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Complex Analysis Notes

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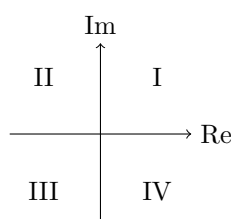
1 Preliminary

1.1 The Complex Plane and the Four Quadrants

The complex plane is a two-dimensional plane where the horizontal axis represents the real part and the vertical axis represents the imaginary part of a complex number. It is divided into four quadrants:

1. **Quadrant I** ($0^\circ < \theta < 90^\circ$): Both x and y are positive.
2. **Quadrant II** ($90^\circ < \theta < 180^\circ$): x is negative, y is positive.
3. **Quadrant III** ($180^\circ < \theta < 270^\circ$): Both x and y are negative.
4. **Quadrant IV** ($270^\circ < \theta < 360^\circ$): x is positive, y is negative.

1.2 Diagram of the Quadrants



1.3 Adjusting Angles Based on Quadrants

To correctly determine θ , adjust the angle returned by $\tan^{-1} \left(\frac{y}{x} \right)$ according to the quadrant where z lies.

1. **Quadrant I** ($x > 0, y > 0$):

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

No adjustment needed since θ is already between 0 and $\frac{\pi}{2}$.

2. **Quadrant II** ($x < 0, y > 0$):

$$\theta = \pi - \tan^{-1} \left(\frac{y}{|x|} \right)$$

Adjust by subtracting from π to place θ between $\frac{\pi}{2}$ and π .

3. **Quadrant III** ($x < 0, y < 0$):

$$\theta = \pi + \tan^{-1} \left(\frac{|y|}{|x|} \right)$$

Add to π to position θ between π and $\frac{3\pi}{2}$.

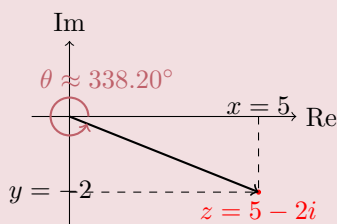
4. **Quadrant IV** ($x > 0, y < 0$):

$$\theta = 2\pi - \tan^{-1} \left(\frac{|y|}{x} \right)$$

Subtract from 2π to set θ between $\frac{3\pi}{2}$ and 2π .

Example: Let $z = 5 - 2i$

$x = 5$ and $y = -2$. Thus, we have: $\theta = 2\pi - \tan^{-1}(2/5) \approx 338.20^\circ$



2 Foundations

2.1 Intro to Complex Numbers

Complex numbers can be written as the sum of a real and imaginary part:

$$z = x + iy$$

We denote the **complex conjugate** (\bar{z}) as:

$$\bar{z} = x - iy$$

Geometrically, \bar{z} is the **reflection of z in the real axis**

With help from Pythagoras' we can now define the distance of z from the origin (**modulus**), that is the length of the vector pointing to z .

$$|z|^2 = x^2 + y^2 \Rightarrow |z| = \sqrt{x^2 + y^2}$$

We notice that:

$$\begin{aligned} z\bar{z} &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy - (iy)(iy) \\ &= x^2 - (i)^2(y^2) \\ &= x^2 - (-1)(y^2) \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

Thus, we have the distance of z from the origin as: $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ We refer to this as the **modulus** of z or the **absolute value** of z .

Letting $z = x + iy$ and $w = u + iv$, we see:

$$|z - w| = \sqrt{(x - u)^2 + (y - v)^2}$$

That is, $|z - w|$ is the distance between z and w in the complex plane.

2.2 Polar Form

Letting $r = |z| = \sqrt{x^2 + y^2}$, we can define x and y as:

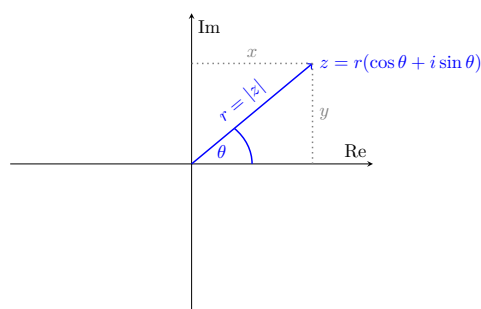
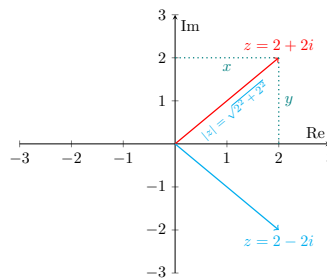
$$\begin{aligned} \cos(\theta) &= \frac{x}{r} \Rightarrow x = r \cos \theta, \\ \sin(\theta) &= \frac{y}{r} \Rightarrow y = r \sin \theta. \end{aligned}$$

