Radboud University

Commutative algebra and symmetric cryptography

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ESCADA

Introduction

- This presentation contains some work of my PhD research under ESCADA
- Main research goal: establishing deeper connections between commutative algebra and symmetric cryprography
- The results are based on three papers
- This presentation is split in three parts
- Each part corresponds to one of the three papers

Outline

Part I: Circulant rings and cyclic shifts

Part II: Semisimple circulant rings

Part III: General circulant rings

Part I: Circulant rings and cyclic shifts

Paper citation

Part I is based on the paper entitled:

An algebraic approach to circulant column parity mixers [Sub23]

Introducing circulant rings

- Let $\vec{m} := (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>0}$, and let k be a field
- We define the circulant ring

$$\mathcal{C}_{\vec{m}/k} := k[X_1, \ldots, X_n]/(X_1^{m_1} - 1, \ldots, X_n^{m_n} - 1)$$

- Circulant rings are interesting both within mathematics and cryptography
  - **Mathematics:** commutative group algebras over finite abelian groups are algebraically isomorphic to circulant rings
  - **Cryptography:** circulant rings are used in constructing the mixing layer of some crypographic primitives like XOODOO (although not in this form)

Univariate circulant rings: cyclic shifts - Part 1

- Let $V_{m/k}$ be the m-dimensional vector space over a field k indexed from 0 to $m-1$
- For $v \in V_{m/k}$, we denote the i-th entry of v by v_i
- Cyclic shift by s: moves entry position i to $i+s$
- We denote this map by $\lll s : V_{m/k} \to V_{m/k},\ v \mapsto v^*$ where $v^*_i = (v_{i-s \bmod m})$

- $\mathcal{C}_{m/k}$ encapsulates also the effect of cyclic shifts
- We have the bijection

$$t_m : V_{m/k} \to \mathcal{C}_{m/k},\ v \to \sum_{0 \le i < m} v_i \cdot X^i$$

- We have the commutative diagram:

$$\begin{array}{ccc} V_{m/k} & \xrightarrow{\lll s} & V_{m/k} \\ \downarrow{\scriptstyle t_m} & & \downarrow{\scriptstyle t_m} \\ \mathcal{C}_{m/k} & \xrightarrow{\cdot X^s} & \mathcal{C}_{m/k} \end{array}$$

- This is equivalent to the theory of circulant matrices!

- Let $V_{(m_1,m_2)/k}$ be an $m_1 \times m_2$ plane with entries in k indexed by (i_1, i_2), where $0 \leq i_1 < m_1$ and $0 \leq i_2 < m_2$
- For an element $v \in V_{(m_1,m_2)/k}$, we denote the entry at position (i_1, i_2) by $v_{(i_1,i_2)}$
- Cyclic shift by (s_1, s_2): moves entry in position (i_1, i_2) to $(i_1 + s_1, i_2 + s_2)$
- We denote this map by

$$\lll (s_1, s_2) : V_{(m_1,m_2)/k} \to V_{(m_1,m_2)/k}, \ v \mapsto v^*$$

where $v^*_{(i_1,i_2)} = v_{(i_1 - s_1 \bmod m_1, i_2 - s_2 \bmod m_2)}$

- $\mathcal{C}_{(m_1,m_2)/k}$ encapsulates also the effect of cyclic shifts
- We have the bijection

$$t_{(m_1,m_2)} : V_{(m_1,m_2)/k} \to \mathcal{C}_{(m_1,m_2)/k}, \ v \to \sum_{(0,0)\leq(i,j)<(m_1,m_2)} v_{(i_1,i_2)} \cdot X_1^{i_1} X_2^{i_2}$$

- We have the commutative diagram:

$$\begin{array}{ccc} V_{(m_1,m_2)/k} & \xrightarrow{\lll(s_1,s_2)} & V_{(m_1,m_2)/k} \\ \downarrow{\scriptstyle t_{(m_1,m_2)}} & & \downarrow{\scriptstyle t_{(m_1,m_2)}} \\ \mathcal{C}_{(m_1,m_2)/k} & \xrightarrow{\cdot X_1^{s_1} X_2^{s_2}} & \mathcal{C}_{(m_1,m_2)/k} \end{array}$$

