MA2287: Complex Analysis

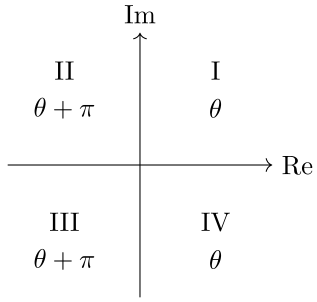
60% Exam
30% Continuous Assessment (Homework)
Robert Davidson

Contents

1 Preliminary

1.1 The Complex Plane and the Four Quadrants

The complex plane is a two-dimensional plane where the horizontal axis represents the real part and the vertical axis represents the imaginary part of a complex number. It is divided into four quadrants:



2 Foundations

2.1 Intro to Complex Numbers

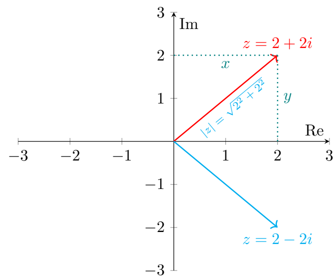
Complex numbers can be written as the sum of a real and imaginary part:

$$z = x + iy$$

We denote the **complex conjugate** ($\overline{z}$) as:

$$\overline{z} = x - iy$$

Geometrically, $\overline{z}$ is the **reflection of z in the real axis**



With help from Pythagoras' we can now define the distance of z from the origin (**modulus**), that is the length of the vector pointing to z.

$$|z|^2 = x^2 + y^2 \Rightarrow |z| = \sqrt{x^2 + y^2}$$

We notice that:

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy - (iy)(iy) \\ &= x^2 - (i)^2(y^2) \\ &= x^2 - (-1)(y^2) \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

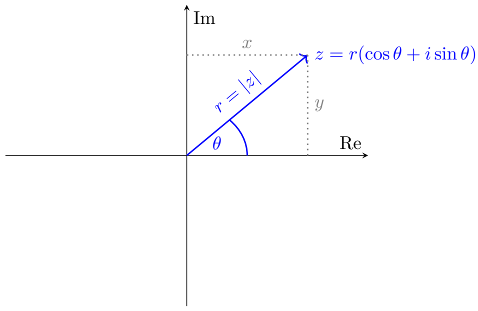
Thus, we have the distance of z from the origin as: $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ We refer to this as the **modulus** of z or the **absolute value** of z.
Letting $z = x + iy$ and $w = u + iv$, we see:

$$|z - w| = \sqrt{(x - u)^2 + (y - z^2)^2}$$

That is, $|z - w|$ is the distance between z and w in the complex plane.

2.2 Polar Form

Letting $r = |z| = \sqrt{x^2 + y^2}$, we can define x and y as:



$$\cos(\theta) = \frac{x}{r} \quad \Rightarrow \quad x = r\cos\theta,$$
$$\sin(\theta) = \frac{y}{r} \quad \Rightarrow \quad y = r\sin\theta.$$

Now:

$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$

To find θ we usualy calculate $\tan^{-1}(y/x)$ and add/subtract π, when appropriate. Recalling $\tan^{-1}(y/x) \in (-\pi/2, \pi/2)$. We denote θ as as the **argument of z**, denoted as $\arg(z)$. Geometrically $\arg(z)$ represent the angle z makes with the positive real axis Thus, the pair (r, arg(z)) is called the **polar coordinates of z**. We introduce the idea that $\arg(z)$ is a version of $\operatorname{Arg}(z)$ that can take multiple values outside of $\operatorname{Arg}(z)$'s bounds, $(-\pi, \pi)$, more precisely:

$$\arg(z) = \operatorname{Arg}(z) + 2n\pi, \quad n \in \mathbb{Z}$$

Example 2.1: Find $\operatorname{Arg}(i)$ and $\arg(i)$

Since $i = 0 + 1i$, we have $x = 0$ and $y = 1$.
Using $\tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \arg(z) = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$
Therefore:

$$\operatorname{Arg}(i) = \frac{\pi}{2} \quad \text{and} \quad \arg(i) = \frac{\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}$$

2.3 De Moivre's Theorem

Theorem: Let $z_1, z_2 \in \mathbb{C}$, be nonzero numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then:

$$\begin{aligned} z_1 z_2 &= r_1 r_2[(cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

Thus, we have:

$$\begin{aligned} |z_1 z_2| &= |z_1||z_2| \\ \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \end{aligned}$$

Theorem Corollary: De Moivre's Theorem

Let $n \in \mathbb{Z}$, and $z = |z|(\cos\theta + i\sin\theta)$, then:

$$z^n = |z|^n = [\cos(n\theta) + i\sin(n\theta)]$$

2.4 Roots of Unity

Roots of unity are solutions to $z^n = 1$, where z is a complex number on the unit circle.
Eulers formula states that $e^{i\alpha} = \cos\alpha + i\sin\alpha$.
Given $z = x + iy$, then:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Since z lies on the unit circle, we know $R = 1$, thus we have

$$z = e^{i\theta}$$

Also, we can rewrite 1 as:

$$\begin{aligned} 1 &= 1 + 0i = \cos(0) + i\sin(0) \\ &= \cos(2\pi) + i\sin(2\pi) = \cos(2\pi k) + i\sin(2\pi k) \quad \text{(Periodic with} 2\pi \text{ k multiples don't change the result)} \\ &= e^{i2\pi k} \quad \text{where } k \in \mathbb{Z} \quad \text{(By Eulers Formula)} \end{aligned}$$

