

MA2287: Complex Analysis Exam Notes

Robert Davidson

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1 Question 1:

1.1 Sketch the region in the complex plane determined by the inequality

- $|z - 4| > 3|z + 4|$ [2023 Q1\(a\)](#)
- $\{z \in \mathbb{C} : |2z - 1| < 2|2z - i|\}$ [2022 Q1\(a\), 2021 Q1\(d\), 2017 Q1\(a\), 2016 Q1\(a\)](#)

1.2 Determine all solutions to roots of unity

- $z^6 - 1 = 0$ and factorize $z^6 - 1$ as a product of linear and quadratic factors [2023 Q1\(b\), 2021 Q1\(c\)](#)
- $z^4 = -81i$ and find a polynomial $p(z)$ with complex coefficients with root w and $p(\bar{w}) \neq 0$ [2022 Q1\(b\), 2018 Q1\(b\)](#)

1.3 Determine and sketch the image under the mapping

- $w = e^z, \{z \in \mathbb{C} : \pi/4 \leq \text{Im}(z) \leq \pi/2\}$ [2023 Q1\(c\), 2021 Q1\(a\), 2017 Q1\(d\)](#)
- $w = \text{Log}(z), \{z : |z| > 1, 0 \leq \text{Arg}(z) \leq \pi/2\}$ [2022 Q1\(d\), 2018 Q1\(d\), 2016 Q1\(d\)](#)

1.4 Find z where the function is 0

- $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$ [2022 Q1\(d\)](#)

1.5 Calculate principal value $\text{Log}(z)$

- $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and prove e^z is the inverse function of $\text{Log}(z)$ [2022 Q1\(c\), 2018 Q1\(c\), 2017 Q1\(c\)](#)

1.6 Prove the following

- Define the complex conjugate (\bar{w}) and prove if w is a zero of a polynomial $p(z) = a_0 + a_1z + \dots + a_nz^n$ then \bar{w} is also a zero of $p(z)$ [2021 Q1\(b\), 2018 Q1\(a\), 2016 Q1\(b\)](#)
- Define the complex exponential function e^z and prove Eulers Formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ 2017 Q1(b)

2 Question 2:

2.1 Determine image of the line

- $f(z) = \frac{1}{z}$ $\{z \in \mathbb{C} : \operatorname{Re}(z) = 2\}$ 2023 Q2(a), 2021 Q2(b)
- $f(z) = \frac{1}{z}$ $\{z \in \mathbb{C} : \operatorname{Re}(z) = 1\}$ 2022 Q2(a), 2018 Q2(a), 2017 Q2(a)

2.2 State and Use Cauchy-Riemann Equations

- State CRE, and use to prove $f(z) = \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ 2023 Q2(a)
- State CRE, and use to prove $f(z) = z^2$ is holomorphic on \mathbb{C} 2022 Q2(b)
- State CRE. Let $f = u + iv$ be holomorphic on $\Omega \subset \mathbb{C}$. Prove ∇u and ∇v are perpendicular of equal length 2016 Q2(b)

2.3 Show that

- If $\overline{f(z)} = f(\bar{z})$ for all $z \in \mathbb{C}$ then $f(x)$ is real for all $x \in \mathbb{R}$. And if in addition f is holomorphic at $x \in \mathbb{R}$ then $f'(x)$ is real. 2023 Q2(c)
- Define that is meant for a function g to be harmonic. If $f = u + iv$ is holomorphic on $\Omega \subset \mathbb{C}$, prove that $v(x, y)$ is a harmonic function, and that ∇u and ∇v are perpendicular of equal length. 2022 Q2(c), 2018 Q2(b)
- If $\overline{f(z)} = f(\bar{z})$ for all $z \in \mathbb{C}$ then $f(x)$ is real for all $x \in \mathbb{R}$. And if in addition f is holomorphic at 0 then the function $f'(0)$ is real. 2021 Q2(a), 2017 Q2(c)
- Let $f(z) = u + iv$ be holomorphic on an open subset Ω of the complex plane and let $h(u, v)$ be a harmonic function of u and v on $f(\Omega)$. Prove that $g(x, y) = h(u(x, y), v(x, y))$ is harmonic on Ω (You may assume $\nabla u, \nabla v$ are equal length and perpendicular) 2021 Q2(c)
- Define what is meant for a function $f(z)$ to be holomorphic at a point $z_0 \in \mathbb{C}$ and prove that $f(z) = z^2$ is holomorphic and find its derivative there. Hence prove that the product uv is harmonic where $f = u + iv$ 2018 Q2(c)
- Define what is meant for a function $f(z)$ to be holomorphic at a point $z_0 \in \mathbb{C}$ and prove that $f(z) = \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and find its derivative there (State any theorems used) 2017 Q2(b)
- Let $h(u, v)$ be a harmonic function of u, v on $f(\Omega)$ (See 2016 Q2(b)). Prove that $g(x, y) = h(u(x, y), v(x, y))$ is harmonic on Ω 2016 Q2(c)