Now:

$$\begin{aligned} z &= x + iy \\ &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta). \end{aligned}$$

To find θ we usually calculate $\tan^{-1}(y/x)$ and add/subtract π , when appropriate. Recalling $\tan^{-1}(y/x) \in (-\pi/2, \pi/2)$. We denote θ as the **argument of z** , denoted as $\arg(z)$. Geometrically $\arg(z)$ represent the angle z makes with the positive real axis. Thus, the pair $(r, \arg(z))$ is called the **polar coordinates of z** . We introduce the idea that $\arg(z)$ is a version of $\text{Arg}(z)$ that can take multiple values outside of $\text{Arg}(z)$'s bounds, $(-\pi, \pi)$, more precisely:

$$\arg(z) = \text{Arg}(z) + 2n\pi, \quad n \in \mathbb{Z}$$



Example: Find $\text{Arg}(i)$ and $\arg(i)$

Since $i = 0 + 1i$, we have $x = 0$ and $y = 1$.
Using $\tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \arg(z) = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$
Therefore:

$$\text{Arg}(i) = \frac{\pi}{2} \quad \text{and} \quad \arg(i) = \frac{\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}$$

2.3 De Moivre's Theorem

Theorem: Let $z_1, z_2 \in \mathbb{C}$, be nonzero numbers

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

Then:

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

Thus, we have:

$$\begin{aligned} |z_1 z_2| &= |z_1| |z_2| \\ \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \end{aligned}$$

Corollary: De Moivre's Theorem

Let $n \in \mathbb{Z}$, and $z = |z|(\cos \theta + i \sin \theta)$, then:

$$z^n = |z|^n [\cos(n\theta) + i \sin(n\theta)]$$

2.4 Roots of Unity

Roots of unity are solutions to $z^n = 1$, where z is a complex number on the unit circle.

Eulers formula states that $e^{i\alpha} = \cos \alpha + i \sin \alpha$.

Given $z = x + iy$, then:

$$z = r(\cos \theta + i \sin \theta) = r e^{i\theta}$$

Since z lies on the unit circle, we know $R = 1$, thus we have

$$z = e^{i\theta}$$

Also, we can rewrite 1 as:

$$\begin{aligned} 1 &= 1 + 0i = \cos(0) + i \sin(0) \\ &= \cos(2\pi) + i \sin(2\pi) = \cos(2\pi k) + i \sin(2\pi k) \quad (\text{Periodic with } 2\pi \text{ k multiples don't change the result}) \\ &= e^{i2\pi k} \quad \text{where } k \in \mathbb{Z} \quad (\text{By Eulers Formula}) \end{aligned}$$

So we have, $z^n = e^{n(i\theta)}$:

$$\begin{aligned} e^{in\theta} &= e^{i2\pi k} \\ in\theta &= i2\pi k \\ n\theta &= 2\pi k \\ \theta &= \frac{2\pi k}{n} \end{aligned}$$

So θ is the angle corresponding to the n -th roots of unity. Using eulers formula again, the solutions are given as:

$$z_k = e_{i\theta} = e^{i\left(\frac{2\pi k}{n}\right)} = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right)$$

2.5 Complex Roots

Recall, square roots can be written as $4^{1/2} = \sqrt{4} = 2$, thus, we can write the n -th root as $x^{1/n}$.

What if we wanted to find the n -th root of a complex number?

Consider $f(z) = z^{1/n}$, where $n \in \mathbb{Z}$. To solve this, we aim to find some w such that $w^n = z$.

$$z = R[\cos(\theta) + i \sin(\theta)] \quad \text{and} \quad w = r[\cos(\phi) + i \sin(\phi)]$$

From De Moivre's Theorem, we have:

$$w^n = r^n[\cos(n\phi) + i \sin(n\phi)] = R[\cos(\theta) + i \sin(\theta)]$$

We see:

$$\begin{aligned} r^n &= R \rightarrow r = \sqrt[n]{R} = R^{1/n} \\ n\phi &= \theta + 2\pi k \rightarrow \phi = \frac{\theta}{n} + \frac{2\pi k}{n} \end{aligned}$$

Note that since \sin and \cos are periodic with 2π , the addition of $2\pi k$ doesn't change the result. So we have:

$$z^{1/n} = R^{1/n}[\cos \phi + i \sin \phi] \quad \text{with} \quad \phi = \frac{\theta + 2k\pi}{n}, \quad k \in (0, 1, 2, \dots, n-1)$$

Note that we reserve the notation $\sqrt[n]{z}$ to denote the **principal root**, defined when $k = 0$.

