

MA283: Linear Algebra

70% Exam

30% Continuous Assessment (Homework)

10% Optional Project (Bonus)

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1 Review of Matrix Algebra

Matrix Addition

If a matrix has m rows and n columns, we say it is $m \times n$. **Two matrices can only be added if they have the same size..** In this case, we just add the entries in each position.

The $m \times n$ **zero** matrix is a matrix with all entries equal to 0. It is the **Identity element** for matrix addition (adding it to any matrix does not change the matrix)

Matrix Multiplication by a Scalar

This simply means multiplying each entry of the matrix by the scalar. For example:

$$\alpha \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \alpha & 2\alpha \\ 3\alpha & 4\alpha \end{bmatrix}$$

Remark: Now that we have addition and scalar multiplication, we can subtract matrices ($A - B = A + (-1)B$), provided they are the same size.

Vector Space

With these operations of addition and scalar multiplication, the set of $m \times n$ matrices is a vector space. A **vector space** algebraic structure whose elements can be added, subtracted and multiplied by scalars.

Linear Combinations

Definition 1.1: Linear Combinations

Suppose v_1, v_2, \dots, v_k are elements that can be added together and multiplied by scalars.

A Linear Combination of v_1, v_2, \dots, v_k is an expression of the form:

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$$

where $a_i \in \mathbb{R}$ are scalars, called **coefficients**.

Matrix-Vector Multiplication

Definition 1.2

Let A be a $m \times n$ matrix, and \mathbf{v} be a column vector with n entries ($n \times 1$ matrix).

Then the matrix vector product Av is the column vector, with m entries, obtained by taking the linear combination of the columns of A with the entries of \mathbf{v} as coefficients.

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix} = 7 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 9 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 41 \\ 33 \end{bmatrix}.$$

Remark: Av , if defined, has the same number of rows as A and the same number of columns as \mathbf{v} .

Matrix-Matrix Multiplication

Definition 1.3

Let A and B be matrices of size $m \times p$ and $p \times n$, respectively. Write v_1, \dots, v_n for the columns of B . Then the product AB is the $m \times n$ matrix whose columns are Av_1, \dots, Av_n .

The entry at row i and column j of the matrix A is given by A_{ij} . The entry in the i, j position of the product AB is the i th entry of the vector Av_j , where the vector v_j is the j th column of B . In other words, the entry in the i, j position of the product AB is given by:

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \dots + A_{ip}B_{pj} = \sum_{k=1}^p A_{ik}B_{kj}$$

Definition 1.4

If A is $m \times p$ with rows u_1, \dots, u_m and B is $p \times n$ with columns v_1, \dots, v_n , then the product AB is:

$$AB = \begin{bmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & \dots & u_1 \cdot v_n \\ u_2 \cdot v_1 & u_2 \cdot v_2 & \dots & u_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m \cdot v_1 & u_m \cdot v_2 & \dots & u_m \cdot v_n \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} \quad AB = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

For matrices A and B , the products AB and BA are generally not equal, even if they are both defined and even if both have the same size.

Linear Transformations

2 Systems of linear equations

2.1 Linear equations and Solution Sets

A linear equation in the variables x and y is an equation of the form

$$2x + y = 3$$

If we replace x and y with some numbers, the statement **becomes true or false**.

Definition 2.1: Solution to a linear equation

A pair, $(x_0, y_0) \in \mathbb{R}$, is a solution to a linear equation if setting $x = x_0$ and $y = y_0$ **makes the equation true**.

Definition 2.2: Solution set

The **solution set** is the set of all solutions to a linear equation.

$$a_1X_1 + a_2X_2 + \dots + a_nX_n = b \quad \text{where } a_i, b \in \mathbb{R}$$

is an **affine hyperplane** in \mathbb{R}^n ; geometrically resembles a copy of \mathbb{R}^{n-1} inside \mathbb{R}^n .

2.1.1 Interpreting Linear Systems as Matrix Equations

$$\begin{array}{rrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 0 \end{bmatrix}$$

2.2 Elementary Row Operations

To solve a system of linear equations we associate an **augmented matrix** to the system of equations. For example:

$$\begin{array}{rrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array} \right]$$

To solve, we can perform the following **Elementary Row Operations (EROs)**:

1. Multiply a row by a non-zero constant.
2. Add a multiple of one row to another row.
3. Swap two rows.

The goal of these operations is to transform the augmented matrix into **row echelon form (REF)** or **reduced row echelon form (RREF)**.

2.2.1 REF and Strategy

We say a matrix is in **row echelon form (REF)** if:

- The first non zero entry in each row is a 1 (called the **leading 1**).
- If a column has a leading 1, then all entries below it are 0.
- The leading 1 in each row is to the right of the leading 1 in the previous row.
- All rows of 0s are at the bottom of the matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Example of REF

We have produced a new system of equations. This is easily solved by back substitution.