2.4 Find Mobius Transformation

- $T(z) : (-1, 1, \infty) \mapsto (-1, -i, 1)$ 2023 Q2(d)
- $T(z) : (2, 1, -1) \mapsto (1, 0, \infty)$ 2022 Q2(d)
- $T(z) : (-i, -1, 1) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation 2021 Q2(d)
- $T(z) : (-i, -1, i) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation 2018 Q2(d), 2017 Q2(d)
- $T(z) : (-1, \frac{1}{2}, 2) \mapsto (1, 0, \infty)$ and find the inverse Mobius Transformation 2016 Q2(d)

3 Worked Examples - Q1

Example 2023 Q1(a)

Given $|z - 4| > 3|z + 4|$
Write $z = x + iy$

$$\begin{aligned} |x + iy - 4| &> 3|x + iy + 4| \\ |(x - 4) + iy| &> 3|(x + 4) + iy| \\ \sqrt{(x - 4)^2 + y^2} &> 3\sqrt{(x + 4)^2 + y^2} \end{aligned}$$

Square both sides

$$\begin{aligned} (x - 4)^2 + y^2 &> 9((x + 4)^2 + y^2) \\ (x^2 - 8x + 16 + y^2) &> 9x^2 + 72x + 144 + 9y^2 \\ x^2 - 8x + 16 + y^2 - 9x^2 - 72x - 144 - 9y^2 &> 0 \\ -8x^2 - 80x - 8y^2 - 128 &> 0 \\ x^2 + 10x + y^2 - 16 &< 0 \end{aligned}$$

Moving all terms to one side

Simplify

Dividing by -8 and reversing inequality

Focus on x and complete the square

$$\begin{aligned} x + bx &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \Rightarrow x^2 + 10x = (x + 5)^2 - 25 \\ (x + 5)^2 - 25 + y^2 + 16 &< 0 \\ (x + 5)^2 + y^2 + 9 &< 0 \\ (x + 5)^2 + y^2 &< -9 \end{aligned}$$

Complete the square

Substitute back into inequality

Simplify

Subtract 9

Recall the equation of a circle

$$(x - a)^2 + (y - b)^2 = r^2 \Rightarrow (x + 5)^2 + y^2 < -9$$

Therefore the region is all the points inside circle with radius 3 and center at (-5, 0)



Example 2022 Q1(a), 2021 Q1(d), 2017 Q1(a), 2016 Q1(a)

Given $\{z \in \mathbb{C} : |2z - 1| < 2|2z - i|\}$
Write $z = x + iy$

$$\begin{aligned} |2x + i2y - 1| &< 2|2x + i2y - i| \\ |(2x - 1) + i2y| &< 2|2x + i(2y - 1)| \\ \sqrt{(2x - 1)^2 + 4y^2} &< 2\sqrt{4x^2 + (2y - 1)^2} \\ (2x - 1)^2 + 4y^2 &< 4[4x^2 + (2y - 1)^2] \\ 4x^2 - 4x + 1 + 4y^2 &< 16x^2 + 16y^2 - 16y + 4 \\ -12x^2 - 4x - 12y^2 + 16y - 3 &< 0 \\ 12x^2 + 4x + 12y^2 - 16y + 3 &> 0 \\ x^2 + \frac{1}{3}x + y^2 - \frac{4}{3}y + \frac{1}{4} &> 0 \end{aligned}$$

Square both sides

Expand

Move all terms to one side

Multiply by -1 and reverse inequality

Divide by 12

Complete square for x

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \Rightarrow x^2 + \frac{1}{3}x = \left(x + \frac{1}{6}\right)^2 - \left(\frac{1}{36}\right)$$

Complete square for y

$$y^2 + by = \left(y - \frac{2}{3}\right)^2 - \left(\frac{4}{9}\right)$$

Substitute back into inequality

$$\begin{aligned} \left(x + \frac{1}{6}\right)^2 - \left(\frac{1}{36}\right) + \left(y - \frac{2}{3}\right)^2 - \left(\frac{4}{9}\right) + \frac{1}{4} &> 0 \\ \left(x + \frac{1}{6}\right)^2 + \left(y - \frac{2}{3}\right)^2 &> \frac{2}{9} \end{aligned}$$