Example: Find the cube roots of $z = -1 + i$

$$R = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

We know z is in the second quadrant, so must adjust θ accordingly:

$$\theta = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

We have $k = 0, 1, 2$ for the cube roots.

Thus, the cubic roots are:

$$w_k = \sqrt[3]{2} \left[\cos\left(\frac{\theta + 2\pi k}{3}\right) + i \sin\left(\frac{\theta + 2\pi k}{3}\right) \right]$$

3 Complex Functions

3.1 Trigonometric Functions

Recall:

$$\begin{aligned}\text{cosine is an even function} &\Rightarrow \cos(-\theta) = \cos(\theta) \\ \text{sine is an odd function} &\Rightarrow \sin(-\theta) = -\sin(\theta)\end{aligned}$$

Also recall Euler's formula states $e^{iz} = \cos(z) + i\sin(z)$ also that:

$$\begin{aligned}e^{-iz} &= \cos(-z) + i\sin(-z) \\ &= \cos(z) - i\sin(z)\end{aligned}$$

If we add these expressions, we get an expression for $\cos(z)$:

$$\begin{aligned}e^{iz} + e^{-iz} &= (\cos(z) + i\sin(z)) + (\cos(z) - i\sin(z)) \\ e^{iz} + e^{-iz} &= 2\cos(z) \Rightarrow \cos(z) = \frac{e^{iz} + e^{-iz}}{2}\end{aligned}$$

If we subtract the expressions, we get an expression for $\sin(z)$:

$$\begin{aligned}e^{iz} - e^{-iz} &= (\cos(z) + i\sin(z)) - (\cos(z) - i\sin(z)) \\ e^{iz} - e^{-iz} &= 2i\sin(z) \Rightarrow \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}\end{aligned}$$

We can now also derive $\tan(z)$ and $\cot(z)$:

$$\begin{aligned}\tan(z) &= \frac{\sin(z)}{\cos(z)} = \frac{\frac{e^{iz} - e^{-iz}}{2i}}{\frac{e^{iz} + e^{-iz}}{2}} = i \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \\ \cot(z) &= \frac{\cos(z)}{\sin(z)} = \frac{\frac{e^{iz} + e^{-iz}}{2}}{\frac{e^{iz} - e^{-iz}}{2i}} = -i \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}\end{aligned}$$

Proposition. Let $z, z_1, z_2 \in \mathbb{C}$

- (i) $\sin(z + 2\pi) = \sin(z)$ and $\cos(z + 2\pi) = \cos(z)$
- (ii) $\cos^2(z) + \sin^2(z) = 1$
- (iii) $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$

3.2 Exponential Functions

Recall the **Taylor Series** for e^x , that is: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
We can now define the exponential function for complex numbers as:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!}$$

Recall also, that $z = rei\theta = e^{i\theta}$ it then follows:

$$z = e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right)}_{\cos \theta} + i \underbrace{\left(1 - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)}_{\sin \theta} = \cos(\theta) + i\sin(\theta)$$

3.3 Complex Logarithms

Recall the log rule: $\log(e^x) = x$. Also recall we defined $\theta = \text{Arg}(z)$ with $\arg(z) = \text{Arg}(z) + 2\pi k$. Lastly, recall the polar form of z :

$$z = |z|(\cos(\theta) + i \sin(\theta)) = e^{i\theta} = |z|e^{i\text{Arg}(z)} = e^{\ln|z| + i\text{Arg}z}$$

We can now define the **Logarithm of a Complex Number**:

$$\begin{aligned}\text{Log}(z) &= \log\left(e^{\ln|z| + i\text{Arg}z}\right) &= \ln|z| + i \text{Arg}(z) \\ \log(z) &= \ln|z| + i \arg z &= \ln|z| + i(\text{Arg}(z) + 2\pi k)\end{aligned}$$

Note: Denote $\text{Log}(z)$ as the **principal branch** of the complex logarithm and denote $\log(z)$ as any branch with $k \neq 0$.

