ matrix which has 1 in the (i, j) position and 0s elsewhere. For example:

$$E_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then:

$$I_3 + 4E_{1,2} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing a row operation on A is the same as multiplying A by an appropriate matrix E on the left. These matrices are called **elementary matrices**. They are **always invertible,** and **their inverses are also elementary matrices**. The statement:

"every matrix can be reduced to RREF through EROs"

is equivalent to saying that

"for every matrix A with m rows, there exists a $m \times m$ matrix B
which is a product of elementary matrices such that BA is in RREF."

2.6.1 Multiplying a Row by a Non-Zero Scalar

When multiplying row i of matrix A by a scalar $\alpha \neq 0$, we can use the matrix:

$$I_m + (\alpha - 1)E_{i,i}$$

This works because it modifies only the (i, i) entry of the identity matrix to be α while keeping all other entries unchanged. When multiplied with A, it scales row i by α and leaves all other rows intact.
Example: If $\alpha = 5$ and $i = 2$, then:

$$I_3 + 4E_{2,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad (I_3 + 4E_{2,2})A = \begin{bmatrix} 1 & 2 & 3 \\ 20 & 25 & 30 \\ 7 & 8 & 9 \end{bmatrix}$$

2.6.2 Switching Two Rows

To swap rows i and k, we use:

$$S = I_m + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k}$$

This works by:

- Removing the 1's at positions (i, i) and (k, k) from the identity matrix
- Adding 1's at positions (i, k) and (k, i)

Example: Swapping rows 1 and 3:

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad SA = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

2.6.3 Adding a Multiple of One Row to Another

To replace row k with row $k + \alpha\times$ row i, use:

$$I_m + \alpha E_{k,i}$$

This adds α times row i to row k while leaving all other rows unchanged because:

- For any row $j \neq k$, the corresponding row in this matrix is just the standard basis row
- Row k becomes the sum of the standard basis row k plus α times the standard basis row i

Example: Adding 3 times row 1 to row 2:

$$I_3 + 3E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad (I_3 + 3E_{2,1})A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \\ 7 & 8 & 9 \end{bmatrix}$$

Example 2.1

Write the inverse of an elementary matrix and show it is an elementary matrix.

Multiplying a row by a nonzero scalar:

- **Operation:** Multiply row i by $\alpha \neq 0$.
- **Elementary Matrix:** $E = I_m + (\alpha - 1)E_{i,i}$
- **Inverse:** To reverse the operation, multiply row i by $1\backslash\alpha$. Hence, the inverse is

$$E^{-1} = I_m + \left(\frac{1}{\alpha} - 1\right) E_{i,i} = I_m + \frac{1-\alpha}{\alpha} E_{i,i}.$$

Swapping two rows:

- **Operation:** Swap rows i and k.
- **Elementary Matrix:** $S = I_m - E_{i,i} - E_{k,k} + E_{i,k} + E_{k,i}$
- **Inverse:** Since swapping the same two rows twice returns them to their original positions,

$$S^{-1} = S.$$

Adding a multiple of one row to another:

- **Operation:** Add α times row i to row k.
- **Elementary Matrix:** $E = I_m + \alpha\, E_{k,i}$
- **Inverse:** To undo the operation, subtract α times row i from row k. Therefore,

$$E^{-1} = I_m - \alpha\, E_{k,i}.$$

Example 2.2

Prove that every invertible matrix in $M_n(\mathbb{R})$ is a product of elementary matrices.

Let A be an invertible matrix in $M_n(\mathbb{R})$. Since A is invertible, we can use Gaussian elimination to transform A into the identity matrix I_n.

Let $E_1, E_2, \ldots, E_k$ be the elementary matrices corresponding to the row operations used in the elimination process. Then, we have:

$$\begin{aligned} \text{Multiplying a row by a scalar:} &\quad I_n + (\alpha - 1)E_{i,i} \\ \text{Swapping two rows:} &\quad I_n + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k} \\ \text{Adding a multiple of one row to another:} &\quad I_n + \alpha\, E_{k,i} \end{aligned}$$

Applying these in sequence to A gives:

$$E_k \cdots E_2 E_1 A = I_n$$

Since $E_k \cdots E_2 E_1 = I_n$, we can multiply both sides by $(E_k \cdots E_2 E_1)^{-1}$ on the left to obtain:

$$A = (E_k \cdots E_2 E_1)^{-1} I_n = (E_k \cdots E_2 E_1)^{-1}$$

Using the property:

$$(E_k \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1},$$

we can express A as a product of elementary matrices:

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Since each E_i is an elementary matrix, its inverse is also an elementary matrix. Therefore, A can be expressed as a product of elementary matrices.

