Homework 1

Due: Friday, January 17, 2014 **CS 103 Winter 2013-2014**

Notes

- ERRATA: We removed the section of problem 2 concerning the case when $-$ has higher precedence than $+$, since it is equivalent to when they have the same predence. Sorry for the confusion!
- These questions require thought, but do not require long answers. Please be as concise as possible.
- Please include your SUNet ID in your submission. (This is your WebAuth login name, like "dill." It is not the 8 or 9-digit number.)
- This problem set is out of 40 points.
- To submit your homework, you must hand in a **physical copy**, either at the beginning of lecture or in the submission cabinet on the Gates first floor "handout hangout" area (near Gates 182). We will only accept e-mailed PDFs from students who have notified the staff in advance that they cannot hand in a physical copy (because they will be out of town on the due date, for example). Submissions will not be accepted after 12:50 p.m. on the due date.

Problem 1 (5 points)

We have some multiple choice and true false questions for you online. We strongly recommend that you do these questions first, since they provide instant feedback and if you have trouble we want you to know before you dive into the rest of the problem set. Go to class.stanford.edu, sign up, and enroll in CS103. Then click on the "Courseware" tab and then go to the "Problem Set 1" section. There will be a "Logic Online Homework" section with the online questions.

If you are trying to answer these on January 10th and it does not work, do not fret since we are still setting it all up.

Problem 2 (6 points)

We investigated parsing, precedence, and syntax trees in the first lecture. Give a single arithmetic expression using only the operators $+$ and $-$ that evaluates to a different result in each of the following cases. Then for each case write out your expression's abstract syntax tree and the number it evaluates to.

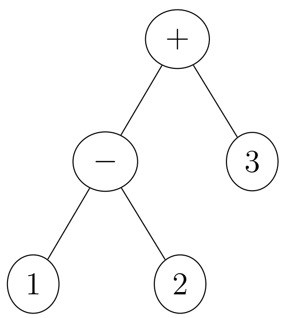
Both $+$ and $-$ are left-associative in all cases.

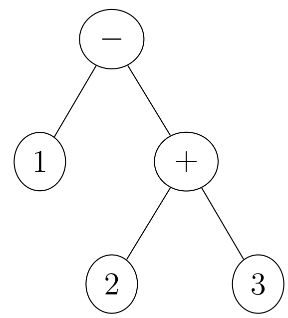
- $+$ and $-$ have the same precedence.
- $+$ has higher precedence than $-$.
- IGNORE THIS CASE. $-$ has higher precedence than $+$.

Solution:

The formula $1 - 2 + 3$, with correct precedence, has the value 2, as expected. The same formula when the precedence of $-$ is less than the precedence of $+$ has the value -4.

The parse tree on the left results when $+$ and $-$ have the same precedence and the parse tree on the right is the result of $-$ having lower precedence than $+$.





Problem 3 (8 points)

Write equational proofs of the following in the style shown in lecture. In addition to the Boolean identities in the lecture, you may use the property $\mathsf{P} \rightarrow \mathsf{Q} \equiv \neg\mathsf{P} \vee \mathsf{Q}$ and call it “implies-or.” These are all very important identities for $\rightarrow$ and $\leftrightarrow$.

In these and future equational proofs, you can use the associative and commutative laws without justification or comment, and you can merge several applications of the same law into one step when what you did is obvious.

a. $\mathsf{P} \rightarrow (\mathsf{Q} \rightarrow \mathsf{R}) \equiv (\mathsf{P} \wedge \mathsf{Q}) \rightarrow \mathsf{R}$

