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Complex Analysis Notes

60% Exam  
40% Continuous Assessment (3 parts)

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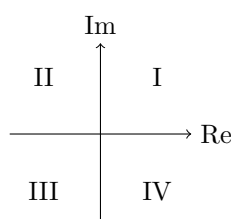
# 1 Preliminary

## 1.1 The Complex Plane and the Four Quadrants

The complex plane is a two-dimensional plane where the horizontal axis represents the real part and the vertical axis represents the imaginary part of a complex number. It is divided into four quadrants:

1. **Quadrant I** ( $0^\circ < \theta < 90^\circ$ ): Both  $x$  and  $y$  are positive.
2. **Quadrant II** ( $90^\circ < \theta < 180^\circ$ ):  $x$  is negative,  $y$  is positive.
3. **Quadrant III** ( $180^\circ < \theta < 270^\circ$ ): Both  $x$  and  $y$  are negative.
4. **Quadrant IV** ( $270^\circ < \theta < 360^\circ$ ):  $x$  is positive,  $y$  is negative.

## 1.2 Diagram of the Quadrants



## 1.3 Adjusting Angles Based on Quadrants

To correctly determine  $\theta$ , adjust the angle returned by  $\tan^{-1} \left( \frac{y}{x} \right)$  according to the quadrant where  $z$  lies.

1. **Quadrant I** ( $x > 0, y > 0$ ):

$$\theta = \tan^{-1} \left( \frac{y}{x} \right)$$

No adjustment needed since  $\theta$  is already between 0 and  $\frac{\pi}{2}$ .

2. **Quadrant II** ( $x < 0, y > 0$ ):

$$\theta = \pi - \tan^{-1} \left( \frac{y}{|x|} \right)$$

Adjust by subtracting from  $\pi$  to place  $\theta$  between  $\frac{\pi}{2}$  and  $\pi$ .

3. **Quadrant III** ( $x < 0, y < 0$ ):

$$\theta = \pi + \tan^{-1} \left( \frac{|y|}{|x|} \right)$$

Add to  $\pi$  to position  $\theta$  between  $\pi$  and  $\frac{3\pi}{2}$ .

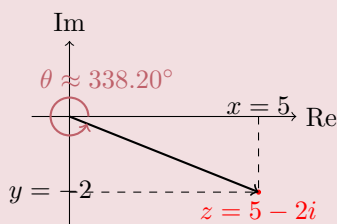
4. **Quadrant IV** ( $x > 0, y < 0$ ):

$$\theta = 2\pi - \tan^{-1} \left( \frac{|y|}{x} \right)$$

Subtract from  $2\pi$  to set  $\theta$  between  $\frac{3\pi}{2}$  and  $2\pi$ .

**Example:** Let  $z = 5 - 2i$

$x = 5$  and  $y = -2$ . Thus, we have:  $\theta = 2\pi - \tan^{-1}(2/5) \approx 338.20^\circ$



## 2 Foundations

### 2.1 Intro to Complex Numbers

Complex numbers can be written as the sum of a real and imaginary part:

$$z = x + iy$$

We denote the **complex conjugate** ( $\bar{z}$ ) as:

$$\bar{z} = x - iy$$

Geometrically,  $\bar{z}$  is the **reflection of  $z$  in the real axis**

With help from Pythagoras' we can now define the distance of  $z$  from the origin (**modulus**), that is the length of the vector pointing to  $z$ .

$$|z|^2 = x^2 + y^2 \Rightarrow |z| = \sqrt{x^2 + y^2}$$

We notice that:

$$\begin{aligned} z\bar{z} &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy - (iy)(iy) \\ &= x^2 - (i)^2(y^2) \\ &= x^2 - (-1)(y^2) \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

Thus, we have the distance of  $z$  from the origin as:  $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$  We refer to this as the **modulus** of  $z$  or the **absolute value** of  $z$ .

Letting  $z = x + iy$  and  $w = u + iv$ , we see:

$$|z - w| = \sqrt{(x - u)^2 + (y - v)^2}$$

That is,  $|z - w|$  is the distance between  $z$  and  $w$  in the complex plane.

