Problem 1

The following fact will be used later to define the “diversification ratio”.

(a) Consider random variables $Y_1, Y_2, \ldots, Y_n$ with covariance matrix $\mathbf{\Sigma}$ with elements $\mathrm{Var}(Y_i) = \sigma_i^2$, and $\mathrm{Cov}(Y_i, Y_j) = \sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, $i, j = 1, \ldots, n$. Show that the standard deviation of the sum $Y_1 + Y_2 + \cdots + Y_n$ is smaller than the sum of the standard deviations $\sigma_1 + \sigma_2 + \cdots + \sigma_n$, that is,

$$\mathrm{Std}(Y_1 + Y_2 + \cdots + Y_n) \leq \sigma_1 + \sigma_2 + \cdots + \sigma_n. \tag{1}$$

Hint: Recall that

$$\begin{aligned} \mathrm{Var}(Y_1 + Y_2 + \cdots + Y_n) &= \sum_{i=1}^{n} \sigma_i^2 + \sum_{i \neq j}\sum \sigma_{ij} &(2) \\ &= \sum_{i=1}^{n} \sigma_i^2 + \sum_{i \neq j}\sum \rho_{ij}\sigma_i\sigma_j, &(3) \end{aligned}$$

and use the fact that $|\rho_{ij}| \leq 1$, $i, j = 1, \ldots, n$.

(b) Now consider a long–only portfolio with N assets. Denote the return of Asset i by R_i, and its portfolio weight by x_i, so that the portfolio return is

$$R_p = \sum_{i=1}^{N} x_i R_i. \tag{4}$$

Using elementary properties of the standard deviation, explain why the result in (a) implies that

$$\sigma_p = \mathrm{Std}(R_p) \leq \sum_{i=1}^{N} x_i \sigma_i, \tag{5}$$

where σ_i is the standard deviation of the return of Asset i. That is, the portfolio standard deviation is smaller than the portfolio–weighted average of the individual standard deviations.