MA2287: Complex Analysis Exam Notes

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1 Question 1:

1.1 Sketch the region in the complex plane determined by the inequality

- $|z-4| > 3|z+4|$ 2023 Q1(a)
- $\{z \in \mathbb{C} : |2z-1| < 2|2z-i|\}$ 2022 Q1(a), 2021 Q1(d), 2017 Q1(a), 2016 Q1(a)

1.2 Determine all solutions to roots of unity

- $z^6 - 1 = 0$ and factorize $x^6 - 1$ as a product of linear and quadratic factors 2023 Q1(b),2021 Q1(c)
- $z^4 = -81i$ and find a polynomial $p(z)$ with complex coefficients with root w and $p(\overline{w}) \neq 0$ 2022 Q1(b),2018 Q1(b)

1.3 Determine and sketch the image under the mapping

- $w = e^z$, $\{z \in \mathbb{C} : \pi/4 \leq \text{Im}(z) \leq \pi/2\}$ 2023 Q1(c), 2021 Q1(a), 2017 Q1(d)
- $w = \text{Log}(z)$, $\{z : |z| > 1, 0 \leq \text{Arg}(z) \leq \pi/2\}$ 2022 Q1(d), 2018 Q1(d), 2016 Q1(d)

1.4 Find z where the function is 0

- $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ 2022 Q1(d)

1.5 Calculate principal value Log(z)

- z $= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and prove e^z is the inverse function of Log(z) 2022 Q1(c), 2018 Q1(c), 2017 Q1(c)

1.6 Prove the following

- Define the complex conjugate $(\overline{w})$ and prove if w is a zero of a polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ then $\overline{w}$ is also a zero of $p(z)$ 2021 Q1(b), 2018 Q1(a), 2016 Q1(b)
- Define the complex exponential function e^z and prove Eulers Formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ 2017 Q1(b)

2 Question 2:

2.1 Determine image of the line

- $f(z) = \frac{1}{z} \quad \{z \in \mathbb{C} : \mathrm{Re}(z) = 2\}$ 2023 Q2(a), 2021 Q2(b)
- $f(z) = \frac{1}{z} \quad \{z \in \mathbb{C} : \mathrm{Re}(z) = 1\}$ 2022 Q2(a), 2018 Q2(a), 2017 Q2(a)

2.2 State and Use Cauchy-Riemann Equations

- State CRE, and use to prove $f(z) = \frac{1}{z}$ is holomoprhic on $\mathbb{C}\backslash\{0\}$ 2023 Q2(a)
- State CRE, and use to prove $f(z) = z^2$ is holomoprhic on $\mathbb{C}$ 2022 Q2(b)
- State CRE. Let $f = u + iv$ be holomoprhic on $\Omega \subset \mathbb{C}$. Prove ∇u and ∇v are perpendicular of equal length 2016 Q2(b)

2.3 Show that

- If $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$ then $f(x)$ is real for all $x \in \mathbb{R}$. And if in addition f is holomoprhic at $x \in \mathbb{R}$ then $f'(x)$ is real. 2023 Q2(c)
- Define that is meant for a function g to be harmonic. If $f = u + iv$ is holomoprhic on $\Omega \subset \mathbb{C}$, prove that $v(x, y)$ is a harmonic function, and that ∇u and ∇v are perpendicular of equal length. 2022 Q2(c), 2018 Q2(b)
- If $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$ then $f(x)$ is real for all $x \in \mathbb{R}$. And if in addition f is holomoprhic at 0 then the function $f'(0)$ is real. 2021 Q2(a), 2017 Q2(c)
- Let $f(z) = u + iv$ be holomoprhic on an open subset Ω of the complex plane and let $h(u, v)$ be a harmonic function of u and v on $f(\Omega)$. Prove that $g(x, y) = h(u(x, y), v(x, y))$ is harmonic on Ω (You may assume $\nabla u, \nabla v$ are equal length and perpendicular) 2021 Q2(c)
- Define what is meant for a function $f(z)$ to be holomoprhic at a point $z_0 \in \mathbb{C}$ and prove that $f(z) = z^2$ is holomoprhic and find its derivative there. Hence prove that the product uv is harmonic where $f = u + iv$ 2018 Q2(c)
- Define what is meant for a function $f(z)$ to be holomoprhic at a point $z_0 \in \mathbb{C}$ and prove that $f(z) = \frac{1}{z}$ is holomoprhic on $\mathbb{C}\backslash 0$ and find its derivative there (State any theorems used) 2017 Q2(b)
- Let $h(u, v)$ be a harmonic function of u, v on $f(\Omega)$ (See 2016 Q2(b)). Prove that $g(x, y) = h(u(x, y), v(x, y))$ is harmonic on Ω 2016 Q2(c)

