MA283: Linear Algebra

70% Exam
30% Continuous Assessment (Homework)
10% Optional Project (Bonus)

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1 Systems of linear equations

1.1 Linear equations and Solution Sets

A linear equation in the variables x and y is an equation of the form

$$2x + y = 3$$

If we replace x and y with some numbers, the statement **becomes true or false**.

Definition 1.1: Solution to a linear equation

A pair, $(x_0, y_0) \in \mathbb{R}$, is a solution to an linear equation if setting $x = x_0$ and $y = y_0$ **makes the equation true.**

Definition 1.2: Solution set

The **solution set** is the set of all solutions to a linear equation.

$$a_1X_1 + a_2X_2 + \ldots + a_nX_n = b \quad \text{where } a_i, b \in \mathbb{R}$$

is an **affine hyperplane** in $\mathbb{R}^n$; geometrically resembles a copy of $\mathbb{R}^{n-1}$ inside $\mathbb{R}^n$.

1.2 Elementary Row Operations

To solve a system of linear equations we associate an **augmented matrix** to the system of equations. For example:

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array}\right]$$

To solve, we can perform the following **Elementary Row Operations (EROs)**:

1. Multiply a row by a non-zero constant.
2. Add a multiple of one row to another row.
3. Swap two rows.

The goal of these operations is to transform the augmented matrix into **row echelon form** (REF) or **reduced row echelon form** (RREF).

1.2.1 REF and Strategy

We say a matrix is in **row echelon form** (REF) if:

- The first non zero entry in each row is a 1 (called the **leading 1**).
- If a column has a leading 1, then all entries below it are 0.
- The leading 1 in each row is to the right of the leading 1 in the previous row.
- All rows of 0s are at the bottom of the matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Example of REF

We have produced a new system of equations. This is easily solved by back substitution.

Concept 1.1: Stategy for Obtaining REF

- Get a 1 as the top left entry
- Use this 1 to clear the entries below it
- Move to the next column and repeat
- Continue until all leading 1s are in place
- Use back substitution to solve the system

1.2.2 Row Reduced Echelon Form

A matrix is in **reduced row echelon form** (RREF) if:

- It is in REF
- The leading 1 in each row is the only non-zero entry in its column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Example of RREF

1.3 Leading variables and free variables

We'll start by an example:

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 & -1 & -1 & 2 & | & 0 \\ 2 & 1 & -1 & 2 & | & 8 \\ 1 & -3 & 2 & 7 & | & 2 \end{bmatrix}$$

Solving this system of equations, we get:

$$\text{RREF:} \quad \begin{bmatrix} 1 & 0 & 0 & 2 & | & 4 \\ 0 & 1 & 0 & -1 & | & 2 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rcrcr} x_1 & + & 2x_4 & = & 4 \\ x_2 & - & x_4 & = & 2 \\ x_3 & + & x_4 & = & 2 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x_1 & = & 4 - 2x_4 \\ x_2 & = & 2 + x_4 \\ x_3 & = & 2 - x_4 \end{array}$$

This RREF tells us how the **leading variables** (x_1, x_2, x_3) depend on the **free variable** (x_4). The free variable can take any value in $\mathbb{R}$. We write the solution set as:

$$x_1 = 4 - 2t, \quad x_2 = 2 + t, \quad x_3 = 2 - t, \quad x_4 = t \quad \text{where } t \in \mathbb{R}$$

$$(x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t); \quad t \in \mathbb{R}$$

Definition 1.3: Leading and Free Variables

- **Leading variable** : A variable whose columns in the RREF contain a leading 1
- **Free variable** : A variable whose columns in the RREF do not contain a leading 1

1.4 Consistent and Inconsistent Systems

Consider the following system of equations:

$$\begin{array}{rcrcrcr} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 3 & 2 & -5 & | & 4 \\ 1 & 1 & -2 & | & 1 \\ 5 & 3 & -8 & | & 6 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 & -2 & | & 1 \\ 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & | & 1 \end{bmatrix} \quad \text{(REF)}$$

We can see the last row of the REF is:

$$0x + 0y + 0z = 1$$

This equation clearly has no solution, and hence the system has no solutions. We say the system is **inconsistent**. Alternatively, we say the system is **consistent** if it has at least one solution.