- For $\vec{m} := (m_1, \ldots, m_n)$, define $V_{\vec{m}/k}$ as the vector space over k indexed by $(i_1, \ldots, i_n)$ where $0 \leq i_j < m_j$ for all $1 \leq j \leq n$
- Shifts by $\vec{s} := (s_1, \ldots, s_n)$ on the indices are defined similarly as earlier, and denoted by $\lll \vec{s}$
- We have the bijection

$$t_{\vec{m}} : V_{\vec{m}/k} \to \mathcal{C}_{\vec{m}/k}, \ v \to \sum_{(0,,\ldots,0) \leq (i_1,\ldots,i_n) < \vec{m}} v_{(i_1,\ldots,i_n)} \cdot \prod_{j=1}^{n} X_j^{i_j}$$

- We have the commutative diagram:

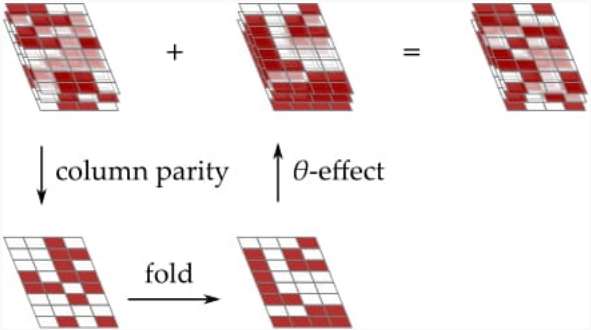
$$\begin{array}{ccc} V_{\vec{m}/k} & \xrightarrow{\lll \vec{s}} & V_{\vec{m}/k} \\ \downarrow{\scriptstyle t_{\vec{m}}} & & \downarrow{\scriptstyle t_{\vec{m}}} \\ \mathcal{C}_{\vec{m}/k} & \xrightarrow{\cdot \prod_{j=1}^{n} X_j^{s_j}} & \mathcal{C}_{\vec{m}/k} \end{array}$$

- Xoodoo is a cryptographic permutation, suitable for lightweight cryptography [DHAK18]
- We use the above insights to investigate the linear layer of Xoodoo from an alternative point of view
- The linear layer consists of a composition of 3 linear invertible maps

$$\rho_{\text{west}} \circ \theta \circ \rho_{\text{east}} : V^3_{(4,32)/\mathbb{F}_2} \to V^3_{(4,32)/\mathbb{F}_2}$$

- The construction of these maps relies heavily on cyclic shifts (next slide demonstrates θ)

$$P \leftarrow A_0 + A_1 + A_2$$
$$E \leftarrow P \lll (1,5) + P \lll (1,14)$$
$$A_y \leftarrow A_y + E, \qquad y \in \{0,1,2\}$$



- Consider the bijection $t^3_{(4,32)} : V^3_{(4,32)/\mathbb{F}_2} \to \mathcal{C}^3_{(4,32)/\mathbb{F}_2}$
- The maps $\rho_{\text{west}}, \theta, \rho_{\text{east}}$ induce corresponding maps

$$\rho^*_{\text{west}}, \theta^*, \rho^*_{\text{east}}, : \mathcal{C}^3_{(4,32)/\mathbb{F}_2} \to \mathcal{C}^3_{(4,32)/\mathbb{F}_2}$$

- Using a commutative diagram:

$$\begin{array}{ccc} V_{(4,32)/\mathbb{F}_2} & \xrightarrow{\rho_{\text{west}}, \theta, \rho_{\text{east}}} & V_{(4,32)/\mathbb{F}_2} \\ \downarrow t^3_{(4,32)} & & \downarrow t^3_{(4,32)} \\ \mathcal{C}_{(4,32)/\mathbb{F}_2} & \xrightarrow{\rho^*_{\text{west}}, \theta^*, \rho^*_{\text{east}}} & \mathcal{C}_{(4,32)/\mathbb{F}_2} \end{array}$$

- How do the maps $\rho^*_{\text{west}}, \theta^*, \rho^*_{\text{east}}$ look like?

Linear layer of Xoodoo - Module automorphism

- Because the cyclic shifts only happen on the level of planes, one can show that $\rho^*_{\text{west}}, \theta^*, \rho^*_{\text{east}}$ are linear under scaling over $\mathcal{C}_{(4,32)/\mathbb{F}_2}$
- Concretely, for $\mathbf{x}, \mathbf{y} \in \mathcal{C}^3_{(4,32)/\mathbb{F}_2}$ and $c \in \mathcal{C}_{(4,32)/\mathbb{F}_2}$:

$$\theta^*(\mathbf{x} + \mathbf{y}) = \theta^*(\mathbf{x}) + \theta^*(\mathbf{y})$$
$$\theta^*(c \cdot \mathbf{x}) = c \cdot \theta^*(\mathbf{x})$$

- Similarly for ρ^*_{east} and ρ^*_{west}
- In commutative algebra terms: these maps are $\mathcal{C}_{(4,32)/\mathbb{F}_2}$-module automorphisms over the free $\mathcal{C}_{(4,32)/\mathbb{F}_2}$-module of rank 3
- As a result, these maps have unique 3×3-matrix representations with entries in $\mathcal{C}_{(4,32)/\mathbb{F}_2}$!