Solution (2 points):

$$\begin{array}{rcll}
\mathsf{P} \rightarrow (\mathsf{Q} \rightarrow \mathsf{R}) & \equiv & \neg\mathsf{P} \vee (\mathsf{Q} \rightarrow \mathsf{R}) & \text{implies-or} \\
& \equiv & \neg\mathsf{P} \vee (\neg\mathsf{Q} \vee \mathsf{R}) & \text{implies-or} \\
& \equiv & (\neg\mathsf{P} \vee \neg\mathsf{Q}) \vee \mathsf{R} & \text{associativity} \\
& \equiv & \neg(\mathsf{P} \wedge \mathsf{Q}) \vee \mathsf{R} & \text{de Morgan} \\
& \equiv & (\mathsf{P} \wedge \mathsf{Q}) \rightarrow \mathsf{R} & \text{implies-or}
\end{array}$$

b. $\mathsf{P} \leftrightarrow \mathsf{Q} \equiv (\neg\mathsf{P} \vee \mathsf{Q}) \wedge (\mathsf{P} \vee \neg\mathsf{Q})$

Solution (2 points):

$$\begin{array}{rcll}
\mathsf{P} \leftrightarrow \mathsf{Q} & \equiv & (\mathsf{P} \rightarrow \mathsf{Q}) \wedge (\mathsf{Q} \rightarrow \mathsf{P}) & \text{by definition} \\
& \equiv & (\neg\mathsf{P} \vee \mathsf{Q}) \wedge (\mathsf{Q} \rightarrow \mathsf{P}) & \text{implies-or} \\
& \equiv & (\neg\mathsf{P} \vee \mathsf{Q}) \wedge (\neg\mathsf{Q} \vee \mathsf{P}) & \text{implies-or} \\
& \equiv & (\neg\mathsf{P} \vee \mathsf{Q}) \wedge (\mathsf{P} \vee \neg\mathsf{Q}) & \text{commutativity}
\end{array}$$

c. $\mathsf{P} \leftrightarrow \mathsf{Q} \equiv (\mathsf{P} \wedge \mathsf{Q}) \vee (\neg\mathsf{P} \wedge \neg\mathsf{Q})$

Solution (2 points):

$$\begin{array}{rcll}
\mathsf{P} \leftrightarrow \mathsf{Q} & \equiv & (\neg\mathsf{P} \vee \mathsf{Q}) \wedge (\mathsf{P} \vee \neg\mathsf{Q}) & \text{1c} \\
& \equiv & (\neg\mathsf{P} \wedge (\mathsf{P} \vee \neg\mathsf{Q})) \vee (\mathsf{Q} \wedge (\mathsf{P} \vee \neg\mathsf{Q})) & \text{distributivity} \\
& \equiv & ((\neg\mathsf{P} \wedge \mathsf{P}) \vee (\neg\mathsf{P} \wedge \neg\mathsf{Q})) \vee ((\mathsf{Q} \wedge \mathsf{P}) \vee (\mathsf{Q} \wedge \neg\mathsf{Q})) & \text{distributivity (twice)} \\
& \equiv & (\mathsf{F} \vee (\neg\mathsf{P} \wedge \neg\mathsf{Q})) \vee ((\mathsf{Q} \wedge \mathsf{P}) \vee \mathsf{F}) & \text{inverse laws (twice)} \\
& \equiv & (\neg\mathsf{P} \wedge \neg\mathsf{Q}) \vee (\mathsf{Q} \wedge \mathsf{P}) & \text{identity (twice)} \\
& \equiv & (\mathsf{P} \wedge \mathsf{Q}) \vee (\neg\mathsf{P} \wedge \neg\mathsf{Q}) & \text{commutativity}
\end{array}$$

d. $\mathsf{P} \leftrightarrow \mathsf{Q} \equiv \neg\mathsf{P} \leftrightarrow \neg\mathsf{Q}$

Solution (2 points):

$$\begin{array}{rcll}
\mathsf{P} \leftrightarrow \mathsf{Q} & \equiv & (\mathsf{P} \rightarrow \mathsf{Q}) \wedge (\mathsf{Q} \rightarrow \mathsf{P}) & \text{by definition} \\
& \equiv & (\neg\mathsf{Q} \rightarrow \neg\mathsf{P}) \wedge (\neg\mathsf{P} \rightarrow \neg\mathsf{Q}) & \text{contrapositive (twice)} \\
& \equiv & (\neg\mathsf{P} \leftrightarrow \neg\mathsf{Q}) & \text{by definition}
\end{array}$$

