

Robert Davidson
Complex Analysis Notes

60% Exam
40% Continuous Assessment (3 parts)

Contents

1	Preliminary	2
1.1	The Complex Plane and the Four Quadrants	2
1.2	Diagram of the Quadrants	2
1.3	Adjusting Angles Based on Quadrants	2
2	Foundations	3
2.1	Intro to Complex Numbers	3
2.2	Polar Form	3
2.3	De Moivre's Theorem	4
2.4	Roots of Unity	4
2.5	Complex Roots	5
3	Complex Functions	6
3.1	Trigonometric Functions	6
3.2	Exponential Functions	6
3.3	Complex Logarithms	7
3.4	Complex Powers	7
4	Geometric Mappings and Transformations	8
4.1	Mappings:	8
4.1.1	Example Mapping 1	8
4.1.2	Example Mapping 2	9
4.2	Circle Preservation Theorem	10
4.3	Prelim to Riemann Sphere	11
4.3.1	Euclidean Space and Compact Sets	11
4.3.2	Compactification of the Complex Plane	11
4.4	Riemann Sphere	11
4.5	Mobius Transform	12
4.6	Matrix Representation	12
4.7	Cayley Transform	12

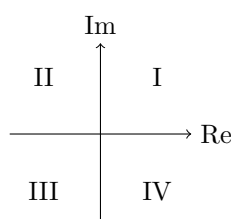
1 Preliminary

1.1 The Complex Plane and the Four Quadrants

The complex plane is a two-dimensional plane where the horizontal axis represents the real part and the vertical axis represents the imaginary part of a complex number. It is divided into four quadrants:

1. **Quadrant I** ($0^\circ < \theta < 90^\circ$): Both x and y are positive.
2. **Quadrant II** ($90^\circ < \theta < 180^\circ$): x is negative, y is positive.
3. **Quadrant III** ($180^\circ < \theta < 270^\circ$): Both x and y are negative.
4. **Quadrant IV** ($270^\circ < \theta < 360^\circ$): x is positive, y is negative.

1.2 Diagram of the Quadrants



1.3 Adjusting Angles Based on Quadrants

To correctly determine θ , adjust the angle returned by $\tan^{-1} \left(\frac{y}{x} \right)$ according to the quadrant where z lies.

1. **Quadrant I** ($x > 0, y > 0$):

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

No adjustment needed since θ is already between 0 and $\frac{\pi}{2}$.

2. **Quadrant II** ($x < 0, y > 0$):

$$\theta = \pi - \tan^{-1} \left(\frac{y}{|x|} \right)$$

Adjust by subtracting from π to place θ between $\frac{\pi}{2}$ and π .

3. **Quadrant III** ($x < 0, y < 0$):

$$\theta = \pi + \tan^{-1} \left(\frac{|y|}{|x|} \right)$$

Add to π to position θ between π and $\frac{3\pi}{2}$.

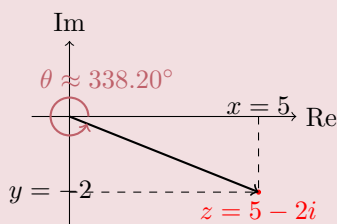
4. **Quadrant IV** ($x > 0, y < 0$):

$$\theta = 2\pi - \tan^{-1} \left(\frac{|y|}{x} \right)$$

Subtract from 2π to set θ between $\frac{3\pi}{2}$ and 2π .

Example: Let $z = 5 - 2i$

$x = 5$ and $y = -2$. Thus, we have: $\theta = 2\pi - \tan^{-1}(2/5) \approx 338.20^\circ$



2 Foundations

2.1 Intro to Complex Numbers

Complex numbers can be written as the sum of a real and imaginary part:

$$z = x + iy$$

We denote the **complex conjugate** (\bar{z}) as:

$$\bar{z} = x - iy$$

Geometrically, \bar{z} is the **reflection of z in the real axis**

With help from Pythagoras' we can now define the distance of z from the origin (**modulus**), that is the length of the vector pointing to z .

$$|z|^2 = x^2 + y^2 \Rightarrow |z| = \sqrt{x^2 + y^2}$$

We notice that:

$$\begin{aligned} z\bar{z} &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy - (iy)(iy) \\ &= x^2 - (i)^2(y^2) \\ &= x^2 - (-1)(y^2) \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

Thus, we have the distance of z from the origin as: $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ We refer to this as the **modulus** of z or the **absolute value** of z .

