Atmospheric Thermodynamics

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April 19, 2021

1 Vertical Structure of the Atmosphere

1.1 Pressure

Under normal atmospheric conditions the atmosphere can be approximated to be hydrostatically balanced, with the vertical pressure gradient being in exact opposition with the gravitational force. This yields the convenient constraint:

$$\frac{\partial p}{\partial z} = -\rho g, \tag{1}$$

where g is the acceleration due to Earth's gravity, z is the altitude, ρ is air density, and p is air pressure. Since air is well approximated by an ideal gas, the ideal gas equation can be imposed as an additional constraint:

$$\rho = \frac{p}{RT}, \tag{2}$$

where R is the individual gas constant of air. Combing these two constraints with the assumption that pressure is solely a function of z, yields:

$$\frac{dp}{p} = -\frac{g}{RT(z)}dz. \tag{3}$$

Assuming air temperature decreases from a ground temperature of T_o at linear rate of $-\Gamma$ with altitude; integrating (3) yields:

$$\frac{R}{g}\int_{p_o}^{p}\frac{dp}{p} = -\int_0^z \frac{d\tilde{z}}{T_o - \Gamma\tilde{z}} \;\;\Rightarrow\;\; \frac{R}{g}\ln\frac{p}{p_o} = \frac{1}{\Gamma}\ln\left(1 - \frac{\Gamma z}{T_o}\right). \tag{4}$$

After some reorganisation, taking the exponential of both sides yields the relationship between pressure and altitude in the form of

$$p(z) = p_o\left(1 - \frac{\Gamma}{T_o}z\right)^{\frac{g}{R\Gamma}}. \tag{5}$$

This relationship is visualised in the figures below using the example surface temperatures $T_o = (30, 10, -10)\,^{\circ}\mathrm{C}$, and lapse rates $\Gamma = (1, 0.6, 0.4)\,\frac{\mathrm{K}}{100\mathrm{m}}$.

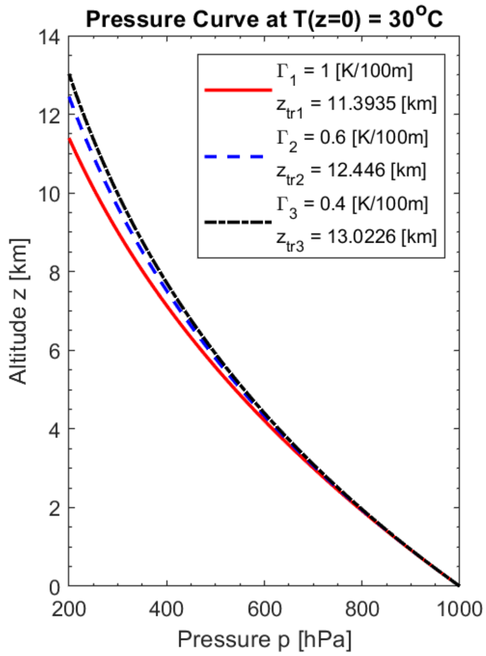

Figure 1. Pressure as a function of altitude given a ground pressure of 1000 hPa and ground temperature of 30 °C

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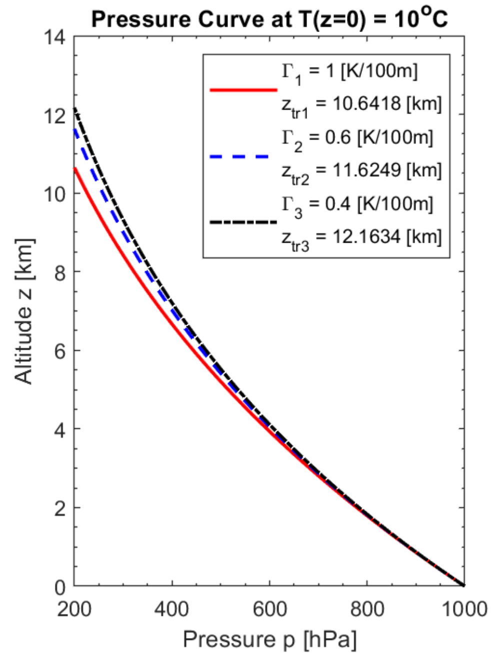

