Complex Analysis

Exams:

60% Exam

40% Continuous Assessment

Contents

1 Week 1: Introduction to Complex Numbers

1.1 Quadratics with Complex Roots

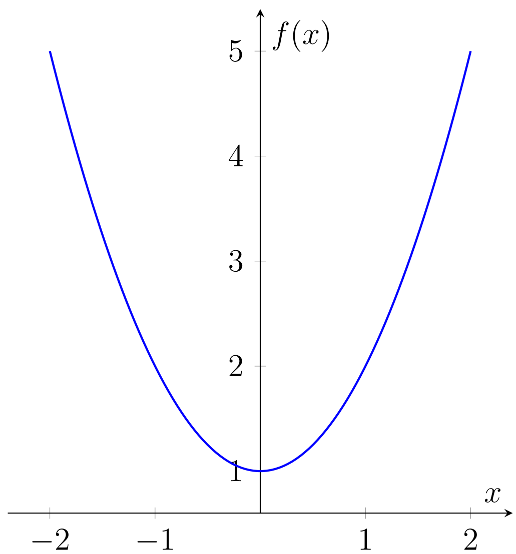
Everybody knows that, for coefficients $a, b, c \in \mathbb{R}$, the quadratic

$$ax^2 + bx + c = 0$$

has real values solutions given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{if } b^2 - 4ac \geq 0$$

but if $b^2 - 4ac < 0$, then we need the roots of negative numbers, and thus the solutions are complex numbers.
For example, the, the plot of $x^2 + 1 = 0$, below implies imaginary soltuons, since there are no real x-values that make y=0



1.2 Real valued solutions of a cubic

Oddly enough, complex numbers are needed to find real-valued solutions of a cubic equation.

Definition

For $p, q \in \mathbb{R}$,

$$x^3 = px + q,$$

has the solution, by Cardano's formula:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

Example

Consider $x^3 = 15x + 4$, staring at this long enough, one could guess that $x = 4$ is a solution, and then factor out $(x - 4)$ to get a quadratic, but that's not the point.
By Cardano's Formula, with $p = 15$ and $q = 4$, we get:

$$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{121}}$$

Setting $i = \sqrt{-1}$, thus $\sqrt{-121} = 11i$
And noticing that:

$$\begin{aligned}
(2 + i)^3 &= 2^3 + 3 \cdot 2^2 \cdot i + 3 \cdot 2 \cdot i^2 + i^3 \\
&= 8 + 12i - 6 - i \\
&= 2 + 11i
\end{aligned}$$

$$\text{Thus} \quad (2 + 1)^3 = 2 + 11i \quad \text{and} \quad (2 - 1)^3 = 2 - 11i$$

Thus, the solution is:

$$\begin{aligned}
&= \sqrt[3]{(2 + i)^3} + \sqrt[3]{(2 - i)^3} \\
&= 2 + i + 2 - i \\
&= 4
\end{aligned}$$

1.3 Definition of Complex Numbers

Definition

The set of complex numbers is defined as:

$$\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\}$$

where a is the real part and yi is the imaginary part, and $i^2 = -1$

1.4 Attributes of Complex Numbers

Given a complex number of the form: $z = x + yi$, we have:

- $\mathrm{Re}(z) = x$ is the real part of z
- $\mathrm{Im}(z) = y$ is the imaginary part of z
- $\bar{z} = x - yi$ is the complex conjugate of z
- $|z| = \sqrt{x^2 + y^2}$ is the modulus of z

Note that:

$$z\bar{z} = x^2 + y^2 = |z|^2 \in \mathbb{R}$$

Also note how this formula is used in the computation of the inverse of z:

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

The geometric meaning of these attributes can be seen below.