Investigating the Acceleration Vector and the Great Circle on S^2
MA3101 Project

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Date: October 17, 2025

1 Why this project?

In my Euclidean and Non Euclidean Geometry class, we learned that if the acceleration vector of a curve is always normal to the tangent plane, then the curve is a geodesic
On the sphere, S^2, every geodesic is a great circle, which is circle formed a plane through the sphere's center.
I was deeply fascintated by this result, and I wanted to find a way to visualise it, which led me to create the website hosted on GitHub. This brief writeup covers the mathematics I needed in order to creat the website.

2 The Mathematics

In order to explore great circles on S^2, and the acceleration vector, we need to understand the geometry of the sphere. We first start by defining the plane:

2.1 The 2-Sphere

We define the 2-sphere, S^2, as the set of all points in $\mathbb{R}^3$ such that:

$$x^2 + y^2 + z^2 = 1$$

This represents a sphere of radius 1 centered at the origin.

The normal vector at a point $\mathbf{s} = (x, y, z)$ on the 2-sphere is the vector perpendicular to the tangent plane at that point.
It is givn by the gradient of the function $f(x, y, z) = x^2 + y^2 + z^2 - 1$:

$$\nabla f(\mathbf{s}) = (2x, 2y, 2z) = 2\mathbf{s}$$

2.2 The Plane

A plane P can be written as:

$$ax + by + cz = d$$

Where $\mathbf{n} = (a, b, c)$ is the normal vector to the plane, so the equation becomes:

$$P : \mathbf{n} \cdot (x, y, z) = d$$

We normalize $\mathbf{n}$ so it represents direction only:

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|}$$

We can then write the plane equation as:

$$P : \hat{\mathbf{n}} \cdot (x, y, z) = \frac{d}{\|\mathbf{n}\|}$$

We let $h = \frac{d}{\|\mathbf{n}\|}$ be the distance from the origin to the plane. So the plane equation becomes:

$$P : \hat{\mathbf{n}} \cdot (x, y, z) = h$$

We note that the plane intersects the sphere when $|h| \leq 1$.

2.3 The Intersection of a Plane and the 2-Sphere

We define a line $L(t)$ a line from the origin in the direction of the planes normal vector:

$$L(t) = t\hat{\mathbf{n}}$$

The line intersects the plane when $t = h$, so the closest point on the plane to the origin is:

$$c = h\hat{\mathbf{n}}$$

Because the sphere is symmetrical, the plane cuts it in a circle centered at c: circle, with center $c = h\hat{\mathbf{n}}$ If we let:

- Q : A point on the circumference of the circle formed by the intersection of the plane and the sphere
- C : The center of the circle
- O : The origin

We form a $\triangle OQC$ with sides:

- $OC = h$: Distance from origin to center of circle
- $OP = R$: Distance from origin to point on circumference
- $CP = r$: Distance from center of circle to point on circumference of planes cut

By the Pythagorean theorem, we have:

$$\begin{aligned}
R^2 &= h^2 + r^2 \\
r &= \sqrt{R^2 - h^2} \\
r &= \sqrt{1 - h^2} \quad \text{Since radius of sphere is 1}
\end{aligned}$$

So wthe circle on the sphere, cut by plane has center, $c = (0, 0, h)$ and radius $r = \sqrt{1 - h^2}$.
For simplicty, in the website, we let $\hat{\mathbf{n}} = (0, 0, 1)$, then we get

$$\begin{aligned}
&P : (0, 0, 1) \cdot (x, y, z) = h \\
&P : z = h \\
&c = h\hat{\mathbf{n}} = (0, 0, h) \\
&r = \sqrt{1 - h^2}
\end{aligned}$$

2.4 Paramertising The Sphere

We choose two vectors, **u** and **v**, that are on the plane, that is:

$$u, v \perp \hat{\mathbf{n}}$$

If we enforce

$$|| \ u \ || = || \ v \ || = 1 \quad \text{and} \quad u \perp v$$

we can define a circle on P:

$$C(t) = \cos(t)u + \sin(t)v \quad t \in [0, 2\pi]$$

Recognising that we let $\hat{\mathbf{n}} = \hat{k} = (0, 0, 1)$, and using the fact:

$$\hat{\imath} \perp \hat{\jmath} \perp \hat{k} \quad \text{and} \quad || \ \hat{\imath} \ || = || \ \hat{\jmath} \ || = || \ \hat{k} \ || = 1$$

We can let $u = \hat{\imath}$ and $v = \hat{\jmath}$, then we get:

$$C(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath} \quad t \in [0, 2\pi]$$

Finally, scaling to radius r and moving to center c we get:

$$C(t) = c + r\left[\cos(t)\hat{\imath} + \sin(t)\hat{\jmath}\right]$$

2.5 Velocity and Acceleration Vectors

We can take the first and second derivatives of $C(t)$ to get the velocity and acceleration vectors:

$$\begin{aligned}
v(t) &= \frac{d}{dt}\left[C(t)\right] \\
v(t) &= r\left[-\sin(t)\hat{\imath} + \cos(t)\hat{\jmath}\right] \\
&\text{and} \\
a(t) &= \frac{d^2}{dt^2}\left[C(t)\right] \\
&= \frac{d}{dt}\left[v(t)\right] \\
&= -r\left[-\cos(t)\hat{\imath} + \sin(t)\hat{\jmath}\right] \\
&= c - c - -r\left[-\cos(t)\hat{\imath} + \sin(t)\hat{\jmath}\right] \\
&= c - C(t)
\end{aligned}$$

2.6 Tieing it all together

We see that if $h = 0$, then the plane cuts the sphere through the origin, which is the definition of a great circle.
We also see that setting $h = 0$ also gives us a circle with center at the origin and radius 1.

$$\begin{aligned} c &= h\hat{\mathbf{n}} \\ c &= 0\hat{\mathbf{n}} \\ c &= 0 \\ &\text{and} \\ r &= \sqrt{1-h^2} \\ r &= \sqrt{1-0^2} \\ r &= 1 \end{aligned}$$

The aceleration vector also simplifies to:

$$\begin{aligned} a(t) &= c - C(t) \\ a(t) &= 0 - C(t) \\ a(t) &= -C(t) \end{aligned}$$