### 2.2 Polar Form

Letting  $r = |z| = \sqrt{x^2 + y^2}$ , we can define  $x$  and  $y$  as:

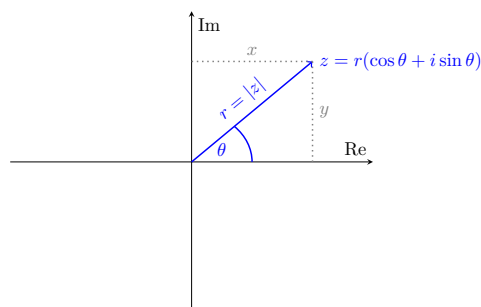
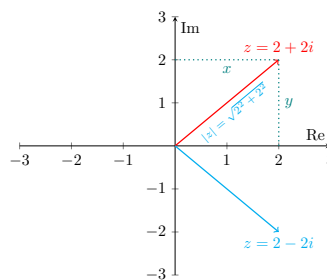
$$\begin{aligned} \cos(\theta) &= \frac{x}{r} \Rightarrow x = r \cos \theta, \\ \sin(\theta) &= \frac{y}{r} \Rightarrow y = r \sin \theta. \end{aligned}$$

Now:

$$\begin{aligned} z &= x + iy \\ &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta). \end{aligned}$$

To find  $\theta$  we usually calculate  $\tan^{-1}(y/x)$  and add/subtract  $\pi$ , when appropriate. Recalling  $\tan^{-1}(y/x) \in (-\pi/2, \pi/2)$ . We denote  $\theta$  as the **argument of  $z$** , denoted as  $\arg(z)$ . Geometrically  $\arg(z)$  represent the angle  $z$  makes with the positive real axis. Thus, the pair  $(r, \arg(z))$  is called the **polar coordinates of  $z$** . We introduce the idea that  $\arg(z)$  is a version of  $\text{Arg}(z)$  that can take multiple values outside of  $\text{Arg}(z)$ 's bounds,  $(-\pi, \pi)$ , more precisely:

$$\arg(z) = \text{Arg}(z) + 2n\pi, \quad n \in \mathbb{Z}$$



**Example: Find  $\text{Arg}(i)$  and  $\arg(i)$**

Since  $i = 0 + 1i$ , we have  $x = 0$  and  $y = 1$ .  
Using  $\tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \arg(z) = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$   
Therefore:

$$\text{Arg}(i) = \frac{\pi}{2} \quad \text{and} \quad \arg(i) = \frac{\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}$$

### 2.3 De Moivre's Theorem

**Theorem:** Let  $z_1, z_2 \in \mathbb{C}$ , be nonzero numbers

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

Then:

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

Thus, we have:

$$\begin{aligned} |z_1 z_2| &= |z_1| |z_2| \\ \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \end{aligned}$$

**Corollary: De Moivre's Theorem**

Let  $n \in \mathbb{Z}$ , and  $z = |z|(\cos \theta + i \sin \theta)$ , then:

$$z^n = |z|^n = [\cos(n\theta) + i \sin(n\theta)]$$

### 2.4 Roots of Unity

**Roots of unity are solutions to  $z^n = 1$** , where  $z$  is a complex number on the unit circle.

**Eulers formula** states that  $e^{i\alpha} = \cos \alpha + i \sin \alpha$ .

Given  $z = x + iy$ , then:

$$z = r(\cos \theta + i \sin \theta) = r e^{i\theta}$$

Since  $z$  lies on the unit circle, we know  $R = 1$ , thus we have

$$z = e^{i\theta}$$

Also, we can rewrite 1 as:

$$\begin{aligned} 1 &= 1 + 0i = \cos(0) + i \sin(0) \\ &= \cos(2\pi) + i \sin(2\pi) = \cos(2\pi k) + i \sin(2\pi k) \quad (\text{Periodic with } 2\pi \text{ k multiples don't change the result}) \\ &= e^{i2\pi k} \quad \text{where } k \in \mathbb{Z} \quad (\text{By Eulers Formula}) \end{aligned}$$

So we have,  $z^n = e^{n(i\theta)}$ :

$$\begin{aligned} e^{in\theta} &= e^{i2\pi k} \\ in\theta &= i2\pi k \\ n\theta &= 2\pi k \\ \theta &= \frac{2\pi k}{n} \end{aligned}$$

So  $\theta$  is the angle corresponding to the  $n$ -th roots of unity. Using eulers formula again, the solutions are given as:

$$z_k = e_{i\theta} = e^{i\left(\frac{2\pi k}{n}\right)} = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right)$$

## 2.5 Complex Roots

Recall, square roots can be written as  $4^{1/2} = \sqrt{4} = 2$ , thus, we can write the  $n$ -th root as  $x^{1/n}$ .