2.7 EROs and Inverses

Elementary Row Operations can be used to find the inverse of a square matrix. Consider a square matrix $A \in M_n(\mathbb{F})$ (that is, an $n \times n$ matrix over a field $\mathbb{F}$). If A is invertible, let

$$A^{-1} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

be its inverse, where each $\mathbf{v}_i$ is the ith column of A^{-1}. By definition of the matrix inverse, we have

$$A\,A^{-1} = A\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n\end{bmatrix} = I_n,$$

the $n \times n$ identity matrix. This implies that

$$A\mathbf{v}_i = \mathbf{e}_i, \quad \text{for each } i = 1, 2, \ldots, n,$$

where $\mathbf{e}_i$ is the ith column of I_n (which has a 1 in the ith row and 0 everywhere else). In other words, each column $\mathbf{v}_i$ of A^{-1} is the unique solution to the linear system

$$A\mathbf{v}_i = \mathbf{e}_i.$$

To find A^{-1} effectively, we form the augmented matrix $[A \mid I_n]$ and apply EROs to trasnform A into I_n. When this is achieved, the augmented portion becomes A^{-1}. Thus, we have

$$\operatorname{RREF}\big([\,A \mid I_n\,]\big) = \big[\,I_n \mid A^{-1}\big].$$

Example 2.3

Find A^{-1} if $A = \begin{bmatrix}3 & 4 & -1\\ 1 & 0 & 3\\ 2 & 5 & -4\end{bmatrix}$.

We form a 3×6 matrix $A' = [A \mid I_3]$:

$$A' = \begin{bmatrix}3 & 4 & -1 & | & 1 & 0 & 0\\ 1 & 0 & 3 & | & 0 & 1 & 0\\ 2 & 5 & -4 & | & 0 & 0 & 1\end{bmatrix}$$

We apply the following EROs to A':

- $R_1 \leftrightarrow R_2$
- $R_2 \to R_2 - 3R_1$
- $R_3 \to R_3 - 2R_1$
- $R_3 \to R_3 + R - 2$
- $R_3 \leftrightarrow R_2$
- $R_3 \to R_3 - 4R_2$
- $R_3 \times (-\frac{1}{10})$
- $R_1 \to R_1 - 3R_3$

To obtain:

$$\begin{bmatrix}1 & 0 & 0 & | & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5}\\ 0 & 1 & 0 & | & -1 & 1 & 1\\ 0 & 0 & 1 & | & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5}\end{bmatrix}$$

That is:

$$A^{-1} = \begin{bmatrix}\frac{3}{2} & -\frac{11}{10} & -\frac{6}{5}\\ -1 & 1 & 1\\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5}\end{bmatrix}$$

It is easily checked that $AA^{-1} = I_3$.

3 Vector Spaces and Subspace Structure

3.1 The Image and Kernel of a Linear Transformation

$T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation defined with:

$$M_T = \begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 5\\ 1 & 1 & 1\end{bmatrix}$$

The **image** of T is the subset of $\mathbb{R}^3$ consisting of all elements $T(\mathbf{v}), \mathbf{v} \in \mathbb{R}^3$. This is the set of all vectors of the form:

$$a\begin{bmatrix}1\\2\\1\end{bmatrix} + b\begin{bmatrix}2\\-1\\1\end{bmatrix} + c\begin{bmatrix}0\\5\\1\end{bmatrix}$$

In matrix terms, this is the **column space** of M_T.
The **kernel** of T is the set of all vectors $\mathbf{v} \in \mathbb{R}^3$ such that $T(\mathbf{v}) = \mathbf{0}$. This is the set of all column vectors, whose entries, a, b, c satisfies:

$$\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 5\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = a\begin{bmatrix}1\\ 2\\ 1\end{bmatrix} + b\begin{bmatrix}2\\ -1\\ 1\end{bmatrix} + c\begin{bmatrix}0\\ 5\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

The kernel is a line and the image is a plane

$$\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 5\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{ccc|c}1 & 2 & 0 & 0\\ 0 & -5 & 5 & 0\\ 1 & 1 & 1 & 0\end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

The kernel (or nullspace) is $(2, 1, 1)t, t \in \mathbb{R}$, which is a line in $\mathbb{R}^3$. The fact that $(-2, 1, 1)$ is in the kernel of T, means that column 3 of M_T is a linear combination of columns 1 and 2.

$$-2\begin{bmatrix}1\\ 2\\ 1\end{bmatrix} + 1\begin{bmatrix}2\\ -1\\ 1\end{bmatrix} + 1\begin{bmatrix}0\\ 5\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix}0\\ 5\\ 1\end{bmatrix} = 2\begin{bmatrix}1\\ 2\\ 1\end{bmatrix} - \begin{bmatrix}2\\ -1\\ 1\end{bmatrix}$$

It follows that every linear combination of all three columns of M_T is just a linear combination of columns 1 and 2.

The column space of M_T is:

$$\left\{ a\begin{bmatrix}1\\ 2\\ 1\end{bmatrix} + b\begin{bmatrix}2\\ -1\\ 1\end{bmatrix} \; : \; a, b \in \mathbb{R} \right\}$$

3.2 Subspaces

Definition 3.1

A non empty subset $\mathbf{V}$ of $\mathbb{R}^n$ is a **subspace** if:

- **Closed under addition**: $u + v \in \mathbb{V}$, $u, v \in \mathbf{V}$
- **Closed under scalar multiplication**: $\alpha u \in \mathbf{V}$, $u \in \mathbf{V}$, $\alpha \in \mathbb{R}$

Examples of subspaces

- $\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$ is not a subspace of $\mathbb{R}^3$. the $[1, 0, 0]$ and $(0, 1, 0)$ vectors are in the set, but their sum $(1, 1, 0)$ is not in the set.
- $\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot (1, 2, 3) = 0\}$ is a subspace of $\mathbb{R}^3$.
- $\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot (1, 2, 3) \neq 0\}$ is not a subspace of $\mathbb{R}^3$.
- The kernel of any linear transformation is a subspace of $\mathbb{R}^n$.
- The image of any linear transformation is a subspace of $\mathbb{R}^n$.

