Homework 2

Due: Friday, January 24, 2013 **CS103, Winter 2013-2014**

- CLARIFICATION: On problem 6 (Pythagorean triples), you may indeed prove the contrapositive by contradiction, if that simplifies matters.
- These questions require thought, but do not require long answers. Please be as concise as possible.
- Please include your SUNet ID in your submission. (This is your WebAuth login name, like "dill." It is not the 8 or 9-digit number.)
- To submit your homework, you must hand in a **physical copy**, either at the beginning of lecture or in the submission cabinet on the Gates first floor "handout hangout" area (near Gates 182). We will only accept e-mailed PDFs from students who have notified the staff in advance that they cannot hand in a physical copy (because they will be out-of-town on the due date, for example). Submissions will not be accepted after 12:50 p.m. on the due date.
- For problems that ask for proofs, please follow the guidelines in lecture. You must use only the definitions and theorems presented in lecture or problem sets. For proofs involving integers, you may assume that the sum, difference, and product of two integers is an integer. (The same goes for natural numbers.) You may also perform simple algebraic manipulations (such as $(x+y)^2 - y^2 = x^2 + 2xy + y^2 - y^2 = x^2 + 2xy$) without justification or comment, as long what you did was obvious.

Problem 1 (5 points)

Once again we have some multiple choice and true false questions for you online. We strongly recommend that you do these questions first, since they provide instant feedback and if you have trouble we want you to know before you dive into the rest of the problem set. Click on the "Courseware" tab and then go to the "Problem Set 2" section. There will be a "Online Homework" section with the online questions.

Problem 2 (3 points):

Find a counterexample, if possible, to these universally quantified statements, where all terms represent real numbers.

a. $\forall x\ (x^2 \neq x)$

b. $\forall x\ (x^2 \neq 2)$

c. $\forall x\ (|x| > 0)$

Problem 3 (4 points)

Show that $\forall x\ P(x) \vee \forall x\ Q(x)$ and $\forall x\ (P(x) \vee Q(x))$ are not logically equivalent. To show that two equations are not logically equivalent, give an example where one is false and the other is true. You should describe a universe of discourse (this is the set of values that terms can represent), and describe what the predicates P and Q mean.

Problem 4 (4 points)

A formula is in **negation normal form** iff negations appear only in front of predicates *and* the only connectives used are $\neg$, $\wedge$, and $\vee$. A formula is in **prenex normal form** iff all of the quantifiers appear at the very front. For each of the two formulas below, first convert it into negation normal form by pushing the negation inward as much as possible, then convert *that* formula into prenex normal form. Please show your work using the same style as an equational proof, but to avoid tedium you do not need to cite which identity you are using at each step. Keep in mind that the x in $P(x)$ is not the same as the x in $Q(x)$.

a. $\neg(\forall x\ P(x) \vee \forall x\ Q(x))$

b. $\neg(\exists x\ P(x) \rightarrow \exists x\ Q(x))$

Problem 5 (6 points)

A number is a multiple of 3 iff it can be written as $3k$ for some integer k. For each integer n, exactly one of the following is true (you do not need to prove this):

- n is a multiple of 3.
- n can be written as $3k + 1$ for some integer k.
- n can be written as $3k + 2$ for some integer k.

a. Prove that if an integer n is a multiple of 3, then n^2 is a multiple of 3.

b. Prove that if n is an integer such that n^2 is a multiple of 3, then n is a multiple of 3. With this proof and the previous proof, you can conclude that an integer n is a multiple of 3 iff n^2 is a multiple of 3.

c. Give a proof by contradiction that $\sqrt{3}$ is irrational. Make sure that you explicitly state what assumption you are making before you derive a contradiction from it. Recall that a rational number is one that can be written as p/q for integers p and q where $q \neq 0$ and p and q have no common divisor other than ± 1.

Problem 6 (5 points)

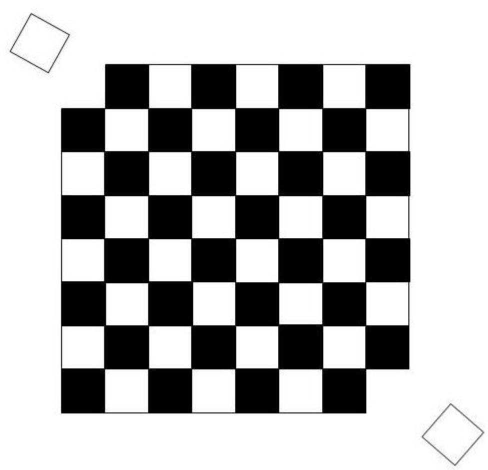
A **Pythagorean triple** is a triple (a, b, c) of positive natural numbers such that $a^2 + b^2 = c^2$. For example, $(3, 4, 5)$ is a Pythagorean triple since $3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

Prove by contrapositive that if (a, b, c) is a Pythagorean triple, then $(a+1, b+1, c+1)$ is **not** a Pythagorean triple.

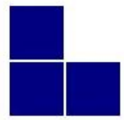
CLARIFICATION: You may indeed prove the contrapositive by contradiction, if that simplifies matters.

Problem 7 (6 points)

Suppose that you have a standard 8 x 8 chessboard with two opposite corners removed:



In the course notes (page 62), there's a proof that it's impossible to tile this chessboard using 2 x 1 dominoes. This question considers what happens if you try to tile the chessboard using right triominoes, L-shaped tiles that look like this:



a. Prove that it is impossible to tile an 8 x 8 chessboard missing two opposite corners with right triominoes.

b. For $n \geq 3$, is it ever possible to tile an n x n chessboard missing two opposite corners with right triominoes? If so, find a number $n \geq 3$ such that it's possible and show how to tile that chessboard with right triominoes. If not, prove that for every $n \geq 3$, it's impossible to tile an n x n chessboard missing two opposite corners with right triominoes.

Problem 8 (6 points)

Prove by cases that $\min(x, \min(y, z)) = \min(\min(x, y), z)$, for all real numbers x, y, and z.

Problem 9: Course Feedback (1 point)

We want this course to be as good as it can be, and we'd appreciate your feedback on how we're doing. Please go to the "HW2 Feedback" link on the course website and answer a few quick survey questions. We'll give you full credit for this problem no matter you write (as long you don't leave anything blank!), but we'd appreciate it if you're honest about how we're doing.

Because we will ask you how many hours you spent on this homework, please do this problem after you've finished the other problems. For this reason, the deadline for submitting your feedback is 5 p.m. on the due date.

Problem 10: Challenge Problem

Note: Please read about the purpose and scoring of challenge problems on the course website before attempting this problem.

Say that a natural number is a *consecutive sum* iff it can be written as the sum of two or more consecutive positive natural numbers. For example, $75 = 24 + 25 + 26$ is a consecutive sum, and so is $100 = 18 + 19 + 20 + 21 + 22$.

Find a number between one vigintillion (10^{63}) and two vigintillion (2×10^{63}) inclusive that is **not** a consecutive sum. Prove that your number is not a consecutive sum and that every other number in the range is a consecutive sum.