Problem 4 (6 points)

In lecture, we covered the five propositional connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. Since we showed that $\mathsf{P} \rightarrow \mathsf{Q} \equiv \neg\mathsf{P} \vee \mathsf{Q}$ and $\mathsf{P} \leftrightarrow \mathsf{Q} \equiv (\mathsf{P} \rightarrow \mathsf{Q}) \wedge (\mathsf{Q} \rightarrow \mathsf{P})$, we can actually express all propositional formulas using three connectives: $\neg$, $\wedge$, and $\vee$. But actually, we can drop down to even fewer connectives.

a. Find a formula that is logically equivalent to $\mathsf{P} \vee \mathsf{Q}$ that uses only the connectives $\neg$ and $\wedge$. Do not use any variables other than P and Q. Show that your formula is correct by writing out its truth table as well as one for $\mathsf{P} \vee \mathsf{Q}$ (the truth tables should end up being the same).

Solution (2 points): One possibility is $\neg(\neg\mathsf{P} \wedge \neg\mathsf{Q})$.

P	Q	$\neg$	$(\neg\mathsf{P}$	$\wedge$	$\neg\mathsf{Q})$
T	**T**	**T**	**F**	**F**	**F**
T	**F**	**T**	**F**	**F**	**T**
F	**T**	**T**	**T**	**F**	**F**
F	**F**	**F**	**T**	**T**	**T**

We can see that this matches the truth table for $\mathsf{P} \vee \mathsf{Q}$ given in lecture.

We are now down to just $\neg$ and $\wedge$, and it might seem like that's as far as we can go. However, it's possible to replace these two connectives with just one. The **nor** connective, denoted $\mathsf{P} \downarrow \mathsf{Q}$, has the same truth table as $\neg(\mathsf{P} \vee \mathsf{Q})$. Intuitively, $\mathsf{P} \downarrow \mathsf{Q}$ means "neither P nor Q."

b. Find a formula logically equivalent to $\neg\mathsf{P}$ that uses just $\downarrow$. Do not use any variables other than P. Show that your formula is correct by writing out its truth table as well as one for $\neg\mathsf{P}$.

Solution (2 points): One possibility is $\mathsf{P} \downarrow \mathsf{P}$.

P	$\mathsf{P} \downarrow \mathsf{P}$
T	**F**
F	**T**

We can see that this matches the truth table for $\neg\mathsf{P}$ given in lecture.

c. Find a formula logically equivalent to $\mathsf{P} \wedge \mathsf{Q}$ that uses just $\downarrow$. Do not use any variables other than P and Q. Show that your formula is correct by writing out its truth table as well as one for $\mathsf{P} \wedge \mathsf{Q}$.

Solution (2 points): One possibility is $(\mathsf{P} \downarrow \mathsf{P}) \downarrow (\mathsf{Q} \downarrow \mathsf{Q})$.

P	Q	$(\mathsf{P} \downarrow \mathsf{P})$	$\downarrow$	$(\mathsf{Q} \downarrow \mathsf{Q})$
T	**T**	**F**	**T**	**F**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**F**	**F**
F	**F**	**T**	**F**	**T**

We can see that this matches the truth table for $\mathsf{P} \wedge \mathsf{Q}$ given in lecture.

You have just shown that every propositional logic formula can be written purely in terms of $\downarrow$. For this reason, it is called a **sole sufficient operator**. There is actually a second sole sufficient operator known as **nand**. Its truth table is the negation of the one for the $\wedge$ connective.

Problem 5 (6 points)

Imagine an island where there are exactly two types of people: truth-tellers and liars. Statements made by truth-tellers are always true, and statements made by liars are always false. Consider two people on this island, A and B. Let TA be the statement "A is a truth-teller" and let TB be the statement "B is a truth-teller." This also means that $\neg\mathsf{TA}$ means "A is a liar" and $\neg\mathsf{TB}$ means "B is a liar."