2.4 Find Mobius Transformation

- $T(z) : (-1, 1, \infty) \mapsto (-1, -i, 1)$ 2023 Q2(d)
- $T(z) : (2, 1, -1) \mapsto (1, 0, \infty)$ 2022 Q2(d)
- $T(z) : (-i, -1, 1) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation 2021 Q2(d)
- $T(z) : (-i, -1, i) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation 2018 Q2(d), 2017 Q2(d)
- $T(z) : (-1, \frac{1}{2}, 2) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation 2016 Q2(d)

3 Worked Examples - Q1

Example 2023 Q1(a)

Given $|z-4| > 3|z+4|$
Write $z = x + iy$

$$|x+iy-4| > 3|x+iy+4|$$
$$|(x-4)+iy| > 3|(x+4)+iy|$$
$$\sqrt{(x-4)^2+y^2} > 3\sqrt{(x+4)^2+y^2}$$

Square both sides

$$(x-4)^2+y^2 > 9((x+4)^2+y^2)$$
$$(x^2-8x+16+y^2) > 9x^2+72x+144+9y^2$$
$$x^2-8x+16+y^2-9x^2-72x-144-9y^2 > 0 \qquad \textit{Moving all terms to one side}$$
$$-8x^2-80x-8y^2-128 > 0 \qquad \textit{Simplify}$$
$$x^2+10x+y^2-16 < 0 \qquad \textit{Dividing by -8 and reversing inequality}$$

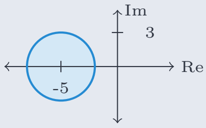
Focus on x and complete the square

$$x+bx = \left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \Rightarrow x^2+10x = (x+5)^2-25 \qquad \textit{Complete the square}$$
$$(x+5)^2-25+y^2+16 < 0 \qquad \textit{Substitute back into inequality}$$
$$(x+5)^2+y^2+9 < 0 \qquad \textit{Simplify}$$
$$(x+5)^2+y^2 < -9 \qquad \textit{Subtract 9}$$

Recall the eqation of a circle

$$(x-a)^2+(y-b)^2 = r^2 \Rightarrow (x+5)^2+y^2 < -9$$

Therefore the region is all the points inside circle with radius 3 and center at (-5, 0)



Example 2022 Q1(a), 2021 Q1(d), 2017 Q1(a), 2016 Q1(a)

Given $\{z \in \mathbb{C} : |2z-1| < 2|2z-i|\}$
Write $z = x + iy$

$$|2x+i2y-1| < 2|2x+i2y-i|$$
$$|(2x-1)+i2y| < 2|2x+i(2y-1)|$$
$$\sqrt{(2x-1)^2+4y^2} < 2\sqrt{4x^2+(2y-1)^2}$$
$$(2x-1)^2+4y^2 < 4[4x^2+(2y-1)^2] \qquad \textit{Square both sides}$$
$$4x^2-4x+1+4y^2 < 16x^2+16y^2-16y+4 \qquad \textit{Expand}$$
$$-12x^2-4x-12y^2+16y-3 < 0 \qquad \textit{Move all terms to one side}$$
$$12x^2+4x+12y^2-16y+3 > 0 \qquad \textit{Multiply by -1 and reverse inequality}$$
$$x^2+\frac{1}{3}x+y^2-\frac{4}{3}y+\frac{1}{4} > 0 \qquad \textit{Divide by 12}$$