**What if we wanted to find the  $n$ -th root of a complex number?**

Consider  $f(z) = z^{1/n}$ , where  $n \in \mathbb{Z}$ . To solve this, we aim to find some  $w$  such that  $w^n = z$ .

$$z = R[\cos(\theta) + i \sin(\theta)] \quad \text{and} \quad w = r[\cos(\phi) + i \sin(\phi)]$$

From De Moivre's Theorem, we have:

$$w^n = r^n[\cos(n\phi) + i \sin(n\phi)] = R[\cos(\theta) + i \sin(\theta)]$$

We see:

$$\begin{aligned} r^n &= R \rightarrow r = \sqrt[n]{R} = R^{1/n} \\ n\phi &= \theta + 2\pi k \rightarrow \phi = \frac{\theta}{n} + \frac{2\pi k}{n} \end{aligned}$$

Note that since  $\sin$  and  $\cos$  are periodic with  $2\pi$ , the addition of  $2\pi k$  doesn't change the result. So we have:

$$z^{1/n} = R^{1/n}[\cos \phi + i \sin \phi] \quad \text{with} \quad \phi = \frac{\theta + 2k\pi}{n}, \quad k \in (0, 1, 2, \dots, n-1)$$

Note that we reserve the notation  $\sqrt[n]{z}$  to denote the **principal root**, defined when  $k = 0$ .

**Example: Find the cube roots of  $z = -1 + i$**

$$R = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

We know  $z$  is in the second quadrant, so must adjust  $\theta$  accordingly:

$$\theta = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

We have  $k = 0, 1, 2$  for the cube roots.

Thus, the cubic roots are:

$$w_k = \sqrt[3]{2} \left[ \cos\left(\frac{\theta + 2\pi k}{3}\right) + i \sin\left(\frac{\theta + 2\pi k}{3}\right) \right]$$

### 3 Complex Functions

#### 3.1 Trigonometric Functions

Recall:

$$\begin{aligned}\text{cosine is an even function} &\Rightarrow \cos(-\theta) = \cos(\theta) \\ \text{sine is an odd function} &\Rightarrow \sin(-\theta) = -\sin(\theta)\end{aligned}$$

Also recall Euler's formula states  $e^{iz} = \cos(z) + i\sin(z)$  also that:

$$\begin{aligned}e^{-iz} &= \cos(-z) + i\sin(-z) \\ &= \cos(z) - i\sin(z)\end{aligned}$$

If we add these expressions, we get an expression for  $\cos(z)$ :

$$\begin{aligned}e^{iz} + e^{-iz} &= (\cos(z) + i\sin(z)) + (\cos(z) - i\sin(z)) \\ e^{iz} + e^{-iz} &= 2\cos(z) \Rightarrow \cos(z) = \frac{e^{iz} + e^{-iz}}{2}\end{aligned}$$

If we subtract the expressions, we get an expression for  $\sin(z)$ :

$$\begin{aligned}e^{iz} - e^{-iz} &= (\cos(z) + i\sin(z)) - (\cos(z) - i\sin(z)) \\ e^{iz} - e^{-iz} &= 2i\sin(z) \Rightarrow \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}\end{aligned}$$

We can now also derive  $\tan(z)$  and  $\cot(z)$ :

$$\begin{aligned}\tan(z) &= \frac{\sin(z)}{\cos(z)} = \frac{\frac{e^{iz} - e^{-iz}}{2i}}{\frac{e^{iz} + e^{-iz}}{2}} = i \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \\ \cot(z) &= \frac{\cos(z)}{\sin(z)} = \frac{\frac{e^{iz} + e^{-iz}}{2}}{\frac{e^{iz} - e^{-iz}}{2i}} = -i \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}\end{aligned}$$

**Proposition.** Let  $z, z_1, z_2 \in \mathbb{C}$

- (i)  $\sin(z + 2\pi) = \sin(z)$  and  $\cos(z + 2\pi) = \cos(z)$
- (ii)  $\cos^2(z) + \sin^2(z) = 1$
- (iii)  $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$

#### 3.2 Exponential Functions

Recall the **Taylor Series** for  $e^x$ , that is:  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$   
We can now define the exponential function for complex numbers as:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!}$$

Recall also, that  $z = rei\theta = e^{i\theta}$  it then follows:

$$z = e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)}_{\cos \theta} + i \underbrace{\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin \theta} = \cos(\theta) + i\sin(\theta)$$

### 3.3 Complex Logarithms

Recall the log rule:  $\log(e^x) = x$ . Also recall we defined  $\theta = \text{Arg}(z)$  with  $\arg(z) = \text{Arg}(z) + 2\pi k$ . Lastly, recall the polar form of  $z$ :

$$z = |z|(\cos(\theta) + i \sin(\theta)) = e^{i\theta} = |z|e^{i\text{Arg}(z)} = e^{\ln|z| + i\text{Arg}z}$$

We can now define the **Logarithm of a Complex Number**:

$$\begin{aligned}\text{Log}(z) &= \log(e^{\ln|z| + i\text{Arg}z}) &= \ln|z| + i \text{Arg}(z) \\ \log(z) &= \ln|z| + i \arg z &= \ln|z| + i(\text{Arg}(z) + 2\pi k)\end{aligned}$$

*Note:* Denote  $\text{Log}(z)$  as the **principal branch** of the complex logarithm and denote  $\log(z)$  as any branch with  $k \neq 0$ .