Substitute back into inequality

Simplify and move constant across

Recall the equation of a circle

$$(x - a)^2 + (y - b)^2 = r^2 \Rightarrow \left(x + \frac{1}{6}\right)^2 + \left(y - \frac{2}{3}\right)^2 < \frac{2}{9}$$

Therefore the region is all the points OUTSIDE the circle with radius $\frac{\sqrt{2}}{3}$ and center at $(-\frac{1}{6}, \frac{2}{3})$



Example Determine all solutions to $z^6 - 1 = 0$ and factor $z^6 - 1$ as a product of linear and quadratic factors

Given $z^6 - 1 = 0$
Write $z = e^{i\theta}$ and $1 = e^{i2\pi k}$ for $k \in \mathbb{Z}$

$$\begin{aligned} z^6 - 1 &= 0 \\ e^{i6\theta} - e^{i2\pi k} &= 0 \\ e^{i6\theta} &= e^{i2\pi k} \\ 6\theta &= 2\pi k \\ \theta &= \frac{\pi k}{3} \end{aligned}$$

Therefore the solutions are

$$z = e^{i\theta} = e^{i\frac{\pi k}{3}} = \cos\left(\frac{\pi k}{3}\right) + i \sin\left(\frac{\pi k}{3}\right) \quad \text{for } k = 0, 1, 2, 3, 4, 5$$

$$k = 0 : w_0 = \cos(0) + i \sin(0) = 1 + i0$$

$$k = 1 : w_1 = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$k = 2 : w_2 = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$k = 3 : w_3 = \cos(\pi) + i \sin(\pi) = -1$$

$$k = 4 : w_4 = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$k = 5 : w_5 = \cos\left(\frac{5\pi}{3}\right) + i \sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$



We can write:

$$z^6 - 1 = (z - w_0)(z - w_1)(z - w_2)(z - w_3)(z - w_4)(z - w_5)$$

Rewriting to group complex conjugates

$$z^6 - 1 = (z - w_0)(z - w_3) \cdot (z - w_1)(z - w_5) \cdot (z - w_2)(z - w_4)$$

Note that

$$\begin{aligned} (w - z)(w - \bar{z}) &= w^2 - w\bar{z} - zw + z\bar{z} \\ &= w^2 - 2(\bar{z} + z) + 1 \end{aligned}$$

We recall that

$$\begin{aligned} z &= x + iy = e^{i\theta} = \cos(\theta) + i \sin(\theta) \\ \bar{z} &= x - iy = e^{-i\theta} = \cos(\theta) - i \sin(\theta) \end{aligned}$$

Then

$$\begin{aligned} \bar{z} + z &= \cos(\theta) + i \sin(\theta) + \cos(\theta) - i \sin(\theta) \\ &= 2 \cos(\theta) \end{aligned}$$

Thus

$$(w - z)(w - \bar{z}) = w^2 - 2 \cos(\theta) + 1$$

We see that $-\frac{5\pi}{3} = \frac{\pi}{3} - 2\pi$, thus:

We see that $-\frac{4\pi}{3} = \frac{\pi}{3} - \pi$, thus:

$$(z - w_1)(z - w_5) = (z - e^{i\frac{\pi}{3}})(z - e^{i\frac{5\pi}{3}})$$

$$(z - w_2)(z - w_4) = (z - e^{i\frac{2\pi}{3}})(z - e^{i\frac{4\pi}{3}})$$

$$(z - w_1)(z - w_5) = z^2 - 2 \cos\left(\frac{\pi}{3}\right) + 1$$

$$(z - w_2)(z - w_4) = z^2 - 2 \cos\left(\frac{2\pi}{3}\right) + 1$$

$$(z - w_1)(z - w_5) = z^2 + z + 1$$

$$(z - w_2)(z - w_4) = z^2 - z + 1$$

Therefore

$$z^6 - 1 = (z + 1)(z - 1)(z^2 + z + 1)(z^2 - z + 1)$$

Example : Determine all solutions to $z^4 = -81i$ and find a polynomial $p(z)$ with complex coefficients with root w and $p(\bar{w}) \neq 0$

Given $z^4 = -81i$, we want to find $z^4\left(\frac{1}{4}\right) = w$

Recall:

$$z^{1/n} = R^{1/n} [\cos \phi + i \sin \phi] \quad \text{with } \phi = \frac{\theta + 2k\pi}{n}, \quad k \in (0, 1, 2, \dots, n-1) \quad \text{and } R = |z|$$

