Solutions to Application problems from Activity 2.1

1. Find the slope of the tangent line to the graph of $h(z) = \sqrt{z} + \frac{1}{z}$ at the point where $z = 4$.

   Solution: The slope of the tangent line is the derivative of h at the point $z = 4$. First, rewrite h to make the root and fraction into exponents:

   $$h(z) = z^{1/2} + z^{-1}$$

   Now take the derivative using the rule for power functions:

   $$h'(z) = \frac{1}{2}z^{-1/2} - z^{-2}$$

   Now evaluate at $z = 4$:

   $$\begin{aligned} h'(4) &= \frac{1}{2}(4)^{-1/2} - 4^{-2} \\ &= \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{16} \\ &= \frac{3}{16}. \end{aligned}$$

2. A population of cells is growing in such a way that its total number (in millions) is given by the function $P(t) = 2(1.37)^t + 32$, where t is measured in days. Find the instantaneous rate at which the population is growing on day 4, and include correct units on your answer.

   Solution: The rate of change is the derivative at $t = 4$. So first take the derivative with respect to t:

   $$\begin{aligned} P'(t) &= \frac{d}{dt}[2(1.37)^t + 32] \\ &= \frac{d}{dt}[2(1.37)^t] + \frac{d}{dt}[32] \quad \text{(Sum rule)} \\ &= 2\frac{d}{dt}[(1.37)^t] + \frac{d}{dt}[32] \quad \text{(Constant multiple rule)} \\ &= 2\frac{d}{dt}[(1.37)^t] + 0 \quad \text{(Derivative of a constant)} \\ &= 2((1.37)^t \ln 1.37) \quad \text{(Derivative of an exponential function)} \end{aligned}$$

   And now evaluate at $t = 4$:

   $$P'(4) = 2((1.37)^4 \ln 1.37) \approx 2.281 \text{ million cells per day}$$

3. Find an equation for the tangent line to the curve $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$ at the point where $a = -1$.

   Solution: To find the equation for the tangent line at $a = -1$, we need the derivative of p at $a = -1$ for the slope, along with the point where the tangent line touches the graph of p at $a = -1$. First let's get the derivative:

   $$p'(a) = 12a^3 - 6a^2 + 14a - 1$$

   Therefore

   $$p'(-1) = 12(-1)^3 - 6(-1)^2 + 14(-1) - (-1) = -12 - 6 - 14 - 1 = -33$$

This is the slope of the tangent line at $a = -1$. The point where the tangent line touches the graph of p is $(-1, p(-1))$, and

$$p(-1) = 3(-1)^4 - 2(-1)^3 + 7(-1)^2 + 12 = 25$$

Therefore the tangent line equation is:

$$y = -33(x - (-1)) + 25 = -33x + -33 + 25 = -33x - 8.$$