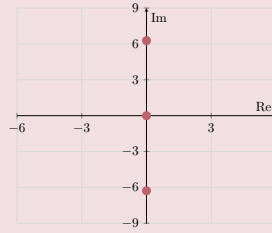
We can also write the **Complex logarithm** as:

$$\begin{aligned}\log(z) &= \ln|z| + i \arg(z) \\ &= \ln|z| + i(\text{Arg}(z) + 2k\pi) \\ &= \ln|z| + i\text{Arg}(z) + 2k\pi i\end{aligned}$$

**Example: Find the log of  $z = 1 + 0i$**

- $z = 1 + 0i = 1 \Rightarrow |z| = 1$
  - $\text{Arg}(z) = \tan^{-1} 0/1 = 0$
- Thus, we have:

$$\begin{aligned}\log(1) &= \ln|1| + i(\text{Arg}(z) + 2k\pi) \\ &= 0 + i(0 + 2k\pi) \\ &= 2k\pi i \quad \text{where } k \in \mathbb{Z}\end{aligned}$$



### 3.4 Complex Powers

Recall the Logarithm Rule:  $\log(a^b) = b \log(a)$ . We want to define  $z^\alpha$ , in such a way that  $\log(z^\alpha) = \alpha \log(z)$ . That is the **Complex Power** is defined as:

$$z^\alpha = e^{\alpha \log(z)} = e^{\alpha(\text{Log}(z) + 2k\pi i)} \quad \text{for } k \in \mathbb{Z}$$

So that we have:

$$\begin{aligned}\log(z^\alpha) &= \log(e^{\alpha(\text{Log}(z) + 2k\pi i)}) \\ &= \alpha(\text{Log}(z) + 2k\pi i) \\ &= \alpha \log(z)\end{aligned}$$

As example, consider  $z = 1 + 0i$ :

$$\begin{aligned}1^\alpha &= e^{\alpha(\text{Log}(1) + 2k\pi i)} \\ &= e^{2k\alpha\pi i}\end{aligned}$$

If  $\alpha \in \mathbb{Z}$  ( $1, 2, 3, \dots$ )

$$1^\alpha = (e^{2k\pi i})^\alpha = (\cos(2\pi k) + i\sin(2\pi k))^\alpha = 1^\alpha = 1$$

If  $\alpha = \frac{m}{n} \in \mathbb{Q}$ , then  $1^\alpha$  is the set of all  $n$ -th roots of unity:

$$1^\alpha = e^{\frac{2k\pi i m}{n}} = \cos\left(\frac{2\pi k m}{n}\right) + i \sin\left(\frac{2\pi k m}{n}\right)$$

If  $\alpha = i$  then we see:

$$1^\alpha = 1^i = e^{2k\pi i \cdot i} = e^{-2k\pi}$$

## 4 Geomtric Mappings and Transformations

### 4.1 Mappings:

Recall we defined the principal branch as

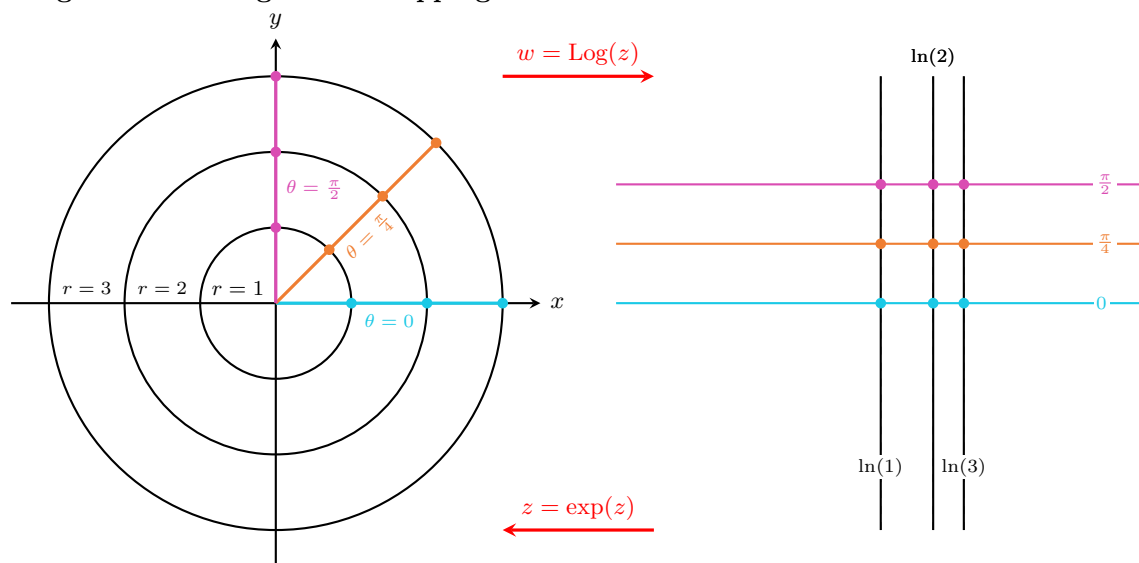
$$\text{Log}(z) = \ln |z| + i \text{Arg}(z)$$

So, when we take the principal branch of the logarithm, we see that it maps to the complex number  $w = u + iv$  where  $u = \ln |z|$  and  $v = \text{Arg}(z)$ .