Concept 2.1: Strategy for Obtaining REF

- Get a 1 as the top left entry
- Use this 1 to clear the entries below it
- Move to the next column and repeat
- Continue until all leading 1s are in place
- Use back substitution to solve the system

2.2.2 Row Reduced Echelon Form

A matrix is in **reduced row echelon form** (RREF) if:

- It is in REF
- The leading 1 in each row is the only non-zero entry in its column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Example of RREF

2.3 Leading variables and free variables

We'll start by an example:

$$\begin{array}{rrrrrrrrcl} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{array} \right]$$

Solving this system of equations, we get:

$$\text{RREF: } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \Rightarrow \begin{array}{rcl} x_1 + 2x_4 & = & 4 \\ x_2 - x_4 & = & 2 \\ x_3 + x_4 & = & 2 \end{array} \Rightarrow \begin{array}{rcl} x_1 & = & 4 - 2x_4 \\ x_2 & = & 2 + x_4 \\ x_3 & = & 2 - x_4 \end{array}$$

This RREF tells us how the **leading variables** (x_1, x_2, x_3) depend on the **free variable** (x_4). The free variable can take any value in \mathbb{R} . We write the solution set as:

$$x_1 = 4 - 2t, \quad x_2 = 2 + t, \quad x_3 = 2 - t, \quad x_4 = t \quad \text{where } t \in \mathbb{R}$$

$$(x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t); \quad t \in \mathbb{R}$$

Definition 2.3: Leading and Free Variables

- **Leading variable** : A variable whose columns in the RREF contain a leading 1
- **Free variable** : A variable whose columns in the RREF do not contain a leading 1

2.4 Consistent and Inconsistent Systems

Consider the following system of equations:

$$\begin{array}{rrrrrrcl} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (\text{REF})$$

We can see the last row of the REF is:

$$0x + 0y + 0z = 1$$

This equation clearly has no solution, and hence the system has no solutions. We say the system is **inconsistent**. Alternatively, we say the system is **consistent** if it has at least one solution.

2.5 Possible Outcomes when solving a system of equations

- The system may be **inconsistent** (no solutions) - i.e:

$$[0 \ 0 \ \dots \ 0 \ | \ a] \quad a \neq 0$$

- The system may be **consistent** which occurs if:

- **Unique Solutions** each column (aside from the rightmost) contains a single leading 1. - i.e:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

- **Infinitely many solutions** at least one variable does not appear as a leading 1 in any row, making it a free variable - i.e:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

2.6 Elementary Row Operations as Matrix Transformations

Elementary row operations may be interpreted as **matrix multiplication**. To see this, first we introduce the **Identity matrix**:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The I_m Identity matrix is an $m \times m$ matrix with 1s on the diagonal and 0s elsewhere. We also introduce the $E_{i,j}$ matrix which has 1 in the (i,j) position and 0s elsewhere. For example:

$$E_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then:

$$I_3 + 4E_{1,2} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing a row operation on A is the same as multiplying A by an appropriate matrix E on the left. These matrices are called **elementary matrices**. They are **always invertible**, and **their inverses are also elementary matrices**. The statement:

"every matrix can be reduced to RREF through EROs"

is equivalent to saying that

"for every matrix A with m rows, there exists a $m \times m$ matrix B which is a product of elementary matrices such that BA is in RREF."

2.6.1 Multiplying a Row by a Non-Zero Scalar

When multiplying row i of matrix A by a scalar $\alpha \neq 0$, we can use the matrix:

$$I_m + (\alpha - 1)E_{i,i}$$

This works because it modifies only the (i,i) entry of the identity matrix to be α while keeping all other entries unchanged. When multiplied with A , it scales row i by α and leaves all other rows intact.

Example: If $\alpha = 5$ and $i = 2$, then:

$$I_3 + 4E_{2,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad (I_3 + 4E_{2,2})A = \begin{bmatrix} 1 & 2 & 3 \\ 20 & 25 & 30 \\ 7 & 8 & 9 \end{bmatrix}$$

2.6.2 Switching Two Rows

To swap rows i and k , we use:

$$S = I_m + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k}$$

This works by:

- Removing the 1's at positions (i,i) and (k,k) from the identity matrix
- Adding 1's at positions (i,k) and (k,i)

Example: Swapping rows 1 and 3:

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad SA = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

2.6.3 Adding a Multiple of One Row to Another

To replace row k with row $k + \alpha \times$ row i , use:

$$I_m + \alpha E_{k,i}$$

This adds α times row i to row k while leaving all other rows unchanged because:

- For any row $j \neq k$, the corresponding row in this matrix is just the standard basis row
- Row k becomes the sum of the standard basis row k plus α times the standard basis row i

Example: Adding 3 times row 1 to row 2:

$$I_3 + 3E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad (I_3 + 3E_{2,1})A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \\ 7 & 8 & 9 \end{bmatrix}$$

Example 2.1

Write the inverse of an elementary matrix and show it is an elementary matrix.