Thus

$$R = |81i| = \sqrt{0^2 + 81^2} = 81$$

$$\theta = -\frac{\pi}{2}$$

$$\phi = \frac{\theta + 2k\pi}{n} = \frac{-\frac{\pi}{2} + 2k\pi}{4} = -\frac{\pi}{8} + \frac{k\pi}{2}$$

Therefore

$$w_k = 81^{1/4} \left[\cos \left(-\frac{\pi}{8} + \frac{k\pi}{2} \right) + i \sin \left(-\frac{\pi}{8} + \frac{k\pi}{2} \right) \right] \quad k \in (0, 1, 2, 3)$$

$$w_0 = 3 \left[\cos \left(-\frac{\pi}{8} \right) + i \sin \left(-\frac{\pi}{8} \right) \right] \approx 2.77 - 1.155i$$

$$w_1 = 3 \left[\cos \left(-\frac{\pi}{8} + \frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{8} + \frac{\pi}{2} \right) \right] \approx 1.155 + 2.77i$$

$$w_2 = 3 \left[\cos \left(-\frac{\pi}{8} + \pi \right) + i \sin \left(-\frac{\pi}{8} + \pi \right) \right] \approx -1.55 + 2.77i$$

$$w_3 = 3 \left[\cos \left(-\frac{\pi}{8} + \frac{3\pi}{2} \right) + i \sin \left(-\frac{\pi}{8} + \frac{3\pi}{2} \right) \right] \approx -2.77 - 1.55i$$

Part 2:

Given $p(z)$ with complex coefficients has root w and $p(\bar{w}) \neq 0$

In other words, we want $p(w) = 0$ and $p(\bar{w}) \neq 0$

Using the most simple polynomial, $p(z) = z - w$ and letting $w = 3e^{i\frac{-\pi}{8}}$ we have

$$p(z) = z - 3e^{i\frac{-\pi}{8}}$$

$$p(w) = w - w$$

$$= 3e^{i\frac{-\pi}{8}} - 3e^{i\frac{-\pi}{8}} \\ = 0$$

$$\begin{aligned} p(\bar{w}) &= \bar{w} - 3e^{i\frac{-\pi}{8}} \\ &= 3e^{-i\frac{\pi}{8}} - 3e^{i\frac{-\pi}{8}} \\ &= 3 \left[\cos \left(\frac{\pi}{8} \right) - i \sin \left(\frac{\pi}{8} \right) - \left(\cos \left(\frac{\pi}{8} \right) + i \sin \left(\frac{\pi}{8} \right) \right) \right] \\ &= 3 \left[\cos \left(\frac{\pi}{8} \right) - i \sin \left(\frac{\pi}{8} \right) - \cos \left(\frac{\pi}{8} \right) - i \sin \left(\frac{\pi}{8} \right) \right] \\ &= 3 \left[-2i \sin \left(\frac{\pi}{8} \right) \right] \\ &= -6i \sin \left(\frac{\pi}{8} \right) \\ &\approx -2.3i \neq 0 \end{aligned}$$



Example Determine and sketch the image under the map $w = e^z$, $\{z \in \mathbb{C} : \pi/4 \leq \text{Im}(z) \leq \pi/2\}$

$$\begin{aligned} w &= e^z = e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x [\cos(y) + i \sin(y)] \end{aligned}$$

Recall the polar form of a complex number $z = |z|[\cos(\theta) + i \sin(\theta)]$

We see, e^x acts as the radius, and is always positive, and $[\cos(y) + i \sin(y)]$ acts draws out a section of the unit circle, thus the mapping $w = e^z$ sends the set to:

$$\left\{ w \in \mathbb{C} : |w| > 0, \frac{\pi}{4} \leq \arg(w) \leq \frac{\pi}{2} \right\}$$

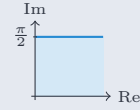


Example Determine and sketch the region $w = \text{Log}(z)$, $\{z : |z| > 1, 0 \leq \text{Arg}(z) \leq \pi/2\}$

$$w = \text{Log}(z) = \ln |z| + i\text{Arg}(z) = u + iv$$

Note that $|z| > 1$ implies $\ln |z| > 0$ Thus:

$$\left\{ w = u + iv \in \mathbb{C} : u > 0, 0 \leq v \leq \frac{\pi}{2} \right\}$$



Example Find where the function is 0 : $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