3.3 The span : how to make subspaces

Definition 3.2

Let $S = \{v_1, \dots, v_k\}$ be any finite subset of $\mathbb{R}^n$

The subset of $\mathbb{R}^n$ consisting of all linear combinations of the elements of S is a subspace of $\mathbb{R}^n$ and is called the **span** of S and is denoted by $\langle(S)\rangle$.

Proof that $\langle S \rangle$ is a subspace of $\mathbb{R}^n$

- **Closed under addition**:
  Let $u, v \in \langle S \rangle$. Then $u = a_1v_1 + a_2v_2 + \cdots + a_kv_k$ and $v = b_1v_1 + b_2v_2 + \cdots + b_kv_k$ for some $a_i, b_i \in \mathbb{R}$. We see that:
  $$u + v = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \cdots + (a_k + b_k)v_k$$
  So S is closed under addition.
- **Closed under scalar multiplication**:
  Let $u \in \langle S \rangle$ and $\alpha \in \mathbb{R}$. We need to show that cu is a linear combination of $v_1, \dots, v_k$. We have $u = a_1v_1 + a_2v_2 + \cdots + a_kv_k$ for some $a_i \in \mathbb{R}$. Then:
  $$cu = c(a_1v_1 + a_2v_2 + \cdots + a_kv_k) = (ca_1)v_1 + (ca_2)v_2 + \cdots + (ca_k)v_k$$
  so $cu \in \langle S \rangle$.

3.4 Spanning sets

Definition 3.3

Let V be a subspace of $\mathbb{R}^n$.

A subset S of V is a **spanning set** for V if $\langle S \rangle = V$.

This means that every element of V can be expressed as a linear combination of the elements of S.

Example
The set $\{e_1, e_2, e_3\}$ is a spanning set of $\mathbb{R}^3$. We know that:

$$e_1 = [1, 0, 0], \quad e_2 = [0, 1, 0], \quad e_3 = [0, 0, 1]$$

We can represent every element of $\mathbb{R}^3$ as a linear combination of e_1, e_2, e_3:

$$\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 2e_1 - 3_2 + 4e_3$$

Remark A set S of three column vectors in $\mathbb{R}^3$ is a spanning set of $\mathbb{R}^3$ if and only if the three vectors are linearly independent.This occurs only if the 3×3 matrix whose columns are the three vectors has S **as an inverse**.

Questions about spanning sets

- Does $\mathbb{R}^3$ have a spanning set fewer than three vectors?
  - **No.** A spanning set for $\mathbb{R}^3$ must contain at least three linearly independent vectors, since the dimension of $\mathbb{R}^3$ is 3. Fewer than three vectors cannot span all of $\mathbb{R}^3$.
- Does every spanning set of $\mathbb{R}^3$ have three vectors?
  - **No.** A spanning set can have more than three vectors, but not necessarily exactly three. Redundant vectors (linearly dependent ones) can be included, so a spanning set might have more than three vectors.
- Does every spanning set of $\mathbb{R}^{\nVDash}$ contain one with exactly three elements?
  - **Yes.** Every spanning set of $\mathbb{R}^3$ contains a basis, and since the dimension is 3, there exists a subset of exactly three linearly independent vectors that still span $\mathbb{R}^3$.
- If V is a subspace of $\mathbb{R}^3$ does V have a spanning set with at most three elements?
  - **Yes.** Any subspace of $\mathbb{R}^3$ has a basis, and since $\mathbb{R}^3$ has dimension 3, the basis of any of its subspaces can have at most 3 elements. Hence, every subspace can be spanned by at most three vectors.
- If V is a proper subspace of $\mathbb{R}^3$, does V have a spanning set with fewer than three elements?
  - **Yes.** A proper subspace of $\mathbb{R}^3$ has dimension less than 3, so it can be spanned by fewer than three vectors.

3.5 Linear Dependence and Linear Independence

Definition 3.4

A set of at least two vectors in $\mathbb{R}^n$ is **linearly dependent** if one of its elements is a linear combination of the others.

A set of vectors in $\mathbb{R}^n$ is **linearly independent** if it is not linearly dependent.

For a subset $\{v_1, \ldots, v_k\}$ of $\mathbb{R}^n$, suppose that v_k is a linear combination of $\{v_1, \ldots, v_{k-1}\}$. Then every linear combination of $\{v_1, \ldots, v_k\}$ **is already a linear combination** of $v_1, \ldots, v_{k-1}$:

$$\langle v_1, \ldots, v_k \rangle = \langle v_1, \ldots, v_{k-1} \rangle$$

If we are interested in the span of $\{v_1, \ldots, v_k\}$, we can throw away v_k and this wouldn't change the span.