Figure 2. Pressure as a function of altitude given a ground pressure of 1000 hPa and ground temperature of 10 °C

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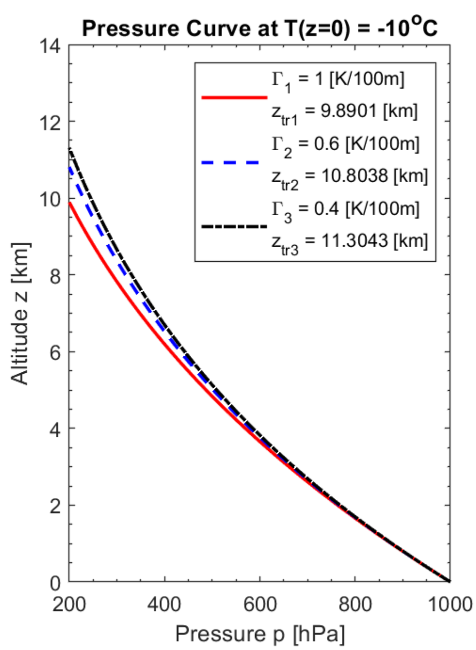

Figure 3. Pressure as a function of altitude given a ground pressure of 1000 hPa and ground temperature of $-10\,°C$.

1.2 Density

Equation (2) allows for (5) to be reformulated in terms of density $\rho(z)$:

$$\rho(z) = \frac{p_o}{R(T_o - \Gamma z)} \left(1 - \frac{\Gamma}{T_o} z\right)^{\frac{g}{R\Gamma}}. \tag{6}$$

Using the sample T_o and Γ values, the following curves are generated.

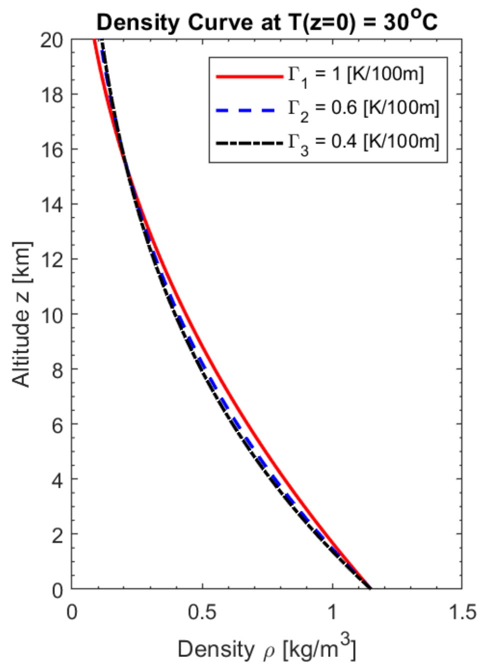

Figure 4. Density as a function of altitude given a ground pressure of 1000 hPa and ground temperature of 30 °C

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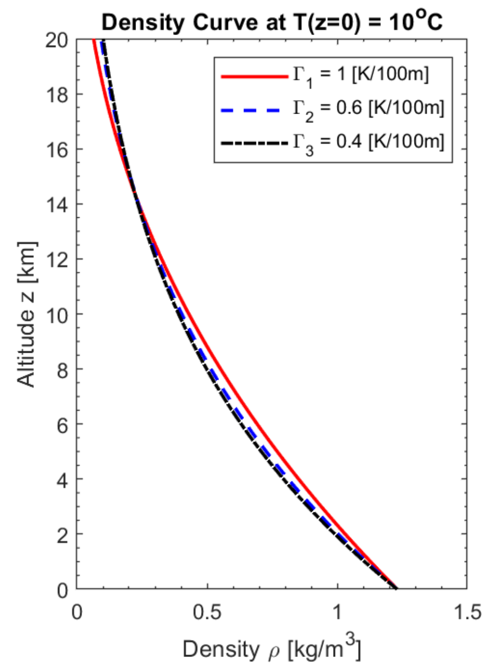

