Q: Prove case 3 of Master Theorem.

Lemma 4.2 states:

Let $a > 0$ and $b > 1$ be constants and let $f(n)$ be a function defined over real numbers $n \geq 1$. Then the recurrence $T(n) = \begin{cases} \Theta(1) & \text{if } 0 \leq n < 1 \\ aT\left(\frac{n}{b}\right) + f(n) & \text{if } n \geq 1 \end{cases}$ has solution $T(n) = \Theta\left(n^{\log_b a}\right) + \sum_{j=0}^{\lfloor \log_b n \rfloor} a^j f\left(\frac{n}{b^j}\right)$.

Lemma 4.3 states:

Let $a > 0$ and $b > 1$ be constants and let $f(n)$ be function defined over real numbers $n \geq 1$. Then the asymptotic behavior of the function $g(n) = \sum_{j=0}^{\lfloor \log_b n \rfloor} a^j f\left(\frac{n}{b^j}\right)$, defined for $n \geq 1$, can be characterized as follows:

Case 3: If there exists a constant c in the range $0 < c < 1$ s.t. $0 < af\left(\frac{n}{b}\right) \leq cf(n)\ \forall n \geq 1$, then $g(n) = \Theta\big(f(n)\big)$.

Proof of case 3:

1. Let $a > 0$, $b > 0$, $\epsilon > 0$ and $c < 1$.
2. Let $f(n) = \Omega(n^{log_b a + \epsilon})$ and $g(n) = \sum_{j=0}^{\lfloor \log_b n \rfloor} a^j f\left(\frac{n}{b^j}\right)$.
3. We can see that $g(n) = \Omega\big(f(n)\big)$ because $f(n)$ is a term of $g(n)$.
4. Choose ϵ s.t. $c = b^{-\epsilon} < 1$.
5. Show that $a^j f\left(\frac{n}{b^j}\right) \leq c^j f(n)$. Proof by induction.
   a. Base case: Let $j = 0$.
      i. $a^0 f\left(\frac{n}{b^0}\right) \leq c^0 f(n) \rightarrow f(n) \leq f(n)$
   b. Inductive case: Assume $a^j f\left(\frac{n}{b^j}\right) \leq c^j f(n)$ is true.
      i. $a^{j+1} f\left(\frac{n}{b^{j+1}}\right) = a^{j+1}\left(\frac{n}{b^{j+1}}\right)^{log_b a + \epsilon} = aa^j\left(\frac{n}{b^j}\right)^{log_b a + \epsilon} b^{-log_b a + \epsilon} = aa^j f\left(\frac{n}{b^j}\right) b^{-\log_b a} b^{-\epsilon} = aa^j f\left(\frac{n}{b^j}\right) a^{-1} b^{-\epsilon} = a^j f\left(\frac{n}{b^j}\right) b^{-\epsilon} \leq c^j f(n) b^{-\epsilon} = c^j f(n) c = c^{j+1} f(n)$.
6. Now show that $g(n) = O\big(f(n)\big)$.
   a. $g(n) = \sum_{j=0}^{\lfloor \log_b n \rfloor} a^j f\left(\frac{n}{b^j}\right)$
   b. Use $a^j f\left(\frac{n}{b^j}\right) \leq c^j f(n)$: $g(n) \leq \sum_{j=0}^{\lfloor \log_b n \rfloor} c^j f(n)$.
   c. Extract $f(n)$ from the sum: $g(n) \leq f(n) \sum_{j=0}^{\lfloor \log_b n \rfloor} c^j$
   d. Increasing the sigma upper bound would make the right side bigger, holding the inequality.
   e. Thus: $g(n) \leq f(n) \sum_{j=0}^{\infty} c^j$.
   f. The infinite geometric series states that $\sum_{j=0}^{\infty} ar^k = \frac{a}{1-r}$ when $0 < r < 1$.
   g. Therefore, we can use it to get: $g(n) \leq f(n)\frac{1}{1-c}$.
   h. Because $\frac{1}{1-c}$ is a constant, we have $g(n) \leq O\big(f(n)\big)$.
7. Therefore, because $g(n) = \Omega\big(f(n)\big) = O\big(f(n)\big)$ then $g(n) = \Theta\big(f(n)\big)$.

Master theorem states:

Let $a > 0$ and $b > 1$ be constants and let $f(n)$ be deriving function that is defined and nonnegative on all sufficiently large reals. Define the algorithmic recurrence $T(n)$ on the positive real numbers by $T(n) = aT\left(\frac{n}{b}\right) + f(n)$. Then the asymptotic behavior of $T(n)$ can be characterized as follows:

Case3: If there exist a constant $\epsilon > 0$ s.t. $f(n) = \Omega(n^{\log_b a+\epsilon})$, and if $f(n)$ additionally satisfies the regularity condition $af\left(\frac{n}{b}\right) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Proof of case 3:

1. Let $a > 0$, $b > 1$ be constants, $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ where $f(n) = \Omega(n^{\log_b a+\epsilon})$ and $af\left(\frac{n}{b}\right) \le cf(n)$ is satisfied for some $c < 1$.
2. By lemma 4.2 we know that if $n \ge 1$, the equation $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ has solution $T(n) = \Theta(n^{\log_b a}) + \sum_{j=0}^{\lfloor \log_b n \rfloor} a^j f\left(\frac{n}{b^j}\right)$.
3. Then, by lemmas 4.3 we know that $\sum_{j=0}^{\lfloor \log_b n \rfloor} a^j f\left(\frac{n}{b^j}\right) = \Theta(f(n))$.
4. Thus, $T(n) \le \Theta(n^{\log_b a}) + \sum_{j=0}^{\lfloor \log_b n \rfloor} c^i f(n) = \Theta(n^{\log_b a}) + \Theta(f(n))$.
5. Using $f(n) = \Omega(n^{\log_b a+\epsilon})$, we have that $\Theta(f(n)) = \Theta\left(\Omega(n^{\log_b a+\epsilon})\right) = \Theta(n^{\log_b a+\epsilon})$.
6. Thus, $T(n) = \Theta(n^{\log_b a}) + \Theta(n^{\log_b a+\epsilon}) = \Theta(n^{\log_b a} + n^{\log_b a+\epsilon}) = \Theta(n^{\log_b a+\epsilon})$.
7. Therefore, $T(n) = \Theta(f(n))$.