Letting $z = x + iy$ and $w = u + iv$, we see:

$$|z - w| = \sqrt{(x - u)^2 + (y - v)^2}$$

That is, $|z - w|$ is the distance between z and w in the complex plane.

2.2 Polar Form

Letting $r = |z| = \sqrt{x^2 + y^2}$, we can define x and y as:

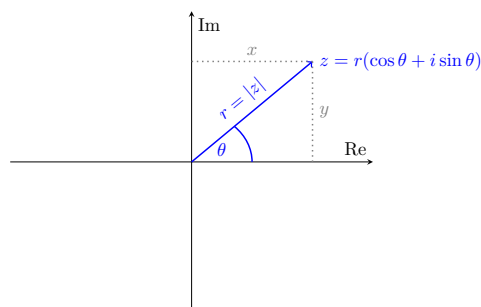
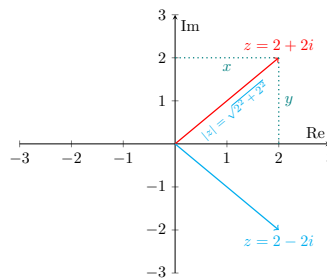
$$\begin{aligned} \cos(\theta) &= \frac{x}{r} \Rightarrow x = r \cos \theta, \\ \sin(\theta) &= \frac{y}{r} \Rightarrow y = r \sin \theta. \end{aligned}$$

Now:

$$\begin{aligned} z &= x + iy \\ &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta). \end{aligned}$$

To find θ we usually calculate $\tan^{-1}(y/x)$ and add/subtract π , when appropriate. Recalling $\tan^{-1}(y/x) \in (-\pi/2, \pi/2)$. We denote θ as the **argument of z** , denoted as $\arg(z)$. Geometrically $\arg(z)$ represent the angle z makes with the positive real axis. Thus, the pair $(r, \arg(z))$ is called the **polar coordinates of z** . We introduce the idea that $\arg(z)$ is a version of $\text{Arg}(z)$ that can take multiple values outside of $\text{Arg}(z)$'s bounds, $(-\pi, \pi)$, more precisely:

$$\arg(z) = \text{Arg}(z) + 2n\pi, \quad n \in \mathbb{Z}$$



Example: Find $\text{Arg}(i)$ and $\arg(i)$

Since $i = 0 + 1i$, we have $x = 0$ and $y = 1$.
Using $\tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \arg(z) = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$
Therefore:

$$\text{Arg}(i) = \frac{\pi}{2} \quad \text{and} \quad \arg(i) = \frac{\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}$$

2.3 De Moivre's Theorem

Theorem: Let $z_1, z_2 \in \mathbb{C}$, be nonzero numbers

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

Then:

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

Thus, we have:

$$\begin{aligned} |z_1 z_2| &= |z_1| |z_2| \\ \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \end{aligned}$$

Corollary: De Moivre's Theorem

Let $n \in \mathbb{Z}$, and $z = |z|(\cos \theta + i \sin \theta)$, then:

$$z^n = |z|^n = [\cos(n\theta) + i \sin(n\theta)]$$

2.4 Roots of Unity

Roots of unity are solutions to $z^n = 1$, where z is a complex number on the unit circle.

Eulers formula states that $e^{i\alpha} = \cos \alpha + i \sin \alpha$.

Given $z = x + iy$, then:

$$z = r(\cos \theta + i \sin \theta) = r e^{i\theta}$$

Since z lies on the unit circle, we know $R = 1$, thus we have

$$z = e^{i\theta}$$

Also, we can rewrite 1 as:

$$\begin{aligned} 1 &= 1 + 0i = \cos(0) + i \sin(0) \\ &= \cos(2\pi) + i \sin(2\pi) = \cos(2\pi k) + i \sin(2\pi k) \quad (\text{Periodic with } 2\pi \text{ k multiples don't change the result}) \\ &= e^{i2\pi k} \quad \text{where } k \in \mathbb{Z} \quad (\text{By Eulers Formula}) \end{aligned}$$

So we have, $z^n = e^{n(i\theta)}$:

$$\begin{aligned} e^{in\theta} &= e^{i2\pi k} \\ in\theta &= i2\pi k \\ n\theta &= 2\pi k \\ \theta &= \frac{2\pi k}{n} \end{aligned}$$

So θ is the angle corresponding to the n -th roots of unity. Using eulers formula again, the solutions are given as:

$$z_k = e_{i\theta} = e^{i\left(\frac{2\pi k}{n}\right)} = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right)$$

2.5 Complex Roots

Recall, square roots can be written as $4^{1/2} = \sqrt{4} = 2$, thus, we can write the n -th root as $x^{1/n}$.