Multiplying a row by a nonzero scalar:

- **Operation:** Multiply row i by $\alpha \neq 0$.
- **Elementary Matrix:** $E = I_m + (\alpha - 1)E_{i,i}$
- **Inverse:** To reverse the operation, multiply row i by $1/\alpha$. Hence, the inverse is

$$E^{-1} = I_m + \left(\frac{1}{\alpha} - 1\right) E_{i,i} = I_m + \frac{1 - \alpha}{\alpha} E_{i,i}.$$

Swapping two rows:

- **Operation:** Swap rows i and k .
- **Elementary Matrix:** $S = I_m - E_{i,i} - E_{k,k} + E_{i,k} + E_{k,i}$
- **Inverse:** Since swapping the same two rows twice returns them to their original positions,

$$S^{-1} = S.$$

Adding a multiple of one row to another:

- **Operation:** Add α times row i to row k .
- **Elementary Matrix:** $E = I_m + \alpha E_{k,i}$
- **Inverse:** To undo the operation, subtract α times row i from row k . Therefore,

$$E^{-1} = I_m - \alpha E_{k,i}.$$

Example 2.2

Prove that every invertible matrix in $M_n(\mathbb{R})$ is a product of elementary matrices.

Let A be an invertible matrix in $M_n(\mathbb{R})$. Since A is invertible, we can use Gaussian elimination to transform A into the identity matrix I_n .

Let E_1, E_2, \dots, E_k be the elementary matrices corresponding to the row operations used in the elimination process. Then, we have:

$$\text{Multiplying a row by a scalar: } I_n + (\alpha - 1)E_{i,i}$$

$$\text{Swapping two rows: } I_n + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k}$$

$$\text{Adding a multiple of one row to another: } I_n + \alpha E_{k,i}$$

Applying these in sequence to A gives:

$$E_k \cdots E_2 E_1 A = I_n$$

Since $E_k \cdots E_2 E_1 = I_n$, we can multiply both sides by $(E_k \cdots E_2 E_1)^{-1}$ on the left to obtain:

$$A = (E_k \cdots E_2 E_1)^{-1} I_n = (E_k \cdots E_2 E_1)^{-1}$$

Using the property:

$$(E_k \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1},$$

we can express A as a product of elementary matrices:

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Since each E_i is an elementary matrix, its inverse is also an elementary matrix. Therefore, A can be expressed as a product of elementary matrices.

2.7 EROs and Inverses

Elementary Row Operations can be used to find the inverse of a square matrix. Consider a square matrix $A \in M_n(\mathbb{F})$ (that is, an $n \times n$ matrix over a field \mathbb{F}). If A is invertible, let

$$A^{-1} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}$$

be its inverse, where each \mathbf{v}_i is the i th column of A^{-1} . By definition of the matrix inverse, we have

$$A A^{-1} = A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} = I_n,$$

the $n \times n$ identity matrix. This implies that

$$A\mathbf{v}_i = \mathbf{e}_i, \quad \text{for each } i = 1, 2, \dots, n,$$

where \mathbf{e}_i is the i th column of I_n (which has a 1 in the i th row and 0 everywhere else). In other words, each column \mathbf{v}_i of A^{-1} is the unique solution to the linear system

$$A\mathbf{v}_i = \mathbf{e}_i.$$

To find A^{-1} effectively, we form the augmented matrix $[A \mid I_n]$ and apply EROs to transform A into I_n . When this is achieved, the augmented portion becomes A^{-1} . Thus, we have

$$\text{RREF}([A \mid I_n]) = [I_n \mid A^{-1}].$$

Example 2.3

Find A^{-1} if $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$.

We form a 3×6 matrix $A' = [A \mid I_3]$:

$$A' = \begin{bmatrix} 3 & 4 & -1 & | & 1 & 0 & 0 \\ 1 & 0 & 3 & | & 0 & 1 & 0 \\ 2 & 5 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

We apply the following EROs to A' :

- $R_1 \leftrightarrow R_2$
- $R_2 \rightarrow R_2 - 3R_1$
- $R_3 \rightarrow R_3 - 2R_1$
- $R_3 \rightarrow R_3 + R - 2$
- $R_3 \leftrightarrow R_2$
- $R_3 \rightarrow R_3 - 4R_2$
- $R_3 \times (-\frac{1}{10})$
- $R_1 \rightarrow R_1 - 3R_3$

To obtain:

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & | & -1 & 1 & 1 \\ 0 & 0 & 1 & | & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

That is:

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

It is easily checked that $AA^{-1} = I_3$.