Atmospheric Thermodynamics - Tutorial 3

Robert Ruta

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1 Saturation Vapour Pressure

Given the Clausius-Clapeyron equation:

$$\frac{de_s}{dT} = \frac{Le_s}{R_v T^2},\tag{1}$$

one can arrive at various formulation of the saturation vapour pressure $e_s(T)$, depending on how latent heat L varies with temperature T. Assuming that latent heat is a constant $L = L_{lv0} = 2501 \frac{\text{kJ}}{\text{kg}}$, we get e_s as:

$$e_s(T) = \frac{L_{lv0}}{R_v}\left(\frac{1}{T_0} - \frac{1}{T}\right).\tag{2}$$

If, however, the latent heat depends on temperature in accordance with the linear formula: $L_{lv}(T) = L_{lv0} - (c_l - c_{\text{pv}})(T - T_0)$, then e_s takes the form:

$$e_s(T) = e_{s0}\left(\frac{T_0}{T}\right)^{\frac{c_p - c_{pv}}{R_v}} \exp\left[\frac{L_{lv0} + T_0(c_l - c_{pv})}{R_v}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right].\tag{3}$$

Plotting Magnus-Tentens formula for e_s in terms of temperature t in °C,

$$e_s(t) = e_{s0}\exp\left(\frac{17.27t}{t + 237.7}\right),\tag{4}$$

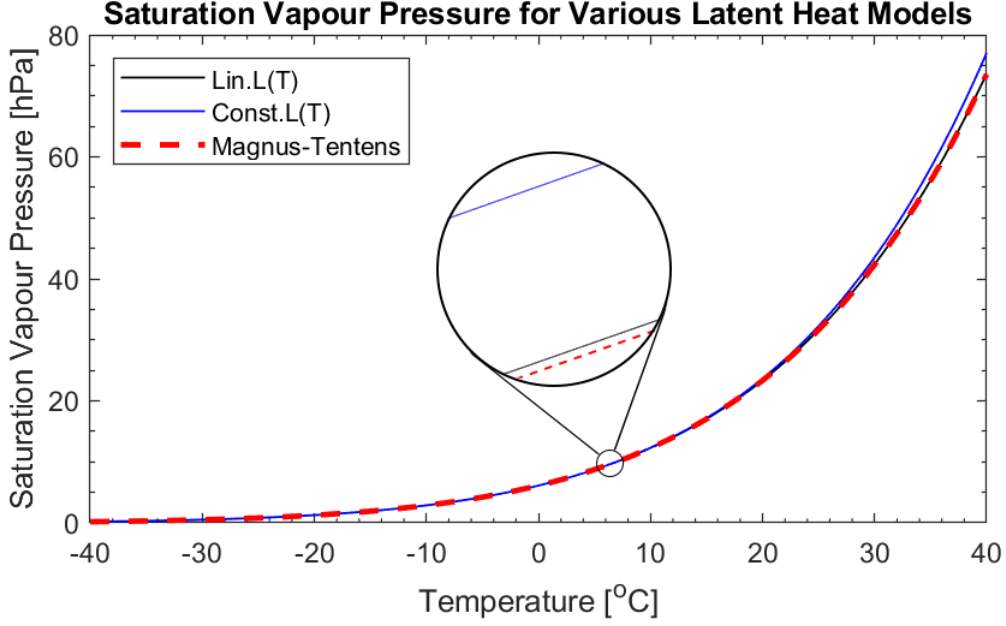alongside equations (2) and (3) yields the graph shown in Fig. 1.

Figure 1. Saturation vapour pressure as generated by various models.

The validity of models (2) and (3) are judged by looking at computing an R-square value in accordance with formula (7),

$$\text{SSE} = \sum_i (O_i - E_i)^2, \tag{5}$$

$$\text{SST} = \sum_i (O_i - \bar{E})^2, \tag{6}$$

$$\text{R}_{\text{sq}} = 1 - \frac{SSE}{SST}. \tag{7}$$

where E_i are the true values given by Magnus-Tenten (4), O_i are the values generated by the model being judged, and $\bar{O} = \text{mean}(O_i)$. The results are, $R_{sq,const} = 0.998129$ and $R_{sq,lin} = 0.999998$, for models (2) and (3) respectively. These results suggest that equation (2) and (3) would serve as excellent fit lines to (5), which is supported by Fig. 1.