What if we wanted to find the n -th root of a complex number?

Consider $f(z) = z^{1/n}$, where $n \in \mathbb{Z}$. To solve this, we aim to find some w such that $w^n = z$.

$$z = R[\cos(\theta) + i \sin(\theta)] \quad \text{and} \quad w = r[\cos(\phi) + i \sin(\phi)]$$

From De Moivre's Theorem, we have:

$$w^n = r^n[\cos(n\phi) + i \sin(n\phi)] = R[\cos(\theta) + i \sin(\theta)]$$

We see:

$$\begin{aligned} r^n &= R \rightarrow r = \sqrt[n]{R} = R^{1/n} \\ n\phi &= \theta + 2\pi k \rightarrow \phi = \frac{\theta}{n} + \frac{2\pi k}{n} \end{aligned}$$

Note that since \sin and \cos are periodic with 2π , the addition of $2\pi k$ doesn't change the result. So we have:

$$z^{1/n} = R^{1/n}[\cos \phi + i \sin \phi] \quad \text{with} \quad \phi = \frac{\theta + 2k\pi}{n}, \quad k \in (0, 1, 2, \dots, n-1)$$

Note that we reserve the notation $\sqrt[n]{z}$ to denote the **principal root**, defined when $k = 0$.

Example: Find the cube roots of $z = -1 + i$

$$R = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

We know z is in the second quadrant, so must adjust θ accordingly:

$$\theta = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

We have $k = 0, 1, 2$ for the cube roots.

Thus, the cubic roots are:

$$w_k = \sqrt[3]{2} \left[\cos\left(\frac{\theta + 2\pi k}{3}\right) + i \sin\left(\frac{\theta + 2\pi k}{3}\right) \right]$$

3 Complex Functions

3.1 Trigonometric Functions

Recall:

$$\begin{aligned}\text{cosine is an even function} &\Rightarrow \cos(-\theta) = \cos(\theta) \\ \text{sine is an odd function} &\Rightarrow \sin(-\theta) = -\sin(\theta)\end{aligned}$$

Also recall Euler's formula states $e^{iz} = \cos(z) + i\sin(z)$ also that:

$$\begin{aligned}e^{-iz} &= \cos(-z) + i\sin(-z) \\ &= \cos(z) - i\sin(z)\end{aligned}$$

If we add these expressions, we get an expression for $\cos(z)$:

$$\begin{aligned}e^{iz} + e^{-iz} &= (\cos(z) + i\sin(z)) + (\cos(z) - i\sin(z)) \\ e^{iz} + e^{-iz} &= 2\cos(z) \Rightarrow \cos(z) = \frac{e^{iz} + e^{-iz}}{2}\end{aligned}$$

If we subtract the expressions, we get an expression for $\sin(z)$:

$$\begin{aligned}e^{iz} - e^{-iz} &= (\cos(z) + i\sin(z)) - (\cos(z) - i\sin(z)) \\ e^{iz} - e^{-iz} &= 2i\sin(z) \Rightarrow \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}\end{aligned}$$

We can now also derive $\tan(z)$ and $\cot(z)$:

$$\begin{aligned}\tan(z) &= \frac{\sin(z)}{\cos(z)} = \frac{\frac{e^{iz} - e^{-iz}}{2i}}{\frac{e^{iz} + e^{-iz}}{2}} = i \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \\ \cot(z) &= \frac{\cos(z)}{\sin(z)} = \frac{\frac{e^{iz} + e^{-iz}}{2}}{\frac{e^{iz} - e^{-iz}}{2i}} = -i \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}\end{aligned}$$

Proposition. Let $z, z_1, z_2 \in \mathbb{C}$

- (i) $\sin(z + 2\pi) = \sin(z)$ and $\cos(z + 2\pi) = \cos(z)$
- (ii) $\cos^2(z) + \sin^2(z) = 1$
- (iii) $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$

3.2 Exponential Functions

Recall the **Taylor Series** for e^x , that is: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
We can now define the exponential function for complex numbers as:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!}$$