Linear independence means that throwing away any element of the set **shrinks the span**

Example 3.1

$$\begin{array}{ccccccc} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 = 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 = 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 = 1 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & -18 & 51 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The three equations of the system form a linearly dependent set. One row was eliminated by adding a linear combination of the other two rows. Thus, all the information in the system was contained in the first two equations.

The non-zero rows of the RREF are linearly independent, they span the rowspace of the matrix. The rowspace is the subspace of $\mathbb{R}^5$ spanned by the rows of the matrix.

3.5.1 Test for linear independence

A set is linearly independent if none of its elements is a linear combination of the others. While this makes sense, to use it as a test would mean checking every element. We have an alternative formulation, which is easier to check:

"A set of vectors is linearly independent if the only way to write the zero vector as a linear combination of the vectors in the set is to use all zero coefficients."

To decide if the set $\{v_1, \dots, v_k\}$ is linearly independent, try to write the zero vector as a linear combination of the vectors in the set:

$$\sum_{i=1}^{k} \alpha_i v_i = \alpha_1 v_1 + \cdots + \alpha_k v_k = 0 \quad \text{for } \alpha_i \in \mathbb{R}$$

If $\forall i \to a_i = 0$, then the set is linearly independent. If not, the set is linearly dependent.

Example 3.2

Decide whether the set $\{[1, 0, 1], [1, 0, -1], [1, 1, 1]\}$ is linearly independent or dependent.

To solve, we use ERO and find:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad a = b = c = 0$$

The set is linearly independent

3.6 Finite Dimensional Spaces

Definition 3.5

A vector space V is finite dimensional if it contain a finite spanning set.

This means a set $\{v_1, \dots, v_k\}$ of elements, with the property that every element of V is a linear combination of $v_1, \dots, v_k$.

Examples

- $\mathbb{R}^n$ is finite dimensional with $\{e_1, \dots, e_n\}$ as a spanning set. The dimension of $\mathbb{R}^n$ is n.
- $M_{m \times n}(\mathbb{R})$ is finite dimensional,with $\{E_{ij}\}_{1 \le i \le m, 1 \le j \le n}$ as a spanning set with mn elements.
- An example of an infinite dimensional space is the set, $\mathbb{R}[x]$, of all polynomials with real coefficients. This set is infinite dimensional because it contains an infinite number of linearly independent vectors, such as $\{1, x, x^2, \dots\}$.

3.7 Basis

Definition 3.6

A **basis** for a vector space is a **linearly independent spanning set**.

- A basis is a minimal spanning set, one in which every element is needed and does not contain a smaller spanning set.

- Example: $\{e_1, \dots, e_n\}$ is a basis for $\mathbb{R}^n$.
- $\{(1,3),(1,4)\}$ is a basis for $\mathbb{R}^2$.
- If S is a finite spanning set of a vector space V, then S contains a basis of V. If S is not linearly independent, then some $v \in S$ is a linear combination of the other elements of S. Throwing away v leaves a smaller set that still spans V. This process can be repeated until a basis is obtained.

3.8 Steinitz Replacement Lemma

Lemma 3.1

Let V be a vector space that has a basis with n elements.

Then every linearly independent set with n elements in V is a basis for V.

Proof (for $n = 3$)
Suppose $B = \{b_1, b_2, b_3\}$ is a basis of V and let $\{y_1, y_2, y_3\}$ be a linearly independent subset of V.

Step 1.
$y_1 = a_1b_1 + a_2b_2 + a_3b_3$ for scalars a_1, a_2, a_3, not all zero. We can assume (after maybe relabelling the b_i), that $a_1 \neq 0$. Then

$$b_1 = a_1^{-1}y_1 - a_1^{-1}a_2b_2 - a_1^{-1}a_3b_3.$$

So $b_1 \in \langle y_1, b_2, b_3\rangle$ and $\{y_1, b_2, b_3\}$ spans V. (Note that we have to use the fact that we can divide by non-zero scalars to write b_1 as a linear combination of y_1, b_2, b_3.)

Step 2.
Now $y_2 \in \langle y_1, b_2, b_3\rangle$ and y_2 is not a scalar multiple of y_1 (because $\{y_1, y_2, y_3\}$ is linearly independent).
So b_2 (or b_3) has non-zero coefficient in any description of y_2 as a linear combination of y_1, b_2, b_3.
Replace again: $\{y_1, y_2, b_2\}$ spans V.

Step 3.
Same reasoning: we can replace b_2 with y_3 to conclude $\{y_1, y_2, y_3\}$ spans V.

Conclusion: $\{y_1, y_2, y_3\}$ is a basis of V.

3.9 Recap of span, linear independence and basis

Let V be a vector space, e.g. $V = \mathbb{R}^n$ and S be a finite subset of V. Let V be a vector space (e.g. $V = \mathbb{R}^n$). Let S be a (finite) subset of V.

1. S is a spanning set of V (or S spans V) if every element of V is a linear combination of the elements of S.
2. The span of S, denoted $\langle S\rangle$, is the set of all linear combinations of elements of S, a subspace of V.
3. S is linearly independent if no element of S is a linear combination of the other elements of S.
   Equivalently, if no proper subset of S spans $\langle S\rangle$.
4. S is a basis of V if S is linearly independent **AND** S spans V.
   A basis is a minimal spanning set.
   A basis is a maximal linearly independent set.
5. Every finite spanning set of V contains a basis of V.
6. Every linearly independent subset of V can be extended to a basis of V (we have not proved this yet!).