Figure 5. Pressure as a function of altitude given a ground pressure of 1000 hPa and ground temperature of 10 °C

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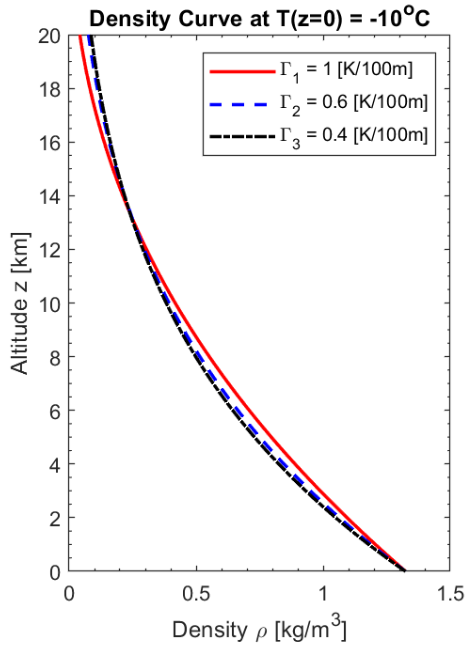

Figure 6. Density as a function of altitude given a ground pressure of 1000 hPa and ground temperature of $-10\,^{\circ}$C.

2 Isolines

Climate and atmospheric structure remain relatively constant along Earth's longitude lines. The same cannot be said about the latitudinal direction. In general and at any given time, climate can be seen to vary significant along Earth's latitude lines. To represent this fact the ground temperature T_o is assumed to vary with latitude ϕ according to:

$$T_o = (20\cos(2\phi) + 10)\ [^{\circ}\mathrm{C}]. \tag{7}$$

This relation presents itself graphically as shown in figure 7.

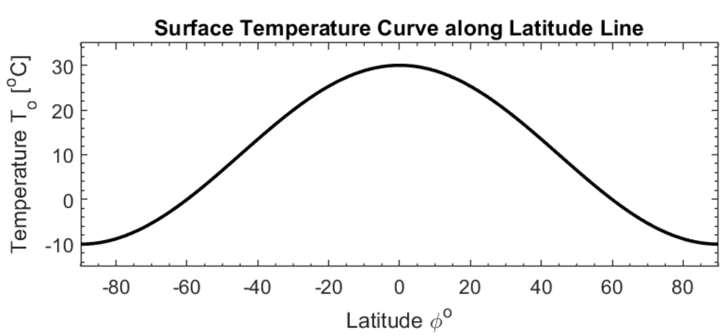

Figure 7. Ground temperature as a function of latitude.

2.1 Pressure

With this model as a foundation, equation (5) can be expressed in terms of z as

$$z_p(T_o) = \frac{T_o(\phi)}{\Gamma}\left(1 - \left(\frac{p}{p_o}\right)^{\frac{R\Gamma}{g}}\right). \tag{8}$$

$\Gamma = 6\,[\frac{\text{K}}{\text{km}}]$ is the example lapse rate used. Using this sample data and equation (8), the isolines can be visualised as shown in Fig. 8.