Recall also, that $z = rei\theta = e^{i\theta}$ it then follows:

$$z = e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)}_{\cos \theta} + i \underbrace{\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin \theta} = \cos(\theta) + i\sin(\theta)$$

3.3 Complex Logarithms

Recall the log rule: $\log(e^x) = x$. Also recall we defined $\theta = \text{Arg}(z)$ with $\arg(z) = \text{Arg}(z) + 2\pi k$. Lastly, recall the polar form of z :

$$z = |z|(\cos(\theta) + i \sin(\theta)) = e^{i\theta} = |z|e^{i\text{Arg}(z)} = e^{\ln|z| + i\text{Arg}z}$$

We can now define the **Logarithm of a Complex Number**:

$$\begin{aligned}\text{Log}(z) &= \log\left(e^{\ln|z| + i\text{Arg}z}\right) &= \ln|z| + i \text{Arg}(z) \\ \log(z) &= \ln|z| + i \arg z &= \ln|z| + i(\text{Arg}(z) + 2\pi k)\end{aligned}$$

Note: Denote $\text{Log}(z)$ as the **principal branch** of the complex logarithm and denote $\log(z)$ as any branch with $k \neq 0$.

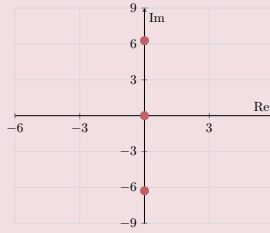
We can also write the **Complex logarithm** as:

$$\begin{aligned}\log(z) &= \ln|z| + i \arg(z) \\ &= \ln|z| + i(\text{Arg}(z) + 2k\pi) \\ &= \ln|z| + i\text{Arg}(z) + 2k\pi i\end{aligned}$$

Example: Find the log of $z = 1 + 0i$

- $z = 1 + 0i = 1 \Rightarrow |z| = 1$
 - $\text{Arg}(z) = \tan^{-1} 0/1 = 0$
- Thus, we have:

$$\begin{aligned}\log(1) &= \ln|1| + i(\text{Arg}(z) + 2k\pi) \\ &= 0 + i(0 + 2k\pi) \\ &= 2k\pi i \quad \text{where } k \in \mathbb{Z}\end{aligned}$$



3.4 Complex Powers

Recall the Logarithm Rule: $\log(a^b) = b \log(a)$. We want to define z^α , in such a way that $\log(z^\alpha) = \alpha \log(z)$. That is the **Complex Power** is defined as:

$$z^\alpha = e^{\alpha \log(z)} = e^{\alpha(\text{Log}(z) + 2k\pi i)} \quad \text{for } k \in \mathbb{Z}$$

So that we have:

$$\begin{aligned}\log(z^\alpha) &= \log(e^{\alpha(\text{Log}(z) + 2k\pi i)}) \\ &= \alpha(\text{Log}(z) + 2k\pi i) \\ &= \alpha \log(z)\end{aligned}$$

As example, consider $z = 1 + 0i$:

$$\begin{aligned}1^\alpha &= e^{\alpha(\text{Log}(1) + 2k\pi i)} \\ &= e^{2k\alpha\pi i}\end{aligned}$$

If $\alpha \in \mathbb{Z}$ ($1, 2, 3, \dots$)

$$1^\alpha = (e^{2k\pi i})^\alpha = (\cos(2\pi k) + i\sin(2\pi k))^\alpha = 1^\alpha = 1$$

If $\alpha = \frac{m}{n} \in \mathbb{Q}$, then 1^α is the set of all n -th roots of unity:

$$1^\alpha = e^{\frac{2k\pi im}{n}} = \cos\left(\frac{2\pi km}{n}\right) + i \sin\left(\frac{2\pi km}{n}\right) \cos\left(\frac{2\pi r}{n}\right) + i \sin\left(\frac{2\pi r}{n}\right)$$