Taking $L = \text{const} = L_{iv0} = 2834 \, \frac{\text{kJ}}{\text{kg}}$ and repeating the above process for temperatures relevant to sublimation yields the plots shown in Fig. 2.
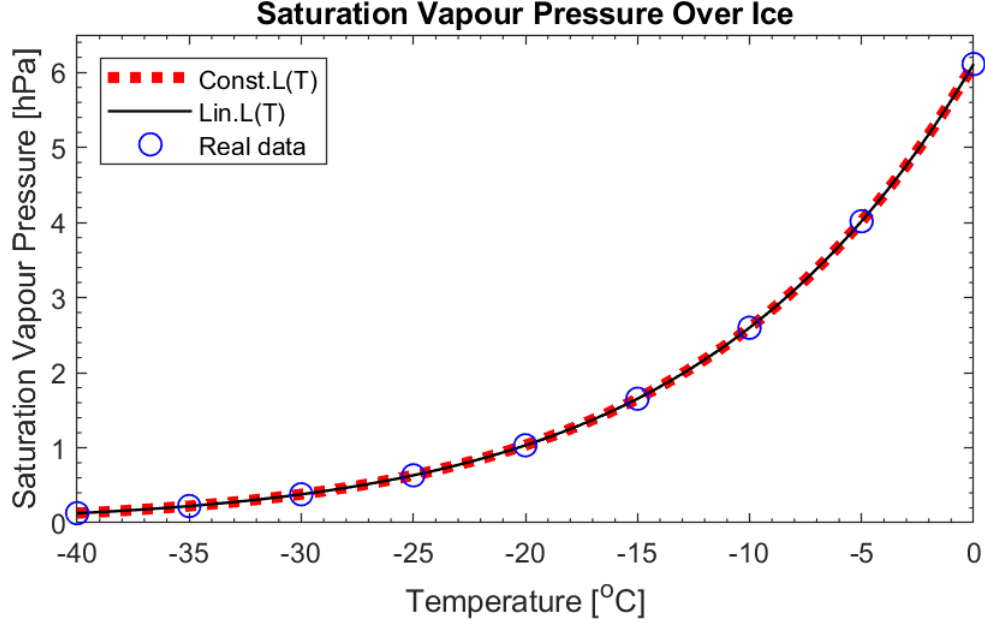
Figure 2. Saturation vapour pressure over ice.

In order to evaluate the validity of models (2) and (3) with a different latent heat constant, published measurements $e_{s,i}$ values (taken from M.K. Yau, R.R. Rogers - A Short Course in Cloud Physics, pg.16) are compared to the values provided by the models. Equation (7) is utilised once more, with E_i being the published data. The results yielded by models (2) and (3) are respectively: $R_{sq,const} = 0.999999$ and $R_{sq,lin} = 0.100000$.

2 Mixing Ratio and Specific Humidity

In order to quantify how moist air is, quantities such as vapour mixing ratio and specific humidity are devised. Mixing ratio is expressed as:

$$w = \epsilon \frac{e}{p - e} \approx \frac{e}{p}, \tag{8}$$

and specific humidity as:

