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Assignment 2 – September 29th, 2023
CS 5350 – Advanced Algorithms

1. The Fibonacci sequence can be defined by $T(1) = T(2) = 1$ and $T(n) = T(n-1) + T(n-2), n \geq 3$. Binet Proposed a closed formula for the Fibonacci sequence:

$$B(n) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}$$

Show that Binet's formula is correct, i.e., that $B(n) = T(n)\ \forall n \geq 1$.

i) It's easy to see that using $T(0) = 0$ and $T(1) = 1$ generate the same sequence.
   a. $T(2) = T(1) + T(0) = 1 + 0 = 1$
   b. Thus, $T(2) = 1$

ii) The Fibonacci sequence is a linear recurrence with constant coefficients equations of degree 2.
   a. Thus, it can be written as $y_t = a_1 y_{t-1} + a_2 y_{t-2} + b$, where $a_1 = a_2 = 1$ and $b = 0$.

iii) A solution to the recurrence relation is $y_n = r^n$ when $t = r$ is a root of the polynomial as shown below:
   a. $r^n = (1)r^{n-1} + (1)r^{n-2} + 0 \to r^n - r^{n-1} - r^{n-2} = 0 \to r^2 - r - 1 = 0.$
   b. Using the quadratic formula, we have $r = \frac{1 \pm \sqrt{5}}{2}$.
      i. Quadratic formula: $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
      ii. Substitution: $r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$.

iv) Because the y_n characteristic roots are distinct real solutions, we can write a general solution as $y_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n$ where C, D are real constants.

v) Solve for C and D using initial conditions:
   a. When $n = 0$:
      i. $0 = C\left(\frac{1+\sqrt{5}}{2}\right)^0 + D\left(\frac{1-\sqrt{5}}{2}\right)^0 = C + D \to C = -D$
   b. When $n = 1$:
      i. $1 = C\left(\frac{1+\sqrt{5}}{2}\right)^1 + D\left(\frac{1-\sqrt{5}}{2}\right)^1 = -D\,\frac{1+\sqrt{5}}{2} + D\,\frac{1-\sqrt{5}}{2} = D\left(\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\right) \to D = \frac{1}{\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}}$
   c. Therefore, $C = \frac{-1}{\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}}$ and $D = \frac{1}{\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}}$

vi) Substituting the values of C and D in y_n equation.
   a. $y_n = \frac{-1}{\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}}\left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{\left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^n}{\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}}$
   b. Multiply numerator and denominator by -1: $y_n = \frac{\left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^n}{\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}} \frac{-1}{-1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}$.

vii) Therefore, $y_n = T(n) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}} = B(n)$.

2. Toom-Cook Multiplication Algorithm split the two input integers a and b, both of size n, into three parts each

$$a = a_h\beta^{\frac{2n}{3}} + a_m\beta^{\frac{n}{3}} + a_l$$
$$b = b_h\beta^{\frac{2n}{3}} + b_m\beta^{\frac{n}{3}} + b_l$$

combines the six parts a_h through b_l with $O(n)$ operations, obtaining intermediate values s_1 through s_5 and r_1 though r_5, each of which has size $\frac{n}{3}$ executes 5 recursive calls to compute five products $t_1 = s_1 * r_1$ through $t_5 = s_5 * r_5$ and finally combines these five products t_1 through t_5 in such a way to obtain the complete product $c = a * b$, using $O(n)$ operations.

a) Using the Master Theorem, show that the complexity of the Toom-Cook Multiplication Algorithm is $\Theta(n^{\log_3 5})$.
   a. The relevant points for time complexity analysis from Toom-Cook Multiplication Algorithm are:
      i. Size decrease by a factor of 3.
      ii. Five recursive calls are performed with the new size.
      iii. Division and merge time take $O(n)$ time.
   b. Thus, the recurrence equation is $T(n) = 5T(n/3) + n$.
   c. Show that case 1 of master theorem apply for this recurrence equation.
      i. Let $\epsilon = 2$ and $f(n) = O(n)$.
      ii. Substitute a, b, ϵ: $O(n^{\log_b a-\epsilon}) = O(n^{\log_b a-\epsilon}) = O(n^{\log_3 5-3}) = O(n^{\log_3 3}) = O(n^1) = O(n)$
   d. Therefore, $T(n) = \Theta(\mathrm{n}^{\log_3 5})$.

b) Determine whether the Toom-Cook or the Karatsuba Algorithm is faster.
   a. Toom-Cook Algorithm: $T(n) = \Theta(\mathrm{n}^{\log_3 5}) \approx \Theta(n^{1.465})$.
   b. Karatsuba Algorithm: $T(n) = \Theta(n^{\log_2 3}) \approx \Theta(n^{1.585})$.
   c. Therefore Toom-Cook multiplication algorithm is faster than Karatsuba multiplication algorithm.

c) Determine whether the Toom-Cook or the Schoolbook Multiplication Algorithm is faster.
   a. Toom-Cook Algorithm: $T(n) = \Theta(\mathrm{n}^{\log_3 5}) \approx \Theta(n^{1.465})$.
   b. Schoolbook Multiplication Algorithm: $T(n) = \Theta(n^2)$.
   c. Therefore Toom-Cook multiplication algorithm is faster than schoolbook multiplication algorithm.