3.10 Consequences of the replacement theorem

Theorem 3.1

Let V be a vector space that has a basis with n elements.

Then ever linearly independent set with n elements in V is a basis for V.

If V has a spanning set with n elements, a linearly independent set in V cannot have more than n elements.

If V has a linearly independent set with n elements, a spanning set in V must have at least n elements. More concisely:

Concept 3.1

The number of elements of a linearly independent set cannot exceed the number in a spanning set. Every spanning set has at least as many elements as the biggest independent set.

3.11 Every basis has the same number of elements

Let V be a finite dimensional vector space and let B and B' be the bases of V. Then:

- B is linearly independent and B' is a spanning set, so B has **at most** as many elements as B'.
- B is a spanning set and B' is linearly independent, so B has **at least** as many elements as B'.

It follows that B and B' have the same number of elements.

Definition 3.7

The dimension of V is the number of elements in a basis of V.

Note: Every vector space that has a finite spanning set has a finite basis (since we can discard elements from a finite spanning set until a basis remains).

Examples:

- The set $\{1, x, x^2, x^3\}$ is a basis for the vector space P_3 of all polynomials of degree at most 3 with real coefficients.

  It is linearly independent because the only way to write the zero polynomial as
  $$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$
  is by taking $a_0 = a_1 = a_2 = a_3 = 0$.

  Another basis of P_3, preferable for some applications, consists of the first four Legendre polynomials:
  $$\{1,\ x,\ \tfrac{1}{2}(3x^2 - 1),\ \tfrac{1}{2}(5x^3 - 3x)\}.$$
- The **row space** of an $m \times n$ matrix is the subspace of $\mathbb{R}^n$ spanned by its rows. When we reduce a matrix to row-reduced echelon form (RREF), we are computing a basis of its row space.
- In $\mathbb{R}^2$, the reflection in the line $y = 2x$ sends:
  $$(1, 0) \mapsto \left(-\tfrac{3}{5}, \tfrac{4}{5}\right), \quad (0, 1) \mapsto \left(\tfrac{4}{5}, \tfrac{3}{5}\right).$$
  Its standard matrix is:
  $$\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$
  The same reflection sends:
  $$(1, 2) \mapsto (1, 2), \quad (2, -1) \mapsto (-2, 1).$$
  It is easier to describe this transformation in terms of the basis:
  $$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

3.12 Row rank and column rank

Let A be an $m \times n$ matrix.

The **row rank** of A, denoted r, is the dimension of the row space of A—the subspace of $\mathbb{R}^n$ spanned by the rows of A.

The **column rank** of A, denoted c, is the dimension of the column space of A—the subspace of $\mathbb{R}^m$ spanned by the columns of A. Equivalently, it is the dimension of the image of the linear transformation represented by A.

- The row rank is the number of linearly independent rows in A.
- The column rank is the number of linearly independent columns in A.

Since the row rank is at most m and the column rank at most n, both values can be strictly less than m or n, respectively.

3.13 Row rank = column rank

Theorem 3.2

The row rank and column rank are the same for every matrix

Therefore, we refer to this common value simply as the **rank** of the matrix.

Let A be an $m \times n$ matrix. The row rank r is the number of non-zero rows in the RREF of A, and the column rank c is the number of linearly independent columns of A. To show that $c \le r$, consider a basis for the row space of A and arrange its vectors as the rows of an $r \times n$ matrix P. Since every row of A is a linear combination of the rows of P, there exists an $m \times r$ matrix Q such that

$$A = QP.$$

It follows that each column of A is a linear combination of the r columns of Q, implying that $\dim(\mathrm{col}(A)) \le r$. Hence, $c \le r$.
Conversely, to show that $r \le c$, take a basis for the column space of A and arrange its vectors as the columns of an $m \times c$ matrix P'. Since every column of A is a linear combination of the columns of P', there exists a $c \times n$ matrix Q' such that

$$A = P'Q'.$$

Therefore, each row of A is a linear combination of the c rows of Q', and $\dim(\mathrm{row}(A)) \le c$. Hence, $r \le c$.
Combining both inequalities, we conclude that $r = c$, i.e., the row rank and column rank are equal.

Example 3.3

Step 1: Determine the Rank. Reduce A to its reduced row echelon form (RREF):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix} \implies \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

There are 2 non-zero rows, so the row rank is $r = 2$. Examining the columns, we observe that the third column is a linear combination of the first two, so the column rank $c = 2$ as well.

Step 2: Show that $c \le r$ via $A = QP$. Choose a basis for the row space from the non-zero rows of RREF

$$A = QP = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}.$$

Thus, each column of A is a linear combination of the $r = 2$ columns of Q, implying that $\dim(\mathrm{col}(A)) \le r$.

Step 3: Show that $r \le c$ via $A = P'Q'$. Take a basis for the column space of A

$$A = P'Q' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}.$$

So, each row of A is a linear combination of the $c = 2$ rows of Q', and $\dim(\mathrm{row}(A)) \le c$.