Complete square for x

$$x^2+bx = \left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \Rightarrow x^2+\frac{1}{3}x = \left(x+\frac{1}{6}\right)^2 - \left(\frac{1}{36}\right)$$

Complete square for y

$$y^2+by = \left(y-\frac{2}{3}\right)^2 - \left(\frac{4}{9}\right)$$

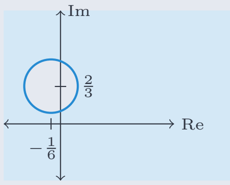
Substitute back into inequality

$$\left(x+\frac{1}{6}\right)^2 - \left(\frac{1}{36}\right) + \left(y-\frac{2}{3}\right)^2 - \left(\frac{4}{9}\right) + \frac{1}{4} > 0 \qquad \textit{Substitute back into inequality}$$
$$\left(x+\frac{1}{6}\right)^2 + \left(y-\frac{2}{3}\right)^2 > \frac{2}{9} \qquad \textit{Simplify and move constant across}$$

Recall the eqation of a circle

$$(x-a)^2+(y-b)^2 = r^2 \Rightarrow \left(x+\frac{1}{6}\right)^2 + \left(y-\frac{2}{3}\right)^2 < \frac{2}{9}$$

Therefore the region is all the points OUTSIDE the circle with radius $\frac{\sqrt{2}}{3}$ and center at $(-\frac{1}{6}, \frac{2}{3})$



Example Determine all solutions to $z^6 - 1 = 0$ and factor $x^6 - 1$ as a product of linear and quadratic factors

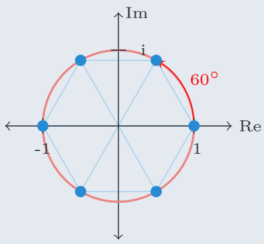
Given $z^6 - 1 = 0$
Write $z = e^{i\theta}$ and $1 = e^{i2\pi k}$ for $k \in \mathbb{Z}$

$$\begin{aligned} z^6 - 1 &= 0 \\ e^{i6\theta} - e^{i2\pi k} &= 0 \\ e^{i6\theta} &= e^{i2\pi k} \\ 6\theta &= 2\pi k \\ \theta &= \frac{\pi k}{3} \end{aligned}$$

Therefore the solutions are

$$z = e^{i\theta} = e^{i\frac{\pi k}{3}} = \cos\left(\frac{\pi k}{3}\right) + i\sin\left(\frac{\pi k}{3}\right) \quad \text{for} \quad k = 0, 1, 2, 3, 4, 5$$

$$\begin{aligned} &k = 0 : w_0 = \cos(0) + i\sin(0) = 1 + i0 \\ &k = 1 : w_1 = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2} \\ &k = 2 : w_2 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ &k = 3 : w_3 = \cos(\pi) + i\sin(\pi) = -1 \\ &k = 4 : w_4 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ &k = 5 : w_5 = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{aligned}$$



We can write:

$$x^6 - 1 = (x - w_0)(x - w_1)(x - w_2)(x - w_3)(x - w_4)(x - w_5)$$

Rewriting to group complex conjugates

$$x^6 - 1 = (z - w_0)(z - w_3) \cdot (z - w_1)(z - w_5) \cdot (z - w_2)(z - w_4)$$

Note that

$$\begin{aligned} (w - z)(w - \overline{z}) &= w^2 - w\overline{z} - zw + z\overline{z} \\ &= w^2 - 2(\overline{z} + z) + 1 \end{aligned}$$

We recall that

$$\begin{aligned} z = x + iy &= e^{i\theta} = \cos(\theta) + i\sin(\theta) \\ \overline{z} = x - iy &= e^{-i\theta} = \cos(\theta) - i\sin(\theta) \end{aligned}$$

Then

$$\begin{aligned} \overline{z} + z &= cos(\theta) + i\sin(\theta) + cos(\theta) - i\sin(\theta) \\ &= 2\cos(\theta) \end{aligned}$$