So we have, $z^n = e^{n(i\theta)}$:

$$\begin{aligned} e^{in\theta} &= e^{i2\pi k} \\ in\theta &= i2\pi k \\ n\theta &= 2\pi k \\ \theta &= \frac{2\pi k}{n} \end{aligned}$$

So θ is the angle corresponding to the n-th roots of unity. Using eulers formula again, the solutions are given as:

$$z^k = e^{i\theta} = e^{i(\frac{2\pi k}{n})} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

Proof: Conjugate Roots Theorem

Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial with real coefficients $a_i \in \mathbb{R}$ for all $i \in \{0, 1, \ldots, n\}$.
Suppose that $w \in \mathbb{C}$ is a root of $p(z)$, meaning that $p(w) = 0$. We aim to prove that the complex conjugate $\overline{w}$ is also a root of $p(z)$, i.e., $p(\overline{w}) = 0$.
Let's evaluate $p(\overline{w})$ step by step:

$$p(\overline{w}) = a_n(\overline{w})^n + a_{n-1}(\overline{w})^{n-1} + \cdots + a_1(\overline{w}) + a_0 \tag{1}$$

We'll use the fundamental property of complex conjugates: for any complex number z and any integer k, $(\overline{z})^k = \overline{z^k}$.
Applying this property to each term:

$$\begin{aligned} p(\overline{w}) &= a_n(\overline{w})^n + a_{n-1}(\overline{w})^{n-1} + \cdots + a_1(\overline{w}) + a_0 && (2)\\ &= a_n\overline{w^n} + a_{n-1}\overline{w^{n-1}} + \cdots + a_1\overline{w} + a_0 && (3) \end{aligned}$$

Now, we use a critical property of real numbers: for any $a \in \mathbb{R}$, we have $\overline{a} = a$. Since all coefficients a_i are real, this means $\overline{a_i} = a_i$ for all i.
For any complex number z and real number a, we have the property $\overline{az} = \overline{a} \cdot \overline{z} = a \cdot \overline{z}$. Using this property:

$$\begin{aligned} p(\overline{w}) &= a_n\overline{w^n} + a_{n-1}\overline{w^{n-1}} + \cdots + a_1\overline{w} + a_0 && (4)\\ &= \overline{a_n w^n} + \overline{a_{n-1} w^{n-1}} + \cdots + \overline{a_1 w} + \overline{a_0} && (5) \end{aligned}$$

Another important property of complex conjugation is that it distributes over addition: $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$. Applying this property:

$$\begin{aligned} p(\overline{w}) &= \overline{a_n w^n} + \overline{a_{n-1} w^{n-1}} + \cdots + \overline{a_1 w} + \overline{a_0} && (6)\\ &= \overline{a_n w^n + a_{n-1} w^{n-1} + \cdots + a_1 w + a_0} && (7)\\ &= \overline{p(w)} && (8) \end{aligned}$$

Since we assumed that $p(w) = 0$, we have:

$$\begin{aligned} p(\overline{w}) &= \overline{p(w)} && (9)\\ &= \overline{0} && (10)\\ &= 0 && (11) \end{aligned}$$

The last step follows because the complex conjugate of zero is zero: $\overline{0} = 0$.
Therefore, we have proven that if w is a root of $p(z)$ (i.e., $p(w) = 0$), then $\overline{w}$ is also a root of $p(z)$ (i.e., $p(\overline{w}) = 0$).
This result has an important corollary: the non-real roots of polynomials with real coefficients always occur in complex conjugate pairs.

2.5 Complex Roots

Recall, square roots can be written as $4^{1/2} = \sqrt{4} = 2$, thus, we can write the n-th root as $x^{1/n}$.
What if we wanted to find the n-th root of a complex number?
Consider $f(z) = z^{1/n}$, where $n \in \mathbb{Z}$. To solve this, we aim to find some w such that $w^n = z$.

$$z = R[\cos(\theta) + i\sin(\theta)] \quad \text{and} \quad w = r[\cos(\phi) + i\sin(\phi)]$$

From De Moivre's Theorem, we have:

$$w^n = r^n[\cos(n\phi) + i\sin(n\phi)] = R[\cos(\theta) + i\sin(\theta)]$$

We see:

$$r^n = R \to r = \sqrt[n]{R} = R^{1/2}$$
$$n\phi = \theta = \theta + 2\pi k \to \phi = \frac{\theta}{n} + \frac{2\pi k}{n}$$

Note that since sin and cos are periodic with 2π, the addition of $2\pi k$ doesn't change the result. So we have:

$$z^{1/n} = R^{1/n}[\cos\phi + i\sin\phi] \quad \text{with} \quad \phi = \frac{\theta + 2k\pi}{n}, \quad k \in (0, 1, 2, \ldots, n-1)$$

Note that we reserve the notation $\sqrt[n]{z}$ to denote the **principal root**, defined when $k = 0$.