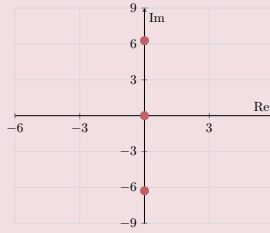
We can also write the **Complex logarithm** as:

$$\begin{aligned}\log(z) &= \ln|z| + i \arg(z) \\ &= \ln|z| + i(\text{Arg}(z) + 2k\pi) \\ &= \ln|z| + i\text{Arg}(z) + 2k\pi i\end{aligned}$$

Example: Find the log of $z = 1 + 0i$

- $z = 1 + 0i = 1 \Rightarrow |z| = 1$
 - $\text{Arg}(z) = \tan^{-1} 0/1 = 0$
- Thus, we have:

$$\begin{aligned}\log(1) &= \ln|1| + i(\text{Arg}(z) + 2k\pi) \\ &= 0 + i(0 + 2k\pi) \\ &= 2k\pi i \quad \text{where } k \in \mathbb{Z}\end{aligned}$$



3.4 Complex Powers

Recall the Logarithm Rule: $\log(a^b) = b \log(a)$. We want to define z^α , in such a way that $\log(z^\alpha) = \alpha \log(z)$. That is the **Complex Power** is defined as:

$$z^\alpha = e^{\alpha \log(z)} = e^{\alpha(\text{Log}(z) + 2k\pi i)} \quad \text{for } k \in \mathbb{Z}$$

So that we have:

$$\begin{aligned}\log(z^\alpha) &= \log(e^{\alpha(\text{Log}(z) + 2k\pi i)}) \\ &= \alpha(\text{Log}(z) + 2k\pi i) \\ &= \alpha \log(z)\end{aligned}$$

As example, consider $z = 1 + 0i$:

$$\begin{aligned}1^\alpha &= e^{\alpha(\text{Log}(1) + 2k\pi i)} \\ &= e^{2k\alpha\pi i}\end{aligned}$$

If $\alpha \in \mathbb{Z}$ ($1, 2, 3, \dots$)

$$1^\alpha = (e^{2k\pi i})^\alpha = (\cos(2\pi k) + i\sin(2\pi k))^\alpha = 1^\alpha = 1$$

If $\alpha = \frac{m}{n} \in \mathbb{Q}$, then 1^α is the set of all n -th roots of unity:

$$1^\alpha = e^{\frac{2k\pi im}{n}} = \cos\left(\frac{2\pi km}{n}\right) + i \sin\left(\frac{2\pi km}{n}\right)$$

If $\alpha = i$ then we see:

$$1^\alpha = 1^i = e^{2k\pi i \cdot i} = e^{-2k\pi}$$

4 Geomtric Mappings and Transformations

4.1 Mappings:

Recall we defined the principal branch as

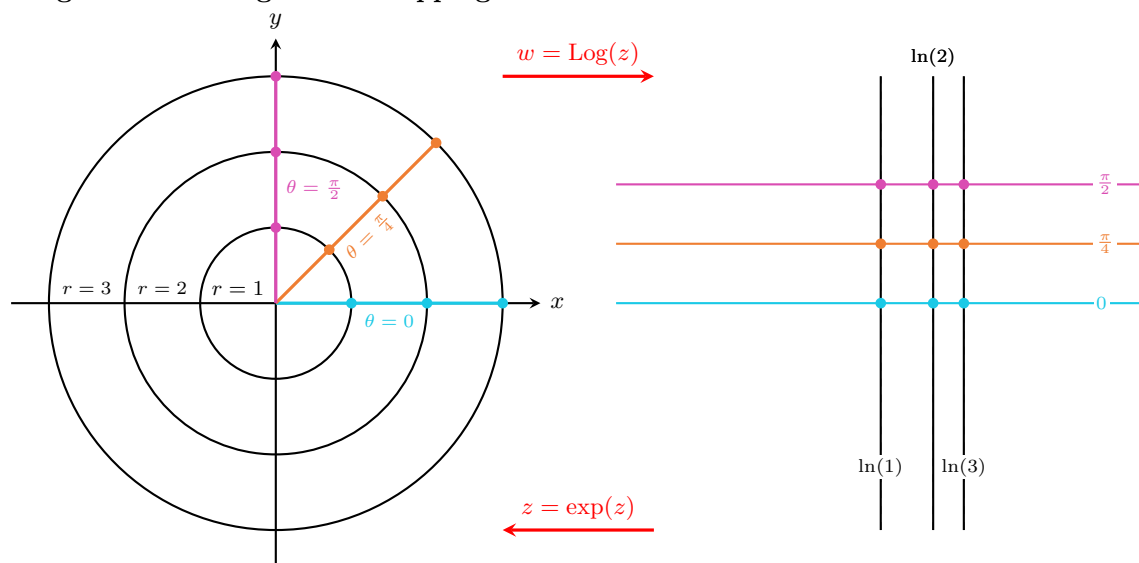
$$\text{Log}(z) = \ln |z| + i \text{Arg}(z)$$

So, when we take the principal branch of the logarithm, we see that it maps to the complex number $w = u + iv$ where $u = \ln |z|$ and $v = \text{Arg}(z)$.