Thus

$$(w - z)(w - \overline{z}) = w^2 - 2\cos(\theta) + 1$$

We see that $-\frac{5\pi}{3} = \frac{\pi}{3} - 2\pi$, thus:

$$\begin{aligned} (z - w_1)(z - w_5) &= (z - e^{i\frac{\pi}{3}})(z - e^{i\frac{5\pi}{3}}) \\ (z - w_1)(z - w_5) &= z^2 - 2\cos\left(\frac{\pi}{3}\right) + 1 \\ (z - w_1)(z - w_5) &= z^2 + z + 1 \end{aligned}$$

We see that $-\frac{4\pi}{3} = \frac{\pi}{3} - \pi$, thus:

$$\begin{aligned} (z - w_2)(z - w_4) &= (z - e^{i\frac{2\pi}{3}})(z - e^{i\frac{4\pi}{3}}) \\ (z - w_2)(z - w_4) &= z^2 - 2\cos\left(\frac{2\pi}{3}\right) + 1 \\ (z - w_2)(z - w_4) &= z^2 - z + 1 \end{aligned}$$

Therefore

$$x^6 - 1 = (x + 1)(x - 1)(x^2 + x + 1)(x^2 - x + 1)$$

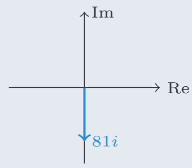
Example : Determine all solutions to $z^4 = -81i$ and find a polynomial $p(z)$ with complex coefficients with root w and $p(\overline{w}) \neq 0$

Given $z^4 = -81i$, we want to find $z^{4\left(\frac{1}{4}\right)} = w$
Recall:

$$z^{1/n} = R^{1/n}[\cos\phi + i\sin\phi] \quad \text{with } \phi = \frac{\theta + 2k\pi}{n}, \ k \in (0,1,2,\ldots,n-1) \quad \text{and } R = |z|$$

Thus

$$R = |81i| = \sqrt{0^2 + 81^2} = 81$$
$$\theta = -\frac{\pi}{2}$$
$$\phi = \frac{\theta + 2k\pi}{n} = \frac{-\frac{\pi}{2} + 2k\pi}{4} = \frac{-\pi}{8} + \frac{k\pi}{2}$$



Therefore

$$\begin{aligned}
w_k &= 81^{1/4}\left[\cos\left(\frac{-\pi}{8} + \frac{k\pi}{2}\right) + i\sin\left(\frac{-\pi}{8} + \frac{k\pi}{2}\right)\right] \quad k \in (0,1,2,3)\\
w_0 &= 3\left[\cos\left(\frac{-\pi}{8}\right) + i\sin\left(\frac{-\pi}{8}\right)\right] \approx 2.77 - 1.155i\\
w_1 &= 3\left[\cos\left(-\frac{\pi}{8} + \frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{8} + \frac{\pi}{2}\right)\right] \approx 1.155 + 2.77i\\
w_2 &= 3\left[\cos\left(-\frac{\pi}{8} + \pi\right) + i\sin\left(-\frac{\pi}{8} + \pi\right)\right] \approx -1.55 + 2.77i\\
w_3 &= 3\left[\cos\left(-\frac{\pi}{8} + \frac{3\pi}{2}\right) + i\sin\left(-\frac{\pi}{8} + \frac{3\pi}{2}\right)\right] \approx -2.77 - 1.55i
\end{aligned}$$

Part 2:
Given $p(z)$ with complex coefficients has root w and $p(\overline{w}) \neq 0$
In other words, we want $p(w) = 0$ and $p(\overline{w}) \neq 0$

Using the most simple polynomial, $p(z) = z - w$ and letting $w = 3e^{i\frac{-\pi}{8}}$ we have

$$p(z) = z - 3e^{i\frac{-\pi}{8}}$$

$$\begin{aligned}
p(w) &= w - w\\
&= 3e^{i\frac{-\pi}{8}} - 3e^{i\frac{-\pi}{8}}\\
&= 0
\end{aligned}$$