In essence. Log maps  $\mathbb{C}$  to the horizontal strip:

$$\{w = u + iv : -\pi < v \leq \pi\}$$

**Diagram of the Logarithm Mapping:**



#### 4.1.1 Example Mapping 1

Let  $f(z) = z^3$ , we see that:

Using exponential rules and polar representation:

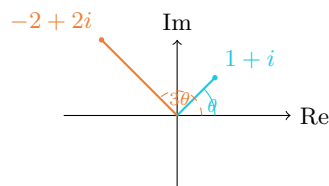
$$\begin{aligned} z &= |z|e^{i\theta} \\ z^3 &= (|z|e^{i\theta})^3 \\ &= |z|^3 e^{i3\theta} \\ &= |z|^3 (\cos(3\theta) + i \sin(3\theta)) \end{aligned}$$

Alternatively, using our definition for complex powers:

$$\begin{aligned} z^3 &= e^{3 \log(z)} \\ &= e^{3(\ln |z| + i \text{Arg}(z))} \\ &= e^{3 \ln |z| + i3 \text{Arg}(z)} \\ &= |z|^3 e^{i3 \text{Arg}(z)} \\ &= |z|^3 (\cos(3\theta) + i \sin(3\theta)) \end{aligned}$$

Letting  $z = 1 + i$ , we see:  $\theta = \tan^{-1}(\frac{1}{1}) = 45^\circ = \frac{\pi}{4}$ , and  $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$ . Thus, we have:

$$\begin{aligned} z^3 &= |z|^3 \cdot [\cos(3\theta) + i \sin(3\theta)] \\ &= (\sqrt{2})^3 \cdot \left[ \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right] \\ &= -2\sqrt{2} + i2\sqrt{2} \end{aligned}$$



In essence, the mapping  $f(z) = z^3$  rotates the complex number  $z$  by  $3\theta$  and scales it by  $|z|^3$ . We can imagine this, for the complex numbers with  $|z| = 1$ , and  $0 < \theta \leq \frac{\pi}{2}$ , as an arc of radius 1, from the angle  $0 \rightarrow 90^\circ$ , mapped to an arc of radius 8, from the angles  $0 \rightarrow 270^\circ$ .



### 4.1.2 Example Mapping 2

We wish to find the image of the line  $x = 1$  under

$$f(z) = \frac{1}{z}, \quad z = x + iy, \quad w = u + iv.$$

For  $z = x + iy$  we have

$$w = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2},$$

so that

$$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}.$$

Setting  $x = 1$  yields

$$u = \frac{1}{1 + y^2}, \quad v = -\frac{y}{1 + y^2}.$$

Since

$$|w|^2 = u^2 + v^2 = \frac{1}{1 + y^2} = u,$$

it follows that

$$u^2 + v^2 = u \implies u^2 - u + v^2 = 0.$$

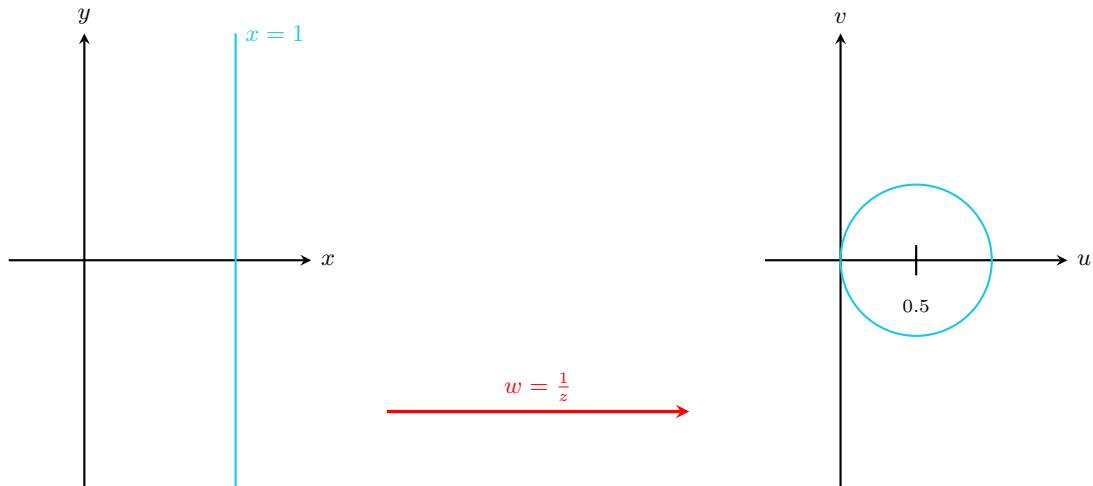
Completing the square in  $u$  by adding and subtracting  $\frac{1}{4}$ :

$$u^2 - u + \frac{1}{4} + v^2 = \frac{1}{4} \implies \left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}.$$