$$q = \epsilon \frac{e}{p - (1 - \epsilon)e} \approx \frac{e}{p}. \tag{9}$$

If we examine these quantities for saturated conditions, one simply takes the vapour pressure e to be saturation vapour pressure e_s. Utilising the descriptions of $e_s(T)$, w and q can be expressed as a 2-dimensional function dependent on atmospheric pressure p and temperature T. This relationship can be presented via a contour plot, as seen in Fig.3 and Fig.4. The difference $d = w - q$ is plotted in Fig.5, providing insight on when specific humidity and mixing ratio can be used interchangeably when it comes to describing moist air.
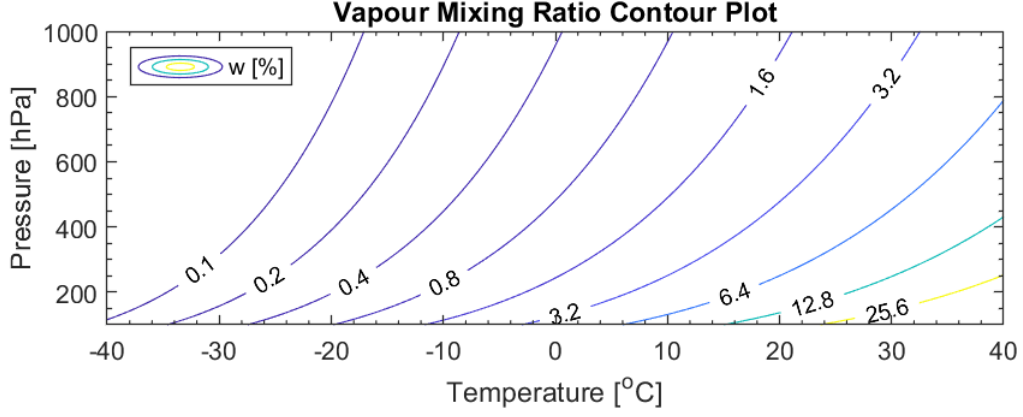
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Figure 3. Relationship between vapour mixing ratio, pressure, and temperature.
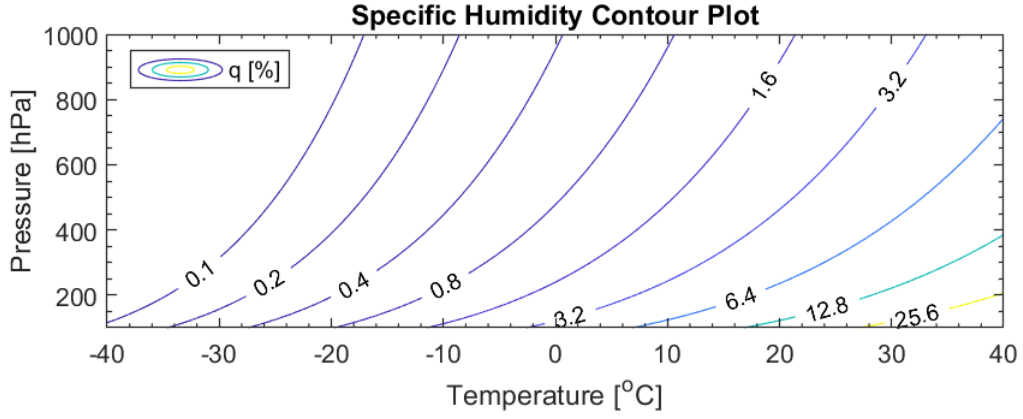

Figure 4. Relationship between specific humidity, pressure, and temperature.
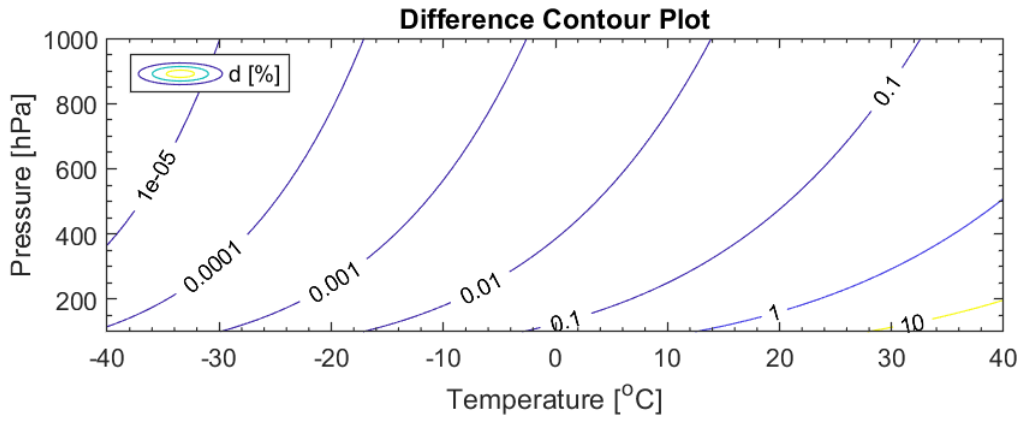

Figure 5. Relationship between the difference $d = w - q$, pressure and temperature.