Conclusion. Since $c \le r$ and $r \le c$, it follows that $r = c = 2$. This example confirms that the row rank equals the column rank.

3.14 Coordinates

Lemma 3.2

If $\{b_1, \ldots, b_n\}$ is a basis of a vector space V, then every element of V has a unique expression of a linear combination of $b_1, \ldots, b_n$:

Proof: Suppose, for some $v \in V$, that:

$$v = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n, \quad a_i \in \mathbb{R}$$
$$v = a_1' b_1 + a_2' b_2 + \cdots + a_n' b_n, \quad a_i' \in \mathbb{R}$$

Then:

$$0_v = (a_1 - a_1')b_1 + (a_2 - a_2')b_2 + \cdots + (a_n - a_n')b_n$$

Since B is linearly independent, we have:

$$a_i - a_i' = 0 \implies a_i = a_i', \quad \forall\, i$$

Example 3.4

In $\mathbb{R}^2$, the standard coordinates of $\begin{bmatrix}4\\3\end{bmatrix}$ are $(4,3)$.

With respect to the basis, $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$ the coordinates of $\begin{bmatrix}4\\3\end{bmatrix}$ are $(2,-1)$. Which is saying:

$$\begin{bmatrix}4\\3\end{bmatrix} = 2\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}2\\-1\end{bmatrix}.$$

3.15 Coordinates with respect to different bases

Let B be the ordered basis of $\mathbb{R}^3$ with elements:

$$b_1 = \begin{bmatrix}1\\0\\4\end{bmatrix}, \quad b_2 = \begin{bmatrix}2\\-1\\0\end{bmatrix}, \quad b_3 = \begin{bmatrix}4\\0\\2\end{bmatrix} \quad \Longrightarrow \quad B = \begin{bmatrix}1 & 2 & 4\\0 & -1 & 0\\4 & 0 & 2\end{bmatrix}$$

Given an element of $\mathbb{R}^3$, say v, how do we find the B-coordinates of v?

$$v = \begin{bmatrix}2\\-3\\4\end{bmatrix}$$

We know

$$v = 2e_1 - 3e_2 + 4e_3 = [v]_B = 2[e_1]_B - 3[e_2]_B + 4[e_3]_B.$$

To find $[e_1]_B$:

$$e_1 = xb_1 + yb_2 + zb_3 = \begin{bmatrix}1\\0\\4\end{bmatrix}x + \begin{bmatrix}2\\-1\\0\end{bmatrix}y + \begin{bmatrix}4\\0\\2\end{bmatrix}z = \begin{bmatrix}1\\0\\0\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 2 & 4\\0 & -1 & 0\\4 & 0 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

This is saying that $[e_1]_B$ is the first column of the inverse of the matrix B. Write a matrix P which has b_1, b_2, b_3 as columns:

$$P = \begin{bmatrix}1 & 2 & 4\\0 & -1 & 0\\4 & 0 & 2\end{bmatrix} \quad \Longrightarrow \quad P^{-1} = \begin{bmatrix}-\frac{1}{7} & -\frac{2}{7} & \frac{2}{7}\\0 & -1 & 0\\\frac{2}{7} & \frac{4}{7} & -\frac{1}{14}\end{bmatrix}$$

To find the B-coordinates of any $v \in \mathbb{R}^3$, we can multiply v on the left by P^{-1}:

$$\begin{bmatrix}2\\-3\\4\end{bmatrix}_B = P^{-1}v = \begin{bmatrix}-\frac{1}{7} & -\frac{2}{7} & \frac{2}{7}\\0 & -1 & 0\\\frac{2}{7} & \frac{4}{7} & -\frac{1}{14}\end{bmatrix}\begin{bmatrix}2\\-3\\4\end{bmatrix} = \begin{bmatrix}\frac{12}{7}\\3\\-\frac{10}{7}\end{bmatrix}$$

This is saying that $v = \frac{12}{7}b_1 + 3b_2 - \frac{10}{7}b_3$. P^{-1} is called the **change of basis matrix** from the standard basis to the basis B.

3.16 The Rank Nullity Theorem

The Rank-Nullity Theorem **relates the dimensions of the kernel, image and domain** of a linear transformation. The dimension of the image of a linear transformation is called the **rank** and the dimension of the the kernel is called the **nullity.** The rank of T is equal to the rank of matrix X, since the image of T is the column space of this matrix.

Theorem 3.3

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Where V and W are finite-dimensional vector spaces, over some field $\mathbb{F}$. Then:

$$\dim(\ker(T)) + \operatorname{rank}(T) = n$$

Informally: The rank-nullity theorem says the full dimension of the domain must be accounted for in the combination of the kernel and image.

Proof:

1. Write k for $\dim(\ker(T))$ and let $\{b_1, \ldots, b_k\}$ be a basis of $\ker(T)$.
2. Extend this to a basis: $\{b_1, \ldots, b_k, b_{k+1}, \ldots, b_n\}$ of $\mathbb{R}^n$.
3. Since T sends each b_i to 0, the image under T of every element of $\mathbb{R}^n$ is a linear combination of $T(c_{k+1}), \ldots, T(c_n)$.