Thus, the image of  $x = 1$  is the circle

$$\boxed{\left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}},$$

centered at  $(\frac{1}{2}, 0)$  with radius  $\frac{1}{2}$



In general,  $f(z) = \frac{1}{z}$  maps circle and lines to circles and lines, respectively.

## 4.2 Circle Preservation Theorem

Consider the equation:

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

We can see that if  $A \neq 0$ , then we can divide by  $A$ :

$$x^2 + y^2 + \frac{B}{A}x + \frac{C}{A}y + \frac{D}{A} = 0$$

Completing the square yields:

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{B^2 + C^2 - 4AD}{4A^2}\right)$$

Thus, if  $A \neq 0$ , we have a circle with center  $(-B/2A, -C/2A)$  and radius  $\sqrt{\frac{B^2 + C^2 - 4AD}{4A^2}}$ .

If  $A = 0$ , then the equation represents a line:

$$Bx + Cy + D = 0$$

If  $D = 0$ , the circle or line contains 0:

$$Bx + Cy + D \big|_{(0,0)} = D = 0$$

### Why is This Important?

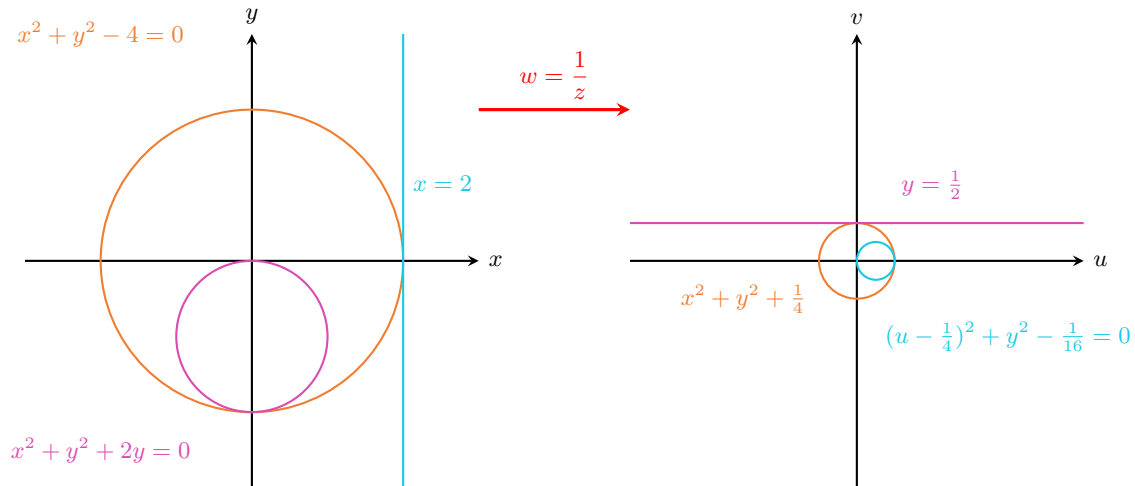
Under the inversion  $f(z) = \frac{1}{z}$  with  $z = x + iy$  and  $w = u + iv$ , one can show that the general equation

$$A(x^2 + y^2) + Bx + Cy + D = 0 \xrightarrow{\text{maps to}} D(u^2 + v^2) + Bu - Cv + A = 0.$$

In this transformed equation:

- If the original set does not contain the origin image is a circle.
- If the original set does contain the origin then the equation becomes linear:
- If the original set is a line (with  $A = 0$ ), if it does not pass through the origin, its inversion is a circle that passes through the origin.

### Examples Illustrating the Inversion Effects



### 4.3 Prelim to Riemann Sphere

Our goal is to define the **Riemann Sphere**, which is the complex plane  $\mathbb{C}$ , together with an extra point at infinity. In essence The Riemann sphere is a way to "wrap up" the entire complex plane into a compact, closed surface that is **homeomorphic** (topologically equivalent) to the sphere  $S^2$  and the connection between them is made via the **stereographic projection**.