- The linear maps has the following matrix representation:

$$\rho^*_{\text{west}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & X_1 & 0 \\ 0 & 0 & X_2^{11} \end{pmatrix} \quad \rho^*_{\text{east}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_1^2 X_2^8 \end{pmatrix} \quad \theta^* = \begin{pmatrix} 1+f & f & f \\ f & 1+f & f \\ f & f & 1+f \end{pmatrix}$$

  where $f = X_1 X_2^5 + X_1 X_2^{14}$

- This approach establishes deeper connections between the behaviour of linear layer of XOODOO, and the algebraic structure of $\mathcal{C}_{(4,32)/\mathbb{F}_2}$
- An example: Using this approach, we managed to provide a mathematical explanation why the order of this linear layer is low (32), using the algebraic properties of $\mathcal{C}_{(4,32)/\mathbb{F}_2}$

Part II: Semisimple circulant rings

Paper citation

Part II is based on the paper entitled:

Wedderburn decomposition of commutative semisimple group algebras using the Combinatorial Nullstellensatz [Sub24]

Intuition: Prime number factorization

- **Prime factorization theorem:** every integer has a unique prime factorization
- Prime numbers are the building blocks for all integers
- Knowing the prime factorization of an integer m gives a lot of information about m
- Can the same ideas be applied to rings?
  - How does one compose rings?
  - What type of rings serve as the "prime numbers"?
  - Which rings admits such a decomposition/factorization?
- **Note:** We only consider commutative rings

(Semi)simple rings

- **Simple rings:** A ring R is simple if it does not contain any proper two-sided ideals
  - Simple rings are natural candidates to serve as fundamental building blocks
  - Simple commutative rings are fields
- **Semisimple rings:** A ring R is semisimple if there exists finitely many simple rings $R_1, \ldots, R_n$ such that

$$R \cong R_1 \oplus R_2 \oplus \ldots \oplus R_n$$

- **Krull-Remak-Schmidt:** Such a decomposition is unique up to isomorphism!
- If a semisimple ring R is commutative, then R is a direct sum of fields

- Are all circulant rings $\mathcal{C}_{\vec{m}/k}$, with $\vec{m} := (m_1, \ldots, m_n)$, always semisimple? NO!
- When is $\mathcal{C}_{\vec{m}/k}$ semisimple?

Maschke's Theorem

$\mathcal{C}_{\vec{m}/k}$ is semisimple if and only if $m_1, \ldots, m_n$ are all coprime to the characteristic of k.

Questions:

- What are the simple components of a semisimple circulant ring $\mathcal{C}_{\vec{m}/k}$?
- What about the case when $k = \mathbb{F}_q$?

Geometric group action

Define $k_{\vec{m}} := k(\mu_{m_1}, \ldots, \mu_{m_n})$, and define $\mathcal{V}(\mathfrak{a}_{\vec{m}}) := \mu_{m_1} \times \cdots \times \mu_{m_n} \subset \overline{k}^n$. For $\mathbf{x} := (x_1, \ldots, x_n) \in \mathcal{V}(\mathfrak{a}_{\vec{m}})$ and $\sigma \in \mathrm{Gal}(k_{\vec{m}}/k)$, define $\sigma(\mathbf{x}) := (\sigma(x_1), \ldots, \sigma(x_n))$. The **Geometric group action** is defined as the group action

$$\alpha_{\vec{m}/k} : \mathrm{Gal}(k_{\vec{m}}/k) \times \mathcal{V}(\mathfrak{a}_{\vec{m}}) \to \mathcal{V}(\mathfrak{a}_{\vec{m}}), \ (\sigma, \mathbf{x}) \mapsto \sigma(\mathbf{x}).$$

Semisimple decomposition over arbitrary fields

Let $\mathcal{C}_{\vec{m}/k}$ be semisimple, and let $S \subset \mathcal{V}(\mathfrak{a}_{\vec{m}})$ be a set of representatives of the orbits of $\alpha_{\vec{m}/k}$. Then we have the isomorphism

$$\mathcal{C}_{\vec{m}/k} \to \bigoplus_{\mathbf{x} \in S} k(\mathbf{x}), \ f \mapsto (f(\mathbf{x}))_{\mathbf{x} \in S}$$