4. Also, $\{T(c_{k+1}), \ldots T(c_n)\}$ is a linearly independent subset of $\mathbb{R}^m$. To see this suppose for some scalar $a_{k+1}, \ldots, a_n$ that $a_{k+1}T(c_{k+1}) + \cdots + a_nT(c_n) = 0$. Then:

$$a_{k+1}c_{k+1} + \cdots + a_nc_n \in \ker T \implies a_{k+1}c_{k+1} + \cdots + a_nc_n \in \langle b_1, \ldots, b_k \rangle.$$

Since $\{b_1, \ldots, b_k, c_{k+1}, \ldots, c_n\}$ is linearly independent in $\mathbb{R}^n$, this means that $a_{k+1}c_{k+1} + a_{k+2}c_{k+2} + \cdots + a_nc_n = 0$ for each $a_j = 0$.
5. It follows that $\{T(c_{k+1}), \ldots, T(c_n)\}$ is a basis for the image of T, so this image has dimensions $n - k$ as required.

3.17 Linear transformations and change of basis

Definition 3.8

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let $B = \{b_1, \ldots, b_n\}$ be a basis of $\mathbb{R}^n$.

The **matrix of** T **with respect to the basis** B is the $n \times n$ matrix that has the B-coordinates of $T(b_1), \ldots, T(b_n)$ as its n columns. This matrix M satisfies

$$[T(v)]_b = M[v]_B, \quad \forall v \in \mathbb{R}^n.$$

Example 3.5

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $v \to Av$ and B be the ordered basis of $\mathbb{R}^3$ with elements:

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \quad ||| \quad b_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \quad \implies \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

A diagonal representation of the matrix A' of T with respect to the basis B

$$T(b_1) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = 2b_1 \implies [T(b_1)]_B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T(b_2) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} = -3b_2 \implies [T(b_2)]_B = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

$$T(b_3) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ 14 \end{bmatrix} = 7b_3 \implies [T(b_3)]_B = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

The matrix of A' of T with respect to B is **diagonal**. For describing this transformation, T, the basis B is preferable to the standard basis.

$$A' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

This means for any $v \in \mathbb{R}^3$:

$$\underbrace{[T(v)]_B}_{\text{B-coordinates of } T(v)} = \underbrace{A'[v]_B}_{\text{matrix-vector product}}$$

3.18 More on Change of Basis

Let P be the matrix with the basis vectors from B as columns. As we've seen, P^{-1} is the change of basis matrix from the standard basis to the basis B.

For any element, $v \in \mathbb{R}^n$, its B-coordinates are given by:

$$[v]_B = P^{-1}v$$

Equivalently, if we start with the B-coordinates, then the standard coordinates of v are given by:

$$v = P[v]_B$$

So P itself, is the change of basis matrix from the basis B to the standard basis.

3.19 Similarity (The relation of A and A')

Starting with A, the matrix $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the standard basis, to we find A', the matrix of T with respect to the basis Bm we:

- **Take a vector $v \in \mathbb{R}^3$ written in B-coordinates as the column** $[v]_B$.
- **Convert to standard coordinates:** take the product $P[v]_B$.
- **Apply T:** Left multiply by A, to get $AP[v]_B$. This column is the standard coordinates of $T(v)$.
- **Convert to B-coordinates:** left multiply by P^{-1} (change of basis matrix from standard to B) to get $P^{-1}AP[v]_B$. This column has the B-coordinates of $T(v)$.
- **Conclusion:** For any element $v \in \mathbb{R}^3$, the B-coordinates of $T(v)$ are given by:

$$[T(v)]_B = P^{-1}AP[v]_B$$

The B matrix of T is: $P^{-1}AP$, where P has the elements of B as columns.

3.20 Similar Matrices

Definition 3.9

Two matrices A and B are **similar** if there exists an invertible matrix P such that:

$$B = P^{-1}AP$$

Notes:

- Two distinct matrices are similar only if they **represent the same linear transformation** with respect to different bases.
- We can't tell if two square matrices are similar just by looking. Instead, we look at the **trace** of the matrix, the sum of the diagonal elements, if two matrices have the same trace, they are similar.
- Similar matrices have some features in common, such as the same determinant, the same eigenvalues, etc (more on this later).
- Our example showed that the matrix:

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 5 & -1 \\ -4 & -8 & 3 \end{bmatrix}$$

  is similar to the diagonal matrix:

$$\operatorname{diag}(2, -3, 7) = A' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

  In this situation, we say that A is **diagonalizable**.

3.21 Two interpretations of diagonalizability

Interpretation 1:
From the diagonal form of A' we have:

$$T(b_1) = 2b_1, \quad T(b_2) = -3b_2, \quad T(b_3) = 7b_3$$

This means that each of the basis elements b_1, b_2, b_3 is mapped by T to a scalar multiple of itself. - each of them is an eigenvector of T

Interpretation 2:
We can rearrange the version of $P^{-1}AP = A'$ to get:

$$AP = P'A$$

Bearing in mind that that:

$$P = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} \quad \text{and} \quad A' = \operatorname{diag}(2, -3, 7)$$

this is saying that:

$$A \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} | & | & | \\ Ab_1 & Ab_2 & Ab_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ 2b_1 & -3b_2 & 7b_3 \\ | & | & | \end{bmatrix}$$

This means that $Ab_1 = 2b_1$, $A_b2 = -3b_2$, $Ab_3 = 7b_3$, so that $B = \{b_1, b_2, b_3\}$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

3.22 Eigenvalues and diagonalizability

Definition 3.10

An **eigenvector** of a square matrix A is a non zero column vector v, for which there exists a scalar λ such that:

$$Av = \lambda v$$

The scalar λ is called the **eigenvalue** of A corresponding to the eigenvector v.

