MA2287: Complex Analysis Exam Notes

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Contents

1 Question 1:

1.1 Sketch the region in the complex plane determined by the inequality

- $|z-4| > 3|z+4|$ 2023 Q1(a)
- $\{z \in \mathbb{C} : |2z-1| < 2|2z-i|\}$ 2022 Q1(a), 2021 Q1(d), 2017 Q1(a), 2016 Q1(a)

1.2 Determine all solutions to roots of unity

- $z^6 - 1 = 0$ and factorize $x^6 - 1$ as a product of linear and quadratic factors 2023 Q1(b),2021 Q1(c)
- $z^4 = -81i$ and find a polynomial $p(z)$ with complex coefficients with root w and $p(\overline{w}) \neq 0$ 2022 Q1(b), 2018 Q1(b)
- $z^3 = 1 + i$, let $n \in \mathbb{N}$ and $w \neq 1$ be an n-th root of unity. Prove $1 + w + w^2 + \ldots + w^{n-1} = 0$ 2016 Q1(c)

1.3 Determine and sketch the image under the mapping

- $w = e^z$, $\{z \in \mathbb{C} : \pi/4 \leq \mathrm{Im}(z) \leq \pi/2\}$ 2023 Q1(c), 2021 Q1(a), 2017 Q1(d)
- $w = \mathrm{Log}(z)$, $\{z : |z| > 1, 0 \leq \mathrm{Arg}(z) \leq \pi/2\}$ 2022 Q1(d), 2018 Q1(d), 2016 Q1(d)

1.4 Find z where the function is 0

- $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ 2022 Q1(d)

1.5 Calculate principal value Log(z)

- z $= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and prove e^z is the inverse function of Log(z) 2022 Q1(c), 2018 Q1(c), 2017 Q1(c)

1.6 Prove the following

- Define the complex conjugate $(\overline{w})$ and prove if w is a zero of a polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ then $\overline{w}$ is also a zero of $p(z)$ 2021 Q1(b), 2018 Q1(a), 2016 Q1(b)
- Define the complex exponential function e^z and prove Eulers Formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ 2017 Q1(b)

2 Question 2:

2.1 Determine image of the line

- $f(z) = \frac{1}{z} \quad \{z \in \mathbb{C} : \mathrm{Re}(z) = 2\}$ 2023 Q2(a), 2021 Q2(b)
- $f(z) = \frac{1}{z} \quad \{z \in \mathbb{C} : \mathrm{Re}(z) = 1\}$ 2022 Q2(a), 2018 Q2(a), 2017 Q2(a)

2.2 State and Use Cauchy-Riemann Equations

- State CRE, and use to prove $f(z) = \frac{1}{z}$ is holomoprhic on $\mathbb{C}\backslash\{0\}$ 2023 Q2(a)
- State CRE, and use to prove $f(z) = z^2$ is holomoprhic on $\mathbb{C}$ 2022 Q2(b)
- State CRE. Let $f = u + iv$ be holomoprhic on $\Omega \subset \mathbb{C}$. Prove ∇u and ∇v are perpendicular of equal length 2016 Q2(b)

2.3 Show that

- If $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$ then $f(x)$ is real for all $x \in \mathbb{R}$. And if in addition f is holomoprhic at $x \in \mathbb{R}$ then $f'(x)$ is real. 2023 Q2(c)
- Define that is meant for a function g to be harmonic. If $f = u + iv$ is holomoprhic on $\Omega \subset \mathbb{C}$, prove that $v(x, y)$ is a harmonic function, and that ∇u and ∇v are perpendicular of equal length. 2022 Q2(c), 2018 Q2(b)
- If $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$ then $f(x)$ is real for all $x \in \mathbb{R}$. And if in addition f is holomoprhic at 0 then the function $f'(0)$ is real. 2021 Q2(a), 2017 Q2(c)
- Let $f(z) = u + iv$ be holomoprhic on an open subset Ω of the complex plane and let $h(u, v)$ be a harmonic function of u and v on $f(\Omega)$. Prove that $g(x, y) = h(u(x, y), v(x, y))$ is harmonic on Ω (You may assume $\nabla u, \nabla v$ are equal length and perpendicular) 2021 Q2(c)
- Define what is meant for a function $f(z)$ to be holomoprhic at a point $z_0 \in \mathbb{C}$ and prove that $f(z) = z^2$ is holomoprhic and find its derivative there. Hence prove that the product uv is harmonic where $f = u + iv$ 2018 Q2(c)
- Define what is meant for a function $f(z)$ to be holomoprhic at a point $z_0 \in \mathbb{C}$ and prove that $f(z) = \frac{1}{z}$ is holomoprhic on $\mathbb{C}\backslash 0$ and find its derivative there (State any theorems used) 2017 Q2(b)
- Let $h(u, v)$ be a harmonic function of u, v on $f(\Omega)$ (See 2016 Q2(b)). Prove that $g(x, y) = h(u(x, y), v(x, y))$ is harmonic on Ω 2016 Q2(c)

2.4 Find Mobius Transformation

- $T(z) : (-1, 1, \infty) \mapsto (-1, -i, 1)$ 2023 Q2(d)
- $T(z) : (2, 1, -1) \mapsto (1, 0, \infty)$ 2022 Q2(d)
- $T(z) : (-i, -1, 1) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation 2021 Q2(d)
- $T(z) : (-i, -1, i) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation 2018 Q2(d), 2017 Q2(d)
- $T(z) : (-1, \frac{1}{2}, 2) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation 2016 Q2(d)

3 Worked Examples - Q1

Example 2023 Q1(a)