Example 2.2: Find the cube roots of $z = -1 + i$

$R = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$
We know z is in the second quadrant, so must adjust θ accordingly:

$$\theta = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

We have $k = 0, 1, 2$ for the cube roots.
Thus, the cubic roots are:

$$w_k = \sqrt[3]{2}\left[\cos\left(\frac{\theta + 2\pi k}{3}\right) + i\sin\left(\frac{\theta + 2\pi k}{3}\right)\right]$$

2.6 Problem Sheet 1

1. Simplify the following (write in form a + ib)

   (a) $3\left(\frac{1+i}{1-i}\right)^2 - 2\left(\frac{1-i}{1+i}\right)^3$

3 Complex Functions

3.1 Trigonemtric Functions

Recall:

$$\text{cosine is an even function} \Rightarrow \cos(-\theta) = \cos(\theta)$$
$$\text{sine is an odd function} \Rightarrow \sin(-\theta) = -\sin(\theta)$$

Also recall Eulers formula states $e^{iz} = \cos(z) + i\sin(z)$ also that:

$$\begin{aligned} e^{-iz} &= \cos(-z) + i\sin(-z) \\ &= \cos(z) - i\sin(z) \end{aligned}$$

If we add these expressions, we get an expression for $\cos(z)$:

$$\begin{aligned} e^{iz} + e^{-iz} &= (\cos(z) + i\sin(z)) + (\cos(z) - i\sin(z)) \\ e^{iz} + e^{-iz} &= 2\cos(z) \Rightarrow \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \end{aligned}$$

If we subtract the expressions, we get an expression for $\sin(z)$:

$$\begin{aligned} e^{iz} - e^{-iz} &= (\cos(z) + i\sin(z)) - (\cos(z) - i\sin(z)) \\ e^{iz} - e^{-iz} &= 2i\sin(z) \Rightarrow \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

We can now also derive $\tan(z)$ and $\cot(z)$:

$$\begin{aligned} \tan(z) &= \frac{\sin(z)}{\cos(z)} = \frac{\frac{e^{iz}-e^{-iz}}{2i}}{\frac{e^{iz}+e^{-iz}}{2}} = i\frac{e^{iz} + e^{-iz}}{e^{iz} + e^{-iz}} \\ \cot(z) &= \frac{\cos(z)}{\sin(z)} = \frac{\frac{e^{iz}+e^{-iz}}{2}}{\frac{e^{iz}-e^{-iz}}{2i}} = -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \end{aligned}$$

Proposition. Let $z, z_1, z_2 \in \mathbb{C}$

$$\begin{aligned} &(i) && \sin(z + 2\pi) = \sin(z) \quad \text{and} \quad \cos(z + 2\pi) = \cos(z) \\ &(ii) && \cos^2(z) + \sin^2(z) = 1 \\ &(iii) && \sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2) \end{aligned}$$

3.2 Exponential Functions

Recall the **Taylor Series** for e^x, that is: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
We can now define the exponential function for complex numbers as:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!}$$

Recall also, that $z = rei\theta = e^{i\theta}$ it then follows:

$$z = e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right)}_{\cos\theta} + i\underbrace{\left(1 - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)}_{\cos\theta} = \cos(\theta) + i\sin(\theta)$$

3.3 Complex Logarithms

Recall the log rule: $\log(e^x) = x$. Also recall we defined $\theta = \text{Arg}(z)$ with $\arg(z) = \text{Arg}(z) + 2\pi k$. Lastly, recall the polar form of z:

$$z = |z|(\cos(\theta) + i\sin(\theta)) = e^{i\theta} = |z|e^{i\text{Arg}(z)} = e^{\ln|z| + i\text{Arg}z}$$

We can now define the **Logarithm of a Complex Number**:

$$\begin{aligned} \text{Log}(z) &= \log\left(e^{\ln|z|+i\text{Arg}z}\right) & &= \ln|z| + i\,\text{Arg}(z) \\ \log(z) &= \ln|z| + i\arg z & &= \ln|z| + i(\text{Arg}(z) + 2\pi k) \end{aligned}$$

Note: Denote $\text{Log}(z)$ as the **principal branch** of the complex logarithm and denote $\log(z)$ as any branch with $k \neq 0$.
We can also write the **Complex logarithm** as:

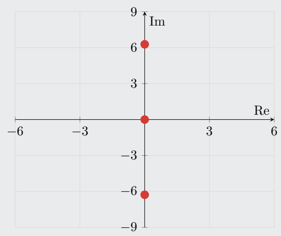
$$\begin{aligned} \log(z) &= \ln|z| + i\arg(z) \\ &= \ln|z| + i(\text{Arg}(z) + 2k\pi) \\ &= \ln|z| + i\text{Arg}(z) + 2k\pi i \end{aligned}$$