The eigenvalues of A are the roots of its **characteristic polynomial**:

$$\det(xI_n - A) = 0$$

The **eigenspace** corresponding to a particular eigenvalue, λ, is the set of all vector, v, satisfying: $Av = \lambda v$. It is a subspace of the relevant $\mathbb{R}^n$ of dimension at least 1.

The matrix $A \in M_n(\mathbb{R})$ is **diagonalizable** only if $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. In this case $P^{-1}AP$ is diagonal, where P is a matrix whose n columns are linearly independent eigenvectors of A. The diagonal entries of $P^{-1}AP$ are the corresponding eigenvalues.

3.23 Two examples of non-diagonalizabilty

For $A \in M_n(\mathbb{R})$, it does not always happen that $\mathbb{R}^n$ has a basis consisting of eigenvectors of A.

Example 1:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The matrix A is diagonalizable in $M_2(\mathbb{C})$, but not in $M_2(\mathbb{R})$. The matrix represents an anti-clockwise rotation of $90°$ in $\mathbb{R}^2$. It does not fix any line in $\mathbb{R}^2$. Its characteristic polynomial is:

$$\det(xI_2 - A) = \det \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} = x^2 + 1 = 0$$

Example 2:

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

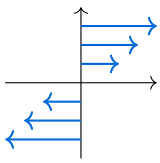
The matrix B is not diagonalizable in $M_2(\mathbb{C})$. The matrix represents **horizontal shear**. Its characteristic polynomial is:

$$\det(xI_2 - B) = \det \begin{bmatrix} x-1 & -1 \\ 0 & x-1 \end{bmatrix} = (x-1)^2 = 0$$

but its 1-dimensional eigenspace only consists of the X-axis. It does not have two linearly independent eigenvectors.

3.24 A Shear in $\mathbb{R}^2$

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



The linear transformation T described by B send $(x, y) \in \mathbb{R}^2$ to $(x + y, y)$. This is a **horizontal shear**: it shifts every point horizontally by its y-coordinate.

For every point $v \in \mathbb{R}^2$, $T(v)$ is on the same horizontal line as v. It follows that $T(v)$ is a scalar multiply of v, only if v lies on the X-axis; in this case, $T(v) = v$.

The characteristic polynomial of B (and T) is:

$$\det(xI_2 - B) = \det \begin{bmatrix} x-1 & -1 \\ 0 & x-1 \end{bmatrix} = (x-1)^2 = (\lambda - 1)^2 = 0$$

The only eigenvalue is $\lambda = 1$, and it has **algebraic multiplicity** 2, meaning it appears twice as a root of the characteristic polynomial.

Its **geometric multiplicity** is only 1, meaning its corresponding eigenspace is only 1-dimensional - just the line $y = 1$.

3.25 Eigenvector for distinct eigenvalues are independent

Theorem 3.4

Let $A \in M_n\mathbb{R}$ and $v_1, \ldots, v_k$ be eigenvector of A in $\mathbb{R}^n$ corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of A.

Then $\{v_1, \ldots, v_k\}$ is a linearly independent subset of $\mathbb{R}^n$.

Proof (for $k = 3$): Note that no two v_1, v_2, v_3 are multiples of each other, since they correspond to distinct eigenvalues.

Suppose: $a_1v_1 + a_2v_2 + a_3v_3 = 0$ for some $a_i \in \mathbb{R}$. We want to show that $a_1 = a_2 = a_3 = 0$.
Multiplying on the left by A gives:

$$a_1Av_1 + a_2Av_2 + a_3Av_3 = 0 \Longrightarrow a_1\lambda_1v_1 + a_2\lambda_2v_2 + a_3\lambda_3v_3 = 0$$

and multiplying the same equation by λ_1 gives:

$$a_1\lambda_1v_1 + a_2\lambda_1v_2 + a_3\lambda_1v_3 = 0$$

Subtract to get:

$$a_2(\lambda_1 - \lambda_2)v_2 + a_3(\lambda_1 - \lambda_3)v_3 = 0$$

Since v_2 and v_3 are linearly independent, and $lambda_1 - \lambda_2 \neq 0$ and $\lambda_1 - \lambda_3 \neq 0$, it follows that $a_2 = a_3 = 0$ and hence $a_1 = 0$.

3.26 At most n distinct eigenvalues

The following consequence of the previous theorem shows that a matrix can have at most n distinct eigenvalues. We already knew this, since the eigenvalues are roots of a polynomial of degree n.

Theorem 3.5: Corollary

A matrix in $M_n(\mathbb{R})$ can have at most n linearly independent eigenvectors.