1.5 Possible Outcomes when solving a system of equations

- The system may be **inconsistent** (no solutions) - i.e:

$$[0\ 0\ \ldots\ 0 \mid a] \quad a \neq 0$$

- The system may be **consistent** which occurs if:
  - **Unique Solutions** each column (aside from the rightmost) contains a single leading 1. - i.e:

$$\begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$$

  - **Infinitely many solutions** at least one variable does not appear as a leading 1 in any row, making it a free variable - i.e:

$$\begin{bmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 0 & 1 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

1.6 Gaussian Elimination and Matrix Algebra

Elementary row operations may be interpreted as **matrix multiplication**. To see this, first we introduce the **Identity matrix:**

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The I_m Identity matrix is an $m \times m$ matrix with 1s on the diagonal and 0s elsewhere. We also introduce the $E_{i,j}$ matrix which has 1 in the (i, j) position and 0s elsewhere. For example:

$$E_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then:

$$I_3 + 4E_{1,2} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing a row operation on A is the same as multiplying A by an appropriate matrix E on the left

Multiplying a row by a non zero scalar

- When you multiply row i of A, by some scalar $\alpha \neq 0$, all other rows remain unchanged, and the ith row is scaled by α.
- The Identity matrix I_m leaved A unchanged if we multiply $I_m A$
- To scale row i by α,you want the (i, i)-entry of this matrix to be α, so the ith row of A is scaled by α.
- Hence $I_m + (\alpha - 1)E_{i,i}$ causes the ith row of A to be scaled by α.

That is what the matrix

$$I_m + (\alpha - 1)E_{i,i}$$

does to A: Observe that I_m has all diagonal entries $= 1$. Adding $(\alpha - 1)E_{i,i}$ makes that diagonal entry become α and all other diagonal entries remain 1. Example:

$$I_3 + 4_E 2, 2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{then} \quad (I_3 + 4E_{2,2})A = \begin{bmatrix} 1 & 2 & 3 \\ 20 & 25 & 30 \\ 7 & 8 & 9 \end{bmatrix}$$

Switching two rows

- Consider the effect of switching two ros in a matrix. The swapping matrix S must do the following:
  - Take row i of A and put it in row k of SA
  - Take row k of A and put it in row i of SA
  - Leave all other rows unchanged
- One way to build this swapping matrix is to start with I_m and then tinker with the rows/columns so that the ith rwo picks out the old kth row and the kth row picks out the old ith row.

The matrix:

$$S = I_m + E_{i,k} + E_{k,i} - E_{i,i} - E_{k,k}$$

does this. Observe that:

- Normally I_m has 1's on the diagonal (j, j)
- We subtract the 1's positions (i, i) and (k, k), so those spots become 0.
- We add 1's in the positions (i, k) and (k, i), so those spots become 1.

$$S = I_3 + E_{1,3} + E_{3,1} - E_{1,1} - E_{3,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{then} \quad SA = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Adding a multiple of one row to another

- Suppose you want to replace row k by with row $k + \alpha \times$ row i.
- Starting with the identity matrix I_m, multiplying on the left by I_m would leave A unchanged.
- To add $\alpha \times$ row i to row k, we want the multiplication to say:
  - Keep j unchanged, for all $j \neq k$
  - Replace row k with itself plus α times row i

The matrix:

$$I_m + \alpha E_{k,i}$$

does this. Observe that:

- Row j of $I_m + \alpha E_{k,i}$ is just the standard basis row whenever $j \neq k$. So row j of A is unchanged.
- Row k of $I_m + \alpha E_{k,i}$ is

$$\text{Row } k \text{ of } I_m + \alpha \text{Row } i \text{ of } I_m$$