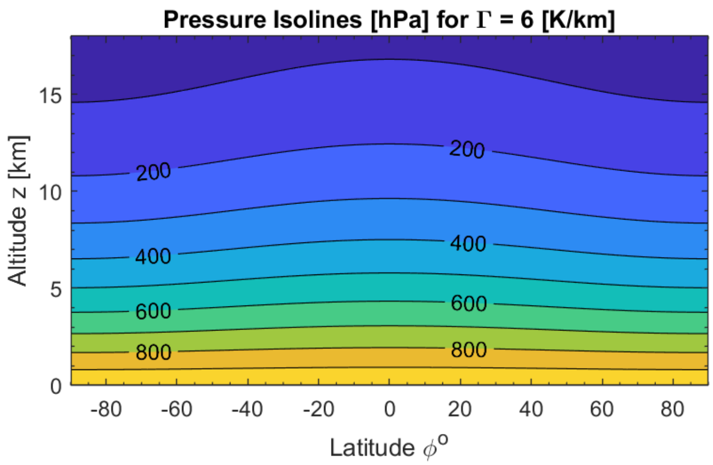

Figure 8. Pressure isolines along latitude lines.

2.2 Temperature

Atmospheric temperature can also be expressed through isolines using:

$$z_T(T_o) = \frac{1}{\Gamma}(T_o(\phi) - T), \tag{9}$$

for various constant T.

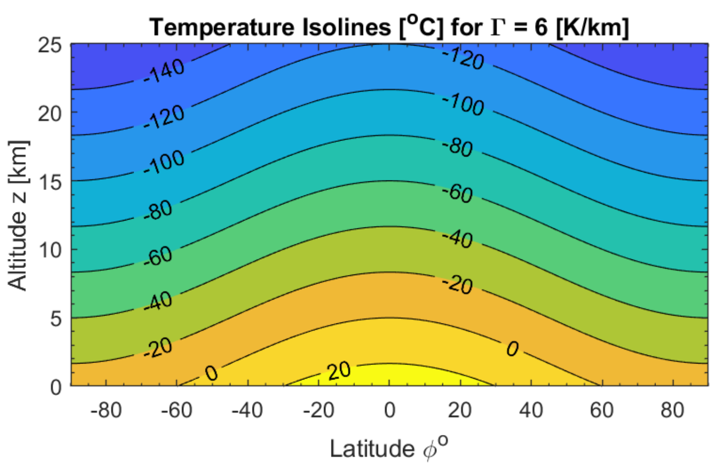

Figure 9. Temperature isolines with lapse rate $\Gamma = 6\,[\text{K/km}]$.

2.3 Potential Temperature

Using equation (5), and the knowledge that potential temperature θ is given by:

$$\theta(z) = T(z)\left(\frac{p_o}{p(z)}\right)^{\kappa}, \tag{10}$$

altitude can be expressed in terms of potential temperature in the form of the equation:

$$z_\theta(T_o) = \frac{T_o(\phi)}{\Gamma}\left(1 - \left(\frac{T_o(\phi)}{\theta}\right)^{\frac{\Gamma}{g}}\right). \tag{11}$$

Using this equation and a sample set potential temperatures $\theta = (300, 320, 340, 360, 380, 400)$ [K], potential temperature isolines can be presented as seen in figure 11.

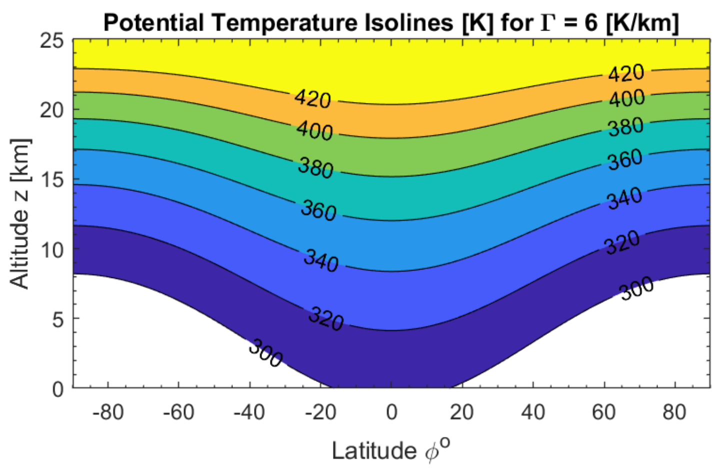

Figure 10. Potential temperature isolines for $\Gamma = 6$ [K/km]