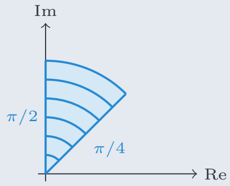
$$\begin{aligned}
p(\overline{w}) &= \overline{w} - 3e^{i\frac{-\pi}{8}}\\
&= 3e^{-i\frac{\pi}{8}} - 3e^{i\frac{-\pi}{8}}\\
&= 3\left[\cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right) - \left(\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right)\right]\\
&= 3\left[\cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right) - cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right)\right]\\
&= 3\left[-2i\sin\left(\frac{\pi}{8}\right)\right]\\
&= -6i\sin\left(\frac{\pi}{8}\right)\\
&\approx -2.3i \neq 0
\end{aligned}$$

Example Determine and sketch the image under the map $w = e^z$, $\{z \in \mathbb{C} : \pi/4 \leq \text{Im}(z) \leq \pi/2\}$

$$\begin{aligned}
w &= e^z = e^{x+iy}\\
&= e^x e^{iy}\\
&= e^x[\cos(y) + i\sin(y)]
\end{aligned}$$

Recall the polar form of a complex number $z = |z|[\cos(\theta) + i\sin(\theta)]$
We see, e^x acts as the radius, and is always positive, and $[\cos(y) + i\sin(y)]$ acts draws out a section of the unit circle, thus the mapping $w = e^z$ sends the set to:

$$\left\{w \in \mathbb{C} : |w| > 0, \frac{\pi}{4} \leq \arg(w) \leq \frac{\pi}{2}\right\}$$

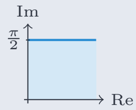


Example Determine and sketch the region $w = \mathrm{Log}(z)$, $\{z : |z| > 1, 0 \leq \mathrm{Arg}(z) \leq \pi/2\}$

$$w = \mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z) = u + iv$$

Note that $|z| > 1$ implies $\ln|z| > 0$ Thus:

$$\left\{ w = u + iv \in \mathbb{C} : u > 0, 0 \leq v \leq \frac{\pi}{2} \right\}$$



Example Find where the function is 0 : $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$

We want $cos(z) = \frac{e^{iz}+e^{-iz}}{2} = 0$, some basic algebra gives us:

$$\begin{aligned} \cos(z) = \frac{e^{iz} + e^{-iz}}{2} &= 0 \\ e^{iz} + e^{-iz} &= 0 \\ e^{iz} &= -e^{-iz} \\ e^{iz} \cdot e^{iz} &= -e^{-iz} \cdot e^{iz} && \textit{Multiply both sides by } e^{iz} \\ e^{2iz} &= -e^{0} && e^a \cdot e^b = e^{a+b} \\ e^{2iz} &= -1 \end{aligned}$$

Recall:

$$-1 = cos(\pi + 2\pi k) + i\sin(\pi + 2\pi k) = e^{i(\pi+2\pi k)}$$

Thus

$$\begin{aligned} e^{2iz} &= e^{i(\pi+2\pi k)} \\ 2iz &= i(\pi + 2\pi k) && \textit{Taking the natrual log of both sides} \\ 2z &= \pi + 2\pi k && \textit{Divide by 2} \\ z &= \frac{\pi}{2} + \pi k && \textit{Divide by } i \end{aligned}$$

Therefore, the zeros of $\cos(z)$ are:

$$z = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}$$

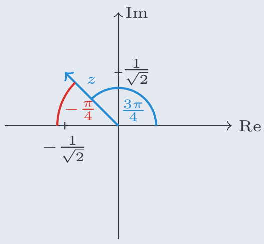
Example Calculate the principal value $\mathrm{Log}(z)$ of $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and prove e^z is the inverse function of $\mathrm{Log}(z)$

Part 1.
Given $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$:

$$\ln|z| = \sqrt{\left(1/\sqrt{2}\right)^2 + \left(1/\sqrt{2}\right)^2} = 1$$

and

$$\begin{aligned} \mathrm{Arg}(z) &= \tan^{-1}(-1) \\ &= -\tan^{-1}(1) \\ &= -\frac{\pi}{4} \Rightarrow \frac{3\pi}{4} \end{aligned}$$