In each of the following parts, we give you a situation describing one or more statements A and B have made. A *solution* to a situation is an assignment of Boolean values to TA and TB such that there are no contradictions. That is, statements by (assigned) truth-tellers are true and statements by (assigned) liars are false. For each situation below:

- Give a propositional formula (using only TA, TB, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\neg$) that is true if and only if the values assigned to TA and TB represent a solution. This formula is thus satisfiable if and only if the situation has a solution. Although we will accept any propositional formula that works (and many do), we encourage you to make your formula as simple and intuitive as possible. For reference, the formulas we came up with each use at most five connectives.

- Create a truth table for your formula. **Please do not** write it out by hand, instead you should plug your propositional logic formula into the automated truth table builder located at cs103.stanford.edu/truthtab/TT.html. Using the table it generates, write down all solutions for each situation (there may be none).

For example, consider this situation: *A says, "B is a truth-teller."* One correct propositional formula is $\mathsf{TA} \leftrightarrow \mathsf{TB}$ (A and B must have the same type). There are exactly two solutions: (1) A and B are both truth-tellers and (2) A and B are both liars. Note that if A and B had different types, then A's statement would create a contradiction.

a. Situation: *B says, "I am a liar."*

Solution (2 points): $\mathsf{TB} \leftrightarrow \neg\mathsf{TB}$, which is equivalent to $\mathsf{TB} \wedge \neg\mathsf{TB}$. Intuitively, this means that B would have to be a liar and a truth-teller at the same time, which is impossible. The truth table will be all Fs, and so there are zero solutions.

b. Situation: *A says, "A and B are opposite types."*

Solution (2 points): $\mathsf{TA} \leftrightarrow (\mathsf{TA} \leftrightarrow \neg\mathsf{TB})$. You should start seeing a general pattern: if the situation just has A saying X, then the formula is $\mathsf{TA} \leftrightarrow \mathsf{X}$. This is because if you know that A is telling the truth, then X must be true ($\mathsf{TA} \rightarrow \mathsf{X}$), and if you know that X is true, then A must be telling the truth ($\mathsf{X} \rightarrow \mathsf{TA}$).

TA	TB	TA $\leftrightarrow$	(TA $\leftrightarrow \neg$TB)
T	**T**	**F**	**F**
T	**F**	**T**	**T**
F	**T**	**F**	**F**
F	**F**	**T**	**T**

There are two solutions, both of which require B to be a liar.

c. Situation: *A says, "Both A and B are truth-tellers" and B says, "A is a liar."*

Solution (2 points): $(\mathsf{TA} \leftrightarrow (\mathsf{TA} \wedge \mathsf{TB})) \wedge (\mathsf{TB} \leftrightarrow \neg\mathsf{TA})$.

TA	TB	(TA $\leftrightarrow$	(TA $\wedge$ TB))	$\wedge$	(TB $\leftrightarrow \neg$TA)
T	**T**	**T**	**T**	**F**	**F**
T	**F**	**F**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**	**T**
F	**F**	**T**	**F**	**F**	**F**

There is only one solution: A is a liar and B is a truth-teller.

Problem 6 (8 points)

For each of the following, you will be given a list of first-order predicates and functions along with an English sentence. In each case, write a statement in first-order logic that expresses the indicated sentence. Your statement may use any first-order construct (equality, connectives, quantifiers, etc.), but you must only use the predicates, functions, and constants provided.

Try to keep your formulas as simple and obvious as possible. In particular, we recommend that your solutions to (b), (c), and (d) **not** have all the quantifiers in the beginning of the formula. In the past, such solutions have often been unintuitive and logically incorrect in subtle ways. Also, if you're having trouble keeping track of all the parentheses in your formula, please consider using multiple lines, indentation, and/or plenty of whitespace to aid readability.

a. Given

$\mathsf{Natural}(\mathsf{x})$, which states that x is a natural number,

$\mathsf{Product}(\mathsf{x}, \mathsf{y})$, which returns the product of x and y,

and the constants 1 and 137, write a statement in first-order logic that says "137 is prime."