Notation:

- For an integer m, Div_m is the set of all divisors of m, incuding 1 and m
- For $\vec{m} := (m_1, \ldots, m_n)$, we define $\mathrm{Div}_{\vec{m}} := \mathrm{Div}_{m_1} \times \ldots \times \mathrm{Div}_{m_n}$
- For $\vec{d} := (d_1, \ldots, d_n) \in \mathrm{Div}_{\vec{m}}$, we define the following expressions:
  - $\nu_{\vec{m}/\mathbb{F}_q}(\vec{d}) := \mathrm{lcm}_{1 \le i \le n}(\mathrm{ord}_{m_i/d_i}(q))$
  - $\eta_{\vec{m}/\mathbb{F}_q}(\vec{d}) := \frac{\prod_{i=1}^{n} \varphi(m_i)}{\nu_{\vec{m}/\mathbb{F}_q}(\vec{d})}$ (φ is Euler's totient function)

Semisimple decomposition over finite fields

Let $\mathcal{C}_{\vec{m}/\mathbb{F}_q}$ be semisimple, then its decomposition equals

$$\mathcal{C}_{\vec{m}/\mathbb{F}_q} \cong \bigoplus_{\vec{d} \in \mathrm{Div}_{\vec{m}}} \left(\mathbb{F}_{q^{\nu_{\vec{m}/\mathbb{F}_q}(\vec{d})}} \right)^{\eta_{\vec{m}/\mathbb{F}_q}(\vec{d})}$$

An example: $\mathcal{C}_{(5,5)/\mathbb{F}_2}$

- Consider: $\mathcal{C}_{(5,5)/\mathbb{F}_2} := \mathbb{F}_2[X_1, X_2]/(X_1^5 - 1, X_2^5 - 1)$
- $\mathrm{Div}_{(5,5)} = \{(1,1), (1,5), (5,1), (5,5)\}$
- For every $\vec{d} \in \mathrm{Div}_{(5,5)}$, compute $\nu_{\vec{m}/\mathbb{F}_q}(\vec{d})$ and $\eta_{\vec{m}/\mathbb{F}_q}(\vec{d})$:

$\vec{d}$	$\nu_{(5,5)/\mathbb{F}_2}(\vec{d})$	$\eta_{(5,5)/\mathbb{F}_2}(\vec{d})$
$(1,1)$	1	1
$(1,5)$	4	1
$(5,1)$	4	1
$(5,5)$	4	4

- $\mathcal{C}_{(5,5)/\mathbb{F}_2} \cong \mathbb{F}_2 \oplus \mathbb{F}_{2^4} \oplus \mathbb{F}_{2^4} \oplus (\mathbb{F}_{2^4})^4 = \mathbb{F}_2 \oplus (\mathbb{F}_{2^4})^6$

Part III: General circulant rings

Paper citation

Part III is based on the paper entitled:

The Krull-Remak-Schmidt decomposition of commutative group algebras

(citation TBA)

- Semisimple circulant rings can be decomposed in simple rings
- Can we do something similar for non-semisimple circulant rings?

(But we need a more general notion of simple rings....)

Indecomposable rings

- **Indecomposable ring:** A ring R is indecomposable if there does not exist other rings R_1 and R_2 such that $R = R_1 \oplus R_2$
- All simple rings are indecomposable
- Reverse statement not true: local rings are not simple, but indecomposable

Krull-Remak-Schmidt

Let R be an Artinian ring, hence also Noetherian. Then there exist indecomposable rings $R_1, \ldots, R_n$ such that

$$R \cong R_1 \oplus \ldots \oplus R_n.$$

This decomposition is unique up to isomorphism.

- Circulant rings are Artinian and Noetherian [Con63]

- When are circulant rings indecomposable?

Indecomposable circulant rings

A circulant ring $\mathcal{C}_{\vec{m}/k}$, with $\vec{m} := (m_1, \ldots, m_n)$, is indecomposable if and only if $m_1, \ldots, m_n$ are all powers of the characteristic of k

- Right-side implication is not too complicated to prove, however the reverse statement is not as immediate
- **Example:** $\mathcal{C}_{(4,32)/\mathbb{F}_2} := \mathbb{F}_2[X_1, X_2]/(X_1^4 - 1, X_2^{32} - 1)$ is indecomposable

