

# Complex Analysis

Exams:

60% Exam

40% Continuous Assessment

# Contents

<b>1</b>	<b>Week 1: Introduction to Complex Numbers</b>	<b>3</b>
1.1	Quadratics with Complex Roots . . . . .	3
1.2	Real valued solutions of a cubic . . . . .	4
1.3	Definition of Complex Numbers . . . . .	5
1.4	Attributes of Complex Numbers . . . . .	5
1.5	Polar Form of a Complex Number . . . . .	6
1.6	Multiplication by $i$ and Geometric Interpretation . . . . .	6
1.7	Products of Complex Numbers and De Moivre's Theorem . . . . .	7
1.8	Roots of Unity . . . . .	8
<b>2</b>	<b>Week 2: Functions of a Complex Variable</b>	<b>8</b>

# 1 Week 1: Introduction to Complex Numbers

## 1.1 Quadratics with Complex Roots

Everybody knows that, for coefficients  $a, b, c \in \mathbb{R}$ , the quadratic

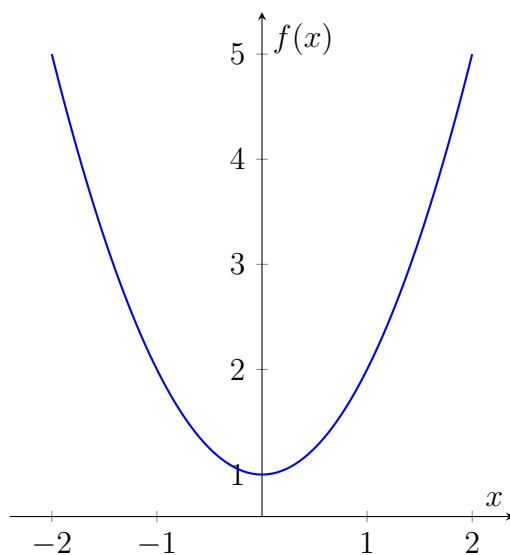
$$ax^2 + bx + c = 0$$

has real values solutions given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{if } b^2 - 4ac \geq 0$$

but if  $b^2 - 4ac < 0$ , then we need the roots of negative numbers, and thus the solutions are complex numbers.

For example, the plot of  $x^2 + 1 = 0$ , below implies imaginary solutions, since there are no real  $x$ -values that make  $y=0$



## 1.2 Real valued solutions of a cubic

Oddly enough, complex numbers are needed to find real-valued solutions of a cubic equation.

### Definition

For  $p, q \in \mathbb{R}$ ,

$$x^3 = px + q,$$

has the solution, by Cardano's formula:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

### Example

Consider  $x^3 = 15x + 4$ , staring at this long enough, one could guess that  $x = 4$  is a solution, and then factor out  $(x - 4)$  to get a quadratic, but that's not the point.

By Cardano's Formula, with  $p = 15$  and  $q = 4$ , we get:

$$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{121}}$$

Setting  $i = \sqrt{-1}$ , thus  $\sqrt{-121} = 11i$

And noticing that:

$$\begin{aligned}(2 + i)^3 &= 2^3 + 3 \cdot 2^2 \cdot i + 3 \cdot 2 \cdot i^2 + i^3 \\ &= 8 + 12i - 6 - i \\ &= 2 + 11i\end{aligned}$$

$$\text{Thus } (2 + i)^3 = 2 + 11i \quad \text{and} \quad (2 - i)^3 = 2 - 11i$$

Thus, the solution is:

$$\begin{aligned}&= \sqrt[3]{(2 + i)^3} + \sqrt[3]{(2 - i)^3} \\ &= 2 + i + 2 - i \\ &= 4\end{aligned}$$

## 1.3 Definition of Complex Numbers

### Definition

The set of complex numbers is defined as:

$$\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\}$$

where  $a$  is the real part and  $yi$  is the imaginary part, and  $i^2 = -1$

## 1.4 Attributes of Complex Numbers

Given a complex number of the form:  $z = x + yi$ , we have:

- $\operatorname{Re}(z) = x$  is the real part of  $z$
- $\operatorname{Im}(z) = y$  is the imaginary part of  $z$
- $\bar{z} = x - yi$  is the complex conjugate of  $z$
- $|z| = \sqrt{x^2 + y^2}$  is the modulus of  $z$

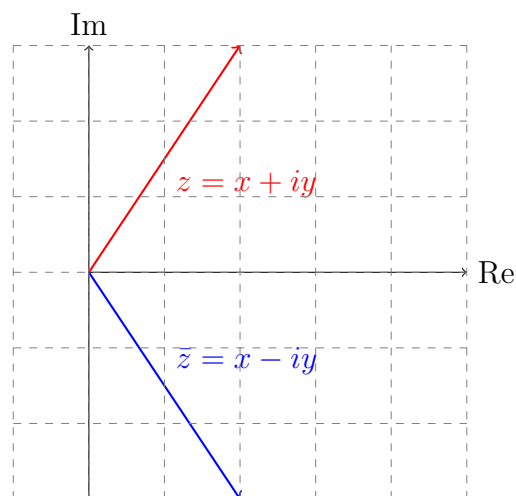
Note that:

$$z\bar{z} = x^2 + y^2 = |z|^2 \in \mathbb{R}$$

Also note how this formula is used in the computation of the inverse of  $z$ :

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

The geometric meaning of these attributes can be seen below.