Example 3.1: Find the log of $z = 1 + 0i$

- $z = 1 + 0i = 1 \Rightarrow |z| = 1$
- $\text{Arg}(z) = \tan^{-1}(\frac{0}{1}) = 0$

Thus, we have:

$$\begin{aligned} \log(1) &= \ln|1| + i\big(\text{Arg}(z) + 2k\pi\big) \\ &= 0 + i\big(0 + 2k\pi\big) \\ &= 2k\pi i \quad \text{where } k \in \mathbb{Z} \end{aligned}$$



3.4 Complex Powers

Recall the Logarithm Rule: $\log(a^b) = b\log(a)$. We want to define z^α, in such a way that $\log(z^\alpha) = \alpha\log(z)$. That is the **Complex Power** is defined as:

$$z^\alpha = e^{\alpha\log(z)} = e^{\alpha(\text{Log}(z)+2k\pi i)} \quad \text{for } k \in \mathbb{Z}$$

So that we have:

$$\begin{aligned} \log(z^\alpha) &= \log(e^{\alpha(\text{Log}(z)+2k\pi i)}) \\ &= \alpha(\text{Log}(z) + 2k\pi i) \\ &= \alpha\log(z) \end{aligned}$$

As example, consider $z = 1 + 0i$:

$$\begin{aligned} 1^\alpha &= e^{\alpha(Log(1)+2k\pi i)} \\ &= e^{2k\alpha\pi i} \end{aligned}$$

If $\alpha \in \mathbb{Z}$ $(1, 2, 3, \dots)$

$$1^\alpha = \left(e^{2k\pi i}\right)^\alpha = (\cos(2\pi k) + \sin(2\pi k))^\alpha = 1^\alpha = 1$$

If $\alpha = \frac{m}{n} \in \mathbb{Q}$, then 1^α is the set of all n-th roots of unity:

$$1^\alpha = e^{\frac{2k\pi im}{n}} = \cos\left(\frac{2\pi km}{n}\right) + i\sin\left(\frac{2\pi km}{n}\right)\cos\left(\frac{2\pi r}{n}\right) + i\sin\left(\frac{2\pi r}{n}\right)$$

If $\alpha = i$ then we see:

$$1^\alpha = 1^i = e^{2k\pi i\cdot i} = e^{-2k\pi}$$

4 Geomtric Mappings and Transformations

4.1 Mappings:

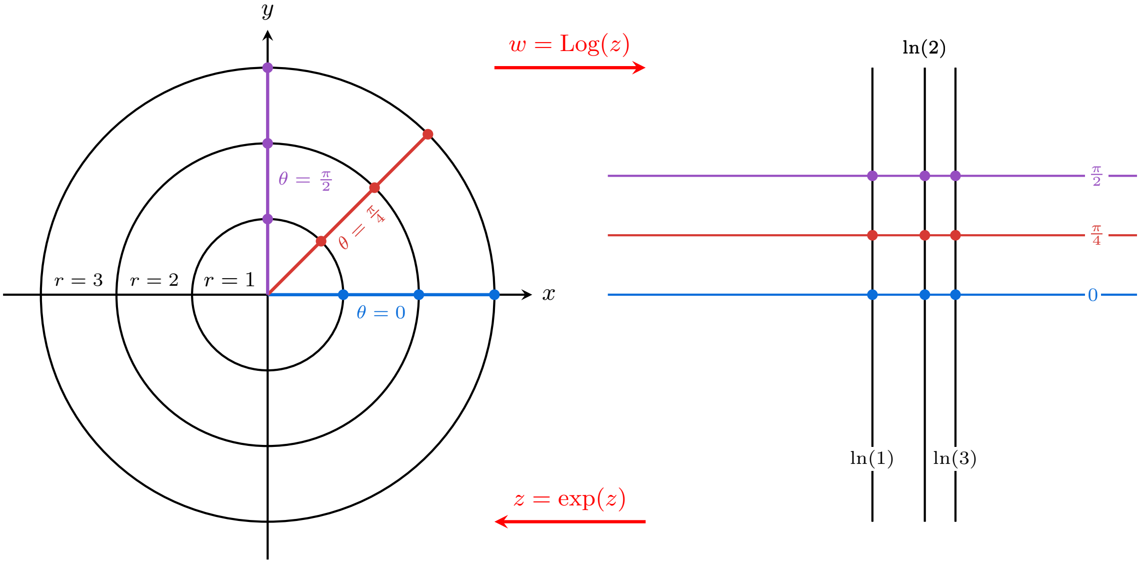
Recall we defined the principal branch as

$$\text{Log}(z) = \ln|z| + i\text{Arg}(z)$$

So, when we take the principal branch of the logarithm, we see that it maps to the complex number $w = u + iv$ where $u = \ln|z|$ and $v = \text{Arg}(z)$.
In essence. Log maps $\mathbb{C}$ to the horizontal strip:

$$\{w = u + iv : -\pi < v \leq \pi\}$$

Diagram of the Logarithm Mapping:



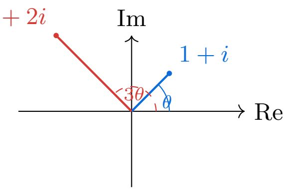
4.1.1 Example Mapping 1 :

Let $f(z) = z^3$
Using exponential rules and polar representation:

$$\begin{aligned} z &= |z|e^{i\theta} \\ z^3 &= \left(|z|e^{i\theta}\right)^3 \\ &= |z|^3 e^{i3\theta} \\ &= |z|^3\Big(\cos(3\theta) + i\sin(3\theta)\Big) \end{aligned}$$

Letting $z = 1 + 1i$, we see: $\theta = \tan^{-1}\left(\frac{1}{1}\right) = 45° = \frac{\pi}{4}$, and $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Thus, we have:

$$\begin{aligned} z^3 &= |z|^3 \cdot [\cos(3\theta) + i\sin(3\theta)] \\ &= (\sqrt{2})^3 \cdot \left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right] \\ &= -2\sqrt{2} + i2\sqrt{2} \end{aligned}$$



In essence, the mapping $f(z) = z^3$ rotates the complex number z by 3θ and scales it by $|z|^3$. We can imagine this, for the complex numbers with $|z| = 1$, and $0 < \theta \leq \frac{\pi}{2}$, as an arc of radius 1, from the angle $0 \to 90°$, mapped to an arc of radius 8, from the angles $0 \to 270°$.

4.1.2 Example Mapping 2

We wish to find the image of the line $x = 1$ under

$$f(z) = \frac{1}{z}, \quad z = x + iy, \quad w = u + iv.$$

For $z = x + iy$ we have

$$w = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2},$$

so that

$$u = \frac{x}{x^2+y^2}, \quad v = -\frac{y}{x^2+y^2}.$$

Setting $x = 1$ yields

$$u = \frac{1}{1+y^2}, \quad v = -\frac{y}{1+y^2}.$$

Since

$$|w|^2 = u^2 + v^2 = \frac{1}{1+y^2} = u,$$

it follows that

$$u^2 + v^2 = u \quad \Longrightarrow \quad u^2 - u + v^2 = 0.$$

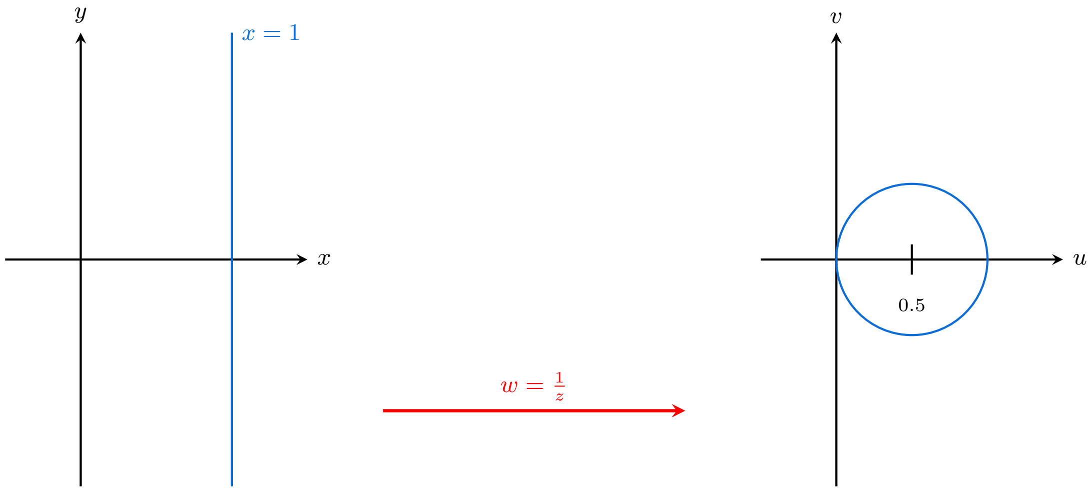
Completing the square in u by adding and subtracting $\frac{1}{4}$:

$$u^2 - u + \frac{1}{4} + v^2 = \frac{1}{4} \quad \Longrightarrow \quad \left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}.$$

Thus, the image of $x = 1$ is the circle

$$\boxed{\left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}},$$

centered at $\left(\frac{1}{2}, 0\right)$ with radius $\frac{1}{2}$



In general, $f(z) = \frac{1}{z}$ maps circle and lines to circles and lines, respectively.