Figure 5 suggests that the difference between specific humidity and mixing ratio begins to become meaningful for greater temperatures and lower pressures. Since atmospheric pressure decreases with altitude along with temperature, most of the atmosphere can be described by the left

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hand side of Fig.5, therefore the convention of interchangeability between w and q can be said to be justified.

Plotting mixing ratio and specific humidity against altitude, given knowledge of the environmental pressure profile and assuming a linearly decreasing temperature with altitude takes the forms shown in Fig.6 and Fig.7.
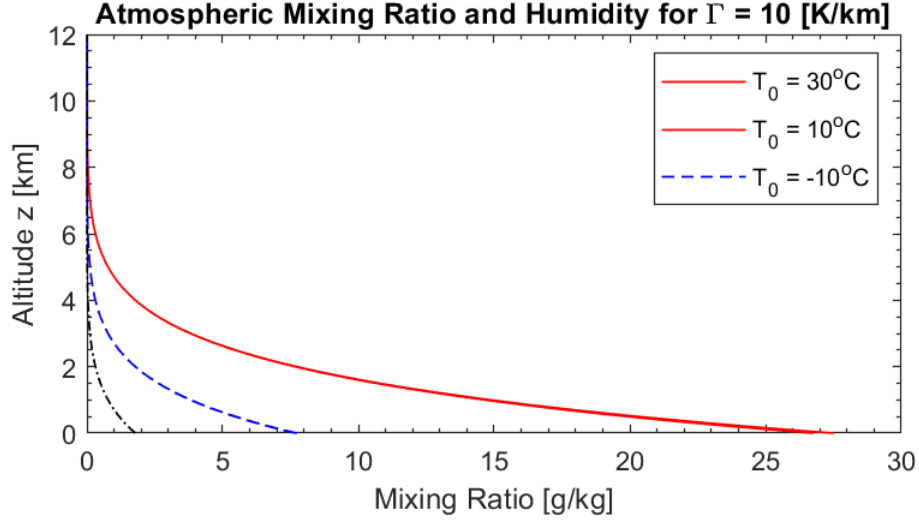

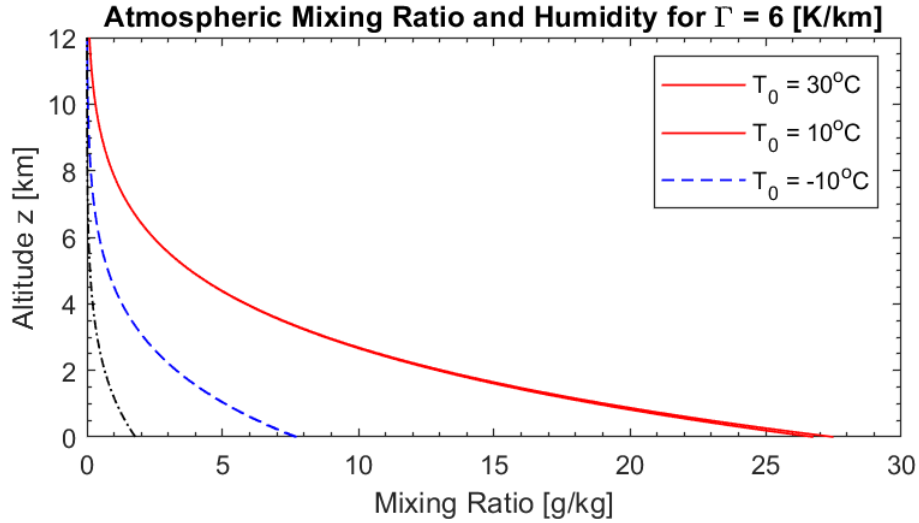**Figure 6.** Specific humidity and vapour mixing ratio environmental profile for $\Gamma = 10\,\mathrm{K/km}$.

Figure 7. Specific humidity and vapour mixing ratio environmental profile for $\Gamma = 6\,\mathrm{K/km}$.

The specific humidity and mixing ratio curves are almost perfectly superimposed at this scale, thus leaving them indistinguishable.