## 1.5 Polar Form of a Complex Number

Given a non zero complex number  $z = x + iy$ :

1. **Magnitude:**  $|z| = \sqrt{x^2 + y^2}$

2. **Argument:** If you think of  $(x, y)$  as a point in the plane,  $\theta$ , is the angle the vector from the origin makes with the positive  $x$ -axis

- In polar form, we write:  $z = |z|(\cos \theta + i \sin \theta)$
- **Principle Argument** ( $\text{Arg } z$ ): This is the "main" angle  $\theta$  chosen to lie in  $(-\pi, \pi]$
- Because angles can differ by full turns ( $2\pi$ ), the general argument of  $z$  can be written as:

$$\arg(z) = \text{Arg}(z) + 2\pi n, \quad n \in \mathbb{Z}$$

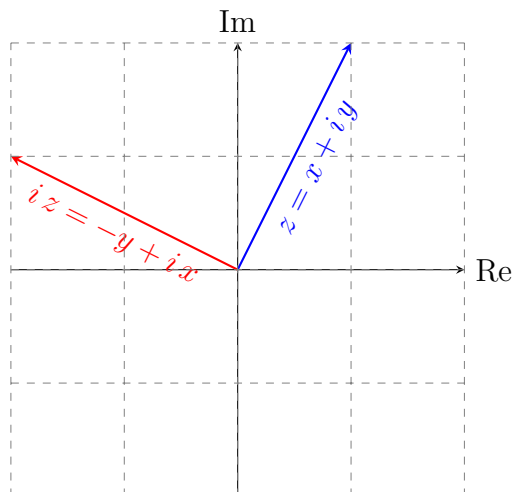
**Why angles are "multi-valued"** A direction in the plane can be expressed by infinitely many angles differing by whole circle ( $2\pi$ ). The "principal" angle is just a standard choice in  $(-\pi, \pi]$

## 1.6 Multiplication by $i$ and Geometric Interpretation

Multiplying a complex number  $z$  by  $i$  corresponds to a rotation by  $\frac{\pi}{2}$  ( $90^\circ$ ) in the complex plane.

Example:

$$\begin{aligned} z &= x + iy \\ iz &= i(x + iy) \\ &= ix - y \\ &= -y + ix \end{aligned}$$



## 1.7 Products of Complex Numbers and De Moivre's Theorem

### Theorem

Let  $z, z' \in \mathbb{C} \setminus \{0\}$ , then:

- $|zz'| = |z||z'|$  and  $\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|}$
- $\text{Arg}(zz') = \arg(z) + \arg(z')$  and  $\arg\left(\frac{z}{z'}\right) = \arg(z) - \arg(z')$

### Proof

Let  $x = x + iy$  and  $x' = x' + iy'$ , thus:

$$z = |z|(\cos \theta + i \sin \theta) \quad \text{and} \quad z' = |z'|(\cos \theta' + i \sin \theta')$$

Then:

$$\begin{aligned} |zz'| &= |z||z'|(\cos \theta + i \sin \theta)(\cos \theta' + i \sin \theta') \\ &= |z||z'|[(\cos \theta \cos \theta' - \sin \theta \sin \theta' + i(\cos \theta \sin \theta' + \sin \theta \cos \theta'))] \\ &= |z||z'|[\cos(\theta + \theta') + i \sin(\theta + \theta')] \end{aligned}$$

Thus,  $|zz'| = |z||z'|$  and  $\arg(zz') = \theta + \theta' = \arg(z) + \arg(z')$

Note that:  $|z||z'|$  acts as a stretch/ shrink,  $\cos(\theta + \theta')$  and  $\sin(\theta + \theta')$  act as a rotation.

### Corollary

**De Moivre's Theorem:** If  $z = |z|(\cos \theta + i \sin \theta)$ , then:

$$z^n = |z|^n \cos(n\theta) + i \sin(n\theta)$$

De Moivre's Theorem is extremely useful for raising a complex number to integer powers and expressing  $n$ th roots of unity (as a special case where  $r = 1$  and  $z^n = 1$ )

## 1.8 Roots of Unity

A **root of unity** is a complex number  $z$ , such that:

$$z^n = 1 \quad \text{for some integer } n \geq 1$$

Geometrically, roots of unity lie on the unit circle in the complex plane.

Specifically:

### Definition

The  $n$ th roots of unity are given by:

$$z_k = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right) \quad \text{for } k = 0, 1, 2, \dots, n-1$$

### Example

Recalling  $z = |z| [\cos \theta + i \sin \theta]$

Letting  $z^4 = 1$ , then we see:

$$z^4 = \cos(4\theta) + i \sin(4\theta) = 1 + 0i$$

We see,  $z$  is one of  $w_k = \cos\left(\frac{2\pi k}{4}\right) + i \sin\left(\frac{2\pi k}{4}\right) = 1, i, -1, i$

And that the 4 roots of unity form a square in the unit circle, and their sum is 0.

## 2 Week 2: Functions of a Complex Variable