MA283: Linear Algebra

70% Exam
30% Continuous Assessment (Homework)
10% Optional Project (Bonus)

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Contents

1 Week 1: Systems of Linear Equations

1.1 Intro to Systems of Linear Equations

They are called linear equations because each variable is raised to the first power. Products of variables, squares, square roots, etc., are not linear. A solution to a system of linear equations is an assignment of numerical values to each variable. Systems can have multiple solutions.

1.2 Augmented Matrices and Elementary Row Operations

$$x + 2y - z = 5, \tag{1}$$
$$3x + y - 2z = 9, \tag{2}$$
$$-x + 4y + 2z = 0 \tag{3}$$

$$\iff \left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array}\right)$$

To solve a system of linear equations, we work with this augmented matrix, applying three types of operations to convert it to a simpler form. These operations include:

1. Adding a scalar multiple of one row to another
2. Multiplying all entries of a row by the same non-zero scalar
3. Swapping two rows

Why does this work? Every elementary row operation (ERO) changes the system, but the new system has exactly the same solutions as the original.

Example

$$\boxed{\begin{aligned} x_1 + 3x_2 + 5x_3 - 9x_4 &= 5, \\ 3x_1 - x_2 - 5x_3 + 13x_4 &= 5, \\ 2x_1 - 3x_2 - 8x_3 + 18x_4 &= 1. \end{aligned}}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{array}\right) \xrightarrow[R2 \to R2 - 3R1]{R3 \to R3 - 2R1} \left(\begin{array}{cccc|c} 1 & 3 & 5 & -9 & 5 \\ 0 & -10 & -20 & 40 & -10 \\ 0 & -9 & -18 & 36 & -9 \end{array}\right)$$

$$\xrightarrow{R2 \times \left(-\frac{1}{10}\right)} \left(\begin{array}{cccc|c} 1 & 3 & 5 & -9 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & -9 & -18 & 36 & -9 \end{array}\right) \xrightarrow{R3 \to R3 + 9R2} \left(\begin{array}{cccc|c} 1 & 3 & 5 & -9 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R1 \to R1 - 3R2} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

1.3 How to Read the Solution from RREF

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ll} \text{Equation 1:} & x_1 - x_3 + 3x_4 = 2, \\ \text{Equation 2:} & x_2 + 2x_3 - 4x_4 = 1, \\ \text{Equation 3:} & 0 = 0. \end{array}$$

Thus:

$$x_1 = 2 + x_3 - 3x_4, \quad x_2 = 1 - 2x_3 + 4x_4.$$

We see x_3 and x_4 can be any real values. Let $s = x_3$ and $t = x_4$. Then

$$(x_1, x_2, x_3, x_4) = (2 + s - 3t,\ 1 - 2s + 4t,\ s,\ t), \quad s, t \in \mathbb{R}.$$

There are infinitely many solutions. For instance, $(2, 1, 0, 0)$ is one particular solution.

1.4 Vocabulary and Definitions

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{array}\right] \xrightarrow[\text{ERO}]{\text{Gauss-Jordan Elimination}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The second matrix is in *reduced row echelon form (RREF)*. A matrix is in RREF if:

1. Every nonzero row has a leading 1 (pivot).
2. Each pivot is the only nonzero entry in its column.
3. The pivots move strictly to the right as you go down the rows.
4. Any all-zero rows appear at the bottom.

Remark: A matrix is in *row echelon form (REF)* if it merely satisfies the "strictly right pivots" and "zeros below pivots," plus any zero rows at the bottom. RREF is the fully reduced form.

$$\textbf{Example (REF but not RREF):} \quad \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 4 & 5 \\ 0 & 1 & 5 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right]$$

1.5 Leading Variables and Free Variables

$$\begin{aligned} x_1 + 3x_2 + 5x_3 - 9x_4 &= 50, \\ 3x_1 - x_2 - 5x_3 + 13x_4 &= 5, \\ 2x_1 - 3x_2 - 8x_3 + 18x_4 &= 1 \end{aligned} \quad \longrightarrow \quad \left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Leading 1's in the RREF appear in columns 1 and 2 (so x_1 and x_2 are leading variables). Columns 3 and 4 have no pivot (so x_3 and x_4 are free).

Def.

A variable is called a **free variable** if its column in the RREF does *not* contain a leading 1.
A variable is called a **leading variable** if its column in the RREF *does* contain a leading 1.

1.6 How to Write the Solution

1. Assign independent parameters to each free variable.
2. Read off the dependencies for the leading variables from the RREF.
3. Optionally, write the solution in vector form:

$$(x_1, x_2, x_3, x_4) \;=\; (2, 1, 0, 0) \;+\; s\,(1, -2, 1, 0) \;+\; t\,(-3, 4, 0, 1).$$

2 Inconsistent Systems

2.1 Gauss Jordan Elimination Algorithm

1. Get a 1 in the upper left corner (unless Column 1 is all zeros).
2. Use this 1 to clear out the rest of Column 1 by adding/subtracting multiples of Row 1 to subsequent rows. (Do not touch Row 1 again until step 5)
3. Move to Column 2. Without disturbing the pattern of zeros in Column 1, get a 1 in the second row. Use this 1 to clear out the rest of Column 2.
4. Proceed through the columns in this manner. (Gives row echelon form.)
5. Use the leading 1s to clear out the columns above them, working left to right

Example

$$\begin{array}{llll} 3x & +2y & -5z & =4 \\ x & +y & -2z & =1 \\ 5x & +3y & -8z & =6 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{array}\right].$$

During Gausian Elimination of the system below, a row is encountered that has all zero entries, excpet the last, i.e. $0x + 0y + 0z = 1$. This is a contradiction, and the system is inconsistent.
A system is inconsistent if its equations can be be simultaneously satisfied, i.e. there is no solution to:

$$x + y = 1 \quad \text{and} \quad x + y = 2$$

2.2 Possible outcomes to solving a linear system

It is not possible for a linear system to have exactly two solutions. The possible outcomes are:

1. **Unique Solution:** The system has exactly one solution.
2. **Infinite Solutions:** The system has infinitely many solutions.
3. **No Solution:** The system is inconsistent.

2.3 Pitfalls of Gaussian Elimination

Algorithms,, based on Gauss Jordan Elimination, for solving systems of linear equations are not foolproof. They can fail in the following ways:

1. **Numerical Instability:** Round-off errors can accumulate and lead to incorrect results.
2. **Computational Complexity:** The number of operations requiredis bounded above by a cubic expression in n. This becomes impractical for large systems, where iterative methods are used.