4.2 Circle Preservation Theorem

Consider the equation:

$$A(x^2+y^2)+Bx+Cy+D=0$$

We can we that if $A \neq 0$, then we can divide by A:

$$x^2+y^2+\frac{B}{A}x+\frac{C}{A}y+\frac{D}{A}=0$$

Completing the square yields:

$$\left(x+\frac{B}{2A}\right)^2+\left(y+\frac{C}{2A}\right)^2=\left(\frac{B^2+C^2-4AD}{4A^2}\right)$$

Thus, if $A \neq 0$, we have a circle with center (-B/2A, -C/2A) and radius $\sqrt{\frac{B^2+C^2-4AD}{4A^2}}$.
If $A=0$, then the equation represents a line:

$$Bx+Cy+D=0$$

If $D=0$, the circle or line contains 0:

$$Bx+Cy+D\mid_{(0,0)}=D=0$$

Why is This Important?

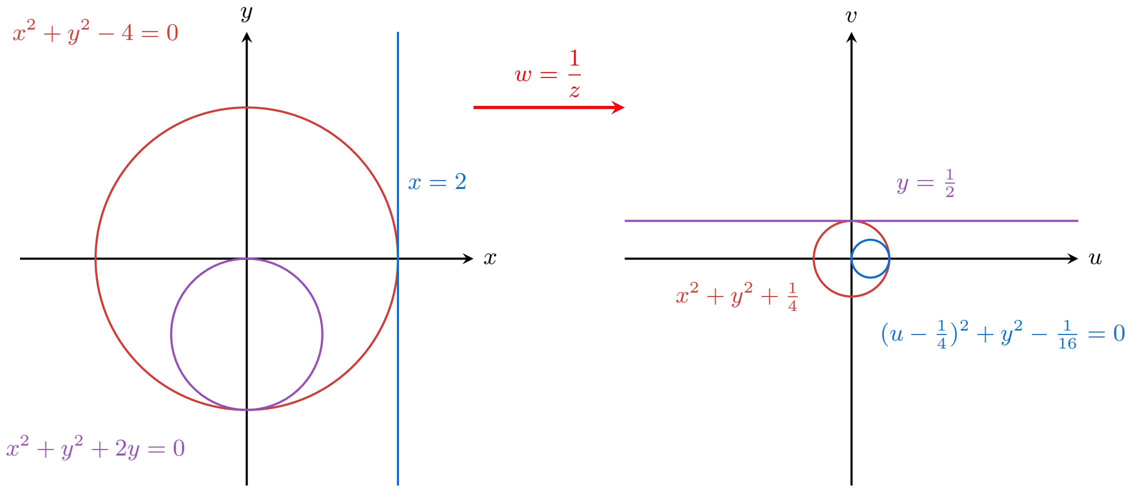
Under the inversion $f(z)=\frac{1}{z}$ with $z=x+iy$ and $w=u+iv$, one can show that the general equation

$$A(x^2+y^2)+Bx+Cy+D=0 \quad \underrightarrow{\text{maps to}} \quad D(u^2+v^2)+Bu-Cv+A=0.$$

In this transformed equation:

- If the original set does not contain the origin image is a circle.
- If the original set does contain the origin then the equation becomes linear:
- If the original set is a line (with $A=0$), if it does not pass through the origin, its inversion is a circle that passes through the origin.

Examples Illustrating the Inversion Effects



4.3 Prelim to Riemann Sphere

Our goal is to define the **Riemann Sphere**, which is the complex plane $\mathbb{C}$, together with an extra point at infinity. In essence The Riemann sphere is a way to "wrap up" the entire complex plane into a compact, closed surface that is **homeomorphic** (toplogically equivalent) to the sphere S^2 and the connection between them is made via the **stereographic projection**.

4.3.1 Euclidean Space and Compact Sets

Euclidean space, denoted as $\mathbb{R}^n$, is the collection of all points in n-dimensional space, where each point is described by n real numbers. In Euclidean spaces (such as the real line $\mathbb{R}$ or the plane $\mathbb{R}^2$), a set is **compact** if it is both: **Closed** (contains all its limit points), and **Bounded** (contained within a finite region).

Examples of Compact Sets:

The closed interval $[0, 1] \subset \mathbb{R}^1$,

A closed disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$

Examples of Non-Compact Sets:

The open interval $(0, 1) \subset \mathbb{R}^1$ (not closed),

The entire real line $\mathbb{R}$ (not bounded)

4.3.2 Compactification of the Complex Plane

The complex plane $\mathbb{C}$ is not compact - it streches out infinitely in all directions. By adding a single point at infinity, we "close" the plane, turning it into a compact set. This new space, is **homeomorphic** (a one-to-one mapping that is continuous in both directions or toplogically equivalent) to to the Riemann Sphere . We define the new space as:

$$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

4.4 Riemann Sphere

Define $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then $\tilde{\mathbb{C}} \overset{1:1}{\longleftrightarrow} S^2\{X = (x, y, z) : x^2 + y^2 + z^2 = 1\}$ (*homeomorphic*) via the sterographic projection, denoted St, defined as follows:

1. Projection from $S^2 \to \tilde{\mathbb{C}}$:
For a point $(x, y, z) \in S^2$, with $z \neq 1$ (the point is not the north pole) the projection is defined as:

$$St(x, y, z) = \frac{1}{1 - x_3}(x_1, x_2) \quad \text{for } z \neq 1$$

This takes a point on the sphere and maps it to a point in the complex plane.