#### 4.3.1 Euclidean Space and Compact Sets

**Euclidean space**, denoted as  $\mathbb{R}^n$ , is the collection of all points in  $n$ -dimensional space, where each point is described by  $n$  real numbers. In Euclidean spaces (such as the real line  $\mathbb{R}$  or the plane  $\mathbb{R}^2$ ), a set is **compact** if it is both: **Closed** (contains all its limit points), and **Bounded** (contained within a finite region).

##### Examples of Compact Sets:

The closed interval  $[0, 1] \subset \mathbb{R}^1$ ,

A closed disk  $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$

##### Examples of Non-Compact Sets:

The open interval  $(0, 1) \subset \mathbb{R}^1$  (not closed),

The entire real line  $\mathbb{R}$  (not bounded)

#### 4.3.2 Compactification of the Complex Plane

The complex plane  $\mathbb{C}$  is not compact - it stretches out infinitely in all directions. By adding a single point at infinity, we "close" the plane, turning it into a compact set. This new space, is **homeomorphic** (a one-to-one mapping that is continuous in both directions or topologically equivalent) to the Riemann Sphere. We define the new space as:

$$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

### 4.4 Riemann Sphere

Define  $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . Then  $\tilde{\mathbb{C}} \xleftrightarrow{1:1} S^2 \{X = (x, y, z) : x^2 + y^2 + z^2 = 1\}$  (*homeomorphic*) via the stereographic projection, denoted  $St$ , defined as follows:

#### 1. Projection from $S^2 \rightarrow \tilde{\mathbb{C}}$ :

For a point  $(x, y, z) \in S^2$ , with  $z \neq 1$  (the point is not the north pole) the projection is defined as:

$$St(x, y, z) = \frac{1}{1 - x_3}(x_1, x_2) \quad \text{for } z \neq 1$$

*This takes a point on the sphere and maps it to a point in the complex plane.*

#### 2. Projection from $\tilde{\mathbb{C}} \rightarrow S^2$ :

For a point  $z \in \mathbb{C}$ , the inverse projection is defined as:

$$St^{-1}(z) = \frac{1}{|z|^2 + 1} \langle 2\text{Re}(z), 2\text{Im}(z), |z|^2 - 1 \rangle$$

*This takes a complex number,  $z$ , written in terms of its real ( $\text{Re}(z)$ ) and imaginary ( $\text{Im}(z)$ ) parts, and maps it to the sphere*

#### 3. Mapping the North Pole:

The projection leaves out the north pole from projection onto  $\mathbb{C}$

$$St(N) = \infty \quad \text{and} \quad St^{-1}(\infty) = N \quad \text{where } N = \langle 0, 0, 1 \rangle$$

*The north pole is mapped to the point at infinity, and vice versa.*

## 5 Complex Analysis

### 5.1 Mobius Transforms

**Recall:** The complex plane  $\mathbb{C}$  can be thought as points  $(x, y) \in \mathbb{R}^2$ , but we usually label a point as  $z = x + iy$ . We can extend  $\mathbb{C}$  by adding a point at infinity, the resulting set is called the **Riemann Sphere**  $\tilde{\mathbb{C}}$ . Visually, we can imagine wrapping the complex plane onto the surface of a sphere, where  $\infty$  is the north pole of the sphere.

Now, letting  $a, b, c, d$  be complex numbers (i.e.  $a = x_a + iy_a$ ), we define a Mobius Transform as a function  $T : \tilde{\mathbb{C}} \rightarrow \tilde{\mathbb{C}}$ :

$$T(z) = \frac{az + b}{cz + d}$$

where  $ad - bc \neq 0$  (that is the determinant  $\neq 0 \rightarrow$  matrix is invertible).

These functions occur on the Riemann Sphere, because we need to define that happens when  $cz + d = 0$  and when  $z = \infty$ :

$$\text{If } c \neq 0 : \quad T(\infty) = \frac{a}{c} \quad \text{and} \quad T\left(-\frac{d}{c}\right) = \infty$$

$$\text{If } c = 0 : \quad T(z) = \frac{az + b}{d} \quad \text{and} \quad T(\infty) = \infty$$

Mobius transforms can be uniquely determined by its action on three distinct points. For example, we'll find a mobius transform that maps three points  $\{z_1, z_2, z_3\}$  to  $\{1, 0, \infty\}$

1. We want  $T(z_2) = 0 : az_2 + b = 0 \Rightarrow b = -az_2$ , then  $T(z)$  becomes:

$$T(z) = \frac{az + b}{cz + d} = \frac{az - az_2}{cz + d} = \frac{a(z - z_2)}{cz + d}$$