Additional notation

- Let $m \in \mathbb{Z}_{>0}$ and p a prime number
  - $v_p(m) := \max\{j \in \mathbb{Z}_{\geq 0} : p^j \mid m\}$ (p-adic valuation)
  - $r_p(m) := m/p^{v_p(m)}$
- Let $\vec{m} := (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>0}$ and p a prime number
  - $v_p(\vec{m}) := (v_p(m_1), \ldots, v_p(m_n)) \in \mathbb{Z}^n_{\geq 0}$
  - $r_p(\vec{m}) := (r_p(m_1), \ldots, r_p(m_n)) \in \mathbb{Z}^n_{\geq 0}$

Krull-Remak-Schmidt of circulant rings over arbitrary field

Let $\mathcal{C}_{\vec{m}/k}$ be any circulant ring over a field k with characteristic p, and let $S \subset \mathcal{V}(\mathfrak{a}_{r_p(\vec{m})})$ be a set of representatives of the orbits of $\alpha_{r_p(\vec{m})/k}$.

For p prime:

$$\mathcal{C}_{\vec{m}/k} \cong \bigoplus_{\mathbf{x} \in S} \left(k(\mathbf{x})[Y_1, \ldots, Y_n]/(Y_1^{p^{v_p(m_1)}} - 1, \ldots, Y_n^{p^{v_p(m_n)}} - 1) \right).$$

For $p = 0$:

$$\mathcal{C}_{\vec{m}/k} \cong \bigoplus_{\mathbf{x} \in S} k(\mathbf{x}).$$

Krull-Remak-Schmidt of circulant rings over finite fields

Let $\mathcal{C}_{\vec{m}/\mathbb{F}_q}$ be any circulant ring over $\mathbb{F}_q$ with characteristic p.

Define the expressions:

- $\nu_{\vec{m}/\mathbb{F}_q}(\vec{d}) := \mathrm{lcm}_{1\le i\le n}(\mathrm{ord}_{r_p(m_i)/d_i}(q))$
- $\eta_{\vec{m}/\mathbb{F}_q}(\vec{d}) := \frac{\prod_{i=1}^{n}\varphi(r_p(m_i))}{\nu_{\vec{m}/\mathbb{F}_q}(\vec{d})}$

Then we have the decomposition

$$\mathcal{C}_{\vec{m}/\mathbb{F}_q} \cong \bigoplus_{\vec{d}\in \mathrm{Div}_{r_p(\vec{m})}} \left(\mathbb{F}_{q^{\nu_{\vec{m}/\mathbb{F}_q}(\vec{d})}}[Y_1,\ldots,Y_n]/(Y_1^{p^{\nu_p(m_1)}}-1,\ldots,Y_n^{p^{\nu_p(m_n)}}-1)\right)^{\nu_{\vec{m}/\mathbb{F}_q}(\vec{d})}.$$

An example: $\mathcal{C}_{(5,5,2^l)/\mathbb{F}_2}$

- Consider $\mathcal{C}_{(5,5,2^l)/\mathbb{F}_2} := \mathbb{F}_2[X_1, X_2, X_3]/(X_1^5 - 1, X_2^5 - 1, X_3^{2^l} - 1)$ for some $l \in \mathbb{Z}_{>0}$
- This structure is used in KECCAK-f related to θ
- Observe: $\mathcal{C}_{r_2(5,5,2^l)/\mathbb{F}_2} = \mathcal{C}_{r_2(5,5,1)/\mathbb{F}_2} \cong \mathcal{C}_{r_2(5,5)/\mathbb{F}_2} \cong \mathbb{F}_2 \oplus (\mathbb{F}_{2^4})^6$
- The indecomposable components of $\mathcal{C}_{(5,5,2^l)/\mathbb{F}_2}$ are up to isomorphism:
  - $\mathbb{F}_2[Y_1, Y_2, Y_3]/(Y_1 - 1, Y_2 - 1, Y_3^{2^l} - 1) \cong \mathbb{F}_2[Y]/(Y^{2^l} - 1)$
  - $\mathbb{F}_{2^4}[Y_1, Y_2, Y_3]/(Y_1 - 1, Y_2 - 1, Y_3^{2^l} - 1) \cong \mathbb{F}_{2^4}[Y]/(Y^{2^l} - 1)$
- We get the decomposition

$$\mathcal{C}_{(5,5,2^l)/\mathbb{F}_2} \cong \left(\mathbb{F}_2[Y]/(Y^{2^l} - 1)\right) \oplus \left(\mathbb{F}_{2^4}[Y]/(Y^{2^l} - 1)\right)^6$$

Conclusion

Conclusion

- We presented an algebraic framework which is suitable for investigating circulant-like linear maps
- Can these insights be used to develop new cryptanalytic techniques?
- And maybe also design “better” linear layers based on these findings?

Thank you for your attention!

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