2. Projection from $\tilde{\mathbb{C}} \to S^2$:
For a point $z \in \mathbb{C}$, the inverse projection is defined as:

$$St^{-1}(z) = \frac{1}{|z|^2 + 1}\langle 2\mathrm{Re}(z), 2\mathrm{Im}(z), |z|^2 - 1\rangle$$

This takes a complex number, z, written in terms of its real (Re(z))
and imaginary (Im(z)) *parts, and maps it to the sphere*

3. Mapping the North Pole:
The projection leaves out the north pole from projection onto $\mathbb{C}$

$$St(N) = \infty \quad \text{and} \quad St^{-1}(\infty) = N \quad \text{where } N = \langle 0, 0, 1\rangle$$

The north pole is mapped to the point at infinity, and vice versa.

5 Complex Analysis

5.1 Mobius Transforms

Recall: The complex plane $\mathbb{C}$ can be throught as points $(x, y) \in \mathbb{R}^2$, but we usually label a point as $z = x + iy$. We can extend $\mathbb{C}$ by adding a point at infinity, the resulting set is called the **Riemann Sphere** $\tilde{\mathbb{C}}$. Visually, we can imaigine wrapping the complex plane onto the surface of a sphere, where ∞ is the north pole of the sphere.

Now, letting a, b, c, d be complex numbers (i.e. $a = x_a + iy_a$), we define a Mobius Transform as a function $T : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$:

$$T(z) = \frac{az + b}{cz + d}$$

where $ad - bc \neq 0$ (that is the determinant $\neq 0 \to$ matrix is invertible).
These functions occur on the Riemann Sphere, because we need to define that happens when $cz + d = 0$ and when $z = \infty$:

$$\text{If } c \neq 0 : \quad T(\infty) = \frac{a}{c} \quad \text{and} \quad T\left(-\frac{d}{c}\right) = \infty$$

$$\text{If } c = 0 : \quad T(z) = \frac{az + b}{d} \quad \text{and} \quad T(\infty) = \infty$$

Mobius transforms can be uniquely determined by its action on three distinct points. For example, we'll find a mobius transform that maps three points $\{z_1, z_2, z_3\}$ to $\{1, 0, \infty\}$

1. We want $T(z_2) = 0 : az_2 + b = 0 \Rightarrow b = -az_2$, then $T(z)$ becomes:

$$T(z) = \frac{az + b}{cz + d} = \frac{az - az_2}{cz + d} = \frac{a(z - z_2)}{cz + d}$$

2. We want $T(z_3) = \infty : cz_3 + d = 0 \Rightarrow d = -cz_3$, then $T(z)$ becomes:

$$T(z) = \frac{a(z - z_2)}{c(z - z_3)}$$

3. We want $T(z_1) = 1$, then $T(z_1)$ becomes:

$$T(z_1) = \frac{a(z_1 - z_2)}{c(z_1 - z_3)} = 1 \Rightarrow \frac{a}{c} = \frac{z_1 - z_3}{z_1 - z_2}$$

Finally, we see that $T(z)$ is:

$$T(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z - z_2}{z - z_3}$$

We can now solve problems, such as : Find the Mobius Transform that maps the 3 points $z_1 = -i, z_2 = -1, z_3 = 1$ to $1, 0, \infty$

$$T(z) = \frac{-1 - 1}{i + 1} \cdot \frac{z + 1}{z - 1} = (-i)\frac{z + 1}{z - 1} = \frac{-iz - i}{z - 1}$$

5.1.1 Matrix Representation of Mobius Transforms

We associate a 2×2 matrix M to a Mobius Transform $T(z)$:

$$T(z) = \frac{az+b}{cz+d} \longleftrightarrow M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Note that: $kM \longleftrightarrow T(z)$ for any $k \in \mathbb{C}, k \neq 0$.

We can also define the **inverse map** T^{-1} as the Mobius transform:

$$T^{-1} \longleftrightarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We can also define the **composition** of two Mobius Transforms, if $T_1(z) = \frac{az+b}{cz+d}$ with matrix M and $T_2(z) = \frac{ez+f}{gz+g}$ with matrix M_2, then:

$$T \circ T_2 \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Putting it all together, we can map any three points to any other three point:

Theorem Three-Point Theorem for Möbius Transformations

If $T \longleftrightarrow M : (z_1, z_2, z_3) \mapsto (1, 0, \infty)$ and if $T_2 \longleftrightarrow M_2 : (z_1', z_2', z_3') \mapsto (1, 0, \infty)$ then:

$$T^{-1} \circ T_2 \longleftrightarrow M^{-1} \cdot : (z_1, z_2, z_3) f \mapsto (z_1', z_2', z'3)$$

This can be visualized like so:

$$\begin{tikzcd}
z_1', z_2', z_3' \arrow[rr, "T^{-1} \circ T'"] \arrow[dr, "T_2 \mapsto M_2"'] & & z_1, z_2, z_3 \arrow[dl, "T \mapsto M"] \\
& 1, 0, \infty &
\end{tikzcd}$$