In essence. Log maps \mathbb{C} to the horizontal strip:

$$\{w = u + iv : -\pi < v \leq \pi\}$$

Diagram of the Logarithm Mapping:



4.1.1 Example Mapping 1

Let $f(z) = z^3$, we see that:

Using exponential rules and polar representation:

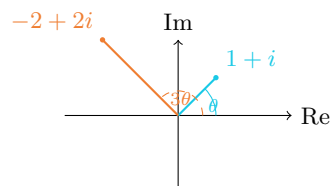
$$\begin{aligned} z &= |z|e^{i\theta} \\ z^3 &= (|z|e^{i\theta})^3 \\ &= |z|^3 e^{i3\theta} \\ &= |z|^3 (\cos(3\theta) + i \sin(3\theta)) \end{aligned}$$

Alternatively, using our definition for complex powers:

$$\begin{aligned} z^3 &= e^{3 \log(z)} \\ &= e^{3(\ln |z| + i \text{Arg}(z))} \\ &= e^{3 \ln |z| + i3 \text{Arg}(z)} \\ &= |z|^3 e^{i3 \text{Arg}(z)} \\ &= |z|^3 (\cos(3\theta) + i \sin(3\theta)) \end{aligned}$$

Letting $z = 1 + i$, we see: $\theta = \tan^{-1}(\frac{1}{1}) = 45^\circ = \frac{\pi}{4}$, and $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Thus, we have:

$$\begin{aligned} z^3 &= |z|^3 \cdot [\cos(3\theta) + i \sin(3\theta)] \\ &= (\sqrt{2})^3 \cdot \left[\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right] \\ &= -2\sqrt{2} + i2\sqrt{2} \end{aligned}$$



In essence, the mapping $f(z) = z^3$ rotates the complex number z by 3θ and scales it by $|z|^3$. We can imagine this, for the complex numbers with $|z| = 1$, and $0 < \theta \leq \frac{\pi}{2}$, as an arc of radius 1, from the angle $0 \rightarrow 90^\circ$, mapped to an arc of radius 8, from the angles $0 \rightarrow 270^\circ$.

4.1.2 Example Mapping 2

We wish to find the image of the line $x = 1$ under

$$f(z) = \frac{1}{z}, \quad z = x + iy, \quad w = u + iv.$$

For $z = x + iy$ we have

$$w = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2},$$

so that

$$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}.$$

Setting $x = 1$ yields

$$u = \frac{1}{1 + y^2}, \quad v = -\frac{y}{1 + y^2}.$$

Since

$$|w|^2 = u^2 + v^2 = \frac{1}{1 + y^2} = u,$$

it follows that

$$u^2 + v^2 = u \implies u^2 - u + v^2 = 0.$$

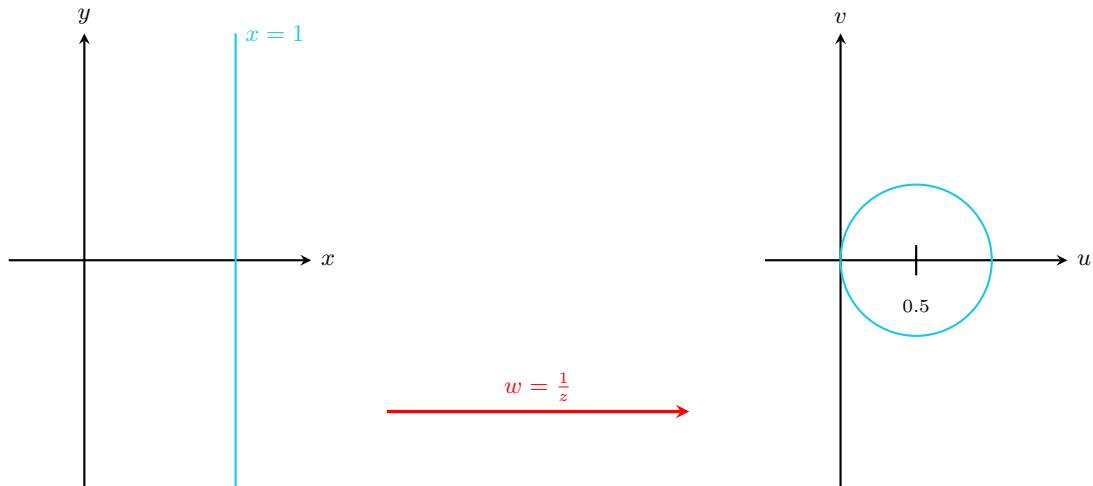
Completing the square in u by adding and subtracting $\frac{1}{4}$:

$$u^2 - u + \frac{1}{4} + v^2 = \frac{1}{4} \implies \left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}.$$

Thus, the image of $x = 1$ is the circle

$$\boxed{\left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}},$$

centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$



In general, $f(z) = \frac{1}{z}$ maps circle and lines to circles and lines, respectively.

4.2 Circle Preservation Theorem

Consider the equation:

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

We can see that if $A \neq 0$, then we can divide by A :

$$x^2 + y^2 + \frac{B}{A}x + \frac{C}{A}y + \frac{D}{A} = 0$$

Completing the square yields:

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{B^2 + C^2 - 4AD}{4A^2}\right)$$

Thus, if $A \neq 0$, we have a circle with center $(-B/2A, -C/2A)$ and radius $\sqrt{\frac{B^2 + C^2 - 4AD}{4A^2}}$.

If $A = 0$, then the equation represents a line:

$$Bx + Cy + D = 0$$

If $D = 0$, the circle or line contains 0:

$$Bx + Cy + D \big|_{(0,0)} = D = 0$$

Why is This Important?

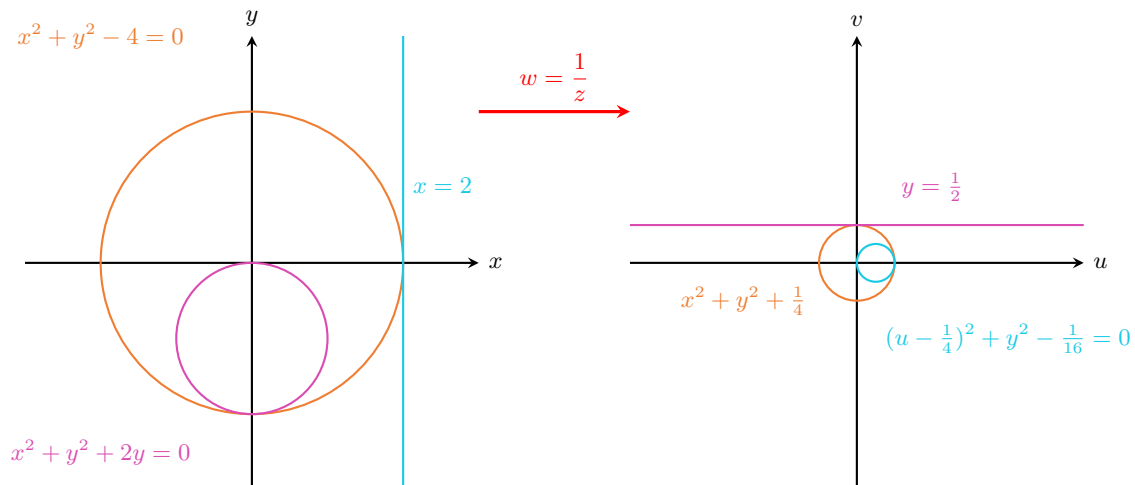
Under the inversion $f(z) = \frac{1}{z}$ with $z = x + iy$ and $w = u + iv$, one can show that the general equation

$$A(x^2 + y^2) + Bx + Cy + D = 0 \xrightarrow{\text{maps to}} D(u^2 + v^2) + Bu - Cv + A = 0.$$

In this transformed equation:

- If the original set does not contain the origin image is a circle.
- If the original set does contain the origin then the equation becomes linear:
- If the original set is a line (with $A = 0$), if it does not pass through the origin, its inversion is a circle that passes through the origin.

Examples Illustrating the Inversion Effects



4.3 Prelim to Riemann Sphere

Our goal is to define the **Riemann Sphere**, which is the complex plane \mathbb{C} , together with an extra point at infinity. In essence The Riemann sphere is a way to "wrap up" the entire complex plane into a compact, closed surface that is **homeomorphic** (topologically equivalent) to the sphere S^2 and the connection between them is made via the **stereographic projection**.