If $\alpha = i$ then we see:

$$1^\alpha = 1^i = e^{2k\pi i \cdot i} = e^{-2k\pi}$$

4 Geomtric Mappings and Transformations

4.1 Mappings:

Recall we defined the principal branch as

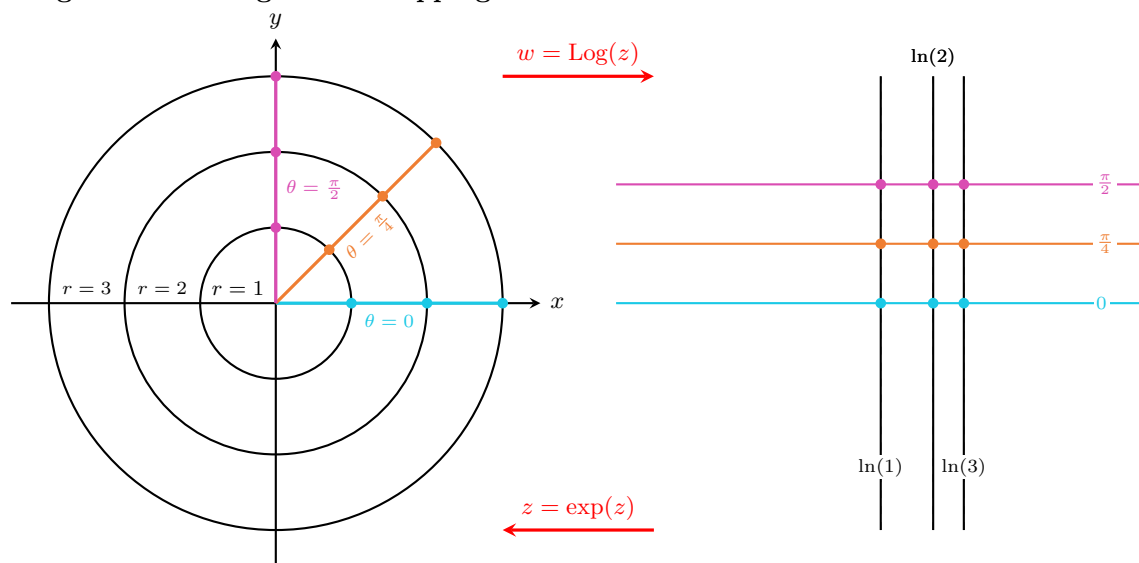
$$\text{Log}(z) = \ln |z| + i \text{Arg}(z)$$

So, when we take the principal branch of the logarithm, we see that it maps to the complex number $w = u + iv$ where $u = \ln |z|$ and $v = \text{Arg}(z)$.

In essence. Log maps \mathbb{C} to the horizontal strip:

$$\{w = u + iv : -\pi < v \leq \pi\}$$

Diagram of the Logarithm Mapping:



4.1.1 Example Mapping 1

Let $f(z) = z^3$, we see that:

Using exponential rules and polar representation:

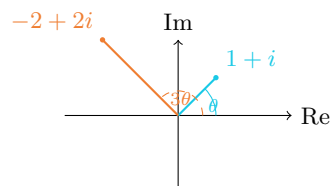
$$\begin{aligned} z &= |z|e^{i\theta} \\ z^3 &= (|z|e^{i\theta})^3 \\ &= |z|^3 e^{i3\theta} \\ &= |z|^3 (\cos(3\theta) + i \sin(3\theta)) \end{aligned}$$

Alternatively, using our definition for complex powers:

$$\begin{aligned} z^3 &= e^{3 \log(z)} \\ &= e^{3(\ln |z| + i \text{Arg}(z))} \\ &= e^{3 \ln |z| + i3 \text{Arg}(z)} \\ &= |z|^3 e^{i3 \text{Arg}(z)} \\ &= |z|^3 (\cos(3\theta) + i \sin(3\theta)) \end{aligned}$$

Letting $z = 1 + i$, we see: $\theta = \tan^{-1}(\frac{1}{1}) = 45^\circ = \frac{\pi}{4}$, and $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Thus, we have:

$$\begin{aligned} z^3 &= |z|^3 \cdot [\cos(3\theta) + i \sin(3\theta)] \\ &= (\sqrt{2})^3 \cdot \left[\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right] \\ &= -2\sqrt{2} + i2\sqrt{2} \end{aligned}$$



In essence, the mapping $f(z) = z^3$ rotates the complex number z by 3θ and scales it by $|z|^3$. We can imagine this, for the complex numbers with $|z| = 1$, and $0 < \theta \leq \frac{\pi}{2}$, as an arc of radius 1, from the angle $0 \rightarrow 90^\circ$, mapped to an arc of radius 8, from the angles $0 \rightarrow 270^\circ$.