We want $\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = 0$, some basic algebra gives us:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = 0$$

$$e^{iz} + e^{-iz} = 0$$

$$e^{iz} = -e^{-iz}$$

$$e^{iz} \cdot e^{iz} = -e^{-iz} \cdot e^{iz}$$

$$e^{2iz} = -e^0$$

$$e^{2iz} = -1$$

Multiply both sides by e^{iz}

$$e^a \cdot e^b = e^{a+b}$$

Recall:

$$-1 = \cos(\pi + 2\pi k) + i \sin(\pi + 2\pi k) = e^{i(\pi + 2\pi k)}$$

Thus

$$e^{2iz} = e^{i(\pi + 2\pi k)}$$

$$2iz = i(\pi + 2\pi k)$$

$$2z = \pi + 2\pi k$$

$$z = \frac{\pi}{2} + \pi k$$

Taking the natural log of both sides

Divide by 2

Divide by i

Therefore, the zeros of $\cos(z)$ are:

$$z = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}$$

Example Calculate the principal value $\text{Log}(z)$ of $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and prove e^z is the inverse function of $\text{Log}(z)$

Part 1.

Given $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$:

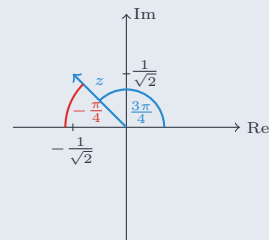
$$\ln |z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

and

$$\text{Arg}(z) = \tan^{-1}(-1)$$

$$= -\tan^{-1}(1)$$

$$= -\frac{\pi}{4} \Rightarrow \frac{3\pi}{4}$$



Therefore

$$\text{Log}(z) = \ln |z| + i\text{Arg}(z) = i \frac{3\pi}{4}$$

Part 2: We need to show that (a) $e^{\text{Log}(z)} = z$ and (b) $\text{Log}(e^z) = z$

(a) Let $z = |z|e^{i\theta}$, $|z| > 0$ and $\theta = \text{Arg}(z)$

$$\text{Log}(z) = \ln |z| + i\theta$$

$$e^{\text{Log}(z)} = e^{\ln |z| + i\theta}$$

$$= e^{\ln |z|} \cdot e^{i\theta}$$

$$= |z| \cdot e^{i\theta}$$

$$e^{\text{Log}(z)} = z$$

Exponentiate both sides

Exponentiation rules

(b) Let $z = x + iy$, $y \in [-\pi, \pi]$

$$e^z = e^{x+iy}$$

$$= e^x \cdot e^{iy}$$

$$\text{Log}(e^z) = \ln |e^x \cdot e^{iy}|$$

$$= \ln |e^x| + \ln |e^{iy}|$$

$$= x + iy$$

$$\text{Log}(e^z) = z$$

Take log of both sides

$$\log(a \cdot b) = \log(a) + \log(b)$$

Example Define the complex conjugate (\bar{w}) and prove if w is a zero of a polynomial $p(z) = a_0 + a_1z + \dots + a_nz^n$ then \bar{w} is also a zero of $p(z)$

Definition: For a complex number $w = a + bi$ the complex conjugate of w is defined as $\bar{w} = a - bi$ (with $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$) This has several properties:

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\overline{zw} = \bar{z} \cdot \bar{w}$$

$$\overline{(w^n)} = (\bar{w})^n$$

Proof: If w is zero of a polynomial $p(z) = a_0 + a_1z + \dots + a_nz^n$ then $p(w) = 0$

$$\text{Assume } p(w) = a_0 + a_1w + \dots + a_nw^n = 0$$

$$\text{Take the conjugate of both sides } \overline{p(w)} = \bar{0} = 0$$

$$\text{Evaluate } p(\bar{w}) = a_0 + a_1\bar{w} + \dots + a_n\bar{w}^n$$

$$= a_0 + a_1\bar{w} + \dots + a_n\bar{w}^n$$

$$= \overline{a_0 + a_1w + \dots + a_nw^n}$$

$$= \overline{a_0 + a_1w + \dots + a_nw^n}$$

$$= \overline{p(w)} = 0$$

Thus, since we assumed $p(w) = 0$:

$$p(\bar{w}) = \overline{p(w)} = 0$$

Example Define the complex exponential function e^z and prove Eulers Formula $e^{i\theta} = \cos(\theta) + i \sin \theta$

Definition : For any $z \in \mathbb{C}$, e^z is defined by its power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

The series converges for all $z \in \mathbb{C}$ and has the following properties:

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$$

$$e^z \cdot e^{-z} = 1$$

Proof of Eulers Formula

$$\text{Eulers Formula } e^{i\theta} = \cos(\theta) + i \sin(\theta)$$