4.3.1 Euclidean Space and Compact Sets

Euclidean space, denoted as \mathbb{R}^n , is the collection of all points in n -dimensional space, where each point is described by n real numbers. In Euclidean spaces (such as the real line \mathbb{R} or the plane \mathbb{R}^2), a set is **compact** if it is both: **Closed** (contains all its limit points), and **Bounded** (contained within a finite region).

Examples of Compact Sets:

The closed interval $[0, 1] \subset \mathbb{R}^1$,

A closed disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$

Examples of Non-Compact Sets:

The open interval $(0, 1) \subset \mathbb{R}^1$ (not closed),

The entire real line \mathbb{R} (not bounded)

4.3.2 Compactification of the Complex Plane

The complex plane \mathbb{C} is not compact - it stretches out infinitely in all directions. By adding a single point at infinity, we "close" the plane, turning it into a compact set. This new space, is **homeomorphic** (a one-to-one mapping that is continuous in both directions or topologically equivalent) to the Riemann Sphere. We define the new space as:

$$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

4.4 Riemann Sphere

Define $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then $\tilde{\mathbb{C}} \xleftrightarrow{1:1} S^2 \{X = (x, y, z) : x^2 + y^2 + z^2 = 1\}$ (*homeomorphic*) via the stereographic projection, denoted St , defined as follows:

1. Projection from $S^2 \rightarrow \tilde{\mathbb{C}}$:

For a point $(x, y, z) \in S^2$, with $z \neq 1$ (the point is not the north pole) the projection is defined as:

$$St(x, y, z) = \frac{1}{1 - x_3}(x_1, x_2) \quad \text{for } z \neq 1$$

This takes a point on the sphere and maps it to a point in the complex plane.

2. Projection from $\tilde{\mathbb{C}} \rightarrow S^2$:

For a point $z \in \mathbb{C}$, the inverse projection is defined as:

$$St^{-1}(z) = \frac{1}{|z|^2 + 1} \langle 2\text{Re}(z), 2\text{Im}(z), |z|^2 - 1 \rangle$$

This takes a complex number, z , written in terms of its real ($\text{Re}(z)$) and imaginary ($\text{Im}(z)$) parts, and maps it to the sphere

3. Mapping the North Pole:

The projection leaves out the north pole from projection onto \mathbb{C}

$$St(N) = \infty \quad \text{and} \quad St^{-1}(\infty) = N \quad \text{where } N = \langle 0, 0, 1 \rangle$$

The north pole is mapped to the point at infinity, and vice versa.

5 Complex Analysis

5.1 Mobius Transforms

Recall: The complex plane \mathbb{C} can be thought as points $(x, y) \in \mathbb{R}^2$, but we usually label a point as $z = x + iy$. We can extend \mathbb{C} by adding a point at infinity, the resulting set is called the **Riemann Sphere** $\tilde{\mathbb{C}}$. Visually, we can imagine wrapping the complex plane onto the surface of a sphere, where ∞ is the north pole of the sphere.

Now, letting a, b, c, d be complex numbers (i.e. $a = x_a + iy_a$), we define a Mobius Transform as a function $T : \tilde{\mathbb{C}} \rightarrow \tilde{\mathbb{C}}$:

$$T(z) = \frac{az + b}{cz + d}$$

where $ad - bc \neq 0$ (that is the determinant $\neq 0 \rightarrow$ matrix is invertible).

These functions occur on the Riemann Sphere, because we need to define that happens when $cz + d = 0$ and when $z = \infty$:

$$\text{If } c \neq 0 : \quad T(\infty) = \frac{a}{c} \quad \text{and} \quad T\left(-\frac{d}{c}\right) = \infty$$

$$\text{If } c = 0 : \quad T(z) = \frac{az + b}{d} \quad \text{and} \quad T(\infty) = \infty$$

Mobius transforms can be uniquely determined by its action on three distinct points. For example, we'll find a mobius transform that maps three points $\{z_1, z_2, z_3\}$ to $\{1, 0, \infty\}$

1. We want $T(z_2) = 0 : az_2 + b = 0 \Rightarrow b = -az_2$, then $T(z)$ becomes:

$$T(z) = \frac{az + b}{cz + d} = \frac{az - az_2}{cz + d} = \frac{a(z - z_2)}{cz + d}$$