4.1.2 Example Mapping 2

We wish to find the image of the line $x = 1$ under

$$f(z) = \frac{1}{z}, \quad z = x + iy, \quad w = u + iv.$$

For $z = x + iy$ we have

$$w = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2},$$

so that

$$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}.$$

Setting $x = 1$ yields

$$u = \frac{1}{1 + y^2}, \quad v = -\frac{y}{1 + y^2}.$$

Since

$$|w|^2 = u^2 + v^2 = \frac{1}{1 + y^2} = u,$$

it follows that

$$u^2 + v^2 = u \implies u^2 - u + v^2 = 0.$$

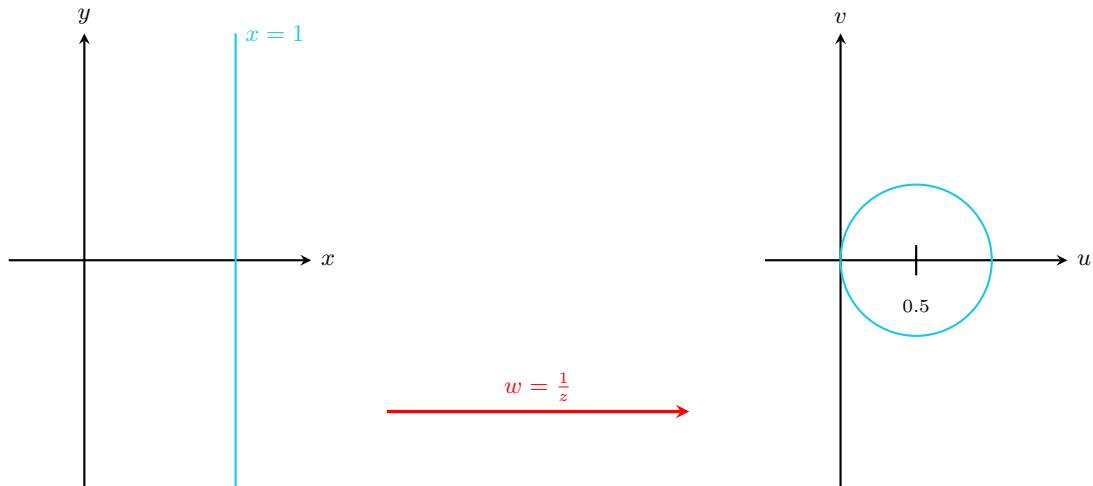
Completing the square in u by adding and subtracting $\frac{1}{4}$:

$$u^2 - u + \frac{1}{4} + v^2 = \frac{1}{4} \implies \left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}.$$

Thus, the image of $x = 1$ is the circle

$$\boxed{\left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}},$$

centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$



In general, $f(z) = \frac{1}{z}$ maps circle and lines to circles and lines, respectively.

4.2 Circle Preservation Theorem

Consider the equation:

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

We can see that if $A \neq 0$, then we can divide by A :

$$x^2 + y^2 + \frac{B}{A}x + \frac{C}{A}y + \frac{D}{A} = 0$$

Completing the square yields:

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{B^2 + C^2 - 4AD}{4A^2}\right)$$

Thus, if $A \neq 0$, we have a circle with center $(-B/2A, -C/2A)$ and radius $\sqrt{\frac{B^2 + C^2 - 4AD}{4A^2}}$.

If $A = 0$, then the equation represents a line:

$$Bx + Cy + D = 0$$

If $D = 0$, the circle or line contains 0:

$$Bx + Cy + D \big|_{(0,0)} = D = 0$$

Why is This Important?

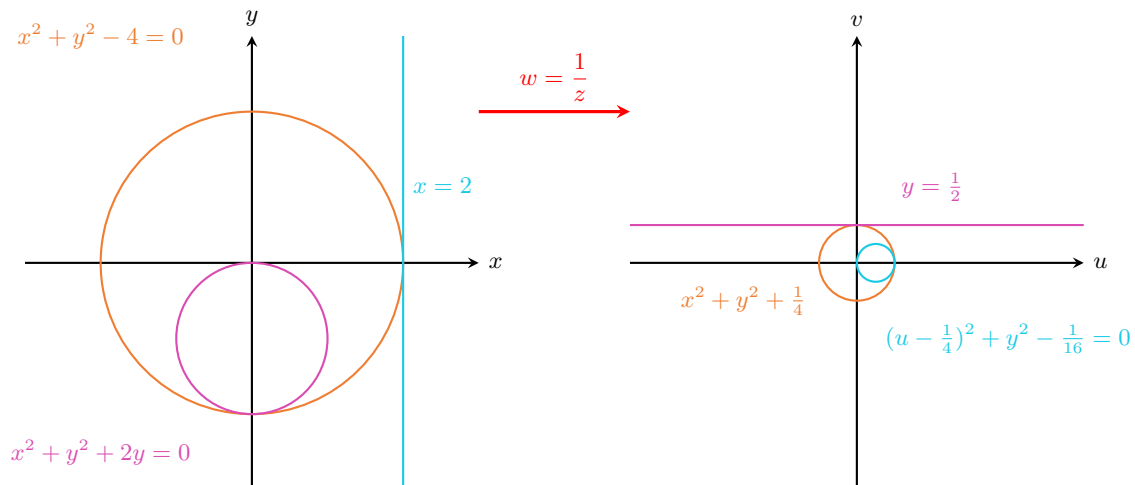
Under the inversion $f(z) = \frac{1}{z}$ with $z = x + iy$ and $w = u + iv$, one can show that the general equation