2. We want  $T(z_3) = \infty : cz_3 + d = 0 \Rightarrow d = -cz_3$ , then  $T(z)$  becomes:

$$T(z) = \frac{a(z - z_2)}{c(z - z_3)}$$

3. We want  $T(z_1) = 1$ , then  $T(z)$  becomes:

$$T(z_1) = \frac{a(z_1 - z_2)}{c(z_1 - z_3)} = 1 \Rightarrow \frac{a}{c} = \frac{z_1 - z_3}{z_1 - z_2}$$

Finally, we see that  $T(z)$  is:

$$T(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z - z_2}{z - z_3}$$

We can now solve problems, such as : Find the Mobius Transform that maps the 3 points  $z_1 = -i, z_2 = -1, z_3 = 1$  to  $1, 0, \infty$

$$T(z) = \frac{-1 - 1}{-i + 1} \cdot \frac{z + 1}{z - 1} = (-i) \frac{z + 1}{z - 1} = \frac{-iz - i}{z - 1}$$

### 5.1.1 Matrix Representation of Möbius Transforms

We associate a  $2 \times 2$  matrix  $M$  to a Möbius Transform  $T(z)$ :

$$T(z) = \frac{az+b}{cz+d} \longleftrightarrow M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Note that:  $kM \longleftrightarrow T(z)$  for any  $k \in \mathbb{C}, k \neq 0$ .

We can also define the **inverse map**  $T^{-1}$  as the Möbius transform:

$$T^{-1} \longleftrightarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We can also define the **composition** of two Möbius Transforms, if  $T_1(z) = \frac{az+b}{cz+d}$  with matrix  $M$  and  $T_2(z) = \frac{ez+f}{gz+h}$  with matrix  $M_2$ , then:

$$T \circ T_2 \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Putting it all together, we can map any three points to any other three point:

#### Three-Point Theorem for Möbius Transformations

If  $T \longleftrightarrow M : (z_1, z_2, z_3) \mapsto (1, 0, \infty)$  and if  $T_2 \longleftrightarrow M_2 : (z'_1, z'_2, z'_3) \mapsto (1, 0, \infty)$  then:

$$T^{-1} \circ T_2 \longleftrightarrow M^{-1} : (z_1, z_2, z_3) \mapsto (z'_1, z'_2, z'_3)$$

This can be visualized like so:

$$\begin{array}{ccc} z'_1, z'_2, z'_3 & \xrightarrow{T^{-1} \circ T'} & z_1, z_2, z_3 \\ T_2 \mapsto M_2 & \searrow & \swarrow T \mapsto M \\ & 1, 0, \infty & \end{array}$$

Note that,  $M, M_2$  and  $T^{-1} \circ T_2$  have matrices: Three-Point Theorem for Möbius Transformations

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad M_2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, \quad T^{-1} \circ T_2 = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

#### Example: Find a Möbius transformation, $T : (0, -i, -1) \mapsto (i, 1, 0)$

If we can find a map  $T_1 : (0, -i, 1) \mapsto (1, 0, \infty)$  and a map  $T_2 : (1, -i, -1) \mapsto (i, 1, 0)$ . Then, by the Theorem above, we can find a  $T$  such that:  $T : (0, -i, -1) \mapsto (i, 1, 0)$ . Recall, we define a general transform  $T$ , that takes 3 points  $(z_1, z_2, z_3) \mapsto (1, 0, \infty)$

$$T(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z - z_2}{z - z_3}$$

$T_1$  becomes:

$$\begin{aligned} T_1(z) &= \frac{0+1}{0+i} \cdot \frac{z+i}{z+1} \\ &= \frac{1}{i} \cdot \frac{z+i}{z+1} \\ &= \frac{z+1}{iz+i} \\ &\Rightarrow \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \end{aligned}$$

$T_2$  becomes:

$$\begin{aligned} T_2(z) &= \frac{i-0}{i-1} \cdot \frac{z-0}{z-0} \\ &= \frac{i}{i-1} \cdot \frac{z-1}{z} \\ &= \frac{iz-i}{(i-1)z} \\ &\Rightarrow \begin{bmatrix} i & -i \\ i-1 & 1 \end{bmatrix} \end{aligned}$$

Thus,  $T$  is:

$$\begin{aligned} T &= T_2^{-1} \circ T_1 \leftrightarrow \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} i & -i \\ i-1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} 0 & i \\ 1 & i-1 \end{bmatrix} = \begin{bmatrix} 1 & i \\ i & i \end{bmatrix} \cdot \begin{bmatrix} i & -i \\ i-1 & 1 \end{bmatrix} = \begin{bmatrix} 0(1) + (i)(i) & (0)(i) + (i)(i) \\ (1-i)(1) + (i)(i) & (1-i)(i) + (i)(i) \end{bmatrix} = \begin{bmatrix} i^2 & i^2 \\ -i & i \end{bmatrix} \\ T(z) &= - \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \longleftrightarrow -i \frac{z+1}{z-1} \end{aligned}$$