2. We want $T(z_3) = \infty : cz_3 + d = 0 \Rightarrow d = -cz_3$, then $T(z)$ becomes:

$$T(z) = \frac{a(z - z_2)}{c(z - z_3)}$$

3. We want $T(z_1) = 1$, then $T(z)$ becomes:

$$T(z_1) = \frac{a(z_1 - z_2)}{c(z_1 - z_3)} = 1 \Rightarrow \frac{a}{c} = \frac{z_1 - z_3}{z_1 - z_2}$$

Finally, we see that $T(z)$ is:

$$T(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z - z_2}{z - z_3}$$

We can now solve problems, such as : Find the Mobius Transform that maps the 3 points $z_1 = -i, z_2 = -1, z_3 = 1$ to $1, 0, \infty$

$$T(z) = \frac{-1 - 1}{-i + 1} \cdot \frac{z + 1}{z - 1} = (-i) \frac{z + 1}{z - 1} = \frac{-iz - i}{z - 1}$$

5.1.1 Matrix Representation of Möbius Transforms

We associate a 2×2 matrix M to a Möbius Transform $T(z)$:

$$T(z) = \frac{az+b}{cz+d} \longleftrightarrow M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Note that: $kM \longleftrightarrow T(z)$ for any $k \in \mathbb{C}, k \neq 0$.

We can also define the **inverse map** T^{-1} as the Möbius transform:

$$T^{-1} \longleftrightarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We can also define the **composition** of two Möbius Transforms, if $T_1(z) = \frac{az+b}{cz+d}$ with matrix M and $T_2(z) = \frac{ez+f}{gz+h}$ with matrix M_2 , then:

$$T \circ T_2 \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Putting it all together, we can map any three points to any other three point:

Three-Point Theorem for Möbius Transformations

If $T \longleftrightarrow M : (z_1, z_2, z_3) \mapsto (1, 0, \infty)$ and if $T_2 \longleftrightarrow M_2 : (z'_1, z'_2, z'_3) \mapsto (1, 0, \infty)$ then:

$$T^{-1} \circ T_2 \longleftrightarrow M^{-1} : (z_1, z_2, z_3) \mapsto (z'_1, z'_2, z'_3)$$

This can be visualized like so:

$$\begin{array}{ccc} z'_1, z'_2, z'_3 & \xrightarrow{T^{-1} \circ T_2} & z_1, z_2, z_3 \\ T_2 \mapsto M_2 \searrow & & \swarrow T \mapsto M \\ & 1, 0, \infty & \end{array}$$

Note that, M, M_2 and $T^{-1} \circ T_2$ have matrices: Three-Point Theorem for Möbius Transformations

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad M_2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, \quad T^{-1} \circ T_2 = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Example: Find a Möbius transformation, $T : (0, -i, -1) \mapsto (i, 1, 0)$

If we can find a map $T_1 : (0, -i, 1) \mapsto (1, 0, \infty)$ and a map $T_2 : (1, -i, -1) \mapsto (i, 1, 0)$. Then, by the Theorem above, we can find a T such that: $T : (0, -i, -1) \mapsto (i, 1, 0)$. Recall, we define a general transform T , that takes 3 points $(z_1, z_2, z_3) \mapsto (1, 0, \infty)$

$$T(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z - z_2}{z - z_3}$$

T_1 becomes:

$$\begin{aligned} T_1(z) &= \frac{0+1}{0+i} \cdot \frac{z+i}{z+1} \\ &= \frac{1}{i} \cdot \frac{z+i}{z+1} \\ &= \frac{z+1}{iz+i} \\ &\Rightarrow \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \end{aligned}$$

T_2 becomes:

$$\begin{aligned} T_2(z) &= \frac{i-0}{i-1} \cdot \frac{z-0}{z-0} \\ &= \frac{i}{i-1} \cdot \frac{z-1}{z} \\ &= \frac{iz-i}{(i-1)z} \\ &\Rightarrow \begin{bmatrix} i & -i \\ i-1 & 1 \end{bmatrix} \end{aligned}$$

Thus, T is:

$$\begin{aligned} T &= T_2^{-1} \circ T_1 \leftrightarrow \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} i & -i \\ i-1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} 0 & i \\ 1 & i-1 \end{bmatrix} = \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} i & -i \\ i-1 & 1 \end{bmatrix} = \begin{bmatrix} 0(1) + (i)(i) & (0)(i) + (i)(i) \\ (1-i)(1) + (i)(i) & (1-i)(i) + (i)(i) \end{bmatrix} = \begin{bmatrix} i^2 & i^2 \\ -i & i \end{bmatrix} \\ T(z) &= - \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \longleftrightarrow -i \frac{z+1}{z-1} \end{aligned}$$