$$A(x^2 + y^2) + Bx + Cy + D = 0 \xrightarrow{\text{maps to}} D(u^2 + v^2) + Bu - Cv + A = 0.$$

In this transformed equation:

- If the original set does not contain the origin image is a circle.
- If the original set does contain the origin then the equation becomes linear:
- If the original set is a line (with $A = 0$), if it does not pass through the origin, its inversion is a circle that passes through the origin.

Examples Illustrating the Inversion Effects



4.3 Prelim to Riemann Sphere

Our goal is to define the **Riemann Sphere**, which is the complex plane \mathbb{C} , together with an extra point at infinity. In essence The Riemann sphere is a way to "wrap up" the entire complex plane into a compact, closed surface that is **homeomorphic** (topologically equivalent) to the sphere S^2 and the connection between them is made via the **stereographic projection**.

4.3.1 Euclidean Space and Compact Sets

Euclidean space, denoted as \mathbb{R}^n , is the collection of all points in n -dimensional space, where each point is described by n real numbers. In Euclidean spaces (such as the real line \mathbb{R} or the plane \mathbb{R}^2), a set is **compact** if it is both: **Closed** (contains all its limit points), and **Bounded** (contained within a finite region).

Examples of Compact Sets:

The closed interval $[0, 1] \subset \mathbb{R}^1$,

A closed disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$

Examples of Non-Compact Sets:

The open interval $(0, 1) \subset \mathbb{R}^1$ (not closed),

The entire real line \mathbb{R} (not bounded)

4.3.2 Compactification of the Complex Plane

The complex plane \mathbb{C} is not compact - it stretches out infinitely in all directions. By adding a single point at infinity, we "close" the plane, turning it into a compact set. This new space, is **homeomorphic** (a one-to-one mapping that is continuous in both directions or topologically equivalent) to the Riemann Sphere. We define the new space as:

$$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

4.4 Riemann Sphere

Define $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then $\tilde{\mathbb{C}} \xleftrightarrow{1:1} S^2 \{X = (x, y, z) : x^2 + y^2 + z^2 = 1\}$ (*homeomorphic*) via the stereographic projection, denoted St , defined as follows:

1. Projection from $S^2 \rightarrow \tilde{\mathbb{C}}$:

For a point $(x, y, z) \in S^2$, with $z \neq 1$ (the point is not the north pole) the projection is defined as:

$$St(x, y, z) = \frac{1}{1 - x_3}(x_1, x_2) \quad \text{for } z \neq 1$$

This takes a point on the sphere and maps it to a point in the complex plane.

2. Projection from $\tilde{\mathbb{C}} \rightarrow S^2$:

For a point $z \in \mathbb{C}$, the inverse projection is defined as:

$$St^{-1}(z) = \frac{1}{|z|^2 + 1} \langle 2\text{Re}(z), 2\text{Im}(z), |z|^2 - 1 \rangle$$

This takes a complex number, z , written in terms of its real ($\text{Re}(z)$) and imaginary ($\text{Im}(z)$) parts, and maps it to the sphere

3. Mapping the North Pole:

The projection leaves out the north pole from projection onto \mathbb{C}

$$St(N) = \infty \quad \text{and} \quad St^{-1}(\infty) = N \quad \text{where } N = \langle 0, 0, 1 \rangle$$

The north pole is mapped to the point at infinity, and vice versa.

- 4.5 Mobius Transform
- 4.6 Matrix Representation
- 4.7 Cayley Transform