Solution (2 points):

$$\forall \mathsf{p}\ \forall \mathsf{q}\ (\mathsf{Natural}(\mathsf{p}) \land \mathsf{Natural}(\mathsf{q}) \land \mathsf{Product}(\mathsf{p}, \mathsf{q}) = 137 \rightarrow$$
$$((\mathsf{p} = 1 \land \mathsf{q} = 137) \lor (\mathsf{p} = 137 \land \mathsf{q} = 1)))$$

b. Given

$\mathsf{Student}(\mathsf{x})$, which states that x is a student,

$\mathsf{Computer}(\mathsf{x})$, which states that x is a computer,

$\mathsf{Owns}(\mathsf{x}, \mathsf{y})$, which states that x owns y,

write a statement in first-order logic that says "some student owns exactly two computers."

Solution (2 points):

$$\exists \mathsf{s}\ (\mathsf{Student}(\mathsf{s}) \land (\exists \mathsf{c}_1 \exists \mathsf{c}_2$$
$$(\mathsf{Computer}(\mathsf{c}_1) \land \mathsf{Computer}(\mathsf{c}_2) \land \mathsf{Owns}(\mathsf{s}, \mathsf{c}_1) \land \mathsf{Owns}(\mathsf{s}, \mathsf{c}_2) \land \mathsf{c}_1 \neq \mathsf{c}_2 \land$$
$$\forall \mathsf{c}\ (\mathsf{Computer}(\mathsf{c}) \land \mathsf{Owns}(\mathsf{s}, \mathsf{c}) \rightarrow (\mathsf{c} = \mathsf{c}_1 \lor \mathsf{c} = \mathsf{c}_2)))))$$

c. Given

$\mathsf{Guards}(\mathsf{x}, \mathsf{y})$, which states that x guards y,

write a statement in first-order logic that says "no one guards the guardians." Note that all that is required to be a guardian is to guard something.

Solution (2 points):

The common interpretation is "every guardian is unguarded":

$$\forall \mathsf{x} \forall \mathsf{y}\ ((\mathsf{Guards}(\mathsf{x}, \mathsf{y})) \rightarrow \neg(\exists \mathsf{z}\ \mathsf{Guards}(\mathsf{z}, \mathsf{x})))$$

Another equally valid interpretation is "no single entity guards all guardians":

$$\neg(\ \exists \mathsf{x}\ (\ \forall \mathsf{y} \forall \mathsf{z}\ (\mathsf{Guards}(\mathsf{y}, \mathsf{z}) \rightarrow \mathsf{Guards}(\mathsf{x}, \mathsf{y}))))$$

d. Given

$\mathsf{Lady(x)}$, which states that x is a lady,

$\mathsf{Glitters(x)}$, which states that x glitters,

$\mathsf{IsSureIsGold(x, y)}$, which states that x is sure y is gold,

$\mathsf{Buying(x, y)}$, which states that x is buying y, and

$\mathsf{StairwayToHeaven(x)}$, which states that x is a stairway to heaven,

write a statement in first-order logic that says "There's a lady who's sure all that glitters is gold, and she's buying a stairway to heaven."

Solution (2 points):

$\exists \mathsf{L}(\mathsf{Lady(L)} \land \forall \mathsf{x}\ (\mathsf{Glitters(x)} \rightarrow \mathsf{IsSureIsGold(L, x)})$

$\land \exists \mathsf{S}\ (\mathsf{StairwayToHeaven(x)} \land \mathsf{Buying(L, S)}))$

Problem 7: Course Feedback (1 point)

We want this course to be as good as it can be, and we'd appreciate your feedback on how we're doing. Please go to the "HW1 Feedback" link on the course website and answer a few quick survey questions. We'll give you full credit for this problem no matter you write (as long you don't leave anything blank!), but we'd appreciate it if you're honest about how we're doing.

Because we will ask you how many hours you spent on this homework, please do this problem after you've finished the other problems. For this reason, the deadline for submitting your feedback is 5 p.m. on the due date.