Given $|z-4| > 3|z+4|$
Write $\boldsymbol{z = x + iy}$

$$|x+iy-4| > 3|x+iy+4|$$
$$|(x-4)+iy| > 3|(x+4)+iy|$$
$$\sqrt{(x-4)^2+y^2} > 3\sqrt{(x+4)^2+y^2}$$

Square both sides

$$(x-4)^2+y^2 > 9((x+4)^2+y^2)$$
$$(x^2-8x+16+y^2) > 9x^2+72x+144+9y^2$$
$$x^2-8x+16+y^2-9x^2-72x-144-9y^2 > 0 \qquad \textit{Moving all terms to one side}$$
$$-8x^2-80x-8y^2-128 > 0 \qquad \textit{Simplify}$$
$$x^2+10x+y^2-16 < 0 \qquad \textit{Dividing by -8 and reversing inequality}$$

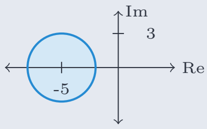
Focus on x and complete the square

$$x+bx = \left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \Rightarrow x^2+10x = (x+5)^2-25 \qquad \textit{Complete the square}$$
$$(x+5)^2-25+y^2+16 < 0 \qquad \textit{Substitute back into inequality}$$
$$(x+5)^2+y^2+9 < 0 \qquad \textit{Simplify}$$
$$(x+5)^2+y^2 < -9 \qquad \textit{Subtract 9}$$

Recall the eqation of a circle

$$(x-a)^2+(y-b)^2 = r^2 \Rightarrow (x+5)^2+y^2 < -9$$

Therefore the region is all the points inside circle with radius 3 and center at (-5, 0)



Example 2022 Q1(a), 2021 Q1(d), 2017 Q1(a), 2016 Q1(a)

Given $\{z \in \mathbb{C} : |2z-1| < 2|2z-i|\}$
Write $z = x + iy$

$$|2x+i2y-1| < 2|2x+i2y-i|$$
$$|(2x-1)+i2y| < 2|2x+i(2y-1)|$$
$$\sqrt{(2x-1)^2+4y^2} < 2\sqrt{4x^2+(2y-1)^2}$$
$$(2x-1)^2+4y^2 < 4[4x^2+(2y-1)^2] \qquad \textit{Square both sides}$$
$$4x^2-4x+1+4y^2 < 16x^2+16y^2-16y+4 \qquad \textit{Expand}$$
$$-12x^2-4x-12y^2+16y-3 < 0 \qquad \textit{Move all terms to one side}$$
$$12x^2+4x+12y^2-16y+3 > 0 \qquad \textit{Multiply by -1 and reverse inequality}$$
$$x^2+\frac{1}{3}x+y^2-\frac{4}{3}y+\frac{1}{4} > 0 \qquad \textit{Divide by 12}$$

Complete square for x

$$x^2+bx = \left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \Rightarrow x^2+\frac{1}{3}x = \left(x+\frac{1}{6}\right)^2 - \left(\frac{1}{36}\right)$$

Complete square for y

$$y^2+by = \left(y-\frac{2}{3}\right)^2 - \left(\frac{4}{9}\right)$$

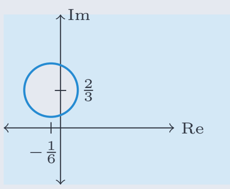
Substitute back into inequality

$$\left(x+\frac{1}{6}\right)^2 - \left(\frac{1}{36}\right) + \left(y-\frac{2}{3}\right)^2 - \left(\frac{4}{9}\right) + \frac{1}{4} > 0 \qquad \textit{Substitute back into inequality}$$
$$\left(x+\frac{1}{6}\right)^2 + \left(y-\frac{2}{3}\right)^2 > \frac{2}{9} \qquad \textit{Simplify and move constant across}$$

Recall the eqation of a circle

$$(x-a)^2+(y-b)^2 = r^2 \Rightarrow \left(x+\frac{1}{6}\right)^2 + \left(y-\frac{2}{3}\right)^2 < \frac{2}{9}$$

Therefore the region is all the points OUTSIDE the circle with radius $\frac{\sqrt{2}}{3}$ **and center at** $(-\frac{1}{6}, \frac{2}{3})$



Example Determine all solutions to $z^6 - 1 = 0$ and factor $x^6 - 1$ as a product of linear and quadratic factors

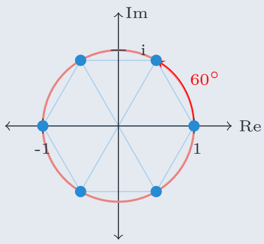
Given $z^6 - 1 = 0$
Write $z = e^{i\theta}$ and $1 = e^{i2\pi k}$ for $k \in \mathbb{Z}$

$$\begin{aligned} z^6 - 1 &= 0 \\ e^{i6\theta} - e^{i2\pi k} &= 0 \\ e^{i6\theta} &= e^{i2\pi k} \\ 6\theta &= 2\pi k \\ \theta &= \frac{\pi k}{3} \end{aligned}$$

Therefore the solutions are

$$z = e^{i\theta} = e^{i\frac{\pi k}{3}} = \cos\left(\frac{\pi k}{3}\right) + i\sin\left(\frac{\pi k}{3}\right) \quad \text{for} \quad k = 0, 1, 2, 3, 4, 5$$

$$\begin{aligned} k = 0: &\quad \cos(0) + i\sin(0) = 1 + i0 \\ k = 1: &\quad \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2} \\ k = 2: &\quad \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ k = 3: &\quad \cos(\pi) + i\sin(\pi) = -1 \\ k = 4: &\quad \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ k = 5: &\quad \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{aligned}$$



Our real roots are when k = 0 and k = 3

$$(x+1) \quad \text{and} \quad (x+1)$$