Note that, M, M_2 and $T^{-1} \circ T_2$ have matrices: Three-Point Theorem for Mobius Transformations

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad M_2 = \begin{bmatrix} e & f \\ g & g \end{bmatrix}, \quad T^{-1} \circ T_2 = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Example 5.1: Find a Mobius transformation, $T : (0, -i, -1) \mapsto (i, 1, 0)$

If we can find a map $T_1 : (0, -i, 1) \mapsto (1, 0, \infty)$ and a map $T_2 : (1, -i, -1) \mapsto (i, 1, 0)$. Then, by the Theorem above, we can find a T such that: $T : (0, -i, -1) \mapsto (i, 1, 0)$ Recall, we define a general transform T, that takes 3 points $(z_1, z_2, z_3) \mapsto (1, 0, \infty)$

$$T(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z - z_2}{z - z_3}$$

T_1 becomes:

$$\begin{aligned} T_1(z) &= \frac{0+1}{0+i} \cdot \frac{z+i}{z+1} \\ &= \frac{1}{i} \cdot \frac{z+i}{z+1} \\ &= \frac{z+1}{iz+i} \\ &\Rightarrow \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \end{aligned}$$

T_2 becomes:

$$\begin{aligned} T_2(z) &= \frac{i-0}{i-1} \cdot \frac{z-1}{z-0} \\ &= \frac{i}{i-1} \cdot \frac{z-1}{z} \\ &= \frac{iz-i}{(i-1)z} \\ &\Rightarrow \begin{bmatrix} i & -i \\ i-1 & 1 \end{bmatrix} \end{aligned}$$

Thus, T is:

$$T = T_2^{-1} \circ T_1 \leftrightarrow \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} i & -i \\ i-1 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} 0 & i \\ 1 & i-1 \end{bmatrix} = \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} i & -i \\ i-1 & 1 \end{bmatrix} = \begin{bmatrix} 0(1)+(i)(i) & (0)(i)+(i)(i) \\ (1-i)(1)+(i)(i) & (1-i)(i)+(i)(i) \end{bmatrix} = \begin{bmatrix} i^2 & i^2 \\ -i & i \end{bmatrix}$$

$$T(z) = -\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \longleftrightarrow = -i\frac{z+1}{z-1}$$

5.2 Complex Differentiation

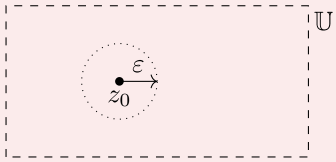
First we must define what is meant for a set to be **open** in the complex plane.

5.2.1 Open Sets in the Complex Plane

Definition 5.1

We say a subset $\mathbb{U} \subseteq \mathbb{C}$ is **open** if $\forall\, z_0 \in \mathbb{U} \quad \exists\, \varepsilon > 0$ such that the open disc centered at z_0 of radius ε is contained in $\mathbb{U}$:

$$D_\varepsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$$



In essence, a set $\mathbb{U}$ in the complex plane is defined as open if for every point z_0 in $\mathbb{U}$, you can draw a small circle around z_0 that fits entirely within $\mathbb{U}$. This radius of this circle is ε. The radius can be very small but must be positive.

5.2.2 Differentiation

Definition 5.2

Let $\mathbb{U} \subseteq \mathbb{C}$ be open, let $f : \mathbb{U} \to \mathbb{C}$ be a function and let $z_0 = x_0 + iy_0 \in \mathbb{U}$.

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

If the limit exists, independant of the direction of approach we say f is **holomorphic** (or complex differentiable / complex analytic) **at** z_0. We also call $f'(z_0)$ the derivative of f at z_0.
Similarly, if f is holomorphic $\forall\, z \in \mathbb{U}$ we say f is holomorphic **on** $\mathbb{U}$.

5.2.3 Cauchy-Riemann Equations

Theorem : Cauchy-Riemann Equations

If $f : \mathbb{U} \to \mathbb{C}$ is holomorphic on $\mathbb{U} \subseteq \mathbb{C}$, then for $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, we have:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

5.2.4 Jacobian Matrix

The Jacobian matrix represents how a function transforms small regions in space. For a function that maps n dimensional space $\to$ m dimensional space, the Jacobian contains all partial derivatives arranged in an $m \times n$ matrix. For example, f as a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, has the Jacobian matrix: $Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ Which for $(x_0, y_0) \in \mathbb{R}^2$ gives an 2×2 matrix:

$$Df(x_0, y_0) = \begin{pmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

Now, f statisfies the Cauchy-Riemann equations:

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Where, the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is the rotation matrix for $\pi/2$ $(90°)$. Meaning that the map Df is $\mathbb{C}$-linear, that is it preserves addition and complex scalar multiplication:

$$f(x + y) = f(x) + f(y)$$

$$f(\alpha x) = \alpha f(x), \quad \forall \alpha \in \mathbb{C}$$

5.3 Complex Integration