Therefore

$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z) = i\frac{3\pi}{4}$$

Part 2: We need to show that (a) $e^{\mathbf{Log(z)}} = z$ and (b) $\mathbf{Log}(e^z) = z$

(a) Let $z = |z|e^{i\theta}, |z| > 0$ and $\theta = \mathrm{Arg}(z)$

$$\begin{aligned} \mathrm{Log}(z) &= \ln|z| + i\theta \\ e^{\mathrm{Log}(z)} &= e^{\ln|z|+i\theta} && \textit{Exponentiate both sides} \\ &= e^{\ln|z|} \cdot e^{i\theta} && \textit{Exponentiation rules} \\ &= |z| \cdot e^{i\theta} \\ e^{\mathrm{Log}(z)} &= z \end{aligned}$$

(b) Let $z = x + iy, y \in [-\pi, \pi]$

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x \cdot e^{iy} \\ \mathrm{Log}(e^z) &= \ln|e^x \cdot e^{iy}| && \textit{Take log of both sides} \\ &= \ln|e^x| + \ln|e^{iy}| && \log(a \cdot b) = \log(a) + \log(b) \\ &= x + iy \\ \mathrm{Log}(e^z) &= z \end{aligned}$$

Example Define the complex conjugate ($\overline{w}$) and prove if w is a zero of a polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ then $\overline{w}$ is also a zero of $p(z)$

Definition: For a complex number $w = a + bi$ the complex conjugate of w is defined as $\overline{w} = a - bi$ (with $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$)
This has several properties:

$$\overline{z+w} = \overline{z} + \overline{w}$$
$$\overline{zw} = \overline{z} \cdot \overline{w}$$
$$\overline{(w^n)} = (\overline{w})^n$$

Proof: If w is zero of a polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ then $p(\overline{w}) = 0$

$$\begin{aligned}
\text{Assume} \quad & p(w) = a_0 + a_1 w + \ldots + a_n w^n = 0 \\
\text{Take the conjugate of both sides} \quad & \overline{p(w)} = \overline{0} = 0 \\
\text{Evalute} \quad & p(\overline{w}) = a_0 + a_1 \overline{w} + \ldots + a_n \overline{w}^n \\
& = a_0 + a_1 \overline{w} + \ldots + a_n \overline{w^n} \\
& = \overline{a_0} + \overline{a_1 w} + \ldots + \overline{a_n w^n} \\
& = \overline{a_0 + a_1 w + \ldots + a_n w^n} \\
& = \overline{p(w)} = 0
\end{aligned}$$

Thus, since we assumed $p(w) = 0$:

$$p(\overline{w}) = \overline{p(w)} = 0$$

Example Define the complex exponential function e^z and prove Eulers Foruma $e^{i\theta} = \cos(\theta) + i \sin\theta$

Defition : For any $z \in mathbbC$, e^z is defined by its power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

The series converges for all $z \in \mathbb{C}$ and has the following properties:

$$e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$$
$$e^z \cdot e^{-z} = 1$$

Proof of Eulers Formula

$$\text{Eulers Formula} \quad e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$\begin{aligned}
e^{i\theta} &= \textstyle\sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} && \textit{Substitute } z = i\theta \\
&= \textstyle\sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} && \textit{Split into even and odd powers} \\
&= \textstyle\sum_{n=0}^{\infty} \frac{(i)^{2n}(\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i)^{2n+1}(\theta)^{2n+1}}{(2n+1)!} && \textit{Seperate powers}
\end{aligned}$$

We note that:

$$i^{2n} = (i^2)^n = (-1)^n$$
$$i^{2n+1} = i \cdot i^{2n} = i(-1)^n$$

Thus:

$$\begin{aligned}
e^{i\theta} &= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n(\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i(-1)^n(\theta)^{2n+1}}{(2n+1)!} \\
&= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n(\theta)^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n(\theta)^{2n+1}}{(2n+1)!} \\
&:= \cos(\theta) + i \sin(\theta)
\end{aligned}$$